Research Article

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A self-adaptive Tseng extragradient method for solving monotone variational inequality and fixed point problems in Banach spaces

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Abstract: In this paper, we introduce a self-adaptive projection method for finding a common element in the solution set of variational inequalities (VIs) and fixed point set for relatively nonexpansive mappings in 2-uniformly convex and uniformly smooth real Banach spaces. We prove a strong convergence result for the sequence generated by our algorithm without imposing a Lipschitz condition on the cost operator of the VIs. We also provide some numerical examples to illustrate the performance of the proposed algorithm by comparing with related methods in the literature. This result extends and improves some recent results in the literature in this direction.

Keywords: variational inequality, monotone operator, self-adaptive, projection method, Banach spaces

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1 Introduction

Let C be a nonempty closed convex subset of a real Banach space E with dual E^* . We denote the duality pairing between E and E^* by $\langle \cdot, \cdot \rangle$. Let $A : C \rightarrow E^*$ be an operator, the variational inequality (VI) with respect to A is formulated as

$$\text{Find } x^+ \in C \text{ such that } \langle Ax^+, y - x^+ \rangle \geq 0, \quad \forall y \in C. \quad (1.1)$$

We denote the solution set of VI (1.1) by $VI(C, A)$. The VI plays a major role as a modelling tool for studying various real life problems such as contact problem, obstacle problem, traffic equilibrium problem and so on (see [1–7]). Several authors have studied the VI in recent years and some notable iterative methods have been introduced for finding its solutions; see, for instance [8–13]. One of the most popular methods for solving the VI is the extragradient method [14], which is based on double projections onto the feasible set C per each iteration. In the case where C is not so simple to project onto, the extragradient method becomes complicated and computationally expensive to implement. Censor et al. [15,16] introduced the subgradient extragradient method (SEM) as a way of overcoming this setback. In particular, they proposed the following algorithm by defining a projection onto a half-space T_n instead of a projection onto the whole feasible set C : let H be real Hilbert space, C be a nonempty closed convex subset of H and $A : C \rightarrow H$ be a monotone and L -Lipschitz continuous operator. Given $x_0 \in H$, compute x_{n+1} by

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$$\begin{cases} x_0 \in C, \lambda > 0, \\ y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_{T_n}(x_n - \lambda Ay_n), \\ \text{where } T_n = \{x \in H : \langle x_n - Ax_n - y_n, x - y_n \rangle \leq 0\}, \quad n \geq 0, \end{cases}$$

where P_C is the metric projection from H onto C . More so, the authors proved that the SEM converges weakly to a solution of VI provided the stepsize λ satisfies $\lambda \in \left(0, \frac{1}{L}\right)$, where L is the Lipschitz constant of A .

Recently, many authors have developed several modifications of the extragradient method and SEM, see for instance [17–22]. One of such modifications of the extragradient method is the single projection method introduced by Tseng [23] as follows:

$$\begin{cases} x_0 \in C, \lambda > 0, \\ y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = y_n - \lambda(Ay_n - Ax_n), \quad n \geq 0. \end{cases} \quad (1.2)$$

It was proved that (1.2) converges weakly to a solution of the VI if the stepsize satisfies $\lambda \in \left(0, \frac{1}{L}\right)$. Note that (1.2) is more efficient than the extragradient method and its modifications due to its single projection onto the feasible set C per each iteration. Moreover, the convergence of SEM and iterative scheme (1.2) depend on the prior estimate of the Lipschitz constant L of the cost operator, which is very difficult to estimate in practice.

On the other hand, the problem of finding a common solution of VI and fixed point problem (FPP) (i.e., finding $x^* \in C$ such that $x^* \in \text{VI}(C, A) \cap \text{Fix}(S)$, where $\text{Fix}(S) = \{x \in C : x = Sx\}$ and $S : C \rightarrow C$ is a non-linear mapping) is very important in convex and nonlinear analysis due to the optimization problems whose constraints can be modelled as VI and FPP. This happened, in particular, in problems such as network distribution, image processing, generalized Nash equilibrium, etc. [1,2,4,5]. Recently, Chidume and Nnakwe [24] introduced a modified SEM for finding common solution of VI and FPP in 2-uniformly convex real Banach space E as follows:

$$\begin{cases} x_0 \in E, \quad \lambda \in \left(0, \frac{1}{L}\right), \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda Ax_n), \\ T_n = \{x \in E : \langle Jx_n - \lambda Ax_n - Jy_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = J^{-1}(\beta Jx_n + (1 - \beta)JS\Pi_{T_n}J^{-1}(Jx_n - \lambda Ay_n)), \quad \forall n \geq 0, \end{cases} \quad (1.3)$$

where $\beta \in (0, 1)$, Π_C is the generalized projection from E onto C and J is the normalized duality mapping from E to 2^{E^*} . They proved that the sequence generated by this algorithm converges weakly to a common solution of the VIP and fixed point of S , where S is a relatively nonexpansive mapping. In order to obtain strong convergence of the SEM in Banach space, Liu [25] combined the Halpern method [26] with the SEM and introduced the following SEM:

$$\begin{cases} x_0 \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ T_n = \{x \in E : \langle Jx_n - \lambda_n Ax_n - Jy_n, x - y_n \rangle \leq 0\}, \\ w_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda_n Ay_n), \\ x_{n+1} = J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)Jw_n), \quad k \geq 0, \end{cases} \quad (1.4)$$

where $\{\alpha_n\} \subset [0, 1]$ satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\lambda_n\} \subset (0, \infty)$ and $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < \frac{1}{L}$. Other similar methods for finding common solution of VI and FPP in Hilbert and Banach spaces can be found in, for instance, [1–5,16,27–31]. An obvious disadvantage of the aforementioned algorithms which impedes their wide usage is the assumption that the Lipschitz constant of the operator A is known or admits a simple estimate. More so, in many practical examples, the cost operators may not even satisfy the Lipschitz condition (see, e.g., Example 3.12 of [32]). Recently, Ma [33] proposed a self-adaptive sub-

gradient extragradient method for solving the VI monotone in 2-uniformly convex Banach space without prior knowledge of the Lipschitz constant as follows:

$$\begin{cases} \text{Given } \lambda_0 > 0, \mu \in (0, 1), x_0 \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda_n Ay_n), \\ \text{where } T_n = \{x \in E : \langle Jx_n - \lambda_n Ax_n - Jy_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)Jz_n), \quad n \geq 0, \end{cases} \quad (1.5)$$

where the stepsize λ_n is updated by using

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\mu(\|x_n - y_n\|^2 + \|z_n - y_n\|^2)}{2\langle Ax_n - Ay_n, z_n - y_n \rangle}, \lambda_n \right\} & \text{if } \langle Ax_n - Ay_n, z_n - y_n \rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases}$$

The author proved a strong convergence theorem using (1.5) with the assumption that $A : E \rightarrow E^*$ is monotone and L -Lipschitz continuous. It should be observed that (1.5) improves (1.3) and (1.4); however, it incurred some setbacks such as A satisfying Lipschitz continuity assumption before its convergence is guaranteed. More so, the authors [34] further introduced the following double projection algorithms using a line search technique for selecting the appropriate stepsize:

$$\begin{cases} \text{Choose: } \gamma \in (0, 1), \sigma \in (0, 1), \rho > 0, \{\alpha_n\}, \{\beta_n\} \subset (0, 1), \quad \text{and } x_1 \in C. \\ \text{Compute:} \\ r(x_n) = x_n - P_C(x_n - Ax_n), \\ y_n = (1 - \eta_n)x_n + \eta_n P_C(x_n - Ax_n), \\ \text{where the stepsize } \eta_n = \rho \gamma^{m_n} \quad \text{and} \\ \langle Ay_n, r(x_n) \rangle \geq \frac{\sigma}{2} \|r(x_n)\|^2, \\ \text{where} \\ \lambda_n = \frac{\langle Ay_n, x_n - y_n \rangle}{\|Ax_n\|^2}, \\ x_{n+1} = \beta_n x_1 + (1 - \beta)Sz_n, \quad n \geq 1, \end{cases} \quad (1.6)$$

where $S : H \rightarrow H$ is a quasi-nonexpansive mapping. They also proved that the sequence generated by (1.6) converges strongly to a solution of the VI and FPP in real Hilbert spaces.

Motivated by the above results, in this paper, we introduce a new self-adaptive projection method for finding a common solution of the VIs and FPP in 2-uniformly convex and uniformly smooth Banach space. Our algorithm performs only a single projection onto the feasible set and no other projection onto any half-space in each iteration. Moreover, a strong convergence result is proved without imposing any Lipschitz condition on the cost operator of the VIs. We also provide some numerical computations to show the accuracy and efficiency of our proposed method with respect to (1.4) and (1.5). This result improves and extends several recent results such as [17,24,25,27,31,33,35–38] in the literature.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and dual E^* . We denote the strong convergence of the sequence $\{x_n\} \subset E$ to a point $x \in E$ by $x_n \rightarrow x$ and the weak convergence by $x_n \rightharpoonup x$.

Let $1 < q \leq 2$ with $\frac{1}{p} + \frac{1}{q} = 1$. The modulus of convexity $\delta_E : [0, 2] \rightarrow [0, 1]$ is defined by

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}.$$

E is said to be uniformly convex if $\delta_E(\varepsilon) > 0$ for any $\varepsilon \in (0, 2]$; p -uniformly convex if there exists a constant $c_p > 0$ such that $\delta_E(\varepsilon) \geq c_p \varepsilon^p$ for any $\varepsilon \in (0, 2]$. It is known that the spaces L^p , ℓ^p , W^p are p -uniformly convex when $p > 2$ and 2-uniformly convex when $1 < p \leq 2$. More so, any Hilbert Space H is uniformly convex and 2-uniformly convex. The modulus of smoothness $\rho_E : [0, \infty) \rightarrow [0, \infty)$ is defined by

$$\rho_E(\tau) = \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1 \right\},$$

E is called smooth if $\rho_E(\tau) > 0$ for any $\tau > 0$ and uniformly smooth if $\lim_{\tau \rightarrow 0} \frac{\rho_E(\tau)}{\tau} = 0$.

The normalized duality mapping $J : E \rightarrow 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2\}.$$

It is known that the normalized duality has the following properties; see, for instance, [39,40].

- (i) If E is a smooth Banach space, then J is single-valued.
- (ii) If E is a strictly convex Banach space, then J is one-to-one and strictly monotone.
- (iii) If E is a uniformly smooth Banach space, then J is uniformly norm-to-norm continuous on each bounded subset of E .
- (iv) If E is a smooth, strictly convex and reflexive Banach space, then J is single-valued, one-to-one and onto.

Let E be a smooth, strictly convex and reflexive Banach space. We defined the Lyapunov functional $\phi : E \times E \rightarrow [0, \infty)$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \quad (2.1)$$

From (2.1), it is easy to see that

- (i) for any $x, y \in E$,

$$\phi(x, y) + \phi(y, x) = 2\langle x - y, Jx - Jy \rangle, \quad (2.2)$$

- (ii) for any $x, y, z \in E$,

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle, \quad (2.3)$$

- (iii) for any $x, y, z \in E$ and $\lambda \in [0, 1]$,

$$\phi(x, J^{-1}(\lambda Jy + (1 - \lambda)Jz)) \leq \lambda \phi(x, y) + (1 - \lambda)\phi(x, z). \quad (2.4)$$

Following Alber [41], we define the function $V : E \times E^* \rightarrow \mathbb{R}$ given by

$$V(x, x^*) = \|x\|^2 - \langle x, x^* \rangle + \|x^*\|^2. \quad (2.5)$$

It follows from the definition of the Lyapunov functional ϕ (2.1) that

$$V(x, x^*) = \phi(x, J^{-1}x^*) \quad \forall x \in E, x^* \in E^*.$$

The generalized projection $\Pi_C : E \rightarrow C$ is defined by

$$\Pi_C(x) = \operatorname{argmin}_{y \in C} \phi(x, y). \quad (2.6)$$

We note that when E is a real Hilbert space H , then $\phi(x, y) = \|x - y\|^2$ and the generalized projection Π_C is the metric projection $P_C : H \rightarrow C$. We need the following properties of ϕ and Π_C in the sequel.

Lemma 2.1. [42] *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty, closed and convex subset of E . Let $x \in E$ and $z \in C$, then the following statements are equivalent:*

- (i) $z = \Pi_C(x)$;
- (ii) $\langle z - y, Jx - Jz \rangle \geq 0$, for all $y \in C$;
- (iii) $\phi(y, z) + \phi(z, x) \leq \phi(y, x)$, for all $y \in C$.

The following lemma was proved for the case of a metric projection in [22] and can also be extended to generalized projection.

Lemma 2.2. *For every $x \in E$ and $\alpha \geq \beta > 0$, the following inequalities hold:*

$$\frac{\|x - P_C(x - \alpha Ax)\|}{\alpha} \leq \frac{\|x - P_C(x - \beta Ax)\|}{\beta} \quad (2.7)$$

and

$$\|x - P_C(x - \beta Ax)\| \leq \|x - P_C(x - \alpha Ax)\|.$$

In the sequel, we assume that C is a nonempty, closed and convex subset of E . Let $S : C \rightarrow C$ be a mapping. A point $x \in C$ is called a fixed point of S if $Sx = x$. A point $p \in C$ is said to be an asymptotic fixed point of S if there exists $\{x_n\}$ in C which converges weakly to p and $\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0$. We denote the set of fixed points and asymptotic fixed points of S by $\text{Fix}(S)$ and $\overline{\text{Fix}}(S)$, respectively. A mapping $S : C \rightarrow C$ is called

- (i) nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$,
- (ii) quasi-nonexpansive if $\text{Fix}(S) \neq \emptyset$ and

$$\|Sx - p\| \leq \|x - p\| \quad \forall x \in C, p \in \text{Fix}(S),$$

- (iii) quasi- ϕ -nonexpansive (see [43]) if $\text{Fix}(S) \neq \emptyset$ and

$$\phi(p, Sx) \leq \phi(p, x) \quad \forall x \in C, p \in \text{Fix}(S),$$

- (iv) relatively-nonexpansive (see [44]) if $\text{Fix}(S) \neq \emptyset$ and

$$\phi(p, Sx) \leq \phi(p, x) \quad \forall x \in C, p \in \text{Fix}(S), \quad \text{and} \quad \text{Fix}(S) = \overline{\text{Fix}}(S).$$

Definition 2.3. Let C be a nonempty, closed and convex subset of E . An operator $A : C \rightarrow E^*$ is called

- (a) strongly monotone with constant $\gamma > 0$ if

$$\langle Ax - Ay, x - y \rangle \geq \gamma \|x - y\|^2 \quad \forall x, y \in C;$$

- (b) monotone on C if

$$\langle Ax - Ay, x - y \rangle \geq 0 \quad \forall x, y \in C;$$

- (c) Lipschitz continuous on C if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L \|x - y\| \quad \forall x, y \in C.$$

It is easy to see that (a) $\Rightarrow$ (b), but converse implication is not always true. Some examples of monotone operators in Banach spaces can be found in, for instance, [45].

The following lemmas will be needed in the sequel.

Lemma 2.4. [46] *Let E be a 2-uniformly convex and smooth Banach space. Then for every $x, y \in E$*

$$\phi(x, y) \geq \tau \|x - y\|^2,$$

where $\tau > 0$ is the 2-uniformly convexity constant of E .

Lemma 2.5. [47] *Let E be a 2-uniformly smooth Banach space. Then for all $x, y \in E$*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx \rangle + 2\kappa^2 \|y\|^2,$$

where κ is the 2-uniformly smoothness constant of E .

Lemma 2.6. [41] Let E be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of E . If $\phi(y_n, z_n) \rightarrow 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $\|y_n - z_n\| \rightarrow 0$.

Lemma 2.7. [40] Let E be a reflexive, strictly convex and smooth Banach space with dual E^* . Then

$$V(x, x^*) + 2\langle J^{-1}x - x^*, y^* \rangle \leq V(x, x^* + y^*),$$

for all $x \in E, x^*, y^* \in E^*$.

Lemma 2.8. [47] Let E be a uniformly convex Banach space and $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \rightarrow [0, +\infty)$ such that $g(0) = 0$ and

$$\|\alpha x + (1 - \alpha)y\|^2 \leq \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)g(\|x - y\|)$$

for all $\alpha \in [0, 1]$ and $x, y \in B_r = \{z \in E : \|z\| \leq r\}$.

Next, we give some results which will be used to establish the convergence of iterates to an element in the solution set.

Lemma 2.9. [48] Let C be a nonempty, closed and convex subset of E . Let $A : C \rightarrow E^*$ be a continuous, monotone mapping and $z \in C$, then

$$z \in \text{VI}(C, A) \quad \text{if and only if} \quad \langle Ax, x - z \rangle \geq 0, \quad \forall x \in C.$$

Lemma 2.10. [49] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\delta_n, \quad n \geq 0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n \rightarrow \infty} \delta_n \leq 0$. Then $\lim_{n \rightarrow \infty} s_n = 0$.

Lemma 2.11. [50] Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ with $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Consider the integer $\{m_k\}$ defined by

$$m_k = \max\{j \leq k : a_j < a_{j+1}\}.$$

Then $\{m_k\}$ is a nondecreasing sequence verifying $\lim_{n \rightarrow \infty} m_n = \infty$, and for all $k \in \mathbb{N}$, the following estimates hold:

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

3 Main results

In this section, we introduce a single projection method with Halpern iterative method for finding common solution of VI and FPP in real Banach spaces. We first give the statement of our algorithm as follows:

Assumption 3.1

- (A1) The feasible set C is a nonempty, closed and convex subset of a 2-uniformly convex real Banach space;
- (A2) $A : E \rightarrow E^*$ is a monotone and uniformly continuous operator;
- (A3) $S : E \rightarrow E$ is a relatively nonexpansive mapping;
- (A4) The solution set $\Gamma := \text{VI}(C, A) \cap \text{Fix}(S)$ is nonempty;
- (A5) The control parameters $\{\beta_n\}$ and $\{\delta_n\}$ satisfy the following:
 - (i) $\{\beta_n\} \subset (0, 1)$, $\lim_{n \rightarrow \infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = +\infty$;
 - (ii) $\{\delta_n\} \subset (0, 1)$ and $0 < \liminf_{n \rightarrow \infty} \delta_n \leq \limsup_{n \rightarrow \infty} \delta_n < 1$.

We now present our iterative algorithm and its convergence analysis.

Algorithm 3.2.

Step 0: Select $\gamma > 0$, $l \in (0, 1)$, $\mu \in (0, 1)$ and arbitrary initial point $x_1 \in E$. Set $n = 1$.

Iterative steps: Given x_n , calculate x_{n+1} as follows:

Step 1: Compute

$$y_n = \Pi_C J^{-1}(Jx_n - \alpha_n Ax_n), \quad (3.1)$$

where $\alpha_n = \gamma l^{k_n}$ with k_n being the smallest nonnegative integer k satisfying

$$\gamma l^k \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|. \quad (3.2)$$

If $x_n - y_n = 0$, set $x_n = z_n$ and go to Step 3. Else, do Step 2.

Step 2: Compute

$$z_n = J^{-1}(Jy_n - \alpha_n(Ay_n - Ax_n))$$

and

$$x_{n+1} = J^{-1}(\beta_n Jx_1 + (1 - \beta_n)(\delta_n Jz_n + (1 - \delta_n)JSz_n)). \quad (3.3)$$

Set $n \leftarrow n + 1$ and go to Step 1.

Remark 3.3. Since E is 2-uniformly convex, then E^* is 2-uniformly smooth Banach space. Also, if $x_n - y_n = 0$, then we are at a solution of the VI. In our convergence analysis, we will implicitly assume that this does not occur after finitely many iterations, so that Algorithm 3.2 generates infinite sequences.

We begin our convergence analysis by showing that the Armijo-line search rule defined in (3.2) is well defined. Consequently, Algorithm 3.2 is well defined.

Lemma 3.4. *There exists a nonnegative integer k satisfying (3.2). In addition, $0 < \alpha_n \leq \gamma$.*

Proof. If $x_n \in \text{VI}(C, A)$, then $x_n = \Pi_C(J^{-1}(Jx_n - \alpha_n Ax_n))$ and $k_n = 0$. Hence, we consider the case where $x_n \notin \text{VI}(C, A)$ and assume the contrary, i.e., for $k > 0$,

$$\gamma l^k \|Ax_n - A\Pi_C(J^{-1}(Jx_n - \gamma l^k Ax_n))\| > \mu \|x_n - \Pi_C(J^{-1}(Jx_n - \gamma l^k Ax_n))\|.$$

This implies that

$$\|Ax_n - A\Pi_C(J^{-1}(Jx_n - \gamma l^k Ax_n))\| > \frac{\|x_n - \Pi_C(J^{-1}(Jx_n - \gamma l^k Ax_n))\|}{\gamma l^k}. \quad (3.4)$$

Next, we consider two possibilities, namely, when $x_n \in C$ and when $x_n \notin C$.

First, if $x_n \in C$, then $x_n = \Pi_C(x_n)$. Since Π_C and A are continuous, then

$$\lim_{k \rightarrow \infty} \|x_n - \Pi_C(J^{-1}(Jx_n - \gamma l^k Ax_n))\| = 0. \quad (3.5)$$

Consequently, by the continuity of A on bounded sets, we get

$$\lim_{k \rightarrow \infty} \|Ax_n - A\Pi_C(J^{-1}(Jx_n - \gamma l^k Ax_n))\| = 0. \quad (3.6)$$

Combining (3.4) and (3.6), we have

$$\lim_{k \rightarrow \infty} \frac{\|x_n - \Pi_C(J^{-1}(Jx_n - \gamma l^k Ax_n))\|}{\gamma l^k} = 0. \quad (3.7)$$

More so from the uniform continuity of J on bounded subsets, we have

$$\lim_{k \rightarrow \infty} \frac{\|Jx_n - J(\Pi_C(J^{-1}((x_n) - \gamma l^k Ax_n)))\|}{\gamma l^k} = 0. \quad (3.8)$$

Now let $z_n = \Pi_C(J^{-1}(Jx_n - \gamma l^k Ax_n))$, then by Lemma 2.1(ii), we obtain

$$\langle Jz_n - Jx_n + \gamma l^k Ax_n, x - z_n \rangle \geq 0 \quad \forall x \in C.$$

This means that

$$\langle Ax_n, x - z_n \rangle \geq \left\langle \frac{Jx_n - Jz_n}{\gamma l^k}, x - z_n \right\rangle \quad \forall x \in C. \quad (3.9)$$

Note that from (3.8), we have $\lim_{n \rightarrow \infty} \|Jx_n - Jz_n\| = 0$, hence from (3.9), we get

$$\langle Ax_n, x - x_n \rangle \geq 0 \quad \forall x \in C.$$

Therefore, $x_n \in VI(C, A)$. This is a contradiction.

On the other hand, if $x_n \notin C$, then

$$\lim_{k \rightarrow \infty} \|x_n - \Pi_C(J^{-1}(Jx_n - \gamma l^k Ax_n))\| = \|x_n - \Pi_C(x_n)\| > 0 \quad (3.10)$$

and

$$\lim_{k \rightarrow \infty} \gamma l^k \|Ax_n - A\Pi_C(J^{-1}(Jx_n - \gamma l^k Ax_n))\| = 0. \quad (3.11)$$

Combining (3.4), (3.10) and (3.11), we get a contradiction. $\square$

Next, we prove that the sequence $\{x_n\}$ generated by Algorithm 3.2 is bounded.

Lemma 3.5. Suppose Assumptions (A1)–(A5) hold, and let $\{x_n\}$ be the sequence generated by Algorithm 3.2. Then $\{x_n\}$ is bounded.

Proof. Let $p \in \Gamma$, then by Lemma 2.5, we have

$$\begin{aligned} \phi(p, z_n) &= \phi(p, J^{-1}(Jy_n - \alpha_n(Ay_n - Ax_n))) \\ &= \|p\|^2 - 2\langle p, Jy_n - \alpha_n(Ay_n - Ax_n) \rangle + \|Jy_n - \alpha_n(Ay_n - Ax_n)\|^2 \\ &\leq \|p\|^2 - 2\langle p, Jy_n \rangle + 2\alpha_n \langle p, Ay_n - Ax_n \rangle + \|Jy_n\|^2 - 2\alpha_n \langle Ay_n - Ax_n, y_n \rangle + 2\alpha_n^2 \kappa^2 \|Ay_n - Ax_n\|^2 \\ &= \|p\|^2 - 2\langle p, Jy_n \rangle + \|Jy_n\|^2 - 2\alpha_n \langle y_n - p, Ay_n - Ax_n \rangle + 2\mu^2 \kappa^2 \|y_n - x_n\|^2 \\ &= \phi(p, y_n) - 2\alpha_n \langle y_n - p, Ay_n - Ax_n \rangle + 2\mu^2 \kappa^2 \|y_n - x_n\|^2. \end{aligned} \quad (3.12)$$

Using (2.3), we have

$$\begin{aligned} \phi(p, y_n) &= \phi(p, x_n) + \phi(x_n, y_n) + 2\langle p - x_n, Jy_n - Jx_n \rangle \\ &= \phi(p, x_n) + \phi(x_n, y_n) + 2\langle x_n - p, Jx_n - Jy_n \rangle. \end{aligned} \quad (3.13)$$

Then substituting (3.13) into (3.12), we have

$$\begin{aligned} \phi(p, z_n) &\leq \phi(p, x_n) + \phi(x_n, y_n) + 2\langle x_n - p, Jx_n - Jy_n \rangle - 2\alpha_n \langle y_n - p, Ay_n - Ax_n \rangle + 2\mu_n^2 \kappa^2 \|y_n - x_n\|^2 \\ &= \phi(p, x_n) + \phi(x_n, y_n) - 2\langle y_n - x_n, Jy_n - Jx_n \rangle + 2\langle y_n - p, Jy_n - Jx_n \rangle - 2\alpha_n \langle Ay_n - Ax_n, y_n - p \rangle \\ &\quad + 2\mu^2 \kappa^2 \|y_n - x_n\|^2. \end{aligned} \quad (3.14)$$

Moreover, from (2.2), we have

$$\phi(y_n, x_n) = 2\langle x_n - y_n, Jx_n - Jy_n \rangle - \phi(x_n, y_n).$$

Then (3.14) becomes

$$\phi(p, z_n) \leq \phi(p, x_n) - \phi(y_n, x_n) + 2\langle y_n - p, Jy_n - Jx_n \rangle - 2\alpha_n \langle Ay_n - Ax_n, y_n - p \rangle + 2\mu^2 \kappa^2 \|y_n - x_n\|^2. \quad (3.15)$$

Since A is monotone and $p \in \Gamma$, then $\langle Ay_n - Ap, y_n - p \rangle \geq 0$ and $\langle Ap, y_n - p \rangle \geq 0$. Therefore, it follows from (3.15) that

$$\phi(p, z_n) \leq \phi(p, x_n) - \phi(y_n, x_n) + 2\kappa^2 \mu^2 \|y_n - x_n\|^2.$$

Using Lemma 2.4, we have

$$\phi(p, z_n) \leq \phi(p, x_n) - \phi(y_n, x_n) + \frac{2\kappa^2 \mu^2}{\tau} \phi(y_n, x_n) = \phi(p, x_n) - \left(1 - \frac{2\kappa^2 \mu^2}{\tau}\right) \phi(y_n, x_n). \quad (3.16)$$

Since $\tau \in \left(0, \frac{1}{\sqrt{2\kappa^2 \mu^2}}\right)$, it follows from (3.16) that

$$\phi(p, z_n) \leq \phi(p, x_n).$$

Therefore,

$$\begin{aligned} \phi(p, x_{n+1}) &= \phi(p, J^{-1}(\beta_n Jx_1 + (1 - \beta_n)(\delta_n Jz_n + (1 - \delta_n)JSz_n))) \\ &= \phi(p, J^{-1}(\beta_n Jx_1 + (1 - \beta_n)\delta_n Jz_n + (1 - \beta_n)(1 - \delta_n)JSz_n)) \\ &\leq \beta_n \phi(p, x_1) + (1 - \beta_n)\delta_n \phi(p, z_n) + (1 - \beta_n)(1 - \delta_n)\phi(p, Sz_n) \\ &\leq \beta_n \phi(p, x_1) + (1 - \beta_n)\delta_n \phi(p, z_n) + (1 - \beta_n)(1 - \delta_n)\phi(p, z_n) \\ &= \beta_n \phi(p, x_1) + (1 - \beta_n)\phi(p, z_n) \\ &\leq \beta_n \phi(p, x_1) + (1 - \beta_n)\phi(p, x_n) \\ &\leq \max\{\phi(p, x_1), \phi(p, x_n)\} \\ &\vdots \\ &\leq \max\{\phi(p, x_1), \phi(p, x_1)\} = \phi(p, x_1). \end{aligned}$$

Hence, $\{\phi(p, x_n)\}$ is bounded. Consequently, $\{x_n\}, \{y_n\}, \{z_n\}, \{Sz_n\}$ are bounded. $\square$

Next, we provide some properties of the iterates under which the strong convergence of our algorithm is proved.

Lemma 3.6. Let $p = \Pi_{\Gamma}x_1$, $s_n = \phi(p, x_n)$ and $b_n = 2\langle Jx_1 - Jp, x_{n+1} - p \rangle$. Then

- (i) $s_{n+1} \leq (1 - \beta_n)s_n + \alpha_n b_n$;
- (ii) $-1 \leq \limsup_{n \rightarrow \infty} b_n < \infty$.

Proof. Let $w_n = J^{-1}(\delta_n Jz_n + (1 - \delta_n)JSz_n)$, then from (2.1), we have

$$\begin{aligned} \phi(p, w_n) &= \phi(p, J^{-1}(\delta_n Jz_n + (1 - \delta_n)JSz_n)) \\ &= \|p\|^2 + 2\langle p, \delta_n Jz_n + (1 - \delta_n)JSz_n \rangle + \|\delta_n Jz_n + (1 - \delta_n)JSz_n\|^2 \\ &\leq \|v\|^2 + 2\delta_n \langle p, Jz_n \rangle + 2(1 - \delta_n) \langle p, JSz_n \rangle + \|\delta_n Jz_n + (1 - \delta_n)JSz_n\|^2. \end{aligned} \quad (3.17)$$

Since $\{z_n\}$ and $\{Sz_n\}$ are bounded, there exists $r > 0$ such that $\{z_n\}, \{Sz_n\} \subset B_r = \{w \in E : \|w\| \leq r\}$ for all $n \geq 0$. Then, by Lemma 2.8 and the definition of J , there exists a strictly increasing, continuous and convex function $g : [0, 2r] \rightarrow [0, \infty)$ satisfying $g(0) = 0$ such that

$$\|\delta_n Jz_n + (1 - \delta_n)JSz_n\|^2 \leq \delta_n \|z_n\|^2 + (1 - \delta_n) \|Sz_n\|^2 - \delta_n(1 - \delta_n)g(\|Jz_n - JSz_n\|). \quad (3.18)$$

Hence from (3.17) and (3.18), we have

$$\begin{aligned} \phi(p, w_n) &\leq \delta_n (\|p\|^2 + 2\langle p, Jz_n \rangle + \|z_n\|^2) + (1 - \delta_n) (\|p\|^2 + 2\langle p, JSz_n \rangle + \|Sz_n\|^2) - \delta_n(1 - \delta_n)g(\|Jz_n - JSz_n\|) \\ &= \delta_n \phi(p, z_n) + (1 - \delta_n)\phi(p, Sz_n) - \delta_n(1 - \delta_n)g(\|Jz_n - JSz_n\|) \\ &\leq \phi(p, z_n) - \delta_n(1 - \delta_n)g(\|Jz_n - JSz_n\|) \\ &\leq \phi(p, x_n) - \delta_n(1 - \delta_n)g(\|Jz_n - JSz_n\|). \end{aligned}$$

Also, using Lemma 2.7, we have

$$\begin{aligned}
 \phi(p, x_{n+1}) &= V(p, \beta_n Jx_1 + (1 - \beta_n)Jw_n) \\
 &\leq V(p, \beta_n Jx_1 + (1 - \beta_n)Jw_n - \beta_n(Jx_1 - Jp)) - 2\langle -\beta_n(Jx_1 - Jp), x_{n+1} - p \rangle \\
 &= V(p, \beta_n Jv + (1 - \beta_n)Jw_n) + 2\beta_n \langle Jx_1 - Jp, x_{n+1} - p \rangle \\
 &= \phi(p, J^{-1}(\beta_n Jp + (1 - \beta_n)Jw_n)) + 2\beta_n \langle Jx_1 - Jp, x_{n+1} - p \rangle \\
 &\leq \beta_n \phi(p, p) + (1 - \beta_n)\phi(p, w_n) + 2\beta_n \langle Jx_1 - Jp, x_{n+1} - p \rangle \\
 &\leq (1 - \beta_n)(\phi(p, x_n) - \delta_n(1 - \delta_n)g(\|Jz_n - JSz_n\|)) + 2\beta_n \langle Jx_1 - Jp, x_{n+1} - p \rangle.
 \end{aligned} \tag{3.19}$$

Since $\{\beta_n\} \subset (0, 1)$, then we have

$$\phi(v, x_{n+1}) \leq (1 - \beta_n)\phi(p, x_n) + 2\beta_n \langle Jx_1 - Jp, x_{n+1} - p \rangle.$$

This established (i).

Now since $\{x_n\}$ is bounded, we have that

$$b_n = 2\langle Jx_1 - Jp, x_{n+1} - p \rangle \leq 2\|Jx_1 - Jp\| \|x_{n+1} - p\| < \infty,$$

and this implies that $\limsup_{n \rightarrow \infty} b_n < \infty$. We next show that $\limsup_{n \rightarrow \infty} b_n \geq -1$. We assume the contrary, i.e., $\limsup_{n \rightarrow \infty} b_n < -1$. Then there exists $n_0 \in \mathbb{N}$ such that $b_n < -1$, for all $n \geq n_0$. Then for all $n \geq n_0$, we get from (i) that

$$s_{n+1} \leq (1 - \beta_n)s_n + \beta_n b_n < (1 - \beta_n)s_n - \beta_n = s_n - \beta_n(s_n + 1) \leq s_n - \beta_n.$$

By induction, we get

$$s_{n+1} \leq s_{n_0} - \sum_{i=1}^{\infty} \alpha_i.$$

Taking $\limsup$ of both sides in the last inequality, we have

$$\limsup_{n \rightarrow \infty} s_n \leq s_{n_0} - \lim_{i \rightarrow \infty} \alpha_i = -\infty.$$

This contradicts the fact that $\{s_n\}$ is a nonnegative real sequence. Therefore, $\limsup_{n \rightarrow \infty} b_n \geq -1$. $\square$

The following lemma will be used to show that $\omega_w(\{x_n\}) \subset \Gamma$, where $\omega_w(\{x_n\})$ is the set of accumulation points of the sequence $\{x_n\}$ generated by Algorithm 3.2.

Lemma 3.7. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences generated by Algorithm 3.2 and assume that $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$. Let $\bar{x}$ be a weak limit of the subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Then $\bar{x} \in VI(C, A)$.*

Proof. Since $y_{n_k} = \Pi_C(J^{-1}(Jx_{n_k} - \alpha_{n_k}Ax_{n_k}))$, then from Lemma 2.1(ii), we have

$$\langle Jx_{n_k} - \alpha_{n_k}Ax_{n_k} - Jy_{n_k}, x - y_{n_k} \rangle \leq 0, \quad \forall x \in C.$$

This means that

$$\frac{1}{\alpha_{n_k}} \langle Jx_{n_k} - Jy_{n_k}, x - y_{n_k} \rangle \geq \langle Ax_{n_k}, x - y_{n_k} \rangle \quad \forall x \in C.$$

Hence,

$$\frac{1}{\alpha_{n_k}} \langle Jx_{n_k} - Jy_{n_k}, x - y_{n_k} \rangle + \langle Ax_{n_k}, y_{n_k} - x_{n_k} \rangle \leq \langle Ax_{n_k}, x - x_{n_k} \rangle, \quad \forall x \in C. \tag{3.20}$$

Now, we show that

$$\liminf_{k \rightarrow \infty} \langle Ax_{n_k}, x - x_{n_k} \rangle \geq 0 \quad \forall x \in C. \tag{3.21}$$

We consider the following possible cases.

Case 1: Suppose $\liminf_{k \rightarrow \infty} \alpha_{n_k} = 0$. Let $v_{n_k} = \Pi_C(J^{-1}(Jx_{n_k} - \alpha_{n_k}l^{-1}Ax_{n_k}))$, we have that $\alpha_{n_k}l^{-1} > \alpha_{n_k}$ and by Lemma 2.2, we obtain

$$\|x_{n_k} - v_{n_k}\| \leq \frac{1}{l} \|x_{n_k} - y_{n_k}\| \rightarrow 0.$$

Also, $v_{n_k} \rightharpoonup \bar{x} \in C$, which implies that $\{v_{n_k}\}$ is a bounded sequence. By the uniform continuity of A , we have

$$\|Ax_{n_k} - Av_{n_k}\| \rightarrow 0. \quad (3.22)$$

Using the Armijo line search, we get

$$\frac{1}{\mu} \|Ax_{n_k} - Av_{n_k}\| > \frac{\|x_{n_k} - v_{n_k}\|}{\alpha_{n_k}l^{-1}}. \quad (3.23)$$

Combining (3.22) and (3.23), we obtain

$$\lim_{k \rightarrow \infty} \frac{\|x_{n_k} - v_{n_k}\|}{\alpha_{n_k}l^{-1}} = 0.$$

More so,

$$\langle Jx_{n_k} - \alpha_{n_k}l^{-1}Ax_{n_k} - Jv_{n_k}, x - v_{n_k} \rangle \leq 0, \quad \forall x \in C.$$

Then

$$\frac{1}{\alpha_{n_k}l^{-1}} \langle Jx_{n_k} - Jv_{n_k}, x - v_{n_k} \rangle + \langle Ax_{n_k}, v_{n_k} - x_{n_k} \rangle \leq \langle Ax_{n_k}, x - x_{n_k} \rangle \quad \forall x \in C.$$

Passing limit to the above inequality, we get

$$\liminf_{k \rightarrow \infty} \langle Ax_{n_k}, x - x_{n_k} \rangle \geq 0 \quad \forall x \in C.$$

Thus, the inequality (3.21) is proven.

Case 2: Suppose that $\liminf_{k \rightarrow \infty} \alpha_{n_k} > 0$. Then there exists a constant $\alpha > 0$ such that $\alpha_{n_k} > \alpha > 0$ holds for all $k \in \mathbb{N}$. Since $\|x_{n_k} - y_{n_k}\| \rightarrow 0$, by the norm-to-norm continuity of J , we have $\|Jx_{n_k} - Jy_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$. Taking the limit of the inequality (3.20), we get

$$\liminf_{k \rightarrow \infty} \langle Ax_{n_k}, x - x_{n_k} \rangle \geq 0, \quad \forall x \in C.$$

Moreover, since A is monotone, we have for any arbitrary $x \in C$ that

$$\langle Ax, x - x_{n_k} \rangle \geq \langle Ax_{n_k}, x - x_{n_k} \rangle \quad \forall k \in \mathbb{N}. \quad (3.24)$$

Taking the $\liminf$ of (3.24), we get

$$\liminf_{k \rightarrow \infty} \langle Ax, x - x_{n_k} \rangle \geq \liminf_{k \rightarrow \infty} \langle Ax_{n_k}, x - x_{n_k} \rangle \geq 0 \quad \forall x \in C.$$

Hence,

$$\langle Ax, x - \bar{x} \rangle = \lim_{k \rightarrow \infty} \langle Ax, x - x_{n_k} \rangle \geq \liminf_{k \rightarrow \infty} \langle Ax_{n_k}, x - x_{n_k} \rangle \geq 0 \quad \forall x \in C. \quad (3.25)$$

In view of Lemma 2.9, (3.25) implies that $\bar{x} \in \text{VI}(C, A)$. $\square$

We now prove the strong convergence of Algorithm 3.2 to a solution $\bar{x} \in \Gamma$.

Theorem 3.8. *Let Assumptions (A1)–(A5) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to a solution $\bar{x} = \Pi_{\Gamma}x_1$.*

Proof. Let $p \in \Gamma$ and $s_n = \phi(p, x_n)$. We divide the proof into two cases.

Case I: Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(p, x_n)\}$ is nonincreasing. Then $\{\phi(p, x_n)\}$ converges and we obtain

$$s_n - s_{n+1} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.26)$$

Using (3.16), we have

$$\begin{aligned} \phi(p, x_{n+1}) &\leq \beta_n \phi(p, x_1) + (1 - \beta_n) \phi(p, z_n) \\ &\leq \beta_n \phi(p, x_1) + (1 - \beta_n) \phi(p, x_n) - (1 - \beta_n) \left(1 - \frac{2\kappa^2 \mu^2}{\tau}\right) \phi(y_n, x_n). \end{aligned}$$

This implies that

$$(1 - \beta_n) \left(1 - \frac{2\kappa^2 \mu^2}{\tau}\right) \phi(y_n, x_n) \leq \beta_n (\phi(p, x_1) - \phi(p, x_n)) + \phi(p, x_n) - \phi(p, x_{n+1}).$$

Since $\beta_n \rightarrow 0$ and $\tau \in (0, \frac{1}{\sqrt{2\kappa\mu}})$, then we have from (3.26)

$$\phi(y_n, x_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Consequently,

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \quad (3.27)$$

Moreover from (3.19), we obtain

$$\phi(p, x_{n+1}) \leq (1 - \beta_n) (\phi(p, x_n) - \delta_n (1 - \delta_n) g(\|Jz_n - JSz_n\|)) + 2\beta_n \langle Jx_1 - Jp, x_{n+1} - p \rangle.$$

This implies that

$$\delta_n (1 - \delta_n) g(\|Jz_n - JSz_n\|) \leq \beta_n \phi(p, x_n) + 2\beta_n \langle Jx_1 - Jp, x_{n+1} - p \rangle + s_n - s_{n+1}.$$

Similarly, since $\beta_n \rightarrow 0$ and condition A5 (ii), we have

$$\lim_{n \rightarrow \infty} \|Jz_n - JSz_n\| = 0.$$

Since J^{-1} is uniformly continuous on bounded subsets of E^* , we obtain

$$\lim_{n \rightarrow \infty} \|z_n - Sz_n\| = 0. \quad (3.28)$$

Also, it is clear that

$$\|Jy_n - Jz_n\| \leq \alpha_n \|Ay_n - Ax_n\| \leq \mu \|y_n - x_n\| \rightarrow 0, \quad n \rightarrow \infty.$$

Therefore, by the uniform continuity of J^{-1} on bounded subsets of E^* , we get

$$\lim_{n \rightarrow \infty} \|y_n - z_n\| = 0.$$

Thus,

$$\|z_n - x_n\| \leq \|z_n - y_n\| + \|y_n - x_n\| \rightarrow 0.$$

From the definition of w_n in Lemma 3.6, we have

$$\|Jw_n - Jz_n\| = (1 - \delta_n) \|JSz_n - Jz_n\| \rightarrow 0.$$

Thus,

$$\lim_{n \rightarrow \infty} \|w_n - z_n\| = 0.$$

Also,

$$\|Jx_{n+1} - Jw_n\| = \beta_n \|Jx_1 - Jw_n\| \rightarrow 0.$$

Thus,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - w_n\| = 0.$$

Therefore,

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - w_n\| + \|w_n - z_n\| + \|z_n - x_n\| \rightarrow 0. \quad (3.29)$$

Now, since $\{x_n\}$ is a bounded sequence, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p$. Since $\|x_n - y_n\| \rightarrow 0$, it follows from Lemma 3.7 that $\bar{x} \in \text{VI}(C, A)$. Also since $\|S z_n - z_n\| \rightarrow 0$ and S is relatively nonexpansive, then $\bar{x} \in F(S)$. Therefore, $\bar{x} \in \Gamma := \text{VI}(C, A) \cap F(S)$.

We now show that $\{x_n\}$ converges strongly to $z = P_{\Gamma} x_1$. It suffices to show that $\limsup_{n \rightarrow \infty} \langle Jx_1 - Jz, x_{n+1} - z \rangle \leq 0$. To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle Jx_1 - Jz, x_{n+1} - z \rangle = \lim_{k \rightarrow \infty} \langle Jx_1 - Jz, x_{n_k+1} - z \rangle.$$

Since $\|x_{n_k+1} - x_{n_k}\| \rightarrow 0$ and $x_{n_k} \rightarrow \bar{x}$, it follows from Lemma 2.1(ii) that

$$\limsup_{n \rightarrow \infty} \langle Jx_1 - Jz, x_{n+1} - z \rangle = \lim_{k \rightarrow \infty} \langle Jx_1 - Jz, x_{n_k+1} - z \rangle = \langle Jx_1 - Jz, \bar{x} - z \rangle \leq 0. \quad (3.30)$$

Therefore, put $z = p$ in Lemma 3.6(i), using Lemma 2.10 and (3.30), we obtain that $\phi(z, x_n) \rightarrow 0$. This implies that $\|x_n - z\| \rightarrow 0$. Therefore, $\{x_n\}$ converges strongly to $z = P_{\Gamma} x_1$.

Case II: Suppose $\{s_n\}$ is not monotonically decreasing, that is, there is a subsequence $\{s_{n_j}\}$ of $\{s_n\}$ such that $s_{n_j} < s_{n_j+1}$ for all $j \in \mathbb{N}$. Then by Lemma 2.11, we can define an integer sequence $\{\tau(n)\}$ for all $n \geq n_0$ by

$$\tau(n) = \max\{k \leq n : s_k < s_{k+1}\}.$$

Moreover, $\{\tau(n)\}$ is a nondecreasing sequence such that $\tau(n) \rightarrow \infty$ as $n \rightarrow \infty$ and $s_{\tau(n)} \leq s_{\tau(n)+1}$ for all $n \geq n_0$. Following similar process as in the proof of Case I, we can show that

$$\lim_{n \rightarrow \infty} \|y_{\tau(n)} - x_{\tau(n)}\| = 0, \quad \lim_{n \rightarrow \infty} \|z_{\tau(n)} - x_{\tau(n)}\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|u_{\tau(n)} - x_{\tau(n)}\| = 0.$$

Also using similar argument as in Case I, we also obtain

$$\limsup_{n \rightarrow \infty} \langle Jx_1 - Jv, x_{\tau(n)+1} - v \rangle \leq 0. \quad (3.31)$$

Also by Lemma 2.11, we get

$$0 \leq s_{\tau(n)+1} - s_{\tau(n)} \leq (1 - \beta_{\tau(n)})\phi(p, x_{\tau(n)}) + 2\beta_{\tau(n)}\langle Jx_1 - Jp, x_{\tau(n)+1} - p \rangle - \phi(p, x_{\tau(n)}).$$

This implies that

$$\phi(p, x_{\tau(n)}) \leq 2\langle Jx_1 - Jp, x_{\tau(n)+1} - p \rangle.$$

Hence from (3.31), we obtain $\limsup_{n \rightarrow \infty} \phi(p, x_{\tau(n)}) \leq 0$, which implies that $\lim_{n \rightarrow \infty} \phi(p, x_{\tau(n)}) = 0$. Consequently, we have

$$0 \leq s_n \leq \max\{s_{\tau(n)}, s_{\tau(n)+1}\} \leq s_{\tau(n)+1} \rightarrow 0.$$

Hence, $\phi(p, x_n) \rightarrow 0$. This implies that $\|x_n - p\| \rightarrow 0$ and thus, $x_n \rightarrow p$. This completes the proof. $\square$

The following result can be obtained as a consequence of our main result.

Corollary 3.9. Let C be a nonempty, closed and convex subset of a real Hilbert space, $A : H \rightarrow H$ be a monotone and uniformly continuous operator and $S : H \rightarrow H$ be a relative nonexpansive mapping. Assume that the solution set $\Gamma = \text{VI}(C, A) \cap \text{Fix}(S)$ is nonempty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ such that

- (i) $\lim_{n \rightarrow \infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = +\infty$;
- (ii) $0 < \liminf_{n \rightarrow \infty} \delta_n \leq \limsup_{n \rightarrow \infty} \delta_n < 1$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.10 converges strongly to a solution $\bar{x} = P_{\Gamma} x_1$.

Algorithm 3.10.

Step 0: Select $\gamma > 0$, $l \in (0, 1)$, $\mu \in (0, 1)$ and arbitrary initial point $x_1 \in E$. Set $n = 1$.

Iterative steps: Given x_n , calculate x_{n+1} as follows:

Step 1: Compute

$$y_n = \Pi_C(x_n - \alpha_n A x_n),$$

where $\alpha_n = \gamma l^{k_n}$ with k_n being the smallest nonnegative integer k satisfying

$$\gamma l^k \|A x_n - A y_n\| \leq \mu \|x_n - y_n\|.$$

If $x_n - y_n = 0$, set $x_n = z_n$ and go to Step 3. Else, do Step 2.

Step 2: Compute

$$z_n = y_n - \alpha_n (A y_n - A x_n)$$

and

$$x_{n+1} = \beta_n x_1 + (1 - \beta_n)(\delta_n z_n + (1 - \delta_n) S z_n).$$

Set $n \leftarrow n + 1$ and go to Step 1.

4 Numerical examples

In this section, we present some numerical experiments to illustrate the performance of our Algorithm 3.2 and compare its efficiency with other methods in the literature. All computations are carried out using a PC with the following specification: Intel(R)core i7-600, CPU 2.48 GHz, RAM 8.0 GB, MATLAB version 9.5 (R2019b).

Example 4.1. Let $E = \mathbb{R}^m$ and $A : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be defined by $A(x) = Mx + q$ with $q \in \mathbb{R}^m$ and

$$M = NN^T + R + D,$$

where N is a $m \times m$ matrix, R is a $m \times m$ skew-symmetric matrix and D is a $m \times m$ diagonal matrix whose entries are nonnegative. The feasible set C is defined by $C = \{x \in \mathbb{R}^m : Bx \leq b\}$ for some random matrix $B \in \mathbb{R}^{m \times k}$ and random vector $b \in \mathbb{R}^k$ ($k = 10$) with nonnegative entries. We note that A is monotone and Lipschitz continuous with $L = \|M\|$, and if $q = 0$, then $0 \in \text{VI}(C, A)$. All entries of q, N, R are generated randomly in $(-2, 2)$ and the diagonal entries of D are in $(0, 2)$. We defined the mapping $S : \mathbb{R}^m \rightarrow \mathbb{R}^m$ by $Sx = \left(\frac{x_1}{2}, \dots, \frac{x_m}{2}\right)$. Since $\phi(x, y) = \|x - y\|^2$ for all $x, y \in \mathbb{R}^m$, it is easy to see that S is nonexpansive and demiclosed at zero. Thus, $\text{Sol} = \{0\}$. First, we show the dependency of the parameter β_n on the convergence of Algorithm 3.2. We take $\gamma = 4$, $l = 0.3$, $\mu = 0.28$, $\delta_n = \frac{3n}{5n+5}$, $\beta_n = \frac{1}{(n+1)^s}$, where $s = 0.2, 0.5, 0.8, 1$. The initial value x_1 is generated randomly in $\mathbb{R}^m$, where $m = 10, 20, 30$. The numerical results are shown in Table 1 and Figure 1.

Furthermore, we compare the performance of the proposed algorithm with that of Algorithms (1.4), (1.5) and (1.6). For Algorithm 3.2, we choose $\gamma = 4$, $l = 0.4$, $\mu = 0.28$, $\delta = \frac{3n}{5n+5}$, $\beta_n = \frac{1}{\sqrt{n+1}}$. Also, for Algorithm (1.4), we take $\lambda = \frac{1}{2L}$, $\alpha_n = \frac{1}{n+1}$; and for Algorithm (1.5), we take $\lambda_0 = 0.24$, $\mu = 0.36$, $\alpha_n = \frac{1}{n+1}$ and for Algorithm (1.6), we take $\gamma = 0.4$, $\sigma = 0.36$, $\rho = 0.28$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{3n}{5n+5}$. The initial value x_1 is generated randomly in $\mathbb{R}^m$ where $m = 10, 40, 70, 100$. We use $\|x_{n+1} - x_n\| < 10^{-4}$ as a stopping criterion for all the algorithms. The numerical results are shown in Table 2 and Figure 2.

Table 1: Computation result for Example 4.1

		$m = 10$	$m = 20$	$m = 30$
$\beta_n = \frac{1}{(n+1)^{0.2}}$	Iter.	9	10	11
	Time (s)	0.0068	0.0027	0.0032
$\beta_n = \frac{1}{(n+1)^{0.5}}$	Iter.	11	14	15
	Time (s)	0.0053	0.0032	0.0030
$\beta_n = \frac{1}{(n+1)^{0.8}}$	Iter.	15	19	20
	Time (s)	0.0080	0.0031	0.0053
$\beta_n = \frac{1}{(n+1)}$	Iter.	17	22	24
	Time (s)	0.0063	0.0046	0.0047

Figure 1: Example 4.1, Experiment 1, top left: $m = 10$; top right: $m = 20$; bottom: $m = 30$.

Table 2: Computation result for Example 4.1

		Algorithm 3.2	Algorithm (1.4)	Algorithm (1.5)	Algorithm (1.6)
$m = 10$	Iter.	25	54	33	40
	Time (s)	0.0043	0.0157	0.0070	0.0114
$m = 40$	Iter.	22	56	37	36
	Time (s)	0.0036	0.0159	0.0068	0.0083
$m = 70$	Iter.	19	52	31	28
	Time (s)	0.0032	0.0117	0.0059	0.0077
$m = 100$	Iter.	19	49	30	32
	Time (s)	0.0031	0.0121	0.0066	0.0078

Figure 2: Example 4.1, Experiment 2, top left: $m = 10$; top right: $m = 20$; bottom: $m = 30$.

Example 4.2. Let $E = L^2([0, 1])$ with inner product $\langle x, y \rangle = \int_0^1 x(t)y(t)dt$ and induced norm $\|x\|^2 = \int_0^1 |x(t)|^2 dt$ for all $x, y \in L^2([0, 1])$. We define the operator $A : L^2([0, 1]) \rightarrow L^2([0, 1])$ by

$$(Ax)(t) = \max\{0, x(t)\}, \quad \forall t \in [0, 1] \quad \text{and} \quad x \in L^2([0, 1]).$$

Then A is monotone and Lipschitz continuous with Lipschitz constant $L = 1$. The feasible set C is defined by $C = \{x \in L^2([0, 1]) : \|x\| \leq 1\}$ and $S : L^2([0, 1]) \rightarrow L^2([0, 1])$ be defined by

$$(Sx)(t) = \int_0^1 tx(s)ds, \quad t \in [0, 1].$$

It is easy to see that S is relative-nonexpansive. The solution set $\text{Sol} = \{0\}$. The projection onto C can be calculated explicitly by using

$$\Pi_C(x) = \begin{cases} \frac{x}{\|x\|^2} & \text{if } \|x\|^2 > 1, \\ x & \text{otherwise.} \end{cases}$$

Similarly as in Example 4.1, we show the dependency of the parameter β_n on the convergence of Algorithm 3.2 for this algorithm. We take $\gamma = 2$, $l = 0.5$, $\mu = 0.85$, $\delta_n = \frac{7n}{10n+9}$, $\beta_n = \frac{1}{(n+1)^s}$, where $s = 0.2, 0.5, 0.8, 1$. The initial value x_1 is chosen as follows: Case I: $x_1 = t^3 - 1$, Case II: $x_1 = \cos(3t)$, Case III: $x_1 = \exp(3t)$. The numerical results are shown in Table 3 and Figure 3.

Table 3: Computation result for Example 4.2, Experiment 1

		Case I	Case II	Case III
$\beta_n = \frac{1}{(n+1)^{0.2}}$	Iter.	5	6	6
	Time (s)	0.9228	0.6057	3.4984
$\beta_n = \frac{1}{(n+1)^{0.5}}$	Iter.	10	11	12
	Time (s)	2.5820	2.2018	5.4784
$\beta_n = \frac{1}{(n+1)^{0.8}}$	Iter.	9	10	11
	Time (s)	1.6335	0.9307	5.2031
$\beta_n = \frac{1}{(n+1)}$	Iter.	8	9	10
	Time (s)	1.3055	0.9419	4.0566

Figure 3: Example 4.2, Experiment 1, top left: Case I; top right: Case II; bottom: Case III.

More so, we compare the performance of the proposed algorithm with Algorithms (1.4), (1.5) and (1.6). For Algorithm 3.2, we choose $\gamma = 2$, $l = 0.4$, $\mu = 0.5$, $\delta = \frac{4n}{8n+5}$, $\beta_n = \frac{1}{n+1}$. Also, for Algorithm (1.4), we take $\lambda = \frac{1}{2L}$, $\alpha_n = \frac{1}{n+1}$; and for Algorithm (1.5), we take $\lambda_0 = 0.4$, $\mu = 0.8$, $\alpha_n = \frac{1}{n+1}$; for Algorithm (1.6), we take $\gamma = 0.2$, $\sigma = 0.49$, $\rho = 2$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{4n}{8n+5}$. The initial value x_1 is chosen as follows:
Case I: $x_1 = \frac{\exp(3t)}{3}$; Case II: $x_1 = \frac{\sin(2t)}{5}$; Case III: $x_1 = 2\frac{\cos(2t)}{2}$; Case IV: $x_1 = t^3 + 2t - 1$.

We use $\|x_{n+1} - x_n\| < 10^{-4}$ as a stopping criterion for all the algorithms. The numerical results are shown in Table 4 and Figure 4.

Table 4: Computation result for Example 4.2, Experiment 2

		Algorithm 3.2	Algorithm (1.4)	Algorithm (1.5)	Algorithm (1.6)
Case I	Iter.	8	41	23	20
	Time (s)	2.3678	7.1300	3.6229	6.5246
Case II	Iter.	6	62	33	28
	Time (s)	3.1102	7.0743	4.4195	7.8965
Case III	Iter.	8	42	23	20
	Time (s)	5.2862	8.4223	4.4196	3.7524
Case IV	Iter.	9	46	25	22
	Time (s)	0.9190	4.1244	2.1402	2.9158

Figure 4: Example 4.2, Experiment 2, top left: Case I; top right: Case II; bottom left: Case III; bottom right: Case IV.

Remark 4.3. In addition, we state the following contributions of our result in comparison with the methods in [34,51–54], which were suggested by one of the reviewers.

- (1) In [51], the authors introduced a subgradient extragradient method for finding a common solution of VI and fixed point of nonexpansive mappings in real Hilbert spaces, where as in this paper, a modified Tseng extragradient method is introduced for solving VI and FPP in real Banach spaces. More so, the method in [51] involves computing a prior estimate of the Lipschitz constant of the cost operator A , which is not easy to compute in general, where as in our method, there is no need for a prior estimate of the Lipschitz constant.

- (2) Although the methods in [52–54] involved an accelerated (or inertial extrapolation) technique, they require solving more than one strongly convex optimization problem per each iteration. One of the aims in this paper is to reduce the number of projections to at most one per each iteration.
- (3) Similar to that in (2), the method in [34] used more than one projection per each iteration for solving the VI and FPP. More so, the method was proposed for solving VI and FPP in real Hilbert spaces.
- (4) Hence, our result improves and extends the result in [34,51–54] from a real Hilbert space to the setting of a real Banach space using a single projection per each iteration.

5 Conclusion

In this paper, we proposed a new self-adaptive projection method for solving monotone VIs and FPPs in 2-uniformly convex and uniformly smooth real Banach spaces. A strong convergence result is proved under some mild conditions and without assuming that the cost operator is Lipschitz continuous. More so, the stepsize is selected via a line search process. We provide some numerical examples to illustrate the performance of the proposed algorithm and compare with other methods in the literature. This results improves and extends the corresponding results of [17,25,27,31,33,35–38] and many other recent results in the literature.

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References

- [1] H. Iiduka, *A new iterative algorithm for the variational inequality problem over the fixed point set of a firmly nonexpansive mapping*, Optimization **59** (2010), 873–885.
- [2] H. Iiduka and I. Yamada, *A use of conjugate gradient direction for the convex optimization problem over the fixed point set of a nonexpansive mapping*, SIAM J. Optim. **19** (2009), 1881–1893.
- [3] H. Iiduka and I. Yamada, *A subgradient-type method for the equilibrium problem over the fixed point set and its applications*, Optimization **58** (2009), 251–261.
- [4] P. E. Maingé, *A hybrid extragradient-viscosity method for monotone operators and fixed point problems*, SIAM J. Control Optim. **47** (2008), 1499–1515.
- [5] P. E. Maingé, *Projected subgradient techniques and viscosity methods for optimization with variational inequality constraints*, Eur. J. Oper. Res. **205** (2010), 501–506.
- [6] A. Iusem and M. Nasri, *Korpelevich's method for variational inequality problem in Banach spaces*, J. Global Optim. **50** (2011), 59–76.
- [7] J. Mashregi and M. Nasri, *Forcing strong convergence of Korpelevich's method in Banach spaces with its applications in game theory*, Nonlinear Anal. **72** (2010), 2086–2099.
- [8] F. Facchinei and J. S. Pang, *Finite-dimensional Variational Inequalities and Complementarity Problems*, Springer, New York, 2003, Springer Series in Operations Research, Vols. I and II.
- [9] R. Glowinski, J. L. Lions, and R. Trémoilières, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981.
- [10] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.
- [11] I. V. Konnov, *Combined Relaxation Methods for Variational Inequalities*, Springer-Verlag, Berlin, 2001.

- [12] P. Marcotte, *Applications of Khobotov's algorithm to variational and network equilibrium problems*, INFOR. Inf. Syst. Oper. Res. **29** (1991), 255–270.
- [13] Y. Censor, A. Gibali, and S. Reich, *Extensions of Korpelevich's extragradient method for variational inequality problems in Euclidean space*, Optimization **61** (2012), 119–1132.
- [14] G. M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Ekon. Mat. Metody. **12** (1976), 747–756, (in Russian).
- [15] Y. Censor, A. Gibali, and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert spaces*, J. Optim. Theory Appl. **148** (2011), 318–335.
- [16] Y. Censor, A. Gibali, and S. Reich, *Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space*, Optim. Methods Softw. **26** (2011), 827–845.
- [17] Y. Shehu, *Single projection algorithm for variational inequalities in Banach spaces with applications to contact problems*, Acta Math. Sci. Ser. B (Engl. Ed.) **40** (2020), no. 4, 1045–1063.
- [18] Y. Shehu, Q. L. Dong, and D. Jiang, *Single projection method for pseudo-monotone variational inequality in Hilbert spaces*, Optimization **68** (2019), 385–409.
- [19] M. V. Solodov and B. F. Svaiter, *A new projection method for variational inequality problems*, SIAM J. Control Optim. **37** (1999), no. 3, 765–776.
- [20] L. C. Ceng, M. Teboulle, and J. C. Yao, *Weak convergence of an iterative method for pseudomonotone variational inequalities and fixed-point problems*, J. Optim. Theory Appl. **146** (2010), 19–31.
- [21] B. S. He, *A class of projection and contraction methods for monotone variational inequalities*, Appl. Math. Optim. **35** (1997), 69–76.
- [22] S. V. Denisov, V. V. Semenov, and L. M. Chabak, *Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators*, Cybern. Syst. **51** (2015), 757–765.
- [23] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim. **38** (2009), 431–446.
- [24] C. E. Chidume and M. O. Nnakwe, *Convergence theorems of subgradient extragradient algorithm for solving variational inequalities and a convex feasibility problem*, J. Fixed Point Theory Appl. **2018** (2018), 16.
- [25] Y. Liu, *Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Banach spaces*, J. Nonlinear Sci. Appl. **10** (2017), 395–409.
- [26] B. Halpern, *Fixed points of nonexpanding maps*, Proc. Amer. Math. Soc. **73** (1967), 957–961.
- [27] R. Kraikaew and S. Saejung, *Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces*, J. Optim. Theory Appl. **163** (2014), 399–412.
- [28] L. O. Jolaoso, A. Taiwo, T. O. Alakoya, and O. T. Mewomo, *A strong convergence theorem for solving pseudo-monotone variational inequalities using projection methods in a reflexive Banach space*, J. Optim. Theory Appl. **185** (2020), no. 3, 744–766.
- [29] L. O. Jolaoso, A. Taiwo, T. O. Alakoya, and O. T. Mewomo, *A unified algorithm for solving variational inequality and fixed point problems with application to the split equality problem*, Comput. Appl. Math. **39** (2020), 38.
- [30] L. O. Jolaoso, A. Taiwo, T. O. Alakoya, and O. T. Mewomo, *A self adaptive inertial subgradient extragradient algorithm for variational inequality and common fixed point of multivalued mappings in Hilbert spaces*, Demonstr. Math. **52** (2019), 183–203.
- [31] Y. C. Yao and M. Postolache, *Iterative methods for pseudomonotone variational inequalities and fixed-point problems*, J. Optim. Theory Appl. **155** (2012), 273–287.
- [32] L. O. Jolaoso and M. Aphane, *Weak and strong convergence Bregman extragradient schemes for solving pseudo-monotone and non-Lipschitz variational inequalities*, J. Ineq. Appl. **2020** (2020), 195.
- [33] F. Ma, *A subgradient extragradient algorithm for solving monotone variational inequalities in Banach spaces*, J. Ineq. Appl. **2020** (2020), 26.
- [34] Y. Shehu and O. S. Iyiola, *Iterative algorithms for solving fixed point problems and variational inequalities with uniformly continuous monotone operators*, Numer. Algorithms **79** (2018), 529–553.
- [35] L. Q. Anh, T. Bantaojai, N. P. Duc, T. Q. Duy, and R. Wangkeeree, *Convergence of solutions to lexicographic equilibrium problems*, J. Appl. Numer. Optim. **1** (2019), 39–51.
- [36] C. E. Chidume, A. Adamu, and L. C. Okereke, *A Krasnoselskii-type algorithm for approximating solutions of variational inequality problems and convex feasibility problems*, J. Nonlinear Var. Anal. **2** (2018), 203–218.
- [37] L. L. Duan, A. F. Shi, L. Wei, and R. R. Agarwal, *Construction techniques of projection sets in hybrid methods for infinite weakly relatively nonexpansive mappings with applications*, J. Nonlinear Funct. Anal. **2019** (2019), 14.
- [38] D. V. Thong, Y. Shehu, and O. S. Iyiola, *A new iterative method for solving pseudomonotone variational inequalities with non-Lipschitz operators*, Comput. Appl. Math. **39** (2020), 108.
- [39] Y. Takahashi, K. Hashimoto, and M. Kato, *On sharp uniform convexity, smoothness, and strong type, cotype inequalities*, J. Nonlinear Convex Anal. **3** (2002), 267–281.
- [40] W. Takahashi and K. Zembayashi, *Strong and weak convergence theorems for equilibrium problems and relatively non-expansive mappings in Banach spaces*, Nonlinear Anal. **70** (2009), 45–57.

- [41] Y. I. Alber, *Metric and generalized projections in Banach spaces: properties and applications*, in: A. G. Kartsatos (ed.), *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, 1996, pp. 15–50.
- [42] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938–945.
- [43] L. C. Ceng and J. C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems*, J. Comput. Applied Math. **214** (2008), 186–201.
- [44] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mapping in a Banach space*, J. Approx. Theory **134** (2005), no. 2, 257–266.
- [45] Y. I. Alber and I. Ryazantseva, *Nonlinear Ill-posed Problems of Monotone Type*, Springer, Dordrecht, 2006.
- [46] K. Nakajo, *Strong convergence for gradient projection method and relatively nonexpansive mappings in Banach spaces*, Appl. Math. Comput. **271** (2015), 251–258.
- [47] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), 1127–1138.
- [48] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.
- [49] H. K. Xu, *Another control condition in an iterative method for nonexpansive mappings*, Bull. Austral. Math. Soc. **65** (2002), 109–113.
- [50] P. E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal. **16** (2008), 899–912.
- [51] V. Dadashi, O. S. Iyiola, and Y. Shehu, *The subgradient extragradient method for pseudomonotone equilibrium problems*, Optimization **69** (2020), no. 4, 901–923.
- [52] Y. Shehu, O. S. Iyiola, D. V. Thong, and N. T. C. Van, *An inertial subgradient extragradient algorithm extended to pseudomonotone equilibrium problems*, Math. Methods Oper. Res. **93** (2021), 213–242.
- [53] O. S. Iyiola, F. U. Ogbuisi, and Y. Shehu, *An inertial type iterative method with Armijo linesearch for nonmonotone equilibrium problems*, Calcolo **55** (2018), 52.
- [54] Y. Shehu, O. S. Iyiola, and F. U. Ogbuisi, *Iterative methods with inertial terms for nonexpansive mappings: application to compressed sensing*, Numer. Algorithms **83** (2020), 1321–1347.