Research Article

Tareq M. Al-shami, Adnan Tercan, and Abdelwaheb Mhemdi*

New soft separation axioms and fixed soft points with respect to total belong and total non-belong relations

<https://doi.org/10.1515/dema-2021-0018>

received February 17, 2021; accepted May 26, 2021

Abstract: In this article, we exploit the relations of total belong and total non-belong to introduce new soft separation axioms with respect to ordinary points, namely tt -soft pre T_i ($i = 0, 1, 2, 3, 4$) and tt -soft pre-regular spaces. The motivations to use these relations are, first, cancel the constant shape of soft pre-open and pre-closed subsets of soft pre-regular spaces, and second, generalization of existing comparable properties on classical topology. With the help of examples, we show the relationships between them as well as with soft pre T_i ($i = 0, 1, 2, 3, 4$) and soft pre-regular spaces. Also, we explain the role of soft hyperconnected and extended soft topological spaces in obtaining some interesting results. We characterize a tt -soft pre-regular space and demonstrate that it guarantees the equivalence of tt -soft pre T_i ($i = 0, 1, 2$). Furthermore, we investigate the behaviors of these soft separation axioms with the concepts of product and sum of soft spaces. Finally, we introduce a concept of pre-fixed soft point and study its main properties.

Keywords: soft pre-open set, tt -soft pre T_i -space, extended soft topology, soft hyperconnected space, additive property, topological property, soft pre-compact space, pre-fixed soft point

MSC 2020: 47H10, 54A05, 54C15, 54D10, 54D15

1 Introduction

Soft set was established by Molodtsov [1], in 1999, as a new technique to approach real-life problems that suffer from vagueness and uncertainty. He investigated merits of soft sets compared with probability theory and fuzzy set theory. Then, the soft set-theoretic concepts have been introduced and investigated by several researchers, and many applications of soft sets have been given on the disciplines of decision-making problems [2], engineering [3] and medical sciences [4].

In 2011, Shabir and Naz [5] as well as Çağman et al. [6] employed soft sets to introduce the concept of soft topological space. However, they used two different methods to define soft topology. On one hand, Shabir and Naz formulated soft topology on the collection of soft sets over a universal crisp set with a fixed set of parameters. On the other hand, Çağman et al. formulated soft topology on the collection of soft sets over an absolute soft set with different sets of parameters that are subsets of the universal set of parameters. In this paper, we continue studying soft topology using the definition given by Shabir and Naz.

Many scholars explored several (crisp) topological concepts via soft topological spaces and examined the validity of some known topological results on soft topological spaces. Soft compactness was defined and

* **Corresponding author: Abdelwaheb Mhemdi**, Department of Mathematics, College of Sciences and Humanities in Aflaj, Prince Sattam bin Abdulaziz University, Riyadh, Saudi Arabia, e-mail: mhemdiabd@gmail.com

Tareq M. Al-shami: Department of Mathematics, Sana'a University, Sana'a, Yemen, e-mail: tareqalshami83@gmail.com

Adnan Tercan: Department of Mathematics, Hacettepe University, Ankara, Turkey, e-mail: adnan.tercan@tubitak.gov.tr

studied in [7], in 2012. Then, Hida [8] distinguished between two types of soft compactness depending on the belong relation. Al-shami et al. [9] studied almost soft compact and approximate soft compact spaces as extensions of a soft compact space. Al-shami and El-Shafei [10] utilized soft pre-open sets to generalize soft compactness. In [11], the behavior of soft closed sets in a soft Hausdorff was revised, and in [12] many alleged results of soft separation axioms were corrected with the help of interesting counterexamples. Babitha and John explored some notions on soft topological spaces in [13] and introduced some methods to generate soft topology from soft relations in [14]. The concept of soft somewhere dense sets was introduced in [15]. Then, it was applied to define new types of soft maps in [16].

The relations of belong and non-belong given in [5] were utilized in the studies of soft set and soft topology. However, the authors of [17], in 2018, described the relations between an ordinary point and soft set by two new relations, namely partial belong and total non-belong. According to these new relations, many soft topological concepts and notions were reformulated; as an example, see the different types of soft separation axioms given in [17–19]. Wardowski [20] tackled the fixed point in the setup of soft topological spaces. Al-shami and Abo-Tabl [21] generalized Wardowski's fixed point using the class of soft α -open sets.

Soft point is one of the important notions that assists to prove some soft topological properties. Zorlutuna and Çakir [22] gave the first definition of a soft point, then, it was redefined by Das and Samanta [23] as well as by Nazmul and Samanta [24]. Novel kinds of soft topologies, namely enriched and extended soft topology, were introduced in [7] and [24], respectively. The equivalence between these two topologies has been recently proved by Al-shami and Kočinac [25].

The layout of this article is as follows: We recall in Section 2 the main principles to make this work self-contained. Section 3 introduces the concepts of tt -soft pre T_i ($i = 0, 1, 2, 3, 4$) and tt -soft pre-regular spaces, which are formulated using total belong and total non-belong relations. The interrelations between them and their essential properties are established with the aid of counterexamples. Section 4 explores a pre-fixed soft point theorem and studies main properties. In particular, we derive the sufficient conditions that keep pre-fixed soft points between a soft topological space and its parametric topological spaces.

2 Preliminaries

We assign this part to mention the necessary concepts and properties to comprehend this manuscript.

2.1 Soft sets

Definition 2.1. [1] For a nonempty set X and a set of parameters M , a map G from M to the power set 2^X of X is said to be a soft set over X . It is denoted by G_M .

We describe a soft set G_M by ordered pairs: $G_M = \{(m, G(m)) : m \in M \text{ and } G(m) \in 2^X\}$.

The family of soft sets defined over X with M as a set of parameters is denoted by $S(X_M)$.

Definition 2.2. [26] A soft set G_M is said to be a subset of a soft set H_M . We write $G_M \subseteq H_M$, reads as G_M is a subset of H_M , if $G(m) \subseteq H(m)$ for each $m \in M$.

We write $G_M = H_M$ if $G_M \subseteq H_M$ and $H_M \subseteq G_M$.

Definition 2.3. [1,17] Let G_M be a soft set over X and $x \in X$. We write that:

- (i) $x \in G_M$ (or x totally belongs to G_M or G_M totally contains x), if $x \in G(m)$ for each $m \in M$.
- (ii) $x \notin G_M$ (or x does not partially belong to G_M or G_M does not partially contain x), if $x \notin G(m)$ for some $m \in M$.
- (iii) $x \in G_M$ (or x partially belongs to G_M or G_M partially contains x), if $x \in G(m)$ for some $m \in M$.
- (iv) $x \notin G_M$ (or x does not totally belong to G_M or G_M does not totally contain x), if $x \notin G(m)$ for each $m \in M$.

Definition 2.4. [27] The complement of a soft set G_M is a soft set G_M^c , where $G^c : M \rightarrow 2^X$ is a mapping defined by $G^c(m) = X \setminus G(m)$ for all $m \in M$.

Definition 2.5. [17,23,24,28] A soft set (G, M) over X is said to be

- (i) A null soft set, denoted by $\tilde{\Phi}$, if $G(m) = \emptyset$ for each $m \in M$. Its complement is called an absolute soft set.
- (ii) A soft point S_m^x if there are $m \in M$ and $x \in X$ such that $G(m) = \{x\}$ and $G(m') = \emptyset$ for each $m' \in M \setminus \{m\}$. We write that $S_m^x \in G_M$ if $x \in G(m)$.
- (iii) A stable soft set if there is a subset A of X such that $G(m) = A$ for each $m \in M$, and it is denoted by $\tilde{A}$. In particular, it is denoted by x_M if $A = \{x\}$.
- (iv) An uncountable (resp. infinite) soft set if $G(m)$ is uncountable (resp. infinite) for some $m \in M$. Otherwise, it is said to be countable (resp. finite).

Definition 2.6. [27–29] Let G_M and H_M be two soft sets over X .

- (i) Their intersection, denoted by $G_M \tilde{\cap} H_M$, is a soft set U_M , where a mapping $U : E \rightarrow 2^X$ is given by $U(m) = G(m) \cap H(m)$.
- (ii) Their union, denoted by $G_M \tilde{\cup} H_M$, is a soft set U_M , where a mapping $U : E \rightarrow 2^X$ is given by $U(m) = G(m) \cup H(m)$.
- (iii) The Cartesian product of G_M and H_F , denoted by $G \times H_{M \times F}$, is defined as $(G \times H)(m, f) = G(m) \times H(f)$ for each $(m, f) \in M \times F$.

Definition 2.7. [4] A soft mapping between $S(X_A)$ and $S(Y_B)$ is a pair (f, ϕ) , denoted also by f_ϕ , of mappings such that $f : X \rightarrow Y$, $\phi : A \rightarrow B$. Let G_A and H_B be subsets of $S(X_A)$ and $S(Y_B)$, respectively. Then, the image of G_A and pre-image of H_B are defined as follows:

- (i) $f_\phi(G_A) = (f_\phi(G))_B$ is a subset of $S(Y_B)$ such that $f_\phi(G)(b) = \bigcup_{a \in \phi^{-1}(b)} f(G(a))$ for each $b \in B$.
- (ii) $f_\phi^{-1}(H_B) = (f_\phi^{-1}(H))_A$ is a subset of $S(X_A)$ such that $f_\phi^{-1}(H)(a) = f^{-1}(H(\phi(a)))$ for each $a \in A$.

Definition 2.8. [22] A soft map $f_\phi : S(X_A) \rightarrow S(Y_B)$ is said to be injective (resp. surjective, bijective) if ϕ and f are injective (resp. surjective, bijective).

2.2 Soft topology

Definition 2.9. [5] A family σ of soft sets over X under a fixed set of parameters M is said to be a soft topology on X if it satisfies the following:

- (i) $\tilde{X}$ and $\tilde{\Phi}$ are members of σ .
- (ii) The intersection of a finite number of soft sets in σ is a member of σ .
- (iii) The union of an arbitrary number of soft sets in σ is a member of σ .

The triple (X, σ, M) is called a soft topological space. We call a subset of $\tilde{X}$ a soft open set if it is a member in σ and call its complement a soft closed set.

Proposition 2.10. [5] In (X, σ, M) , a family $\sigma_m = \{G(m) : G_M \in \sigma\}$ is a classical topology on X for each $m \in M$. σ_m is called a parametric topology, and (X, σ_m) is called a parametric topological space.

Definition 2.11. [5] Let (X, σ, M) be a soft topological space and $\emptyset \neq Y \subseteq X$. A family $\sigma_Y = \{\tilde{Y} \tilde{\cap} G_M : G_M \in \sigma\}$ is called a soft relative topology on Y and the triple (Y, σ_Y, M) is called a soft subspace of (X, σ, M) .

Definition 2.12. [30] A subset G_M of (X, σ, M) is called soft pre-open if $G_M \subseteq \text{int}(\text{cl}(G_M))$.

Theorem 2.13. [30]

- (i) Every soft open set is soft pre-open.
- (ii) The family of soft pre-open sets is closed under arbitrary soft union.

Definition 2.14. [30] Let G_M be a subset of (X, σ, M) . Then, $\overline{G_M}^{pre}$ is the intersection of all soft pre-closed sets containing G_M .

It is clear that $x \in \overline{G_M}^{pre}$ iff $G_M \cap U_M \neq \Phi$ for each soft pre-open set U_M totally containing x ; and $S_m^x \in \overline{G_M}^{pre}$ iff $G_M \cap U_M \neq \Phi$ for each soft pre-open set U_M containing S_m^x .

Definition 2.15. [31] (X, σ, M) which does not contain disjoint soft open sets is called soft hyperconnected.

Theorem 2.16. [10] The family of soft pre-open sets in a soft hyperconnected space is closed under finite soft intersection.

The above result will assist us to initiate some properties of soft pre-separation axioms and soft pre-compact spaces, see Theorem 3.15 and Proposition 3.22. It means that the class of soft pre-open sets in a soft hyperconnected space (X, σ, M) constructs a new soft topology σ^{pre} over X containing σ .

Proposition 2.17. [10] Let $\tilde{Y}$ be soft open subset of (X, σ, M) . Then,

1. If (H, M) is a soft pre-open and $\tilde{Y}$ is a soft open in (X, σ, M) , then $(H, M) \cap \tilde{Y}$ is a soft pre-open subset of (Y, σ_Y, M) .
2. If $\tilde{Y}$ is a soft open in (X, σ, M) and (H, M) is a soft pre-open in (Y, σ_Y, M) , then (H, M) is a soft pre-open subset of (X, σ, M) .

Definition 2.18. [30,32] (X, σ, M) is said to be:

- (i) Soft pre T_0 if there is a soft pre-open set U_M for every $x \neq y \in X$ such that $x \in U_M$ and $y \notin U_M$; or $y \in U_M$ and $x \notin U_M$.
- (ii) Soft pre T_1 if there are two soft pre-open sets U_M and V_M for every $x \neq y \in X$ such that $x \in U_M$ and $y \notin U_M$; and $y \in V_M$ and $x \notin V_M$.
- (iii) Soft pre T_2 if there are two disjoint soft pre-open sets U_M and V_M for every $x \neq y \in X$ such that $x \in G_M$ and $y \in F_M$.
- (iv) Soft pre-regular if for every soft pre-closed set H_M and $x \in X$ such that $x \notin H_M$, there are two disjoint soft pre-open sets U_M and V_M such that $H_M \subseteq U_M$ and $x \in V_M$.
- (v) Soft pre-normal if for every two disjoint soft pre-closed sets H_M and F_M , there are two disjoint soft pre-open sets U_M and V_M such that $H_M \subseteq U_M$ and $F_M \subseteq V_M$.
- (vi) Soft pre T_3 (resp. soft pre T_4) if it is both soft pre-regular (resp. soft pre-normal) and soft pre T_1 -space.

Definition 2.19. [10] A family $\{G_{i_M} : i \in I\}$ of soft pre-open subsets of (X, σ, M) is said to be a soft pre-open cover of $\tilde{X}$ if $\tilde{X} = \bigcup_{i \in I} G_{i_M}$.

Definition 2.20. [10] (X, σ, M) is said to be:

- (i) Soft pre T_2' if for every $S_m^x \neq S_{m'}^y \in \tilde{X}$, there are two disjoint soft pre-open sets U_M and V_M containing S_m^x and $S_{m'}^y$, respectively.
- (ii) Soft pre-compact if every soft pre-open cover of $\tilde{X}$ has a finite subcover.

Proposition 2.21. [10]

- (i) A soft pre-compact subset of a soft pre T_2' -space is soft pre-closed.
- (ii) A stable soft pre-compact subset of a soft pre T_2 -space is soft pre-closed.

Definition 2.22. $g_\varphi : (X, \sigma, M) \rightarrow (Y, \tau, M)$ is called soft pre*-continuous (or soft pre-irresolute [10]) if the inverse image of each soft pre-open set is soft pre-open.

Proposition 2.23. [10] *The soft pre*-continuous image of a soft pre-compact set is soft pre-compact.*

The definitions of soft pre-continuous, soft pre-open, soft pre-closed and soft pre-homeomorphism mappings were introduced in [30].

Definition 2.24. (See, [25]) A soft topology σ on X is said to be extended if $\sigma = \{G_M : G(m) \in \sigma_m \text{ for each } m \in M\}$; equivalently, if all soft sets G_M such that $G(m) = \emptyset$ or X are members of σ . It is also called an enriched soft topology.

For more details of extended (or enriched) soft topologies, see [25].

Theorem 2.25. [25] *A subset (F, M) of an extended soft topological space (X, σ, M) is soft pre-open if and only if each m -approximate element of (F, M) is pre-open.*

Proposition 2.26. [33] *Let $\{(X_i, \sigma_i, M) : i \in I\}$ be a family of pairwise disjoint soft topological spaces and $X = \bigcup_{i \in I} X_i$. Then, the collection*

$$\sigma = \{(G, M) \subseteq \widetilde{X} : (G, M) \widetilde{\cap} \widetilde{X}_i \text{ is a soft open set in } (X_i, \sigma_i, M) \text{ for every } i \in I\}$$

defines a soft topology (called sum of soft topological spaces) on X with a fixed set of parameters M .

Theorem 2.27. [33] *A soft set $(G, M) \subseteq \widehat{\bigoplus_{i \in I} \widetilde{X}_i}$ is soft pre-open (resp. soft pre-closed) in $(\bigoplus_{i \in I} X_i, \sigma, M)$ if and only if all $(G, M) \widetilde{\cap} \widetilde{X}_i$ are soft pre-open (resp. soft pre-closed) in (X_i, σ_i, M) .*

Proposition 2.28. [20] *Let $g_\varphi : (X, \sigma, M) \rightarrow (X, \sigma, M)$ be a soft map such that $\bigcap_{n \in \mathbb{N}} g_\varphi^n(\widetilde{X})$ is a soft point S_m^x . Then, S_m^x is a unique fixed point of g_φ .*

Theorem 2.29. [22] *Let (X, σ, A) and (Y, τ, B) be two soft topological spaces and $\Omega = \{G_A \times F_B : G_A \in \sigma \text{ and } F_B \in \tau\}$. Then, the family of all arbitrary union of elements of Ω is a soft topology over $X \times Y$ under a fixed set of parameters $A \times B$.*

Lemma 2.30. *Let (G, M) and (H, N) be two subsets of (X_1, σ_1, M) and (X_2, σ_2, N) , respectively. Then,*

- (i) $cl(G, M) \times cl(H, N) = cl((G, M) \times (H, N))$.
- (ii) $int(G, M) \times int(H, N) = int((G, M) \times (H, N))$.

3 Pre-soft separation axioms

Herein, we define new family of soft spaces called tt -soft pre T_i ($i = 0, 1, 2, 3, 4$) and tt -soft pre-regular spaces, where tt is the abbreviation of the initials of the phrases “total belong” and “total non-belong” used in the definitions of these spaces. We provide several counterexamples to elucidate the relationships between them and validate the obtained findings. Moreover, we explain whether they are hereditary, topological, and additive properties or not.

Definition 3.1. (X, σ, M) is said to be:

- (i) tt -soft pre T_0 if there exists a soft pre-open set U_M for every $x \neq y$ in X such that $x \in U_M$ and $y \notin U_M$ or $y \in U_M$ and $x \notin U_M$.

- (ii) tt -soft pre T_1 if there exist soft pre-open sets U_M and V_M for every $x \neq y$ in X such that $x \in U_M$ and $y \notin U_M$; and $y \in V_M$ and $x \notin V_M$.
- (iii) tt -soft pre T_2 if there exist two disjoint soft pre-open sets U_M and V_M for every $x \neq y$ in X such that $x \in U_M$ and $y \notin U_M$; and $y \in V_M$ and $x \notin V_M$.
- (iv) tt -soft pre-regular if for every soft pre-closed set H_M and $x \in X$ such that $x \notin H_M$, there exist disjoint soft pre-open sets U_M and V_M such that $H_M \subseteq U_M$ and $x \in V_M$.
- (v) tt -soft pre T_3 (resp. tt -soft pre T_4) if it is both tt -soft pre-regular (resp. soft pre-normal) and tt -soft pre T_1 .

Remark 3.2. Note that if F_M and G_M are disjoint soft sets, then the two relations $x \in F_M$ and $x \notin G_M$ are equivalent. This means that the concepts of tt -soft pre T_2 and soft pre T_2 -spaces are equivalent.

We can say that (X, σ, M) is tt -soft pre T_2 if there exist two disjoint soft pre-open sets U_M and V_M for every $x \neq y \in X$ such that $x \in U_M$ and $y \in V_M$.

Remark 3.3. The soft pre-regular spaces imply a strict condition on the shape of soft pre-open and soft pre-closed subsets. To clarify this matter, let F_M be a soft pre-closed set such that $x \notin H_M$. Then, we obtain two cases as follows:

- (i) There are $m, m' \in M$ such that $x \notin H(m)$ and $x \in H(m')$. This case is out of the question because there are not two disjoint soft sets U_M and V_M containing x and H_M , respectively.
- (ii) For all $m \in M$, we have $x \notin H(m)$. This means that H_M is stable.

We derive from (i) and (ii) above that all soft pre-closed and soft pre-open subsets of a soft pre-regular space are stable. But, this property does not hold on the tt -soft pre-regular spaces because we replace a partial non-belong relation by a total non-belong relation. Therefore, a tt -soft pre-regular space need not be stable.

Proposition 3.4.

- (i) Every tt -soft pre T_i -space is soft pre T_i for $i = 0, 1, 4$.
- (ii) Every soft pre-regular space is tt -soft pre-regular.
- (iii) Every soft pre T_3 -space is tt -soft pre T_3 .

Proof. The proofs of (i) and (ii) are straightforward.

(iii): It suffices to prove that a soft pre T_i -space is tt -soft pre T_i when (X, σ, M) is soft pre-regular. Suppose $x \neq y \in X$. Then, there exist two soft pre-open sets U_M and V_M such that $x \in U_M$ and $y \notin U_M$; and $y \in V_M$ and $x \notin V_M$. Since U_M and V_M are soft pre-open subsets of a soft pre-regular space, they are stable. So, $y \notin U_M$ and $x \notin V_M$. Hence, (X, σ, M) is tt -soft pre T_i , as required. $\square$

We supply the following counterexamples to clarify that the converse of the above proposition fails.

Example 3.5. Let $M = \{m_1, m_2\}$. A family $\sigma = \{\widetilde{\Phi}, \widetilde{X}, G_{iM} : i = 1, 2, \dots, 5\}$ is a soft topology on $X = \{x, y\}$, where

$$\begin{aligned} G_{1M} &= \{(m_1, \{x\}), (m_2, X)\}; \\ G_{2M} &= \{(m_1, \{y\}), (m_2, X)\}; \\ G_{3M} &= \{(m_1, \emptyset), (m_2, X)\}; \\ G_{4M} &= \{(m_1, \emptyset), (m_2, \{x\})\} \quad \text{and} \\ G_{5M} &= \{(m_1, \emptyset), (m_2, \{y\})\}. \end{aligned}$$

It can be checked that a soft subset of (X, σ, M) is soft open iff it is soft pre-open. Then, the soft sets $\{(m_1, \{x\}), (m_2, \{x\})\}$ and $\{(m_1, \{y\}), (m_2, \{y\})\}$ are not soft pre-open subsets of (X, σ, M) . Therefore, it is not tt -soft pre T_0 . However, it is clear that (X, σ, M) is a soft pre T_1 -space. Also, it is soft pre-normal. Therefore, it is soft pre T_4 .

Example 3.6. Let $M = \{m_1, m_2\}$. A family $\sigma = \{\widetilde{\Phi}, \widetilde{X}, G_{i_M} : i = 1, 2, 3, 4\}$ is a soft topology over $X = \{x, y\}$, where

$$\begin{aligned} G_{1_M} &= \{(m_1, \{x\}), (m_2, \{x\})\}; \\ G_{2_M} &= \{(m_1, \{y\}), (m_2, \{y\})\}; \\ G_{3_M} &= \{(m_1, \{x\}), (m_2, X)\} \quad \text{and} \\ G_{4_M} &= \{(m_1, \emptyset), (m_2, \{y\})\}. \end{aligned}$$

Since there exists an unstable soft pre-open subset of (X, σ, M) , (X, σ, M) is not soft pre-regular. Therefore, it is not soft pre T_3 . However, (X, σ, M) is a *tt*-soft pre-regular space. Also, it is a *tt*-soft pre T_1 -space. Hence, it is a *tt*-soft pre T_3 -space.

Before we display the relationship between *tt*-soft pre T_i -spaces, we will prove the following helpful lemma.

Lemma 3.7. (X, σ, M) is a *tt*-soft pre T_1 -space if and only if x_M is soft pre-closed for every $x \in X$.

Proof. *Necessity:* For each $y_i \in X \setminus \{x\}$, there is a soft pre-open set G_{i_M} such that $y_i \in G_{i_M}$ and $x \notin G_{i_M}$. Therefore, $X \setminus \{x\} = \bigcup_{i \in I} G_i(m)$ and $x \notin \bigcup_{i \in I} G_i(m)$ for each $m \in M$. Thus, $\bigcup_{i \in I} G_{i_M} = \widetilde{X \setminus \{x\}}$ is soft pre-open. Hence, x_M is soft pre-closed.

Sufficiency: Let $x \neq y$. By hypothesis, x_M and y_M are soft pre-closed sets. Then, x_M^c and y_M^c are soft pre-open sets such that $x \in (y_M)^c$ and $y \in (x_M)^c$. Obviously, $y \notin (y_M)^c$ and $x \notin (x_M)^c$. Hence, (X, σ, M) is *tt*-soft pre T_1 . $\square$

Proposition 3.8. For $i = 1, 2, 3, 4$, a *tt*-soft pre T_i -space is *tt*-soft pre T_{i-1} .

Proof. For $i = 3$, let $x \neq y$ in a *tt*-soft pre T_3 -space (X, σ, M) . Then, x_M is soft pre-closed. Since $y \notin x_M$ and (X, σ, M) is *tt*-soft pre-regular, there are disjoint soft pre-open sets G_M and F_M such that $x_M \subseteq G_M$ and $y \in F_M$. Therefore, (X, σ, M) is *tt*-soft pre T_2 .

For $i = 4$, let H_M be a soft pre-closed set such that $x \notin H_M$, where $x \in X$. Since (X, σ, M) is *tt*-soft pre T_1 , x_M is soft pre-closed. Since $x_M \cap H_M = \widetilde{\Phi}$ and (X, σ, M) is soft pre-normal, we can find two disjoint soft pre-open sets G_M and F_M satisfying that $H_M \subseteq G_M$ and $x_M \subseteq F_M$. Hence, (X, σ, M) is *tt*-soft pre T_3 . $\square$

The three counterexamples below elucidate that the converse of the above proposition fails.

Example 3.9. Let $M = \{m_1, m_2\}$. A family $\sigma = \{\widetilde{\Phi}, \widetilde{X}, \{(m_1, \{x\}), (m_2, \{x\})\}\}$ is a soft topology on $X = \{x, y\}$. A family of all soft pre-open subsets of (X, σ, M) is $S(X_M) \setminus \{G_{i_M} : i = 1, 2, 3\}$, where

$$\begin{aligned} G_{1_M} &= \{(m_1, \{y\}), (m_2, \{y\})\}; \\ G_{2_M} &= \{(m_1, \{y\}), (m_2, \emptyset)\} \quad \text{and} \\ G_{3_M} &= \{(m_1, \emptyset), (m_2, \{y\})\}. \end{aligned}$$

Now, $x \neq y$. We have $x \in \{(m_1, \{x\}), (m_2, \{x\})\}$ and $y \notin \{(m_1, \{x\}), (m_2, \{x\})\}$. Therefore, (X, σ, M) is *tt*-soft pre T_0 . However, there does not exist a soft pre-open set such that x does not totally belong to it. Hence, (X, σ, M) is not *tt*-soft pre T_1 .

It is well known that the identification between soft topological and (crisp) topological spaces occurs if E is a singleton. Then, the following counterexample is *tt*-soft pre T_1 , but not *tt*-soft pre T_2 .

Example 3.10. Let $M = \{m_1\}$, $X = Y \cup \{x, y\}$, where Y is any infinite set and x, y are two distinct points not in Y . Let σ be the family of subsets of X such that

- (i) $A \in \sigma$ if $A \subseteq Y$;
- (ii) $A \in \sigma$ if x or y belong to A such that A^c contains only a finite number of Y .

Then, σ is a soft topology on X . Now, let $a \neq b \in X$. Then, we have three cases as follows:

- (i) $a, b \in Y$. Then, $\{a\}$ and $\{b\}$ are two soft pre-open sets containing a and b , respectively.
- (ii) $a \in Y$ and $b = x$ or y . Say, $b = x$. Then, $\{a\}$ and $\{x\} \cup [Y \setminus \{a\}]$ are two soft pre-open sets containing a and b , respectively. Similarly, if $b \in Y$ and $a = x$ or y .
- (iii) $a = x$ and $b = y$. Then, $X \setminus \{a\}$ and $X \setminus \{b\}$ are two soft pre-open sets containing a and b , respectively.

Hence, (X, σ, M) is tt -soft pre T_1 . To prove (X, σ, M) is not tt -soft pre T_2 , it suffices to show that $G \cap H \neq \emptyset$ for each soft pre-open set G containing x and each soft pre-open set H containing y . We note that the soft pre-openness of G gives that the soft open set $\text{int}(cl(G))$ contains all points of Y except finitely many. So, from $\text{int}(cl(G)) \subseteq cl(G)$, it follows that $cl(G)$ contains all points of Y except possibly a finite number of points of Y . Next, we see that G is not soft closed. If possible, let G be soft closed. Then, G^c is soft open. Again, the soft closedness together with the soft pre-openness of G gives that G is soft open. Hence, by the definition of σ , G^c containing y contains only a finite number of Y . This contradicts the assertion that G^c is soft open. So, G is not soft closed. Now, $[G \cup \{y\}]^c$ being a subset of Y is soft open, by the definition of σ . So, $G \cup \{y\}$ is soft closed, and it is the smallest soft closed set containing G . Therefore, $cl(G) = G \cup \{y\}$ which implies that $G \cup \{y\}$ and hence G contains all points of Y except possibly finitely many. Similarly, it can be shown that H contains all points of Y except possibly many. Hence, $G \cap H \neq \emptyset$.

Example 3.11. To illustrate the remaining cases, it suffices to consider examples that satisfy a (crisp) pre T_2 -space (resp. [crisp] pre T_3 -space) but not (crisp) pre T_3 (resp. not [crisp] pre T_4).

Now, we establish some findings concerning tt -soft pre T_i and tt -soft pre-regular.

Lemma 3.12. Let U_M be a subset of (X, σ, M) and $x \in X$. Then, $x \notin \overline{U_M}^{pre}$ iff there exists a soft pre-open set V_M totally containing x such that $U_M \widetilde{\cap} V_M = \widetilde{\Phi}$.

Proof. Let $x \notin \overline{U_M}^{pre}$. Then, $x \in (\overline{U_M}^{pre})^c = V_M$. So, $U_M \widetilde{\cap} V_M = \widetilde{\Phi}$. Conversely, if there exists a soft pre-open set V_M totally containing x such that $U_M \widetilde{\cap} V_M = \widetilde{\Phi}$, then $U_M \subseteq V_M^c$. Therefore, $\overline{U_M}^{pre} \subseteq V_M^c$. Since $x \notin V_M^c$, $x \notin \overline{U_M}^{pre}$. $\square$

Proposition 3.13. If (X, σ, M) is a tt -soft pre T_0 -space, then $\overline{x_M}^{pre} \neq \overline{y_M}^{pre}$ for every $x \neq y \in X$.

Proof. Let $x \neq y$ in a tt -soft pre T_0 -space. Then, there is a soft pre-open set U_M such that $x \in U_M$ and $y \notin U_M$ or $y \in U_M$ and $x \notin U_M$. Say, $x \in U_M$ and $y \notin U_M$. Now, $y_M \widetilde{\cap} U_M = \widetilde{\Phi}$. So, by the above lemma, $x \notin \overline{y_M}^{pre}$. But $x \in \overline{x_M}^{pre}$. Hence, we obtain the desired result. $\square$

Corollary 3.14. If (X, σ, M) is a tt -soft pre T_0 -space, then $\overline{S_m^x}^{pre} \neq \overline{S_{m'}^y}^{pre}$ for all $x \neq y$ and $m, m' \in M$.

Theorem 3.15. Let M be a finite set and (X, σ, M) be soft hyperconnected. Then, (X, σ, M) is a tt -soft pre T_1 -space if and only if $x_M = \widetilde{\cap} \{U_M : x \in U_M \in \sigma^{pre}\}$ for each $x \in X$.

Proof. $\Rightarrow$: Let $y \in X$. Then, for each $x \in X \setminus \{y\}$, we have a soft pre-open set U_M such that $x \in U_M$ and $y \notin U_M$. Therefore, $y \notin \widetilde{\cap} \{U_M : x_M \subseteq U_M \in \sigma^{pre}\}$, as required.

$\Leftarrow$: Let $x \neq y$ and $|M| = n$. Since $y \notin x_M$, then for each $j = 1, 2, \dots, n$ there is a soft pre-open set U_{i_m} such that $y \notin U_{i_m}(e_j)$ and $x \in U_{i_m}$. Since (X, σ, M) is soft hyperconnected, it follows from Theorem 2.16 that $\widetilde{\cap}_{i=1}^n U_{i_m}$ is a soft pre-open set such that $y \notin \widetilde{\cap}_{i=1}^n U_{i_m}$ and $x \in \widetilde{\cap}_{i=1}^n U_{i_m}$. Similarly, we can get a soft pre-open set V_M such that $y \in V_M$ and $x \notin V_M$. Thus, (X, σ, M) is a tt -soft pre T_1 -space. $\square$

Theorem 3.16. If (X, σ, M) is an extended tt -soft pre T_1 -space, then S_m^x is soft pre-closed for all $S_m^x \in \widetilde{X}$.

Proof. By Lemma 3.7, $\widetilde{X \setminus \{x\}}$ is a soft pre-open set. Since (X, σ, M) is extended, a soft set H_M , where $H(m) = \emptyset$ and $H(m') = X$ for each $m' \neq m$, is a soft pre-open set. Therefore, $\widetilde{X \setminus \{x\}} \cup H_M$ is soft pre-open. Thus, $(\widetilde{X \setminus \{x\}} \cup H_M)^c = S_m^x$ is soft pre-closed. $\square$

Corollary 3.17. *If (X, σ, M) is an extended tt -soft pre T_1 -space, then the intersection of all soft pre-open sets containing U_M is exactly U_M for each $U_M \subseteq \widetilde{X}$.*

Proof. Let U_M be a soft subset of $\widetilde{X}$. Since S_m^x is a soft pre-closed set for every $S_m^x \in U_M^c$, $\widetilde{X} \setminus S_m^x$ is a soft pre-open set containing U_M . Therefore, $U_M = \bigcap \{\widetilde{X} \setminus S_m^x : S_m^x \in U_M^c\}$, as required. $\square$

Theorem 3.18. *Let (X, σ, M) be finite soft hyperconnected. Then, (X, σ, M) is tt -soft pre T_2 if and only if it is tt -soft pre T_1 .*

Proof. By Proposition 3.8 the necessary part holds. Conversely, for each $x \neq y$, we have x_M and y_M are soft pre-closed sets. Since X is finite, $\bigcup_{y \in X \setminus \{x\}} y_M$ and $\bigcup_{x \in X \setminus \{y\}} x_M$ are soft pre-closed sets. Since (X, σ, M) is soft hyperconnected, $(\bigcup_{y \in X \setminus \{x\}} y_M)^c = x_M$ and $(\bigcup_{x \in X \setminus \{y\}} x_M)^c = y_M$ are soft pre-open sets. The disjointness of x_M and y_M end the proof that (X, σ, M) is tt -soft pre T_2 . $\square$

Remark 3.19. In Example 3.10, note that x_M is not a soft pre-open set for each $x \in \mathbb{N}$. This clarifies that a soft set x_M in a tt -soft pre T_1 -space need not be soft pre-open if the universal set is infinite.

Theorem 3.20. *(X, σ, M) is tt -soft pre-regular iff for every soft pre-open subset F_M of (X, σ, M) totally containing x , there is a soft pre-open set V_M such that $x \in V_M \subseteq \overline{V_M}^{pre} \subseteq F_M$.*

Proof. Let $x \in X$ and F_M be a soft pre-open set totally containing x . Then, F_M^c is pre-soft closed and $x_M \cap F_M^c = \emptyset$. Therefore, there are disjoint soft pre-open sets U_M and V_M such that $F_M^c \subseteq U_M$ and $x \in V_M$. Thus, $V_M \subseteq U_M^c \subseteq F_M$. Hence, $\overline{V_M}^{pre} \subseteq U_M^c \subseteq F_M$. Conversely, let F_M^c be a soft pre-closed set. Then, for each $x \notin F_M^c$, we have $x \in F_M$. By hypothesis, there is a soft pre-open set V_M totally containing x such that $\overline{V_M}^{pre} \subseteq F_M$. Therefore, $F_M^c \subseteq (\overline{V_M}^{pre})^c$ and $V_M \cap (\overline{V_M}^{pre})^c = \emptyset$. Thus, (X, σ, M) is tt -soft pre-regular, as required. $\square$

Theorem 3.21. *If (X, σ, M) is a tt -soft pre-regular space, then the concepts of tt -soft pre T_2 , tt -soft pre T_1 and tt -soft pre T_0 -spaces are equivalent.*

Proof. It suffices to prove that a tt -soft pre T_0 -space is tt -soft pre T_2 . To do so, let $x \neq y$ in a tt -soft pre T_0 -space (X, σ, M) . Then, there exists a soft pre-open set G_M such that $x \in G_M$ and $y \notin G_M$ or $y \in G_M$ and $x \notin G_M$. Say, $x \in G_M$ and $y \notin G_M$. Obviously, $x \notin G_M^c$ and $y \in G_M^c$. Since (X, σ, M) is tt -soft pre-regular, there exist two disjoint soft pre-open sets U_M and V_M such that $x \in U_M$ and $y \in G_M^c \subseteq V_M$. Hence, (X, σ, M) is tt -soft pre T_2 . $\square$

Proposition 3.22. *Let (X, σ, M) be finite soft hyperconnected. If (X, σ, M) is a tt -soft pre T_2 -space, then it is tt -soft pre-regular.*

Proof. Let H_M be a soft pre-closed set and $x \in X$ such that $x \notin H_M$. Then, $x \neq y$ for each $y \in H_M$. By hypothesis, there are two disjoint soft pre-open sets U_{i_M} and V_{i_M} such that $x \in U_{i_M}$ and $y \in V_{i_M}$. Since $\{y : y \in X\}$ is a finite set, there is a finite number of soft pre-open sets V_{i_M} such that $H_M \subseteq \bigcup_{i=1}^n V_{i_M}$. Since (X, σ, M) is soft hyperconnected, it follows from Theorem 2.16 that $\bigcap_{i=1}^n U_{i_M}$ is a soft pre-open set containing x . Since $[\bigcup_{i=1}^n V_{i_M}] \cap [\bigcap_{i=1}^n U_{i_M}] = \emptyset$, (X, σ, M) is tt -soft pre-regular. $\square$

Corollary 3.23. *The following properties are equivalent if (X, σ, M) is finite soft hyperconnected.*

- (i) *A tt -soft pre T_3 -space.*
- (ii) *A tt -soft pre T_2 -space.*
- (iii) *A tt -soft pre T_1 -space.*

Proof. The directions (i) $\rightarrow$ (ii) and (ii) $\rightarrow$ (iii) follow from Proposition 3.8.

The direction (iii) $\rightarrow$ (ii) follows from Theorem 3.18.

The direction (ii) $\rightarrow$ (i) follows from Proposition 3.22. $\square$

Theorem 3.24. *The property of being a tt -soft pre T_i -space ($i = 0, 1, 2, 3$) is a soft open hereditary.*

Proof. For $i = 3$.

Let (Y, σ_Y, M) be a soft open subspace of a tt -soft pre T_3 -space (X, σ, M) . To prove that (Y, σ_Y, M) is tt -soft pre T_1 , let $x \neq y \in Y$. Since (X, σ, M) is tt -soft pre T_1 , there exist two soft pre-open sets G_M and F_M such that $x \in G_M$ and $y \notin G_M$; and $y \in F_M$ and $x \notin F_M$. Therefore, $x \in U_M = \tilde{Y} \cap G_M$ and $y \in V_M = \tilde{Y} \cap F_M$ such that $y \notin U_M$ and $x \notin V_M$. By Proposition 2.17, U_M and V_M are soft pre-open subsets of (Y, σ_Y, M) ; thus, (Y, σ_Y, M) is tt -soft pre T_1 .

To prove that (Y, σ_Y, M) is tt -soft pre-regular, let $y \in Y$ and F_M be a soft pre-closed subset of (Y, σ_Y, M) such that $y \notin F_M$. Then, $F_M \cup \tilde{Y}^c$ is a soft pre-closed subset of (X, σ, M) such that $y \notin F_M \cup \tilde{Y}^c$. Therefore, there exist disjoint soft pre-open subsets U_M and V_M of (X, σ, M) such that $F_M \cup \tilde{Y}^c \subseteq U_M$ and $y \in V_M$. Now, $U_M \cap \tilde{Y}$ and $V_M \cap \tilde{Y}$ are disjoint soft pre-open subsets of (Y, σ_Y, M) such that $F_M \subseteq U_M \cap \tilde{Y}$ and $y \in V_M \cap \tilde{Y}$. Thus, (Y, σ_Y, M) is tt -soft pre-regular.

Hence, (Y, σ_Y, M) is tt -soft pre T_3 , as required. $\square$

Theorem 3.25. *Let (X, σ, M) be extended and $i = 0, 1, 2, 3, 4$. Then, (X, σ, M) is tt -soft pre T_i iff (X, σ_m) is pre T_i for each $m \in M$.*

Proof. For $i = 4$.

$\Rightarrow$: Let $x \neq y$ in X . Then, there exist two soft pre-open sets U_M and V_M such that $x \in U_M$ and $y \notin U_M$; and $y \in V_M$ and $x \notin V_M$. Obviously, $x \in U(m)$ and $y \notin U(m)$; and $y \in V(m)$ and $x \notin V(m)$. Since (X, σ, M) is extended, Theorem 2.25 shows that $U(m)$ and $V(m)$ are pre-open subsets of (X, σ_m) for each $m \in M$. Thus, (X, σ_m) is a pre T_1 -space. To prove that (X, σ_m) is pre-normal, let F_m and H_m be two disjoint pre-closed subsets of (X, σ_m) . Let F_M and H_M be two soft sets given by $F(m) = F_m$, $H(m) = H_m$ and $F(m') = H(m') = \emptyset$ for each $m' \neq m$. By Theorem 2.25, we obtain F_M and H_M are two disjoint soft pre-closed subsets of (X, σ, M) . By hypothesis, there exist two disjoint soft pre-open sets G_M and W_M such that $F_M \subseteq G_M$ and $H_M \subseteq W_M$. This implies that $F(m) = F_m \subseteq G(m)$ and $H(m) = H_m \subseteq W(m)$. Since (X, σ, M) is extended, Theorem 2.25 shows that $G(m)$ and $W(m)$ are pre-open subsets of (X, σ_m) . Thus, (X, σ_m) is a pre-normal space. Hence, it is a pre T_4 -space.

$\Leftarrow$: Let $x \neq y$ in X . Then, there exists two pre-open subsets U_m and V_m of (X, σ_m) such that $x \in U_m$ and $y \notin U_m$; and $y \in V_m$ and $x \notin V_m$. Let U_M and V_M be two soft sets given by $U(m) = U_m$, $V(m) = V_m$ for each $m \in M$. Since (X, σ, M) is extended, Theorem 2.25 shows that U_M and V_M are soft pre-open subsets of (X, σ, M) such that $x \in U_M$ and $y \notin U_M$; and $y \in V_M$ and $x \notin V_M$. Thus, (X, σ, M) is a tt -soft pre T_1 -space. To prove that (X, σ, M) is soft pre-normal, let F_M and H_M be two disjoint soft pre-closed subsets of (X, σ, M) . Since (X, σ, M) is extended, Theorem 2.25 shows that $F(m)$ and $H(m)$ are two disjoint pre-closed subsets of (X, σ_m) . By hypothesis, there exist two disjoint pre-open subsets G_m and W_m of (X, σ_m) such that $F(m) \subseteq G_m$ and $H(m) \subseteq W_m$. Let G_M and W_M be two soft sets given by $G(m) = G_m$ and $W(m) = W_m$ for each $m \in M$. Since (X, σ, M) is extended, Theorem 2.25 shows that G_M and W_M are two disjoint soft pre-open subsets of (X, σ, M) such that $F_M \subseteq G_M$ and $H_M \subseteq W_M$. Thus, (X, σ, M) is soft pre-normal. Hence, it is a tt -soft pre T_4 -space. $\square$

In the next counterexamples, we illustrate that there is no relationship between soft topological space and their parametric topological spaces in terms of separation axioms, if an extended condition given in the above theorem does not exist.

Example 3.26. Let (X, σ, M) be the same as in Example 3.5. We clarified that (X, σ, M) is not tt -soft pre T_0 . In contrast, σ_{m_1} and σ_{m_2} are the discrete topology on X . Hence, (X, σ_{m_1}) and (X, σ_{m_2}) are pre T_4 .

Example 3.27. Let $M = \{m_1, m_2\}$. A family $\sigma = \{\widetilde{\Phi}, \widetilde{X}, \{(m_1, \{x\}), (m_2, \{y\})\}\}$ is a soft topology on $X = \{x, y\}$. A family of all soft pre-open subsets of (X, σ, M) is $S(X_M) \setminus \{G_{i_M} : i = 1, 2, 3\}$, where

$$\begin{aligned} G_{1_M} &= \{(m_1, \{y\}), (m_2, \{x\})\}; \\ G_{2_M} &= \{(m_1, \{y\}), (m_2, \emptyset)\} \quad \text{and} \\ G_{3_M} &= \{(m_1, \emptyset), (m_2, \{x\})\}. \end{aligned}$$

It can be checked that (X, σ, M) is tt -soft pre T_4 . On the other hand, the two parametric topological spaces (X, σ_{m_1}) and (X, σ_{m_2}) are not pre T_1 .

Theorem 3.28. The property of being a tt -soft pre T_i -space ($i = 0, 1, 2$) is preserved under a finite product soft spaces.

Proof. For $i = 2$.

Let (X_1, σ_1, M_1) and (X_2, σ_2, M_2) be two tt -soft pre T_2 -spaces and let $(x, y) \neq (x', y')$ in $X_1 \times X_2$. Then, $x \neq x'$ or $y \neq y'$. Without loss of generality, let $x \neq x'$. Then, there exist two disjoint soft pre-open subsets G_{M_1} and H_{M_1} of (X_1, σ_1, M_1) such that $x \in G_{M_1}$ and $x' \notin G_{M_1}$; and $x' \in H_{M_1}$ and $x \notin H_{M_1}$. Obviously, $G_{M_1} \times \widetilde{X}_2$ and $H_{M_1} \times \widetilde{X}_2$ are two disjoint soft pre-open subsets $X_1 \times X_2$ such that $(x, y) \in G_{M_1} \times \widetilde{X}_2$ and $(x', y') \notin G_{M_1} \times \widetilde{X}_2$; and $(x', y') \in H_{M_1} \times \widetilde{X}_2$ and $(x, y) \notin H_{M_1} \times \widetilde{X}_2$. Hence, $X_1 \times X_2$ is a tt -soft pre T_2 -space. $\square$

Theorem 3.29. The property of being a tt -soft pre T_i -space ($i = 0, 1, 2, 3, 4$) is an additive property.

Proof. For $i = 2$.

Let $x \neq y \in \oplus_{i \in I} X_i$. Then, we have the following two cases:

1. There exists $i_0 \in I$ such that $x, y \in X_{i_0}$. Since $(X_{i_0}, \sigma_{i_0}, M)$ is tt -soft pre T_2 , there exist two disjoint soft pre-open subsets G_M and H_M of $(X_{i_0}, \sigma_{i_0}, M)$ such that $x \in G_M$ and $y \in H_M$. By Theorem 2.27, G_M and H_M are disjoint soft pre-open subsets of $(\oplus_{i \in I} X_i, \sigma, M)$.
2. There exists $i_0 \neq j_0 \in I$ such that $x \in X_{i_0}$ and $y \in X_{j_0}$. Now, $\widetilde{X}_{i_0}$ and $\widetilde{X}_{j_0}$ are soft pre-open subsets of $(X_{i_0}, \sigma_{i_0}, M)$ and $(X_{j_0}, \sigma_{j_0}, M)$, respectively. By Theorem (2.27), $\widetilde{X}_{i_0}$ and $\widetilde{X}_{j_0}$ are disjoint soft pre-open subsets of $(\oplus_{i \in I} X_i, \sigma, M)$.

It follows from the two cases above that $(\oplus_{i \in I} X_i, \sigma, M)$ is a tt -soft pre T_2 -space.

For $i = 3$ and $i = 4$, we suffice by proving the tt -soft pre-regularity and soft pre-normality.

First, we prove the tt -soft pre-regularity property. Let F_M be a soft pre-closed subset of $(\oplus_{i \in I} X_i, \sigma, M)$ such that $x \notin F_M$. It comes from Theorem 2.27 that $F_M \widetilde{\cap} \widetilde{X}_i$ is soft pre-closed in (X_i, σ_i, M) for each $i \in I$. Since $x \in \oplus_{i \in I} X_i$, there is only $i_0 \in I$ such that $x \in X_{i_0}$. This implies that there are disjoint soft pre-open subsets G_M and H_M of $(X_{i_0}, \sigma_{i_0}, M)$ such that $F_M \widetilde{\cap} \widetilde{X}_{i_0} \subseteq G_M$ and $y \in H_M$. Now, $G_M \bigcup_{i \neq i_0} \widetilde{X}_i$ is a soft pre-open subset of $(\oplus_{i \in I} X_i, \sigma, M)$ containing F_M . The disjointness of $G_M \bigcup_{i \neq i_0} \widetilde{X}_i$ and H_M ends the proof that $(\oplus_{i \in I} X_i, \sigma, M)$ is a tt -soft pre-regular space.

Second, we prove the soft pre-normality property. Let F_M and H_M be two disjoint soft pre-closed subsets of $(\oplus_{i \in I} X_i, \sigma, M)$. By Theorem 2.27, we obtain $F_M \widetilde{\cap} \widetilde{X}_i$ and $H_M \widetilde{\cap} \widetilde{X}_i$ are soft pre-closed in (X_i, σ_i, M) for every $i \in I$. Since (X_i, σ_i, M) is soft pre-normal for each $i \in I$, there exist two disjoint soft pre-open subsets U_{i_M} and V_{i_M} of (X_i, σ_i, M) such that $F_M \widetilde{\cap} \widetilde{X}_i \subseteq U_{i_M}$ and $H_M \widetilde{\cap} \widetilde{X}_i \subseteq V_{i_M}$. This implies that $F_M \subseteq \bigcup_{i \in I} U_{i_M}$, $H_M \subseteq \bigcup_{i \in I} V_{i_M}$ and $\left[\bigcup_{i \in I} U_{i_M} \right] \widetilde{\cap} \left[\bigcup_{i \in I} V_{i_M} \right] = \widetilde{\Phi}$. Hence, $(\oplus_{i \in I} X_i, \sigma, M)$ is a soft pre-normal space. $\square$

In the following, we prove the behaviors of tt -soft pre T_i -spaces under some soft maps.

Definition 3.30. A map $f_\varphi : (X, \sigma, M) \rightarrow (Y, \tau, B)$ is said to be

1. Soft pre*-continuous if $f_\varphi^{-1}(V_B)$ is a soft pre-open set for each soft pre-open set V_B .
2. Soft pre*-open (resp. soft pre*-closed) if $f_\varphi(U_M)$ is a soft pre-open (resp. soft pre-closed) set for each soft pre-open (resp. soft pre-closed) set U_M .
3. Soft pre*-homeomorphism if it is bijective, soft pre*-continuous and soft pre*-open.

Proposition 3.31. Let $f_\varphi : (X, \sigma, M) \rightarrow (Y, \tau, B)$ be a soft pre-continuous map such that f is injective. Then, if (Y, τ, B) is a p -soft T_i -space ($i = 0, 1, 2$), then (X, σ, M) is a tt -soft pre T_i -space.

Proof. For $i = 2$.

Let $f_\varphi : (X, \sigma, M) \rightarrow (Y, \tau, B)$ be a soft pre-continuous map and $a \neq b \in X$. Since f is injective, there are two distinct points x and y in Y such that $f(a) = x$ and $f(b) = y$. Since (Y, τ, B) is a p -soft T_2 -space, there are two disjoint soft open sets G_B and F_B such that $x \in G_B$ and $y \in F_B$. Now, $f_\varphi^{-1}(G_B)$ and $f_\varphi^{-1}(F_B)$ are two disjoint soft pre-open subsets of (X, σ, M) such that $a \in f_\varphi^{-1}(G_B)$ and $b \in f_\varphi^{-1}(F_B)$. Thus, (X, σ, M) is a tt -soft pre T_2 -space. $\square$

In a similar way, one can prove the following result.

Proposition 3.32. Let $f_\varphi : (X, \sigma, M) \rightarrow (Y, \tau, B)$ be a soft pre*-continuous map such that f is injective. Then, if (Y, τ, B) is a tt -soft pre T_i -space ($i = 0, 1, 2$), then (X, σ, M) is a tt -soft pre T_i -space.

Proposition 3.33. Let $f_\varphi : (X, \sigma, M) \rightarrow (Y, \tau, B)$ be a bijective soft pre-open map. Then, if (X, σ, M) is a p -soft T_i -space, then (Y, τ, B) is a tt -soft pre T_i -space for $i = 0, 1, 2$.

Proof. For $i = 2$.

Let $f_\varphi : (X, \sigma, M) \rightarrow (Y, \tau, B)$ be a soft pre-open map and $x \neq y \in Y$. Since f is bijective, there are two distinct points a and b in X such that $a = f^{-1}(x)$ and $b = f^{-1}(y)$. Since (X, σ, M) is a p -soft T_2 -space, there are two disjoint soft open sets U_M and V_M such that $x \in U_M$ and $y \in V_M$. Now, $f_\varphi(U_M)$ and $f_\varphi(V_M)$ are two disjoint soft pre-open subsets of (Y, τ, B) such that $x \in f_\varphi(U_M)$ and $y \in f_\varphi(V_M)$. Thus, (Y, τ, B) is a tt -soft pre T_2 -space. $\square$

Proposition 3.34. Let $f_\varphi : (X, \sigma, M) \rightarrow (Y, \tau, B)$ be a bijective soft pre*-open map. Then, if (X, σ, M) is a tt -soft pre T_i -space ($i = 0, 1, 2$), then (Y, τ, B) is a tt -soft pre T_i -space.

Proposition 3.35. The property of being tt -soft pre T_i ($i = 0, 1, 2, 3, 4$) is preserved under a soft pre*-homeomorphism map.

We close this part by studying the relationships between tt -soft pre T_i -spaces ($i = 2, 3, 4$) and soft pre-compact spaces.

Proposition 3.36. A stable soft pre-compact subset of a tt -soft pre T_2 -space is soft pre-closed.

Proof. It follows from Proposition 2.21 and Remark 3.2. $\square$

Theorem 3.37. Let H_M be a soft pre-compact subset of a soft hyperconnected tt -soft pre T_2 -space. If $x \notin H_M$, then there are disjoint soft pre-open sets U_M and V_M such that $x \in U_M$ and $H_M \subseteq V_M$.

Proof. Let $x \notin H_M$. Then, $x \neq y$ for each $y \in H_M$. Since (X, σ, M) is a tt -soft pre T_2 -space, there are disjoint soft pre-open sets U_{i_M} and V_{i_M} such that $x \in U_{i_M}$ and $y \in V_{i_M}$. Therefore, $\{V_{i_M}\}$ forms a soft pre-open cover of H_M . Since H_M is soft pre-compact, $H_M \subseteq \bigcup_{i=1}^n V_{i_M}$. By the soft hyperconnectedness of (X, σ, M) , we obtain $\bigcap_{i=1}^n U_{i_M} = U_M$ is a soft pre-open set. Hence, we obtain the desired result. $\square$

Theorem 3.38. *A soft hyperconnected, soft pre-compact and tt -soft pre T_2 -space is tt -soft pre-regular.*

Proof. Let H_M be a soft pre-closed subset of soft pre-compact and tt -soft pre T_2 -space (X, σ, M) such that $x \notin H_M$. Then, H_M is soft pre-compact. By Theorem 3.37, there exist disjoint soft pre-open sets U_M and V_M such that $x \in U_M$ and $H_M \subseteq V_M$. Thus, (X, σ, M) is tt -soft pre-regular. $\square$

Corollary 3.39. *A soft hyperconnected, soft pre-compact and tt -soft pre T_2 -space is tt -soft pre T_3 .*

Lemma 3.40. *Let F_M be a soft pre-open subset of a soft pre-regular space. Then, for each $S_m^x \in F_M$, there exists a soft pre-open set G_M such that $S_m^x \in \overline{G_M}^{pre} \subseteq F_M$.*

Proof. Let F_M be a soft pre-open set such that $S_m^x \in F_M$. Then, $x \notin F_M^c$. Since (X, σ, M) is soft pre-regular, there exist two disjoint soft pre-open sets G_M and W_M totally containing x and F_M^c , respectively. Thus, $x \in G_M \subseteq W_M^c \subseteq F_M$. Hence, $S_m^x \in G_M \subseteq \overline{G_M}^{pre} \subseteq W_M^c \subseteq F_M$. $\square$

Theorem 3.41. *Let H_M be a soft pre-compact subset of a soft pre-regular space and F_M be a soft pre-open set containing H_M . Then, there exists a soft pre-open set G_M such that $H_M \subseteq G_M \subseteq \overline{G_M}^{pre} \subseteq F_M$.*

Proof. Let the given conditions be satisfied. Then, for each $S_m^x \in H_M$, we have $S_m^x \in F_M$. Therefore, there is a soft pre-open set $W_{x \in M}$ such that $S_m^x \in W_{x \in M} \subseteq \overline{W_{x \in M}}^{pre} \subseteq F_M$. Now, $\{W_{x \in M} : S_m^x \in F_M\}$ is a soft pre-open cover of H_M . Since H_M is soft pre-compact, $H_M \subseteq \bigcup_{i=1}^{i=n} W_{x \in M}$. Putting $G_M = \bigcup_{i=1}^{i=n} W_{x \in M}$. Thus, $H_M \subseteq G_M \subseteq \overline{G_M}^{pre} \subseteq F_M$. $\square$

Corollary 3.42. *If (X, σ, M) is soft pre-compact and soft pre T_3 , then it is tt -soft pre T_4 .*

Proof. Suppose that F_{1_M} and F_{2_M} are two disjoint soft pre-closed sets. Then, $F_{2_M} \subseteq F_{1_M}^c$. Since (X, σ, M) is soft pre-compact, F_{2_M} is soft pre-compact and since (X, σ, M) is soft pre-regular, there is a soft pre-open set G_M such that $F_{2_M} \subseteq G_M \subseteq \overline{G_M}^{pre} \subseteq F_{1_M}^c$. Obviously, $F_{2_M} \subseteq G_M$, $F_{1_M} \subseteq (\overline{G_M}^{pre})^c$ and $G_M \cap (\overline{G_M}^{pre})^c = \emptyset$. Thus, (X, σ, M) is soft pre-normal. Since, (X, σ, M) is soft pre T_3 , it is tt -soft pre T_1 . Hence, it is tt -soft pre T_4 . $\square$

4 Pre-fixed soft points of soft mappings

In this part, we define a pre-fixed soft point property and scrutinize main characterizations. Also, we point out its transmission between soft topology and parametric topologies.

Theorem 4.1. *Let $\{\mathcal{B}_n : n \in \mathbb{N}\}$ be a class of soft sets in a soft pre-compact space (X, σ, M) satisfying the following:*

- (i) $\mathcal{B}_n \neq \emptyset$ for each $n \in \mathbb{N}$;
- (ii) $\mathcal{B}_n$ is a soft pre-closed set for each $n \in \mathbb{N}$;
- (iii) $\mathcal{B}_{n+1} \subseteq \mathcal{B}_n$ for each $n \in \mathbb{N}$.

Then, $\bigcap_{n \in \mathbb{N}} \mathcal{B}_n \neq \emptyset$.

Proof. Suppose that $\bigcap_{n \in \mathbb{N}} \mathcal{B}_n = \widetilde{\Phi}$. Then, $\bigcup_{n \in \mathbb{N}} \mathcal{B}_n^c = \widetilde{X}$. It follows from (ii) that $\{\mathcal{B}_n^c : n \in \mathbb{N}\}$ is a soft pre-open cover of $\widetilde{X}$. By hypothesis of soft pre-compactness, there exist $i_1, i_2, \dots, i_k \in \mathbb{N}$, $i_1 < i_2 < \dots < i_k$ such that $\widetilde{X} = \mathcal{B}_{i_1}^c \bigcup \mathcal{B}_{i_2}^c \bigcup \dots \bigcup \mathcal{B}_{i_k}^c$. It follows from (iii) that $\mathcal{B}_{i_k} \subseteq \widetilde{X} = \mathcal{B}_{i_1}^c \bigcup \mathcal{B}_{i_2}^c \bigcup \dots \bigcup \mathcal{B}_{i_k}^c = [\mathcal{B}_{i_1} \widetilde{\cap} \mathcal{B}_{i_2} \widetilde{\cap} \dots \widetilde{\cap} \mathcal{B}_{i_k}]^c = \mathcal{B}_{i_k}^c$. This yields a contradiction. Thus, we obtain the proof that $\bigcap_{n \in \mathbb{N}} \mathcal{B}_n \neq \widetilde{\Phi}$. $\square$

Proposition 4.2. Let (X, σ, M) be a soft pre-compact and soft pre T'_2 -space and $g_\varphi : (X, \sigma, M) \rightarrow (X, \sigma, M)$ be soft pre*-continuous. Then, there is a unique soft point $S_m^x \in \widetilde{X}$ of g_φ .

Proof. Let $\{\mathcal{B}_1 = g_\varphi(\widetilde{X})$ and $\mathcal{B}_n = g_\varphi(\mathcal{B}_{n-1}) = g_\varphi^n(\widetilde{X})$ for each $n \in \mathbb{N}\}$ be a family of soft subsets of (X, σ, M) . It is clear that $\mathcal{B}_{n+1} \subseteq \mathcal{B}_n$ for each $n \in \mathbb{N}$. Since g_φ is soft pre*-continuous, $\mathcal{B}_n$ is a soft pre-compact set for each $n \in \mathbb{N}$ and since (X, σ, M) is soft pre T'_2 , $\mathcal{B}_n$ is also a soft pre-closed set for each $n \in \mathbb{N}$. It follows from Theorem 4.1 that $(H, M) = \bigcap_{n \in \mathbb{N}} \mathcal{B}_n$ is a non-null soft set. Note that $g_\varphi(H, M) = g_\varphi(\bigcap_{n \in \mathbb{N}} g_\varphi^n(\widetilde{X})) \subseteq \bigcap_{n \in \mathbb{N}} g_\varphi^{n+1}(\widetilde{X}) \subseteq \bigcap_{n \in \mathbb{N}} g_\varphi^n(\widetilde{X}) = (H, M)$. To show that $(H, M) \subseteq g_\varphi(H, M)$, suppose that there is a $S_m^x \in (H, M)$ such that $S_m^x \notin g_\varphi(H, M)$. Let $C_n = g_\varphi^{-1}(S_m^x) \widetilde{\cap} \mathcal{B}_n$. Obviously, $C_n \neq \widetilde{\Phi}$ and $C_n \subseteq C_{n-1}$ for each $n \in \mathbb{N}$. Now, C_n is a soft pre-closed set for each $n \in \mathbb{N}$; and by Theorem 4.1, there exists a soft point S_m^y such that $S_m^y \in g_\varphi^{-1}(S_m^x) \widetilde{\cap} \mathcal{B}_n$. Therefore, $S_m^x = g_\varphi(S_m^y) \in g_\varphi(H, M)$. This is a contradiction. Thus, $g_\varphi(H, M) = (H, M)$. Hence, the proof is complete. $\square$

Definition 4.3.

- (i) (X, σ, M) is said to have a pre-fixed soft point property if every soft pre*-continuous map $g_\varphi : (X, \sigma, M) \rightarrow (X, \sigma, M)$ has a fixed soft point.
- (ii) A property that is preserved by any soft pre*-homeomorphism map is called a pre*-soft topological property.

Proposition 4.4. The property of being a pre-fixed soft point is a pre*-soft topological property.

Proof. Let (X, σ, M) and (Y, τ, M) be a soft pre*-homeomorphic. Then, there is a bijective soft map $f_\varphi : (X, \sigma, M) \rightarrow (Y, \tau, M)$ such that f_φ and f_φ^{-1} are soft pre*-continuous. Since (X, σ, M) has a pre-fixed soft point property, every soft pre*-continuous map $g_\varphi : (X, \sigma, M) \rightarrow (X, \sigma, M)$ has a pre-fixed soft point. Now, let $h_\varphi : (Y, \tau, M) \rightarrow (Y, \tau, M)$ be a soft pre*-continuous. Obviously, $h_\varphi \circ f_\varphi : (X, \sigma, M) \rightarrow (Y, \tau, M)$ is a soft pre*-continuous. Also, $f_\varphi^{-1} \circ h_\varphi \circ f_\varphi : (X, \sigma, M) \rightarrow (X, \sigma, M)$ is a soft pre*-continuous. Since (X, σ, M) has a pre-fixed soft point property, $f_\varphi^{-1}(h_\varphi(f_\varphi(S_m^x))) = S_m^x$ for some $S_m^x \in \widetilde{X}$. Consequently, $f_\varphi(f_\varphi^{-1}(h_\varphi(f_\varphi(S_m^x)))) = f_\varphi(S_m^x)$. This implies that $h_\varphi(f_\varphi(S_m^x)) = f_\varphi(S_m^x)$. Thus, $f_\varphi(S_m^x)$ is a pre-fixed soft point of h_φ . Hence, (Y, τ, M) has a pre-fixed soft point property, as required. $\square$

Theorem 4.5. Let σ be an extended soft topology on X . Then, a soft map $g_\varphi : (X, \sigma, M) \rightarrow (Y, \tau, M)$ is soft pre*-continuous if and only if a map $g : (X, \sigma_m) \rightarrow (Y, \tau_{\phi(m)})$ is pre*-continuous.

Proof. *Necessity:* Let U be a pre-open subset of $(Y, \tau_{\phi(m)})$. Then, there exists a soft pre-open subset G_M of (Y, τ, M) such that $G(\phi(m)) = U$. Since g_φ is a soft pre*-continuous map, $g_\varphi^{-1}(G_M)$ is a soft pre-open set. From Definition (2.7), it follows that a soft subset $g_\varphi^{-1}(G_M) = (g_\varphi^{-1}(G))_M$ of (X, σ, M) is given by $g_\varphi^{-1}(G)(m) = g^{-1}(G(\phi(m)))$ for each $m \in M$. By hypothesis, σ is extended, Theorem 2.25 shows a subset $g^{-1}(G(\phi(m))) = g^{-1}(U)$ of (X, σ_m) is pre-open. Hence, a map g is pre*-continuous.

Sufficiency: Let G_M be a soft pre-open subset of (Y, τ, M) . According to Definition 2.7, a soft subset $g_\varphi^{-1}(G_M) = (g_\varphi^{-1}(G))_M$ of (X, σ, M) is given by $g_\varphi^{-1}(G)(m) = g^{-1}(G(\phi(m)))$ for each $m \in M$. Since a map g is pre*-continuous, a subset $g^{-1}(G(\phi(m)))$ of (X, σ_m) is pre-open. By hypothesis, σ is extended, we obtain from Theorem 2.25 that $g_\varphi^{-1}(G_M)$ is a soft pre-open subset of (X, σ, M) . Hence, a soft map g_φ is soft pre*-continuous. $\square$

Definition 4.6. (X, σ) is said to have a pre-fixed point property if every pre*-continuous map $g : (X, \sigma) \rightarrow (X, \sigma)$ has a fixed point.

Proposition 4.7. (X, σ, M) has the property of a pre-fixed soft point iff (X, σ_m) has the property of a pre-fixed point for each $m \in M$.

Proof. *Necessity:* Let (X, σ, M) has the property of a pre-fixed soft point. Then, every soft pre*-continuous map $g_\varphi : (X, \sigma, M) \rightarrow (X, \sigma, M)$ has a fixed soft point. Say, S_m^x . The above theorem shows that $g_m : (X, \sigma_m) \rightarrow (X, \tau_{\phi(m)})$ is pre*-continuous. Since S_m^x is a fixed soft point of g_φ , we obtain $g_m(x) = x$. Thus, g_m has a fixed point.

Sufficiency: Let (X, σ_m) has the property of a pre-fixed point for each $m \in M$. Then, every pre*-continuous map $g_m : (X, \sigma_m) \rightarrow (X, \tau_{\phi(m)})$ has a fixed point. Say, x . The above theorem shows that $g_\varphi : (X, \sigma, M) \rightarrow (X, \tau, M)$ is soft pre*-continuous. Since x is a fixed point of g_m , we get $g_\varphi(S_m^x) = S_m^x$. Thus, g_φ has a fixed soft point. Hence, the proof is complete. $\square$

5 Conclusion

Classification of soft spaces is one of the interesting applications of soft separation axioms. To contribute to this scope, we allocate this paper to explore new families of soft topologies. We have formulated the concepts of tt -soft pre T_i -spaces using the relations of total belong and total non-belong. We have initiated their main properties and elucidated the relationships between them with the aid of several counterexamples. Also, we have investigated a pre-fixed soft point theorem and established main results.

One of the divergences between soft set and crisp set is the diversity of belong and non-belong relations between ordinary points and soft sets. This widely opens the door to establish several classes of separation axiom on soft setting. Therefore, in the upcoming works, we will discuss the presented soft separation axioms using another type of belong and total non-belong relations. Also, we define new types of soft separation axioms using different kinds of generalized soft open sets.

Acknowledgment: The authors would like to thank the editor and the referees for their valuable comments, which help us to improve the manuscript. This publication was supported by the Deanship of Scientific Research at Prince Sattam bin Abdulaziz University, Alkharij, Saudi Arabia.

Conflict of interest: The authors declare that there is no conflict of interest regarding the publication of this article.

References

- [1] D. Molodtsov, *Soft set theory-first results*, Comput. Math. Appl. **37** (1999), 19–31.
- [2] P. K. Maji, R. Biswas, and R. Roy, *An application of soft sets in a decision making problem*, Comput. Math. Appl. **44** (2002), 1077–1083.
- [3] F. Karaaslan, *Soft classes and soft rough classes with applications in decision making*, Math. Probl. Eng. **2016** (2016), 1584528.
- [4] A. Kharal and B. Ahmad, *Mappings on soft classes*, New Math. Nat. Comput. **7** (2011), no. 3, 471–481.
- [5] M. Shabir and M. Naz, *On soft topological spaces*, Comput. Math. Appl. **61** (2011), 1786–1799.
- [6] N. Çağman, S. Karataş, and S. Enginoglu, *Soft topology*, Comput. Math. Appl. **62** (2011), 351–358.
- [7] A. Aygünöğlu and H. Aygün, *Some notes on soft topological spaces*, Neural Comput. & Appl. **21** (2012), 113–119.
- [8] T. Hida, *A comparison of two formulations of soft compactness*, Ann. Fuzzy Math. Inform. **8** (2014), no. 4, 511–524.
- [9] T. M. Al-shami, M. E. El-Shafei, and M. Abo-Elhamayel, *Almost soft compact and approximately soft Lindelöf spaces*, J. Taibah Univ. Sci. **12** (2018), no. 5, 620–630.

- [10] T. M. Al-shami and M. E. El-Shafei, *On soft compact and soft Lindelöf spaces via soft pre-open sets*, Ann. Fuzzy Math. Inform. **17** (2019), no. 1, 79–100.
- [11] T. M. Al-shami, *Comments on “Soft mappings spaces”*, Sci. World J. **2019** (2019), 6903809.
- [12] T. M. Al-shami, *Comments on some results related to soft separation axioms*, Afr. Mat. **31** (2020), no. 7, 1105–1119.
- [13] K. V. Babitha and S. J. John, *Studies on soft topological spaces*, J. Intell. Fuzzy Syst. **28** (2015), 1713–1722.
- [14] K. V. Babitha and S. J. John, *Soft topologies generated by soft set relations*, in: S. J. John (ed.), Handbook of Research on Generalized and Hybrid Set Structures and Applications for Soft Computing, IGI Global Pub, Springer International Publishing; 2016, pp. 118–126.
- [15] T. M. Al-shami, *Soft somewhere dense sets on soft topological spaces*, Commun. Korean Math. Soc. **33** (2018), no. 4, 1341–1356.
- [16] T. M. Al-shami, I. Alshammari, and B. A. Asaad, *Soft maps via soft somewhere dense sets*, Filomat **34** (2020), no. 10, 3429–3440.
- [17] M. E. El-Shafei, M. Abo-Elhamayel, and T. M. Al-shami, *Partial soft separation axioms and soft compact spaces*, Filomat **32** (2018), no. 13, 4755–4771.
- [18] T. M. Al-shami and M. E. El-Shafei, *Partial belong relation on soft separation axioms and decision making problem: two birds with one stone*, Soft Comput. **24** (2020), 5377–5387.
- [19] M. E. El-Shafei and T. M. Al-shami, *Applications of partial belong and total non-belong relations on soft separation axioms and decision-making problem*, Comput. Appl. Math. **39** (2020), no. 3, 138, DOI: <https://doi.org/10.1007/s40314-020-01161-3>.
- [20] D. Wardowski, *On a soft mapping and its fixed points*, Fixed Point Theory Appl. **2013** (2013), 182, DOI: <https://doi.org/10.1186/1687-1812-2013-182>.
- [21] T. M. Al-shami and E. A. Abo-Tabl, *Soft α -separation axioms and α -fixed soft points*, AIMS Mathematics **6** (2021), no. 6, 5675–5694.
- [22] I. Zorlutuna and H. Çakir, *On continuity of soft mappings*, Appl. Math. Inf. Sci. **9** (2015), no. 1, 403–409.
- [23] S. Das and S. K. Samanta, *Soft metric*, Ann. Fuzzy Math. Inform. **6** (2013), no. 1, 77–94.
- [24] S. Nazmul and S. K. Samanta, *Neighbourhood properties of soft topological spaces*, Ann. Fuzzy Math. Inform. **6** (2013), no. 1, 1–15.
- [25] T. M. Al-shami and L. D. R. Kočinac, *The equivalence between the enriched and extended soft topologies*, Appl. Comput. Math. **18** (2019), no. 2, 149–162.
- [26] F. Feng, Y. M. Li, B. Davvaz, and M. I. Ali, *Soft sets combined with fuzzy sets and rough sets: a tentative approach*, Soft Comput. **14** (2010), 899–911.
- [27] M. I. Ali, F. Feng, X. Liu, W. K. Min, and M. Shabir, *On some new operations in soft set theory*, Comput. Math. Appl. **57** (2009), 1547–1553.
- [28] P. K. Maji, R. Biswas, and R. Roy, *Soft set theory*, Comput. Math. Appl. **45** (2003), 555–562.
- [29] K. V. Babitha and S. J. John, *Soft set relations and functions*, Comput. Math. Appl. **60** (2010), 1840–1849.
- [30] I. Arockiarani and A. A. Lancy, *Generalized soft $g\beta$ -closed sets and soft $gs\beta$ -closed sets in soft topological spaces*, Int. J. Math. Arch. **4** (2013), no. 2, 1–7.
- [31] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh, and A. M. Abd El-latif, *Soft connectedness via soft ideals*, J. New Results Sci. **4** (2014), 90–108.
- [32] M. Akdag and A. Ozkan, *On Soft pre-open sets and soft pre separation axioms*, Gazi Univ. J. Sci. **27** (2014), no. 4, 1077–1083.
- [33] T. M. Al-shami, L. D. R. Kočinac, and B. A. Asaad, *Sum of soft topological spaces*, Mathematics **8** (2020), no. 6, 990, DOI: <https://doi.org/10.3390/math8060990>.