Research Article

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An iterative approximation of common solutions of split generalized vector mixed equilibrium problem and some certain optimization problems

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Abstract: In this paper, we study the problem of finding a common solution of split generalized vector mixed equilibrium problem (SGVMEP), fixed point problem (FPP) and variational inequality problem (VIP). We propose an inertial-type iterative algorithm, which uses a projection onto a feasible set and a linesearch, which can be easily calculated. We prove a strong convergence of the sequence generated by the proposed algorithm to a common solution of SGVMEP, fixed point of a quasi- ϕ -nonexpansive mapping and VIP for a general class of monotone mapping in 2-uniformly convex and uniformly smooth Banach space E_1 and a smooth, strictly convex and reflexive Banach space E_2 . Some numerical examples are presented to illustrate the performance of our method. Our result improves some existing results in the literature.

Keywords: split feasibility problem, vector equilibrium problem, generalized mixed equilibrium problem, quasi-monotone mapping, strong convergence, Banach space

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1 Introduction

Let C denote a nonempty, closed and convex subset of a real Banach space E with norm $\|\cdot\|$ and $J : E \rightarrow 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2, \forall x \in E\}, \quad (1)$$

where E^* is the dual space of E and $\langle \cdot, \cdot \rangle$ is the duality pairing between the elements of E and E^* .

Let C and X be nonempty, closed and convex subsets of real Banach spaces E_1 and E_2 , respectively, then the split feasibility problem (SFP) (see [1,2]), consists of finding a point

$$x \in C \text{ such that } Lx \in X, \quad (2)$$

where $L : E_1 \rightarrow E_2$ is a bounded linear operator. The SFP is a variant of the inverse problem and finds real life applications in image processing, radiation therapy and remote sensing [3–6]. For approximating the solutions of (2) and related optimization problems in Hilbert, Banach, Hadamard and p -uniformly convex metric spaces, researchers have developed several iterative methods that adopt a fixed point approach

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(see [3,7–9] and references therein). It has been shown that a point $x \in C$ is a solution of (2) if and only if x is a fixed point of the operator $P_C(I - \gamma L^*(I - P_X)L)$ (see [10]), where P_C, P_X are metric projections onto C and X , respectively, L is a bounded linear operator with adjoint L^* and γ is a positive parameter. Let $T : C \rightarrow C$ be a mapping. A point $x \in C$ is called a fixed point of T , if $x = Tx$. We shall denote the set of fixed points of T by $\text{Fix}(T)$, that is $\text{Fix}(T) := \{x \in C : x = Tx\}$.

Let X be a nonempty, convex subset of a real Banach space E . Assume that P is a proper, pointed, closed, convex cone of a real Hausdorff Y and $e \in \text{int}P$. Let $f : X \times X \rightarrow Y$ be a bifunction, $\psi : X \rightarrow E$ be a nonlinear mapping and $\phi : X \rightarrow Y$ be a function. The generalized vector mixed equilibrium problem (GVMEP) is the problem of finding a point $x \in X$ such that

$$f(x, y) + \langle y - x, \psi(x) \rangle + \phi(y) - \phi(x) \in P, \quad \forall y \in X. \quad (3)$$

Problem (3) so defined is called strong GVMEP. The problem is however said to be weak if the notation $\in P$ in (3) is replaced by the notation $\notin \text{int}P$. Now, suppose in (3) we let $\psi = 0$, then problem reduces to the generalized vector equilibrium problem (GVEP) studied by Kazmi and Farid (see [11]). Also, if we set $\psi = \phi = 0$, then (3) reduces to a vector equilibrium problem (VEP), where VEP consists of finding $x \in X$ such that

$$f(x, y) \in P, \quad \forall y \in X. \quad (4)$$

Moreover, if we set $Y = \mathbb{R}$ and $e = 1$, then Problem (3) reduces to the generalized mixed equilibrium problem considered by Peng and Yao [12]. Furthermore, problem (4) reduces to the classical equilibrium due to Blum and Oettli [13]. Vector equilibrium represents a unified framework for studying several problems, including vector optimization, vector variational inequality, vector complementarity problems and so on [14,15]. In recent years, iterative algorithms for obtaining the equilibrium problems, zero points problems and related optimization problems have been studied in the literature (see [14–20] and references therein).

Let C be a nonempty, closed and convex subset of a real Banach space E with dual E^* and $F : C \rightarrow E^*$ be a mapping. The variational inequality problem (VIP) is to find a point $x \in C$ such that

$$\langle y - x, F(x) \rangle \geq 0, \quad \forall y \in C. \quad (5)$$

We shall denote the solution set of (5) by $\text{VIP}(C, F)$. Closely related to (5) (see [21]) is the problem of finding $y \in C$ such that

$$\langle y - x, F(y) \rangle \geq 0, \quad \forall y \in C. \quad (6)$$

Following [21], we shall refer to (6) as the dual variational inequality problem (DVIP) of (5). The VIP is one of the central problems in nonlinear analysis (see [22–24]) with monotonicity playing a major role in its study. For instance, monotone operators are important tools in the study of several problems in the domain of optimization, nonlinear analysis, differential equation and other related fields. However, there have also been studies of variational inequalities with weaker monotonicity conditions such as pseudomonotone, quasimonotone, strictly quasimonotone, etc. In 2019, Chang et al. [25] studied an iterative approximation of solution of VIP for a semistrictly quasimonotone operator in the framework of infinite-dimensional Hilbert spaces (see [26] and references therein). It is known that extragradient methods for solving VIP require projections onto a set which are difficult to evaluate especially when the structure of the set is not simple. He et al. [27] introduced a totally relaxed self subgradient extragradient method (TRSSEM) involving feasible sets which are easily defined for solving the VIP. Let $C^j := \{z \in E : h_j(z) \leq 0\}$, where $h_j : E \rightarrow \mathbb{R}$ for all $j = 1, 2, \dots, m$ are convex and differentiable functions. For the TRSSEM, the feasible set is defined as

$$C := \bigcap_{j=1}^m C^j.$$

Furthermore, obtaining a common element in the solution set of a fixed point problem (FPP), VIP and EP has recently been considered by authors in the literature due to its various applications, see [28–30]. In 2012, Shan and Huang [31] introduced the concept of generalized mixed vector equilibrium problem (GMVEP). They obtain the existence result in the framework of Hilbert space for this problem. They further

proposed an iterative algorithm for obtaining a common element in the solution set of GMVEP, VIP and fixed point of a nonexpansive mapping. Very recently, Farid and Kazmi [32] introduced and studied a general iterative algorithm for approximating a common solution of split generalized equilibrium problem (SGEP), VIP and FPP. They proved a strong convergence theorem for the sequences generated by the proposed algorithm. For more references on this see [26].

In this paper, motivated by Shan and Huang [31], Chang et al. [25] and He et al. [27], we study an iterative approximation of a common solution of split generalized vector mixed equilibrium problem (SGVMEP), VIP and fixed point of quasi- ϕ -nonexpansive mapping. We proposed an inertial-type iterative algorithm which uses projection onto a feasible set and a linesearch with Halpern method. We prove a strong convergence theorem for the sequence generated by this algorithm to a common solution of these problems in the frame work of 2-uniformly convex and uniformly smooth Banach space E_1 and a smooth, strictly convex and reflexive Banach space E_2 . Finally, some numerical examples are presented to illustrate the performance of our method.

The rest of the paper is organized as follows: We first recall some basic definitions, required assumptions and results in Section 2. We give an explicit statement of the problem and show that its solution set is well defined, and we also propose an iterative process and prove a strong convergence of the method to a solution of the problem in Section 3. Some numerical experiments of our results are given in Section 4. We give concluding remarks in Section 5.

2 Preliminaries

In this section, we give some important definitions, results and restrictions which are useful in establishing our main results. Throughout this paper, we denote the weak and strong convergence of a sequence $\{x_k\}$ in a real Banach space E to a point $x \in E$ by $x_k \rightharpoonup x$ and $x_k \rightarrow x$, respectively.

Let E be a real Banach space, a function $h : E \rightarrow \mathbb{R}$ is said to be:

- (a) Gâteaux differentiable at a point $x \in E$, if there exists an element in E denoted by $h'(x)$ or $\nabla h(x)$ such that

$$\lim_{t \rightarrow 0} \frac{h(x + ty) - h(x)}{t} = \langle y, h'(x) \rangle, \quad \forall y \in E,$$

where $h'(x)$ or $\nabla h(x)$ is called Gâteaux differential or gradient of h at x . h is said to be Gâteaux differentiable on E if it is Gâteaux differentiable on every $x \in E$;

- (b) weakly lower semicontinuous at $x \in E$ if $\{x_k\} \subset E$, $x_k \rightharpoonup x$ implies $h(x) \leq \liminf_{k \rightarrow \infty} h(x_k)$. Also, h is weakly lower semicontinuous on E if h is weakly lower semicontinuous for each $x \in E$;
- (c) if h is such that $h((1 - \lambda)x + \lambda y) \leq (1 - \lambda)h(x) + \lambda h(y)$, for each $x, y \in E$ and $\lambda \in (0, 1)$, then h is said to be a convex function. h is said to be differentiable if the set

$$\partial h(x) := \{w \in E^* : h(y) - h(x) \geq \langle w, y - x \rangle, \quad \forall y \in E\} \neq \emptyset. \quad (7)$$

Each element $\partial h(x)$ is called a subgradient of h at x or the subdifferential of h and inequality (7) is said to be the subdifferential inequality of h at x . We say that the function h is subdifferentiable at E if it is subdifferentiable at every point of E . It is known that if h is Gâteaux differentiable at x , then h is subdifferentiable at x and $\{h'(x)\} = \partial h(x)$, which implies $\partial h(x)$ is singleton (see [33]). For more details on these, see [4,34,35] and references therein.

Following [36], Albert introduced a generalized projection operator $\Pi_C : E \rightarrow C$ defined by

$$\Pi_C(x) = \arg \min_{y \in C} \phi(y, x), \quad \forall x \in E,$$

where $\phi : E \times E \rightarrow \mathbb{R}$ is the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$

The functional ϕ is known to satisfy the following properties:

- (P1) $(\|x\|^2 - \|y\|^2) \leq \phi(x, y) \leq (\|x\|^2 + \|y\|^2)$;
- (P2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$;
- (P3) $\phi(z, \lambda Jx + (1 - \lambda)Jy) \leq \lambda\phi(z, x) + (1 - \lambda)\phi(z, y)$, for all $x, y, z \in E$ and $\lambda \in (0, 1)$;
- (P4) $\phi(x, y) + \phi(y, x) = \langle x - y, Jx - Jy \rangle$, $\forall x, y \in E$;
- (P5) $\phi(x, y) \leq \|x\| \|Jx - Jy\| + \|y\| \|x - y\|$.

In Hilbert space, $\Pi_C = P_C$, the metric projection and $\phi(x, y) = \|x - y\|^2$, see [37] for details on P_C .

Remark 2.1. [38, 39] If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.

We also require the functional $V : E \times E^* \rightarrow \mathbb{R}$ defined by

$$V(u, v) = \|u\|^2 - 2\langle u, v \rangle + \|v\|^2, \quad \text{for each } u \in E \text{ and } v \in E^*.$$

It is easy to see that $V(u, v) = \phi(u, J^{-1}v)$ for all $u \in E$ and $v \in E^*$. It is well known (see [40]) that if E is a reflexive, strictly convex and smooth Banach space, then

$$V(u, v) \leq V(u, v + w) - 2\langle J^{-1}v - u, w \rangle, \quad (8)$$

for all $u \in E$ and $v, w \in E^*$.

Let C be a nonempty, closed and convex subset of a real Banach space and $T : C \rightarrow C$ be a mapping. A point $\bar{x} \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_k\}$ such that $x_k \rightarrow \bar{x}$ and $\|x_k - Tx_k\| \rightarrow 0$ as $k \rightarrow \infty$. We denote by $\widehat{\text{Fix}}(T)$ the asymptotic fixed point set of T . A mapping T from C into itself is said to be relatively nonexpansive [41,42], if $\widehat{\text{Fix}}(T) = \text{Fix}(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in \text{Fix}(T)$. Furthermore, T is said to be ϕ -nonexpansive, if $\phi(Tx, Ty) \leq \phi(x, y)$ for $x, y \in C$. T is said to be quasi- ϕ -nonexpansive if $\text{Fix}(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in \text{Fix}(T)$. The class of quasi- ϕ -nonexpansive mappings is more general than the class of relatively nonexpansive mappings [41,43] as the latter requires the strong restriction $\widehat{\text{Fix}}(T) = \text{Fix}(T)$.

Let E a real Banach space. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\rho_E(t) := \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t \right\}.$$

E is uniformly smooth if and only if

$$\lim_{t \rightarrow 0} \frac{\rho_E(t)}{t} = 0.$$

Let E be a real Banach space with a dimension greater or equal to 2, the modulus of convexity of E is the function $\delta_E : (0, 2] \rightarrow [0, 1]$ defined by

$$\delta_E(\varepsilon) := \inf \left\{ 1 - \frac{1}{2}\|x + y\| : \|x\| = 1 = \|y\|; \|x - y\| = \varepsilon \right\}.$$

E is uniformly convex if for any $\varepsilon \in (0, 2]$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in E$ with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$, then $\frac{1}{2}\|x + y\| \leq 1 - \delta$. Equivalently, E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. E is said to be 2-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\varepsilon) > c\varepsilon^2$ for every $\varepsilon \in (0, 2]$, where c is the 2-uniformly convexity constant of E . It is known [38,44,45] that every 2-uniformly convex Banach space is uniformly convex and reflexive. The space E is strictly convex, if for each $x \neq y \in E$ and $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1 - \lambda)y\| < 1$ for all $\lambda \in (0, 1)$.

Lemma 2.2. [46] Given a number $s > 0$. A real Banach space E is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|),$$

for all $x, y \in E$, $\lambda \in [0, 1]$, with $\|x\| < s$ and $\|y\| < s$.

Lemma 2.3. [36] Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E . If $x \in E$ and $q \in C$, then

$$q = \Pi_C x \Leftrightarrow \langle y - q, Jx - Jq \rangle \leq 0, \quad \forall y \in C \quad (9)$$

and

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C, x \in E. \quad (10)$$

Lemma 2.4. [47] Let E be a smooth and uniformly convex real Banach space and let $\{x_k\}$ and $\{y_k\}$ be two sequences in E . If either $\{x_k\}$ or $\{y_k\}$ is bounded and $\phi(x_k, y_k) \rightarrow 0$ as $k \rightarrow \infty$, then $\|x_k - y_k\| \rightarrow 0$ as $k \rightarrow \infty$.

Remark 2.5. [48] Using P5, it is easy to see that converse of Lemma 2.4 is also true whenever both $\{x_k\}$ and $\{y_k\}$ are bounded.

Lemma 2.6. [49] Let E be a 2-uniformly convex and smooth Banach space. Then for every $x, y \in E$

$$\phi(x, y) \geq v\|x - y\|^2,$$

where $v > 0$ is the 2-uniformly convexity constant of E .

Lemma 2.7. [46] Let E be a 2-uniformly smooth Banach space with the best smoothness constant $d > 0$. Then, the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + \langle y, Jx \rangle + 2\|dy\|^2, \quad \forall x, y \in E.$$

Definition 2.8. [20,50] Let X and Y be two Hausdorff topological spaces and let D be a nonempty, convex subset of X and P be a pointed, proper, closed and convex cone of Y with $\text{int}P \neq \emptyset$. Let 0 be the zero point of Y , $N(0)$ be the neighbourhood set of 0 , $N(x_0)$ be the neighbourhood set of x_0 and $f : D \rightarrow Y$ be a mapping.

(1) If, for any $V \in N(0) \subset Y$, there exists $\mathcal{U} \in N(x_0)$ such that

$$f(x) \in f(x_0) + V + P, \quad \forall x \in \mathcal{U} \cap D,$$

then f is called upper P -continuous on x_0 . If f is upper P -continuous for all $x \in D$, then f is called upper P -continuous on D ;

(2) If, for any $V \in N(0) \subset Y$, there exists $\mathcal{B} \in N(x_0)$ such that

$$f(x) \in f(x_0) + V - P, \quad \forall x \in \mathcal{B} \cap D,$$

then f is called lower P -continuous on x_0 . f is called lower P -continuous on D , if f is lower P -continuous at every $x \in D$;

(3) If, for any $x, y \in D$ and $\lambda \in [0, 1]$, the mapping f satisfies $f(x) \in f(\lambda x + (1-\lambda)y) + P$ or $f(y) \in f(\lambda x + (1-\lambda)y) + P$, then f is called proper P -quasiconvex;

(4) If, for $x, y \in D$ and $\lambda \in [0, 1]$, the mapping f satisfies

$$\lambda f(x) + (1-\lambda)f(y) \in f(\lambda x + (1-\lambda)y) + P,$$

then f is called P -convex.

Lemma 2.9. [51] Let X and Y be two real Hausdorff topological spaces, D is a nonempty, compact, convex subset of X and P is a pointed, proper, closed and convex cone of Y with $\text{int}P \neq \emptyset$. Assume $f : D \times D \rightarrow Y$ and $\Phi : D \rightarrow Y$ are two vector mappings. Suppose f and Φ satisfy

- (a) $f(x, x) \in P$, for all $x \in D$;
- (b) Φ is upper P -continuous on D ;
- (c) $f(\cdot, y)$ is lower P -continuous for all $y \in D$;
- (d) $f(x, \cdot) + \Phi(\cdot)$ is proper P -quasiconvex for all $x \in D$.

Then, there exists a point $x \in D$ that satisfies $G(x, y) \in P \setminus \{0\}$, $\forall y \in D$ where

$$G(x, y) = f(x, y) + \Phi(y) - \Phi(x), \quad \forall x, y \in D.$$

For solving the GMVEP, we give the following assumptions: Let $X \subset E_2$ be a nonempty, compact, convex subset of real Banach space E_2 and Y a real Hausdorff topological space, $P \subset Y$ is a proper, closed and convex cone. Let $f : X \times X \rightarrow Y$ and $\psi : X \rightarrow Y$ be two mappings. For any $x \in E_2$, define a mapping $\Psi_x : X \times X \rightarrow Y$ as follows:

$$\Psi_x(y, z) = f(z, y) + \psi(z) - \psi(y) + \frac{e}{r} \langle y - z, Jz - Jx \rangle, \quad (11)$$

where r is a positive number in $\mathbb{R}$ and $e \in \text{int}P$. Let Ψ_x, f and ψ satisfy the following conditions:

- (R1) For all $x \in X$, $f(x, x) = 0$;
- (R2) f is monotone, that is $f(x, y) + f(y, x) \in -P$, $\forall x, y \in X$;
- (R3) $f(\cdot, \cdot)$ is continuous, $\forall y \in X$;
- (R4) $f(x, \cdot)$ is weakly continuous and P -convex, that is,

$$\lambda f(x, y) + (1 - \lambda)f(x, z) \in f(x, \lambda y + (1 - \lambda)z) + P, \quad \forall x, y, z \in X \quad \text{and} \quad \forall \lambda \in [0, 1];$$

- (R5) $\Psi_x(\cdot, y)$ is lower P -continuous for all $y \in X$ and $z \in E_2$;
- (R6) $\psi_x(\cdot)$ is P -continuous;
- (R7) $\Psi_x(x, \cdot)$ is proper P -quasiconvex for all $x \in X$ and $z \in E_2$.

The following result was proved in [31] in the framework of Hilbert space, but can easily be adapted for this study.

Lemma 2.10. [31] Let f and ψ satisfy restrictions (R1)–(R7). Define a mapping $K_r : E_2 \rightarrow X$ as follows:

$$K_r(x) = \{z \in E_2 : f(z, y) + \psi(z) - \psi(y) + \frac{e}{r} \langle y - z, Jz - Jx \rangle \in P, \quad \forall y \in X\}. \quad (12)$$

Then,

- (i) $K_r(x) \neq \emptyset$ for all $x \in E_2$;
- (ii) K_r is single valued;
- (iii) K_r is firmly nonexpansive-type mapping, that is for all $x, y \in E_2$,

$$\langle K_r(x) - K_r(y), JK_r(x) - JK_r(y) \rangle \leq \langle K_r x - K_r y, Jx - Jy \rangle;$$

- (iv) $\text{Fix}(K_r) = \text{Sol}(\text{GV EP})$;
- (v) $\text{Sol}(\text{GV EP})$ is closed and convex.

Let E be a real Banach space. Given $x, y \in E$, define the open segment

$$(x, y) = \{tx + (1 - t)y : 0 < t < 1\}.$$

The segments $(x, y]$, $[x, y)$ and $[x, y]$ are defined analogously.

Definition 2.11. [25] A mapping $F : E \rightarrow E^*$ is said to be

- (a) weakly hemicontinuous if F is upper semicontinuous from line segments to the weak topology of E ;
- (b) sequentially weakly semicontinuous if for each sequence $\{x_k\}$ in E with $x_k \rightarrow q$, then $Fx_k \rightarrow Fq$.

It is easy to check that (b) implies (a).

Lemma 2.12. [21] *A solution of DVIP is always a solution of VIP, if the operator F is weakly hemicontinuous.*

Remark 2.13. It is well known that $p \in C$ is a solution of (5) if and only if p is a fixed point of the operator $P_C(I - \lambda F)$ for all $\lambda > 0$.

Definition 2.14. Let C be a nonempty closed and subset of a real Banach space E with dual E^* . The mapping $F : C \rightarrow E^*$ is said to be:

(i) m -inverse strongly monotone on C with constant $m > 0$ if for each $u, v \in C$, there holds

$$\langle u - v, F(u) - F(v) \rangle \geq m \|F(u) - F(v)\|^2;$$

(ii) strongly monotone on C with constant $\tau > 0$ if for each $u, v \in C$, there holds

$$\langle u - v, F(u) - F(v) \rangle \geq \tau \|u - v\|^2;$$

(iii) strictly monotone on C if for all distinct $u, v \in C$, there holds

$$\langle u - v, F(u) - F(v) \rangle > 0;$$

(iv) monotone on C if for each $u, v \in C$, there holds

$$\langle u - v, F(u) - F(v) \rangle \geq 0;$$

(v) pseudo-monotone on C if for each $u, v \in C$, there holds

$$\langle u - v, F(v) \rangle \geq 0 \Rightarrow \langle u - v, F(u) \rangle \geq 0;$$

(vi) quasi-monotone on C if for each $u, v \in C$, there holds

$$\langle u - v, F(v) \rangle > 0 \Rightarrow \langle u - v, F(u) \rangle \geq 0;$$

(vii) (see [52]) semistrictly quasi-monotone on C if F is quasi-monotone on C and for all distinct of points $u, v \in C$, we have that

$$\langle u - v, F(v) \rangle > 0 \Rightarrow \langle u - v, F(w) \rangle > 0,$$

for some $w \in \left(\frac{1}{2}(u + v), u\right)$.

Lemma 2.15. [53] *Let C be a nonempty, closed and convex subset of E and $F : C \rightarrow E$ be a weakly hemicontinuous and semistrictly quasi-monotone mapping. Then $DVI(C, F)$ at least has one solution.*

Lemma 2.16. [54] *Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{k+1} \leq (1 - \alpha_k)a_k + \alpha_k \sigma_k + \gamma_k, \quad k \geq 0,$$

where

(a) $\{\alpha_k\} \subset [0, 1]$, $\lim_{k \rightarrow \infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$;

(b) $\limsup_{k \rightarrow \infty} \sigma_k \leq 0$;

(c) $\gamma_k \geq 0$, $(k \geq 1)$ and $\sum_{k=1}^{\infty} \gamma_k < \infty$.

Then, $\lim_{k \rightarrow \infty} a_k = 0$.

Lemma 2.17. [9,42] *Let $\{a_k\}$ be a sequence of real numbers such that there exists a subsequence $\{k_j\}$ of $\{k\}$ such that $a_{k_j} < a_{k_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing subsequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \rightarrow \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$: $a_{m_k} < a_{m_k+1}$ and $a_k < a_{m_k+1}$. In fact, $m_k = \max\{i \leq k : a_i < a_{i+1}\}$.*

3 Main result

In this section, we prove our main result. First, we explicitly state the problem considered in this paper, then we introduce a linesearch algorithm for obtaining the solution of this problem and finally discuss its convergence analysis.

Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E_1 , X be a nonempty, compact and convex subset of a smooth, strictly convex and reflexive Banach space E_2 and $L : E_1 \rightarrow E_2$ be a bounded linear operator with $L^* : E_2^* \rightarrow E_1^*$ its adjoint. Let P be a pointed, proper, closed and convex cone of a real Hausdorff topological space Y . Let $f : X \times X \rightarrow Y$ and $A : X \rightarrow Y$ functions satisfying assumptions (R1)–(R7). Let $\psi : X \rightarrow E_2$ be an m -inverse strongly monotone mapping and $T : C \rightarrow C$ be a quasi- ϕ -nonexpansive mapping. Also, let $F : C \rightarrow E^*$ be a semistrictly quasi-monotone, sequentially weakly continuous mapping. We consider the problem of finding a point $p \in C$ such that

$$p \in \text{VIP}(C, F) \cap \text{Fix}(T) \quad \text{such that } Lp \in \text{GVMEP}(f, A, \psi, X). \quad (13)$$

Assume $\Gamma \neq \emptyset$, where Γ denotes the solution set of problem (13). We note that Γ is closed and convex. Indeed, following [41], the fixed point of quasi- ϕ -nonexpansive mapping T is closed and convex. Also by Lemma 2.10(v), we have that $\text{GVMEP}(f, A, \psi, X)$ is closed and convex and finally by Lemma 2.15, $\text{VIP}(C, F)$. Hence, the solution of (13) is well defined. To obtain the solution of (13), we consider the following iterative algorithm:

Algorithm 3.1. Pick $x_0, x_1, u \in C$ and choose the parameters $\rho \in (0, 1)$, $\eta \in (0, \nu)$, where ν is the 2-uniform convexity constant of E_1 and suppose that $\{\alpha_k\} \subset [a, e]$ for some $0 < a < e < 1$, $\{\beta_k\} \subseteq [d, b]$ for some $0 < d < b < 1$, $\{\eta_k\}$ be a sequence of positive real numbers and $\gamma \in \left(0, \frac{1}{d^2 \|L\|^2}\right)$ and d is the best smoothness constant of E_2 . For $j = 1, 2, \dots, m$, let $h_j : E \rightarrow R$ be a family of convex, weakly lower semicontinuous and Gâteaux differentiable functions such that $h'_j(\cdot)$ is K_j -Lipschitz continuous with $K = \max_{1 \leq j \leq m} K_j$. The feasible set C is defined by

$$C := \bigcap_{i=1}^m C^i,$$

where

$$C^j := \{z \in E : h_j(z) \leq 0\}.$$

We also assume the following conditions are satisfied:

$$(B1) \quad \lim_{k \rightarrow \infty} \beta_k = 0, \quad \sum_{k=1}^{\infty} \beta_k = \infty;$$

$$(B2) \quad \liminf_{k \rightarrow \infty} \alpha_k(1 - \alpha_k) > 0;$$

$$(B3) \quad \sum_{k=1}^{\infty} \phi(x_k, x_{k-1}) < \infty;$$

$$(B4) \quad \lim_{k \rightarrow \infty} \frac{\theta_k}{\beta_k} = 0.$$

For each $k > 0$, having the k -iterate $\{x_k\}$, compute the following steps:

Step I: For $j = 1, 2, \dots, m$ and given the current iterate, construct the family of half spaces

$$C_k^j = \{w \in E_1 : h_j(w_k) + \langle h'_j(w_k), w - w_k \rangle \leq 0\}$$

and set

$$C_k = \bigcap_{j=1}^m C_k^j.$$

Compute

$$\begin{cases} w_k = J_1^{-1}(J_1 x_k + \theta_k(J_1 x_{k-1} - J_1 x_k)), \\ z_k = \Pi_{C_k J_1^{-1}}(J_1 w_k - \lambda_k F w_k), \end{cases} \quad (14)$$

where $\lambda_k = \rho^{l_k}$ and l_k is the smallest nonnegative integer such that

$$\lambda_k \|Fw_k - Fz_k\| \leq \eta \|w_k - z_k\|. \quad (15)$$

Step II: If $w_k = z_k$, ($w_k \in \text{VIP}(C, F)$), then $w_k = y_k$ and go to step III. Otherwise, compute the next iterate by

$$y_k = \Pi_{Q_k} J_1^{-1}(J_1 w_k - \lambda_k Fz_k), \quad (16)$$

where $Q_k = \{w \in E_1 : \langle w - z_k, J_1 w_k - \lambda_k Fw_k - J_1 z_k \rangle \leq 0\}$.

Step III: Compute

$$\begin{cases} u_k = J_1^{-1}(\alpha_k J_1 y_k + (1 - \alpha_k) J_1 T y_k), \\ v_k = J_1^{-1}(J_1 u_k + \gamma L^* J_2((K_{r_k}(I - rA) - I) L u_k)), \\ x_{k+1} = J_1^{-1}(\beta_k J_1 u + (1 - \beta_k) J_1 v_k). \end{cases} \quad (17)$$

Step IV: Set $k := k + 1$ and go to step I.

Lemma 3.2. *If $w_k = z_k$, then $w_k \in \text{VIP}(C, F)$.*

Proof. Suppose $w_k = z_k$, then by the characteristics of Π_{C_k} and (15), we have

$$\langle Jw_k - \lambda_k F(w_k) - Jw_k, y - z_k \rangle \leq 0, \quad \forall y \in C_k$$

or equivalently

$$\lambda_k \langle F(w_k), y - w_k \rangle \geq 0, \quad \forall y \in C_k. \quad (18)$$

Since $\lambda_k > 0$, we obtain

$$\langle F(w_k), y - w_k \rangle \geq 0, \quad \forall y \in C_k. \quad (19)$$

Hence, $w_k \in \text{VIP}(C_k, F)$. We conclude therefore from this, $w_k \in C$ and $C \in C_k$ that $w_k \in \text{VIP}(C, F)$. $\square$

In what follows, we shall show that the Armijo Linesearch rule (15) is well defined.

Lemma 3.3. *There exists a nonnegative integer l_k satisfying (15) of Algorithm 3.1.*

Proof. If $w_k \in \text{VIP}(C, F)$, then $w_k = \Pi_{C_k} J_1^{-1}(J_1 w_k - \lambda_k Fw_k)$ and $l_k = 0$. Hence, we consider the case where $w_k \notin \text{VIP}(C, F)$ and assume the contrary, that is for $l > 0$

$$\lambda_k \|Fw_k - Fz_k\| > \eta \|w_k - z_k\|.$$

This implies

$$\|Fw_k - Fz_k\| > \frac{\eta \|w_k - z_k\|}{\lambda_k}. \quad (20)$$

Next, we consider the following possibilities, $w_k \in C_k$ and $w_k \notin C_k$. First, let $w_k \in C$, then $w_k \in C_k$ and $w_k = \Pi_{C_k}(w_k)$. Using the continuity of F and Π_{C_k} , then $\lim_{k \rightarrow \infty} \|w_k - z_k\| = 0$. Consequently,

$$\lim_{k \rightarrow \infty} \|Fw_k - Fz_k\| = 0. \quad (21)$$

We have from (20) and (21) that

$$\lim_{k \rightarrow \infty} \frac{\|w_k - z_k\|}{\lambda_k} = 0.$$

Using the continuity of J_1 on bounded subsets of E_1 , we get

$$\lim_{k \rightarrow \infty} \frac{\|J_1 w_k - J_1 z_k\|}{\lambda_k} = 0. \quad (22)$$

By (9), we obtain

$$\langle J_1 z_k - J_1 w_k + \lambda_k F w_k, y - z_k \rangle \geq 0, \quad \forall y \in C_k,$$

this implies

$$\langle y - z_k, F w_k \rangle \geq \frac{\langle J_1 w_k - J_1 z_k, y - z_k \rangle}{\lambda_k}, \quad \forall y \in C_k. \quad (23)$$

Letting $k \rightarrow \infty$ and using (22), we get

$$\langle y - w_k, F w_k \rangle \geq 0, \quad \forall y \in C_k, \quad (24)$$

which implies $w_k \in \text{VIP}(C_k, F)$. It is easy to see from $w_k \in C$ and $C \subset C_k$ that $w_k \in \text{VIP}(C, F)$, a contradiction. On the other hand, suppose $w_k \in C_k$, then

$$\lim_{k \rightarrow \infty} \|w_k - z_k\| = \|w_k - \Pi_{C_k} w_k\| > 0 \quad (25)$$

and

$$\lim_{k \rightarrow \infty} \lambda_k \|F w_k - F z_k\| = 0. \quad (26)$$

By using (20), (25) and (26), we obtain a contradiction. Therefore, the linesearch (15) is well defined. $\square$

For our convergence analysis, we will assume that $\sup_{k \geq 1} l_k < \infty$, which implies $\inf_{k \geq 1} \lambda_k > 0$.

In the following result, we prove the boundedness of the sequence generated by our proposed method.

Lemma 3.4. *Let $\{x_k\}$ be the sequence given by Algorithm 3.1 and $x^* \in \Gamma$. Then, $\{\phi(x^*, x_k)\}$ is bounded. Consequently, the sequences $\{x_k\}$, $\{y_k\}$, $\{w_k\}$, $\{u_k\}$, $\{v_k\}$ and $\{L u_k\}$ are bounded.*

Proof. Fix $p \in \Gamma$, then from Lemma 2.3, we have that

$$\begin{aligned} \phi(p, y_k) &= \phi(p, \Pi_{Q_k} J_1^{-1}(J_1 w_k - \lambda_k F z_k)) \\ &\leq \phi(p, J_1^{-1}(J_1 w_k - \lambda_k F z_k)) - \phi(y_k, J_1^{-1}(J_1 w_k - \lambda_k F z_k)) \\ &= \|p\|^2 - 2\langle p, J_1 w_k - \lambda_k F z_k \rangle + \|J_1 w_k - \lambda_k F z_k\|^2 - (\|y_k\|^2 - 2\langle p, J_1 w_k - \lambda_k F z_k \rangle + \|J_1 w_k - \lambda_k F z_k\|^2) \\ &= \|p\|^2 - 2\langle p, J_1 w_k - \lambda_k F z_k \rangle - \|y_k\|^2 + 2\langle y_k, J_1 w_k - \lambda_k F z_k \rangle \\ &= \|p\|^2 - 2\langle p, J_1 w_k \rangle + \langle p, \lambda_k F z_k \rangle - \|y_k\|^2 + 2\langle y_k, J_1 w_k \rangle - \langle y_k, \lambda_k F z_k \rangle \\ &= \|p\|^2 - 2\langle p, J_1 w_k \rangle + \|w_k\|^2 - (\|y_k\|^2 - 2\langle y_k, J_1 w_k \rangle + \|w_k\|^2) + 2\lambda_k \langle p - y_k, F z_k \rangle \\ &= \phi(p, w_k) - \phi(y_k, w_k) + 2\lambda_k \langle p - y_k, F z_k \rangle, \end{aligned}$$

which by (P2) implies

$$\begin{aligned} \phi(p, y_k) &= \phi(p, w_k) - (\phi(y_k, z_k) + \phi(z_k, w_k) + 2\langle y_k - z_k, J_1 z_k - J_1 w_k \rangle) + 2\lambda_k [\langle p - z_k, F z_k \rangle + \langle z_k - y_k, F z_k \rangle] \\ &= \phi(p, w_k) - \phi(y_k, z_k) - \phi(z_k, w_k) - 2\langle y_k - z_k, J_1 z_k - J_1 w_k \rangle + 2\lambda_k \langle p - z_k, F z_k \rangle - 2\lambda_k \langle y_k - z_k, F z_k \rangle \\ &= \phi(p, w_k) - \phi(y_k, z_k) - \phi(z_k, w_k) + 2\langle y_k - z_k, J_1 w_k - \lambda_k F z_k - J_1 z_k \rangle + 2\lambda_k \langle p - z_k, F z_k \rangle \\ &\leq \phi(p, w_k) - \phi(y_k, z_k) - \phi(z_k, w_k) + 2\langle y_k - z_k, J_1 w_k - \lambda_k F z_k - J_1 z_k \rangle. \end{aligned} \quad (27)$$

We obtained the last inequality by using the fact that $p \in \Gamma$ and the definition of F . Indeed, for any $p \in \Gamma$, we have $\langle y - p, F(p) \rangle \geq 0$, $\forall y \in C$. Replacing y by z_k , we get $\langle z_k - p, F(p) \rangle \geq 0$. Since F is semi-quasi strictly monotone, we obtain $\langle z_k - p, F(p) \rangle > 0 \Rightarrow \langle z_k - p, F(z_k) \rangle > 0$. Thus, $\langle p - z_k, F(z_k) \rangle \leq 0$.

By the definition of Q_k and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} 2\langle y_k - z_k, J_1 w_k - \lambda_k F z_k - J_1 z_k \rangle &= 2\langle y_k - z_k, J_1 w_k - \lambda_k F w_k - J_1 z_k \rangle + 2\lambda_k \langle y_k - z_k, F w_k - F z_k \rangle \\ &\leq 2\lambda_k \langle y_k - z_k, F w_k - F z_k \rangle \end{aligned}$$

and by (15), we get

$$2\langle y_k - z_k, J_1 w_k - 2\lambda_k F z_k - J_1 z_k \rangle \leq 2\eta \|y_k - z_k\| \cdot \|w_k - z_k\|.$$

Using Lemma 2.6, one gets

$$2\eta\|y_k - z_k\| \cdot \|w_k - z_k\| \leq 2\eta\sqrt{\frac{\phi(y_k, z_k)}{\nu} \times \frac{\phi(z_k, w_k)}{\nu}} \leq \frac{\eta}{\nu}(\phi(y_k, z_k) + \phi(z_k, w_k)),$$

that is

$$2\langle y_k - z_k, J_1 w_k - 2\lambda_k Fz_k - J_1 z_k \rangle \leq \frac{\eta}{\nu}(\phi(y_k, z_k) + \phi(z_k, w_k)). \quad (28)$$

Therefore, from (27) and (28), we obtain that

$$\phi(p, y_k) \leq \phi(p, w_k) - \left(1 - \frac{\eta}{\nu}\right)(\phi(y_k, z_k) + \phi(z_k, w_k)). \quad (29)$$

Now, from (17) and Lemma 2.7, we have

$$\begin{aligned} \phi(p, u_k) &= \phi(p, J_1^{-1}(\alpha_k J_1 y_k + (1 - \alpha_k) J_1 T y_k)) \\ &= \|p\|^2 - 2\langle p, \alpha_k J_1 y_k + (1 - \alpha_k) J_1 T y_k \rangle + \|\alpha_k J_1 y_k + (1 - \alpha_k) J_1 T y_k\|^2 \\ &= \|p\|^2 - 2\alpha_k \langle p, J_1 y_k \rangle - 2(1 - \alpha_k) \langle p, J_1 T y_k \rangle + \alpha_k \|y_k\|^2 + (1 - \alpha_k) \|T y_k\|^2 \\ &\quad - \alpha_k (1 - \alpha_k) g(\|J_1 y_k - J_1 T y_k\|) \\ &\leq \alpha_k \phi(p, y_k) + (1 - \alpha_k) \phi(p, T y_k) - \alpha_k (1 - \alpha_k) g(\|J_1 y_k - J_1 T y_k\|) \\ &= \alpha_k \phi(p, y_k) + (1 - \alpha_k) \phi(p, y_k) - \alpha_k (1 - \alpha_k) g(\|J_1 y_k - J_1 T y_k\|) \\ &\leq \phi(p, y_k) - \alpha_k (1 - \alpha_k) g(\|J_1 y_k - J_1 T y_k\|) \\ &\leq \phi(p, y_k). \end{aligned} \quad (30)$$

Again by using (17) and Lemma 2.7, we have

$$\begin{aligned} \phi(p, v_k) &= \phi(p, J_1^{-1}(J_1 u_k + \gamma L^* J_2((K_{r_k}(I - r_k A) - I) Lu_k))) \\ &= \|p\|^2 - 2\langle p, J_1 u_k + \gamma L^* J_2((K_{r_k}(I - r_k A) - I) Lu_k) \rangle + \|J_1 u_k + \gamma L^* J_2((K_{r_k}(I - r_k A) - I) Lu_k)\|^2 \\ &= \|p\|^2 - 2\langle p, J_1 u_k \rangle - 2\gamma \langle Lp, J_2((K_{r_k}(I - r_k A) - I) Lu_k) \rangle + \|u_k\|^2 \\ &\quad + 2\gamma \langle Lu_k, J_2((K_{r_k}(I - r_k A) - I) Lu_k) \rangle + 2\|d\gamma L^* J_2(K_{r_k}(I - r_k A) - I) Lu_k\|^2 \\ &= \|p\|^2 - 2\langle p, J_1 u_k \rangle - 2\gamma \langle Lp - Lu_k, J_2((K_{r_k}(I - r_k A) - I) Lu_k) \rangle + \|u_k\|^2 \\ &\quad + 2d^2 \gamma^2 \|L\|^2 \|(K_{r_k}(I - r_k A) - I) Lu_k\|^2 \\ &= \phi(p, u_k) - 2\gamma_k \langle Lp - Lu_k, J_2((K_{r_k}(I - r_k A) - I) Lu_k) \rangle + 2d^2 \gamma^2 \|L\|^2 \|(K_{r_k}(I - r_k A) - I) Lu_k\|^2. \end{aligned} \quad (31)$$

Since $(K_{r_k}(I - r_k A) - I)$ is firmly nonexpansive type, we have

$$\begin{aligned} &\langle Lp - Lu_k, J_2((K_{r_k}(I - r_k A) - I) Lu_k) \rangle \\ &= \langle Lp - K_{r_k}(I - r_k A) Lu_k, J_2((K_{r_k}(I - r_k A) - I) Lu_k) \rangle + \|(K_{r_k}(I - r_k A) - I) Lu_k\|^2 \\ &\geq \|(K_{r_k}(I - r_k A) - I) Lu_k\|^2, \end{aligned}$$

using this in (31), we get that

$$\begin{aligned} \phi(p, v_k) &\leq \phi(p, u_k) - 2\gamma \|(K_{r_k}(I - r_k A) - I) Lu_k\|^2 + 2d^2 \gamma^2 \|L\|^2 \|(K_{r_k}(I - r_k A) - I) Lu_k\|^2 \\ &= \phi(p, u_k) - 2\gamma(1 - \gamma d^2 \|L\|^2) \|(K_{r_k}(I - r_k A) - I) Lu_k\|^2 \\ &\leq \phi(p, u_k). \end{aligned} \quad (32)$$

Furthermore, from (17), $0 < \eta < \nu$ and (P3), we have

$$\begin{aligned} \phi(p, x_{k+1}) &= \phi(p, J_1^{-1}(\beta_k J_1 u + (1 - \beta_k) J_1 v_k)) \\ &\leq \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, v_k) \\ &\leq \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, u_k) \end{aligned} \quad (33)$$

$$\begin{aligned}
&\leq \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, y_k) \\
&\leq \beta_k \phi(p, u) + (1 - \beta_k) \left(\phi(p, w_k) - \left(1 - \frac{\eta}{v} \right) (\phi(y_k, z_k) + \phi(w_k, z_k)) \right) \\
&= \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, w_k) - (1 - \beta_k) \left(1 - \frac{\eta}{v} \right) (\phi(y_k, z_k) + \phi(w_k, z_k)) \\
&\leq \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, w_k).
\end{aligned} \tag{34}$$

Again from (P3) and (14), we have

$$\begin{aligned}
\phi(p, w_k) &= \phi(p, J_1^{-1}(Jx_k + \theta_k(J_1x_{k-1} - J_1x_k))) \\
&= \phi(p, J_1^{-1}(\theta_k J_1x_{k-1} + (1 - \theta_k)J_1x_k)) \\
&\leq (1 - \theta_k) \phi(p, x_k) + \theta_k \phi(p, x_{k-1}),
\end{aligned} \tag{35}$$

hence

$$\begin{aligned}
\phi(p, x_{k+1}) &\leq \beta_k \phi(p, u) + (1 - \beta_k) [(1 - \theta_k) \phi(p, x_k) + \theta_k \phi(p, x_{k-1})] \\
&\leq \beta_k \phi(p, u) + (1 - \beta_k) [\phi(p, x_k) + \phi(p, x_{k-1})] \\
&\leq \max\{\phi(p, u), (\phi(p, x_k) + \phi(p, x_{k-1}))\} \\
&\leq \vdots \\
&\leq \max\{\phi(p, u), (\phi(p, x_1) + \phi(p, x_0))\}, \quad \forall k \geq 1.
\end{aligned} \tag{36}$$

Thus, $\{\phi(p, x_k)\}$ is bounded. Therefore, $\{x_k\}$ is bounded and the conclusion of the lemma holds. $\square$

In what follows, we obtain a result, which is a consequence of the boundedness of $\{x_k\}$.

Lemma 3.5. *Let $\{x_{k_i}\}$ be a subsequence of the sequence $\{x_k\}$ defined by Algorithm 3.1 such that $x_{k_i} \rightarrow q$. Let $\|w_{k_i} - z_{k_i}\| \rightarrow 0$ and $\|w_{k_i} - x_{k_i}\| \rightarrow 0$ as $i \rightarrow \infty$ hold. Then $q \in \text{VIC}(C, F)$.*

Proof. First, we show that $q \in C$. Indeed, it follows from $z_{k_i} \in C_{k_i}$ that

$$h_j(w_{k_i}) + \langle h'_j(w_{k_i}), z_{k_i} - w_{k_i} \rangle \leq 0.$$

By using the Cauchy-Schwartz inequality, we have

$$h_j(w_n) \leq \langle h'_j(w_{k_i}), w_{k_i} - z_{k_i} \rangle \leq \|h'_j(w_{k_i})\| \cdot \|w_{k_i} - z_{k_i}\|.$$

Since h'_j is Lipschitz continuous and $\{w_{k_i}\}$ is bounded, we have that $\{h'_j(w_{k_i})\}$ is bounded. Thus, there exists $K_j > 0$ such that $\|h'_j(w_{k_i})\| \leq K_j$ for each i . Therefore, we obtain

$$h_j(w_{k_i}) \leq K \cdot \|w_{k_i} - z_{k_i}\|,$$

where $K = \max_{1 \leq j \leq m} \{K_j\}$. Hence, by the weakly continuity of h_j , we have

$$h_j(q) \leq \liminf_{k \rightarrow \infty} h_j(w_{n_k}) \leq \lim_{k \rightarrow \infty} K \cdot \|w_{k_i} - z_{k_i}\| = 0.$$

Thus, $q \in C$.

By the definition of z_k and characterization of Π_{C_k} , we have

$$\langle J_1 w_{k_i} - \lambda_{k_i} F w_{k_i} - J_1 z_{k_i}, w - z_{k_i} \rangle \leq 0, \quad \forall w \in C_{k_i}$$

or

$$\langle J_1 w_{k_i} - J_1 z_{k_i}, w - z_{k_i} \rangle \leq \lambda_{k_i} \langle F w_{k_i}, w - z_{k_i} \rangle, \quad \forall w \in C_{k_i}.$$

This implies that

$$\langle J_1 w_{k_i} - J_1 z_{k_i}, w - z_{k_i} \rangle + \lambda_{k_i} \langle F w_{k_i}, z_{k_i} - w_{k_i} \rangle \leq \lambda_{k_i} \langle F w_{k_i}, w - w_{k_i} \rangle, \quad \forall w \in C_{k_i}. \tag{37}$$

Fix $w \in C_{k_i}$ and let $i \rightarrow \infty$ in (37), by hypothesis, $\lambda_{k_i} > 0$ and uniform continuity of J_1 on bounded subsets of E_1 , we have

$$\liminf_{i \rightarrow \infty} \langle w - w_{k_i}, Fw_{k_i} \rangle \geq 0, \quad \forall w \in C_{k_i}. \quad (38)$$

Thus, we have from (38), the fact that $w_{k_i} \in C$ and $C \subset C_{k_i}$, that

$$\liminf_{i \rightarrow \infty} \langle w - w_{k_i}, Fw_{k_i} \rangle \geq 0, \quad \forall w \in C. \quad (39)$$

Let $\{\varepsilon_i\}$ be a sequence of positive numbers such that $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$. From (39), we can find N large enough such that

$$\langle w - w_{k_i}, Fw_{k_i} \rangle + \varepsilon_i > 0, \quad \forall k \geq N. \quad (40)$$

For some $q_{k_i} \in E_1$ satisfying $\langle Fw_{k_i}, q_{k_i} \rangle = 1$, we can write (40) as

$$\langle w + \varepsilon_i q_{k_i} - w_{k_i}, Fw_{k_i} \rangle > 0, \quad \forall k \geq N. \quad (41)$$

Since F is semistrictly quasi-monotone, we have that

$$\langle w + \varepsilon_i q_{k_i} - w_{k_i}, F(w + \varepsilon_i q_{k_i}) \rangle > 0 \quad (42)$$

from which we get

$$\langle w - w_{k_i}, Fw \rangle > \langle w + \varepsilon_i q_{k_i} - w_{k_i}, Fw - F(w + \varepsilon_i q_{k_i}) \rangle - \langle \varepsilon_i q_{k_i}, Fw \rangle, \quad \forall k \geq N. \quad (43)$$

Letting $i \rightarrow \infty$, we obtain

$$\liminf_{i \rightarrow \infty} \langle w - w_{k_i}, Fw \rangle \geq 0.$$

Hence, for all $w \in C$

$$\langle w - p, Fw \rangle = \lim_{i \rightarrow \infty} \langle w - w_{k_i}, Fw \rangle = \liminf_{i \rightarrow \infty} \langle w - w_{k_i}, Fw \rangle \geq 0. \quad (44)$$

It follows from Lemma 2.12 and Remark 2.13 that $q \in \text{VIC}(C, F)$. $\square$

Theorem 3.6. Let $\{x_k\}$ be the sequence generated by Algorithm 3.1 such that assumptions (B1)–(B4) are satisfied. Then $\{x_k\}$ converges strongly to a point $\Pi_\Gamma u = p \in \Gamma$, where Π_Γ is the generalized projection of E_1 onto Γ .

Proof. As in Lemma 3.4, let $p \in \Gamma$. Then from (8) and (17), we have

$$\begin{aligned} \phi(p, x_{k+1}) &= \phi(p, J_1^{-1}(\beta_k J_1 u + (1 - \beta_k) J_1 v_k)) \\ &= V(p, \beta_k J_1 u + (1 - \beta_k) J_1 v_k - \beta_k (J_1 u - J_1 p)) - 2\langle -\beta_k (J_1 u - J_1 p), J_1^{-1}(\beta_k J_1 u + (1 - \beta_k) J_1 v_k) - p \rangle \\ &\leq V(p, \beta_k J_1 p + (1 - \beta_k) J_1 v_k) + 2\beta_k \langle J_1 u - J_1 p, x_{k+1} - p \rangle \\ &\leq \beta_k V(p, J_1 p) + (1 - \beta_k) V(p, J_1 v_k) + 2\beta_k \langle J_1 u - J_1 p, x_{k+1} - p \rangle \\ &= \beta_k \phi(p, p) + (1 - \beta_k) \phi(p, v_k) + 2\beta_k \langle J_1 u - J_1 p, x_{k+1} - p \rangle \\ &= (1 - \beta_k) \phi(p, v_k) + 2\beta_k \langle J_1 u - J_1 p, x_{k+1} - p \rangle \\ &\leq (1 - \beta_k) \phi(p, u_k) + 2\beta_k \langle J_1 u - J_1 p, x_{k+1} - p \rangle \\ &\leq (1 - \beta_k) \phi(p, y_k) + 2\beta_k \langle J_1 u - J_1 p, x_{k+1} - p \rangle \\ &\leq (1 - \beta_k) \phi(p, w_k) + 2\beta_k \langle J_1 u - J_1 p, x_{k+1} - p \rangle \\ &\leq (1 - \beta_k) ((1 - \theta_k) \phi(p, x_k) + \theta_k \phi(p, x_{k-1})) + 2\beta_k \langle J_1 u - J_1 p, x_{k+1} - p \rangle \\ &\leq (1 - \beta_k) \phi(p, x_k) + \beta_k \left(\frac{\theta_k}{\beta_k} \phi(p, x_{k-1}) + 2 \langle J_1 u - J_1 p, x_{k+1} - p \rangle \right), \end{aligned} \quad (45)$$

for each $k \in N$.

Now, consider the following two possible cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(p, x_k)\}_{k \geq n_0}$ is either non-increasing or non-decreasing. Then, by the boundedness of $\{\phi(p, x_k)\}$, it follows that $\{\phi(p, x_k)\}$ is convergent and

$$\phi(p, x_k) - \phi(p, x_{k+1}) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

From (30), (33) and (35), we have

$$\begin{aligned} \phi(p, x_{k+1}) &\leq \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, u_k) \\ &\leq \beta_k \phi(p, u) + (1 - \beta_k) [\phi(p, y_k) - \alpha_k(1 - \alpha_k)g(\|J_1 y_k - J_1 T y_k\|)] \\ &\leq \beta_k \phi(p, u) + (1 - \beta_k) [\phi(p, w_k) - \alpha_k(1 - \alpha_k)g(\|J_1 y_k - J_1 T y_k\|)] \\ &\leq \beta_k \phi(p, u) + (1 - \beta_k) [(1 - \theta_k) \phi(p, x_k) + \theta_k \phi(p, x_{k-1}) - \alpha_k(1 - \alpha_k)g(\|J_1 y_k - J_1 T y_k\|)], \end{aligned}$$

which implies

$$\alpha_k(1 - \alpha_k)(1 - \beta_k)g(\|J_1 y_k - J_1 T y_k\|) \leq \beta_k \left(\frac{\theta_k}{\beta_k} \phi(p, x_{k-1}) + \phi(p, u) \right) + (1 - \beta_k) \phi(p, x_k) - \phi(p, x_{k+1}).$$

By letting $k \rightarrow \infty$, we get $\alpha_k(1 - \alpha_k)(1 - \beta_k)g(\|J_1 y_k - J_1 T y_k\|) \rightarrow 0$, since $\alpha_k(1 - \alpha_k)(1 - \beta_k) > 0$ we obtain

$$g(\|J_1 y_k - J_1 T y_k\|) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Using the property of g and the uniform continuity of J_1^{-1} on bounded subsets of E_1^* , we have

$$\lim_{k \rightarrow \infty} \|y_k - T y_k\| = 0 = \phi(y_k, T y_k), \quad (46)$$

from Remark 2.5. Observe from (17) that

$$\begin{aligned} \phi(y_k, u_k) &= \phi(y_k, J_1^{-1}(\alpha_k J_1 y_k + (1 - \alpha_k) J_1 T y_k)) \\ &= \|y_k\|^2 - 2\langle y_k, \alpha_k J_1 y_k + (1 - \alpha_k) J_1 T y_k \rangle + \|\alpha_k J_1 y_k + (1 - \alpha_k) J_1 T y_k\|^2 \\ &= \|y_k\|^2 - 2\alpha_k \langle y_k, J_1 y_k \rangle - 2(1 - \alpha_k) \langle y_k, J_1 T y_k \rangle + \alpha_k \|y_k\|^2 + (1 - \alpha_k) \|T y_k\|^2 - \alpha_k(1 - \alpha_k)g(\|J_1 y_k - J_1 T y_k\|) \\ &= \alpha_k \phi(y_k, y_k) + (1 - \alpha_k) \phi(y_k, T y_k) - \alpha_k(1 - \alpha_k)g(\|J_1 y_k - J_1 T y_k\|), \end{aligned}$$

which implies by using (46) that $\phi(y_k, u_k) \rightarrow 0$ as $k \rightarrow \infty$. By invoking Lemma 2.4, we get

$$\lim_{k \rightarrow \infty} \|y_k - u_k\| = 0. \quad (47)$$

Also, from (34), we have

$$\begin{aligned} (1 - \beta_k) \left(1 - \frac{\eta}{\nu} \right) (\phi(y_k, z_k) + \phi(z_k, w_k)) &\leq \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, w_k) - \phi(p, x_{k+1}) \\ &\leq \beta_k \left(\frac{\theta_k}{\beta_k} \phi(p, x_{k-1}) + \phi(p, u) \right) + (1 - \beta_k) \phi(p, x_k) - \phi(p, x_{k+1}), \end{aligned}$$

thus by condition (B2), we get $(\phi(y_k, z_k) + \phi(z_k, w_k)) \rightarrow 0$ as $k \rightarrow \infty$, that is

$$\lim_{k \rightarrow \infty} \phi(y_k, z_k) = \lim_{k \rightarrow \infty} \phi(y_k, z_k) = 0,$$

Lemma 2.4 ensures

$$\lim_{k \rightarrow \infty} \|y_k - z_k\| = \lim_{k \rightarrow \infty} \|y_k - z_k\| = 0. \quad (48)$$

Next, we show that $\|(K_{r_k}(I - r_k A) - I)Lu_k\| \rightarrow 0$ as $k \rightarrow \infty$. Indeed, we have from (32) and (33) that

$$\begin{aligned} \phi(p, x_{k+1}) &\leq \beta_k \phi(p, u) + (1 - \beta_k) \phi(p, v_k) \\ &\leq \beta_k \phi(p, u) + (1 - \beta_k) [\phi(p, u_k) - 2\gamma(1 - \gamma_k d^2 \|L\|^2) \|(K_{r_k}(I - r_k A) - I)Lu_k\|^2], \end{aligned}$$

by further calculations, we get

$$2\gamma(1 - \gamma d^2 \|L\|^2) \|(K_{r_k}(I - r_k A) - I)Lu_k\|^2 \leq \beta_k \left(\frac{\theta_k}{\beta_k} \phi(p, x_{k-1}) + \phi(p, u) \right) + (1 - \beta_k) \phi(p, x_k) - \phi(p, x_{k+1}).$$

Thus, we obtain

$$\lim_{k \rightarrow \infty} \|(K_{r_k}(I - r_k A) - I)Lu_k\| = 0. \quad (49)$$

Observe from (17) that

$$\begin{aligned} \phi(u_k, v_k) &= \phi(u_k, J_1^{-1}(J_1 u_k + \gamma L^* J_2((K_{r_k}(I - r_k A) - I)Lu_k))) \\ &= \|u_k\|^2 - 2\langle u_k, J_1 u_k + \gamma L^* J_2((K_{r_k}(I - r_k A) - I)Lu_k) \rangle + \|J_1 u_k + \gamma L^* J_2((K_{r_k}(I - r_k A) - I)Lu_k)\|^2 \\ &= \|u_k\|^2 - 2\langle u_k, J_1 u_k \rangle - 2\gamma \langle u_k, L^* J_2((K_{r_k}(I - r_k A) - I)Lu_k) \rangle + \|u_k\|^2 \\ &\quad + 2\gamma \langle u_k, L^* J_2((K_{r_k}(I - r_k A) - I)Lu_k) \rangle + 2d^2 \gamma^2 \|L\|^2 \|(K_{r_k}(I - r_k A) - I)Lu_k\|^2 \\ &= \phi(u_k, u_k) + 2d^2 \gamma^2 \|L\|^2 \|(K_{r_k}(I - r_k A) - I)Lu_k\|^2, \end{aligned}$$

which by (49) and Lemma 2.4, that

$$\lim_{k \rightarrow \infty} \phi(v_k, u_k) = \lim_{k \rightarrow \infty} \|v_k - u_k\| = 0. \quad (50)$$

Furthermore, we obtain the limit of $\|x_{k+1} - x_k\|$ as $k \rightarrow \infty$. Observe first from (14) and condition (B3) that

$$\lim_{k \rightarrow \infty} \phi(w_k, x_k) = \lim_{k \rightarrow \infty} \theta_k \phi(x_k, x_{k-1}) = 0, \quad (51)$$

that is, by Lemma 2.4, we have

$$\|w_k - x_k\| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (52)$$

It is also easy to see that

$$\begin{cases} \lim_{k \rightarrow \infty} \|v_k - y_k\| = \lim_{k \rightarrow \infty} (\|v_k - u_k\| + \|u_k - y_k\|) = 0, \\ \lim_{k \rightarrow \infty} \|y_k - x_k\| = \lim_{k \rightarrow \infty} (\|y_k - w_k\| + \|w_k - x_k\|) = 0, \\ \lim_{k \rightarrow \infty} \|v_k - x_k\| = \lim_{k \rightarrow \infty} (\|v_k - y_k\| + \|y_k - x_k\|) = 0. \end{cases} \quad (53)$$

Now,

$$\begin{aligned} \phi(x_k, x_{k+1}) &= \phi(x_k, J_1^{-1}(\beta_k J_1 u + (1 - \beta_k) J_1 v_k)) \\ &= \|x_k\|^2 - 2\langle x_k, \beta_k J_1 u + (1 - \beta_k) J_1 v_k \rangle + \|\beta_k J_1 u + (1 - \beta_k) J_1 v_k\|^2 \\ &= \beta_k \phi(x_k, u) + (1 - \beta_k) \phi(x_k, v_k) - \beta_k (1 - \beta_k) g(\|J_1 v_k - J_1 u\|), \end{aligned}$$

thus, we obtain

$$\lim_{k \rightarrow \infty} \phi(x_k, x_{k+1}) = \lim_{k \rightarrow \infty} \|x_{k+1} - x_k\| = 0, \quad (54)$$

where we have used Lemma 2.4.

Since $\{x_k\}$ is bounded by Lemma 3.4, there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \rightharpoonup q$. It is easy to see that there exists $\{v_{k_i}\}$, $\{w_{k_i}\}$, $\{u_{k_i}\}$ and $\{y_{k_i}\}$ all converge weakly to q . Hence, by (48) and Lemma 3.5, we have that $q \in \text{VIP}(C, F)$. Also, by (46) and demiclosedness of $(I - T)$, we get that $q \in \text{Fix}(T)$. Hence, $q \in \text{VIP}(C, F) \cap \text{Fix}(T)$. On the other hand, by the linearity of L we obtain that $Lu_{k_i} \rightharpoonup Lq$. We now show that $Lq \in \text{GVMEP}(f, \psi, \phi, X)$. Indeed, suppose $v_k = K_{r_k}(I - r_k A)Lu_k$, then by

$$f(v_k, z) + \psi(z) - \psi(v_k) + e\langle z - v_k, ALu_k \rangle + \frac{e}{r_k} \langle z - v_k, v_k - Lu_k \rangle \in P, \quad \forall z \in X, \quad (55)$$

which implies that

$$0 \in f(z, v_k) - \left\{ \psi(z) - \psi(v_k) + e\langle z - v_k, ALu_k \rangle + \frac{e}{r_k} \langle z - v_k, v_k - Lu_k \rangle \right\} + P, \quad \forall z \in X. \quad (56)$$

Let $z_\lambda = (1 - \lambda)Lq + \lambda z$ for all $\lambda \in (0, 1]$, then $z_\lambda \in E_2$. By (55), we have

$$\begin{aligned} e\langle z_\lambda - v_{k_i}, Az_\lambda \rangle &\in f(z_\lambda, v_{k_i}) - (\psi(z_\lambda) - \psi(v_{k_i})) + e\langle z_\lambda - v_{k_i}, Az_\lambda \rangle - e\langle z_\lambda - v_{k_i}, ALu_{k_i} \rangle \\ &\quad - \frac{e}{r_k} \langle z_\lambda - v_{k_i}, v_{k_i} - Lu_{k_i} \rangle + P \\ &= f(z_\lambda, v_{k_i}) + e\langle z_\lambda - v_{k_i}, Az_\lambda - Av_{k_i} \rangle + e\langle z_\lambda - v_{k_i}, Av_{k_i} - ALu_{k_i} \rangle - e\left\langle z_\lambda - v_{k_i}, \frac{v_{k_i} - Lu_{k_i}}{r_{k_i}} \right\rangle \\ &\quad - (\psi(z_\lambda) - \psi(v_{k_i})) + P. \end{aligned} \quad (57)$$

Using (49), the properties of A and f , we have

$$\|Av_{k_i} - ALu_{k_i}\| \rightarrow 0, \quad \frac{\|v_{k_i} - Lu_{k_i}\|}{r_{k_i}} \rightarrow 0, \quad \text{and} \quad \langle z_\lambda - v_{k_i}, Av_{k_i} - ALu_{k_i} \rangle \rightarrow 0.$$

Let $i \rightarrow \infty$, we get

$$e\langle z_\lambda - Lq, Az_\lambda \rangle \in f(z_\lambda, Lq) - (\psi(z_\lambda) - \psi(Lq)) + P. \quad (58)$$

It follows from (R1), (R2) and (R6) that

$$\lambda f(z_\lambda, z) + (1 - \lambda)f(z_\lambda, Lq) + \lambda\psi(z) + (1 - \lambda)\psi(Lq) - \psi(z_\lambda) \in f(z_\lambda, z_\lambda) + \psi(z_\lambda) - \psi(z_\lambda) + P \in P, \quad (59)$$

that is

$$-\lambda[f(z_\lambda, z) + \psi(z) - \psi(z_\lambda)] - (1 - \lambda)[f(z_\lambda, Lq) + \psi(Lq) - \psi(z_\lambda)] \in -P. \quad (60)$$

Using this and (58), we have

$$-\lambda[f(z_\lambda, z) + \psi(z) - \psi(z_\lambda)] \in (1 - \lambda)[f(z_\lambda, Lq) + \psi(Lq) - \psi(z_\lambda)] - P \in (1 - \lambda)e\langle z_\lambda - Lq, Az_\lambda \rangle - P$$

and so

$$-\lambda[f(z_\lambda, z) + \psi(z) - \psi(z_\lambda)] - e(1 - \lambda)\langle z - Lq, Az_\lambda \rangle \in -P, \quad (61)$$

thus

$$[f(z_\lambda, z) + \psi(z) - \psi(z_\lambda)] + e(1 - \lambda)\langle z - Lq, Az_\lambda \rangle \in P. \quad (62)$$

Letting $\lambda \rightarrow 0_+$, we get

$$f(Lq, z) + e\langle z - Lq, ALq \rangle + \psi(z) - \psi(Lq) \in P, \quad \forall z \in E_2$$

and so $Lq \in \text{GVMEP}(f, A, \psi, X)$.

To conclude Case 1, we show that $x_k \rightarrow \Pi_\Gamma u = p$. Let $\{x_{k_i}\}$ be a subsequence of $\{x_k\}$ such that $x_{k_i} \rightarrow q$ and

$$\limsup_{k \rightarrow \infty} \langle J_1 u - J_1 p, x_{k+1} - p \rangle = \lim_{k \rightarrow \infty} \langle J_1 u - J_1 p, x_{k_i+1} - p \rangle. \quad (63)$$

Since $\|x_{k+1} - x_k\| \rightarrow 0$ as $k \rightarrow \infty$, it follows that $x_{k_i+1} \rightarrow q$. By using Lemma 2.3 (9), we obtain

$$\limsup_{k \rightarrow \infty} \langle J_1 u - J_1 p, x_{k+1} - p \rangle = \lim_{k \rightarrow \infty} \langle J_1 u - J_1 p, x_{k_i+1} - p \rangle = \langle J_1 u - J_1 p, q - p \rangle \leq 0. \quad (64)$$

It follows from (45) and (64) that $\phi(p, x_k) \rightarrow$ as $k \rightarrow \infty$. Therefore, by Lemma 2.4, we obtain $\|x_k - p\| \rightarrow 0$ as $k \rightarrow \infty$. Hence, $x_k \rightarrow p$.

Case 2. Suppose that $\{\phi(p, x_k)\}$ is not a monotone sequence. Then there exists a subsequence $\{k_i\}$ of $\{k\}$ such that

$$\phi(p, x_{k_i}) < \phi(p, x_{k_i+1}), \quad \forall i \in \mathbb{N}.$$

For some N large enough, let $\tau : \mathbb{N} \rightarrow \mathbb{N}$ be a mapping defined for all $k \geq N$ by

$$\tau(k) := \max\{i \in \mathbb{N} : i \leq k, \phi(p, x_{k_i}) \leq \phi(p, x_{k_i+1})\}.$$

By Lemma 2.17, $\tau(k)$ is non-decreasing with $\tau(k) \rightarrow \infty$ as $k \rightarrow \infty$ and

$$\phi(p, x_{\tau(k)}) \leq \phi(p, x_{\tau(k)+1}) \quad \text{and} \quad \phi(p, x_k) \leq \phi(p, x_{\tau(k)+1}).$$

Following similar argument as above in Case 1, we can conclude that

$$\begin{cases} \lim_{k \rightarrow \infty} \|(K_{r_{\tau(k)}}(I - r_{\tau(k)}A) - I)Lu_{\tau(k)}\| = 0, \\ \lim_{k \rightarrow \infty} \|y_{\tau(k)} - z_{\tau(k)}\| = \lim_{k \rightarrow \infty} \|w_{\tau(k)} - z_{\tau(k)}\| = 0, \\ \lim_{k \rightarrow \infty} \|y_{\tau(k)} - Ty_{\tau(k)}\| = 0, \\ \lim_{k \rightarrow \infty} \|x_{\tau(k)+1} - x_{\tau(k)}\| = 0 \end{cases} \quad (65)$$

and

$$\lim_{k \rightarrow \infty} \langle J_1 u - J_1 p, x_{\tau(k)+1} - p \rangle \leq 0. \quad (66)$$

Since $\{x_{\tau(k)}\}$ is bounded, we can find a subsequence of $\{x_{\tau(k)}\}$, still denoted $\{x_{\tau(k)}\}$ such that $x_{\tau(k)} \rightharpoonup q \in \text{Fix}(T) \cap \text{VIP}(C, F)$ and $Lq \in \text{GVMEP}(f, A, \psi, X)$. It follows from (45) that

$$\phi(p, x_{\tau(k)+1}) \leq (1 - \beta_{\tau(k)})\phi(p, x_{\tau(k)}) + \beta_{\tau(k)} \left(\frac{\theta_{\tau(k)}}{\beta_{\tau(k)}} \phi(p, x_{\tau(k)-1}) + 2\langle J_1 u - J_1 p, x_{\tau(k)+1} - p \rangle \right) \quad (67)$$

for each $\tau(k) \in \mathbb{N}$. Since $\phi(p, x_{\tau(k)}) \leq \phi(p, x_{\tau(k)+1})$, we have that $\phi(p, x_{\tau(k)}) - \phi(p, x_{\tau(k)+1}) \leq 0$. Thus, we obtain from (67) that

$$\begin{aligned} \beta_{\tau(k)}\phi(p, x_{\tau(k)}) &\leq \phi(p, x_{\tau(k)}) - \phi(p, x_{\tau(k)+1}) + \beta_{\tau(k)} \left(\frac{\theta_{\tau(k)}}{\beta_{\tau(k)}} \phi(p, x_{\tau(k)-1}) + 2\langle J_1 u - J_1 p, x_{\tau(k)+1} - p \rangle \right) \\ &\leq \beta_{\tau(k)} \left(\frac{\theta_{\tau(k)}}{\beta_{\tau(k)}} \phi(p, x_{\tau(k)-1}) + 2\langle J_1 u - J_1 p, x_{\tau(k)+1} - p \rangle \right), \end{aligned} \quad (68)$$

that is

$$\phi(p, x_{\tau(k)}) \leq \left(\frac{\theta_{\tau(k)}}{\beta_{\tau(k)}} \phi(p, x_{\tau(k)-1}) + 2\langle J_1 u - J_1 p, x_{\tau(k)+1} - p \rangle \right), \quad (69)$$

since $\beta_{\tau(k)} > 0$.

Using condition (B4) and (66), we get that $\phi(p, x_{k_i}) \rightarrow 0$ as $k \rightarrow \infty$. This together with (67) implies that $\phi(p, x_{k_i+1}) \rightarrow 0$ as $k \rightarrow \infty$. But $\phi(p, x_k) \leq \phi(p, x_{k_i+1})$ for all $k \in \mathbb{N}$, we get that $\phi(p, x_k) \rightarrow 0$. By Lemma 2.4, we obtain $x_k \rightarrow p$ as $k \rightarrow \infty$. Therefore, from the above two cases, we can conclude that $\{x_k\}$ converges strongly to a point $p = \Pi_{\Gamma} u$. Thus, completing the proof. $\square$

The following is a direct consequence of our main result: Suppose we set $Y = \mathbb{R}$ and $e = 1$, then we have the following result for obtaining a common solution of SFP for variational inequality, FPP and GMEP.

Corollary 3.7. *Let C and Q be nonempty, closed and convex subsets of a 2-uniformly convex and uniformly smooth Banach space E_1 , and smooth, strictly convex and reflexive Banach space E_2 with dual space E_1^* and E_2^* , respectively. Let $L : E_1 \rightarrow E_2$ be a bounded linear operator with $L^* : E_2^* \rightarrow E_1^*$ its adjoint. Let $f : Q \times Q \rightarrow \mathbb{R}$ satisfying assumptions (A1)–(A4) (see [55]) and $A : Q \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuous function. Let $\psi : Q \rightarrow E_2$ be an m -inverse strongly monotone mapping and $T : C \rightarrow C$ be a quasi- ϕ -nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$. Let $F : C \rightarrow E^*$ be a semistrictly quasi-monotone, sequentially weakly continuous mapping. Let $\Gamma = \{p \in \text{VIP}(C, F) \cap \text{Fix}(T) : Lp \in \text{GMEP}(f, \phi, \psi)\} \neq \emptyset$. Then, the sequence $\{x_k\}$ generated by Algorithm 3.1 converges strongly to $x^* \in \Gamma$.*

4 Numerical examples

In this section, we give some numerical examples to illustrate the performance of our method.

First, we give an example in ℓ_p ($1 \leq p < \infty$) with $p \neq 2$, which is not Hilbert space. It is well known that the dual space $(\ell_p)^*$ is isomorphic to ℓ_q provided $\frac{1}{p} + \frac{1}{q} = 1$ (see for instance [56, Lemma 2.2]).

Example 4.1. Let $E_1 = E_2 = X = Y = \ell_3(\mathbb{R})$ where $\ell_3(\mathbb{R}) = \{x = (x_1, x_2, x_3, \dots) \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^3 < \infty\}$ with norm $\|x\|_{\ell_3} = \left(\sum_{i=1}^{\infty} |x_i|^3\right)^{\frac{1}{3}}$ and inner product $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$ for all $x = (x_1, x_2, x_3, \dots)$, $y = (y_1, y_2, y_3, \dots) \in E$.

Let $L : \ell_3 \rightarrow \ell_3$ be defined by $L(x) = \left(\frac{2x_1}{5}, \frac{2x_2}{5}, \frac{2x_3}{5}, \dots\right)$ for every $x \in \ell_3$. Then, $L^*(y) = \left(\frac{2y_1}{5}, \frac{2y_2}{5}, \frac{2y_3}{5}, \dots\right)$ for each $y \in \ell_3$. Let $C = \{x = (x_1, x_2, x_3, \dots) \in \ell_3 : \|x\| \leq 4\}$, define the mapping $T : C \rightarrow C$ by $T(x) = \left(\frac{x_1^2}{1+x_1}, \frac{x_2^2}{1+x_2}, \frac{x_3^2}{1+x_3}, \dots\right)$ for all $x \in \ell_3$. Define the function $F : C \rightarrow E_1^*$ by $F(x) = (6 - \|x\|)x$. Then, F is semistrictly quasi-monotone and weakly sequentially continuous on C , see [25].

Also, define the mappings $f : X \times X \rightarrow Y$ by $f(x) = -x^2 + y^2$, $\forall x, y \in \ell_3$, $A(x) = \left(\frac{x_1}{5}, \frac{x_2}{5}, \dots, \frac{x_i}{5}, \dots\right)$, $\forall x \in \ell_3$ and $\psi(x) = 0$, $\forall x \in \ell_3$. It is easy to see that the mappings f , A and ψ satisfy conditions (R1)–(R7) of Theorem 3.6. Let $r_k = 1$ for all k , by Lemma 2.10, we can easily find $z \in E_2$ such that

$$f(z, y) + \langle Az, y - z \rangle \psi(y) - \psi(x) + \frac{e}{r_k} \langle y - z, Jz - Jx \rangle \in P, \quad \forall y \in X.$$

It can be easily checked that

$$(K_{r_k}(I - r_k A) - I)Lu_k = \left(\frac{5Lu_k^1}{16}, \frac{5Lu_k^2}{16}, \frac{5Lu_k^3}{16}, \dots\right).$$

It follows therefore that $\Gamma = \{0\} \neq \emptyset$. Choose the sequences $\alpha_k = \frac{3k}{5k^2+7}$, $\beta_k = \frac{1}{(k+13)}$, $\theta_k = \frac{1}{3} - \frac{2k}{k^2+3k}$, $\rho = 0.9$, $\gamma = 0.25/14$, $\lambda_k = \rho^2$ and $u = [-0.2345, 0.8943, 0, \dots]^T$. Take $\varepsilon = 1 \times 10^{-5}$ and choose the following initial values:

(Case 1) $x_0 = [0.15, 0.25, 0.75, \dots]^T$, $x_1 = [0.75, 0.5, 0.25, \dots]^T$, $u = [0.1, 0.9, 0.3, \dots]^T$;

(Case 2) $x_0 = [1.5, 2.5, 3.5, \dots]^T$, $x_1 = [7.5, 5, 2.5, \dots]^T$, $u = [10, 5, 3, \dots]^T$;

(Case 3) $x_0 = [-5, 0, 5, \dots]^T$, $x_1 = [-10, -20, 10, \dots]^T$, $u = [10, 10, 10, \dots]^T$.

We plot the graphs of errors against the number of iterations in each case. The numerical results can be found in Figure 1.

Example 4.2. Let $E_1 = E_2 = X = Y = \ell_2(\mathbb{R})$ be the space of square-summable sequences of real numbers $\{x_i\}_{i=1}^{\infty}$, that is

$$\ell_2(\mathbb{R}) = \left\{x = (x_1, x_2, \dots, x_i, \dots), x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} |x_i|^2 < \infty\right\},$$

with inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \rightarrow \mathbb{R}$ and $\|\cdot\| : \ell_2 \rightarrow \mathbb{R}$ defined by $\langle u, v \rangle := \sum_{i=1}^{\infty} u_i v_i$ and $\|u\| := \left(\sum_{i=1}^{\infty} |u_i|^2\right)^{\frac{1}{2}}$, respectively, for all $u, v \in \ell_2$.

Let $L : \ell_2 \rightarrow \ell_2$ be defined by $L(x) = \left(\frac{2x_1}{5}, \frac{2x_2}{5}, \dots, \frac{2x_i}{5}, \dots\right)$ for every $x \in \ell_2$. Then, $L^*(y) = \left(\frac{2y_1}{5}, \frac{2y_2}{5}, \dots, \frac{2y_i}{5}, \dots\right)$ for each $y \in \ell_2$. Let $C = \{x \in E_1 : \|x\| \leq 4\}$, define the mapping $T : C \rightarrow C$ by $T(x) = \left(\frac{2x_1}{7}, \frac{2x_2}{7}, \dots, \frac{2x_i}{7}, \dots\right)$ for all $x \in \ell_2$. Define the function $F : C \rightarrow E_1^*$ by $F(x) = (6 - \|x\|)x$. Then, F is semistrictly quasi-monotone and weakly sequentially continuous on C , see [25].

Also, define the mappings $f : X \times X \rightarrow Y$ by $f(x) = -x^2 + y^2$, $\forall x, y \in \ell_2$, $A(x) = \left(\frac{x_1}{5}, \frac{x_2}{5}, \dots, \frac{x_i}{5}, \dots\right)$, $\forall x \in \ell_2$ and $\psi(x) = 0$, $\forall x \in \ell_2$. It is easy to see that the mappings f , A and ψ satisfy conditions (R1)–(R7) of Theorem 3.6 and $0 \in \text{GVMEP}(f, A, \psi, X)$. Let $r_k = 1$ for all k , by Lemma 2.10, we can easily find $z \in E_2$ such that

$$f(z, y) + \langle Az, y - z \rangle \psi(y) - \psi(x) + \frac{e}{r_k} \langle y - z, Jz - Jx \rangle \in P, \quad \forall y \in X.$$

Figure 1: Example 4.3, top left: Case (1); top right: Case (2); and bottom: Case (3).

It can be easily checked that

$$(K_{r_k}(I - r_k A) - I)Lu_k = \left(\frac{5Lu_k^1}{16}, \frac{5Lu_k^2}{16}, \dots, \frac{5Lu_k^i}{16}, \dots \right).$$

It follows therefore that $\Gamma = \{0\} \neq \emptyset$. Choose the sequences $\alpha_k = \frac{3k}{4k^2 + 7}$, $\beta_k = \frac{1}{1000(k+13)}$, $\theta_k = \frac{1}{3} - \frac{2k}{k^2 + 3k}$, $\rho = 0.07$, $\lambda_k = \rho^4$ and $u = [-0.2345, 0.8943, 0, \dots, 0, \dots]^T$. By taking $\varepsilon = 1 \times 10^{-6}$, we choose the following initial values:

(Case 1) $x_0 = [4.1285, -2.9018, 0, \dots, 0, \dots]^T$ and $x_1 = [3.2158, -5.8091, 0, \dots, 0, \dots]^T$,

(Case 2) $x_0 = [3.9015, 2.0345, 0, \dots, 0, \dots]^T$ and $x_1 = [1.2456, 2.3125, 0, \dots, 0, \dots]^T$,

(Case 3) $x_0 = [0.4563, 1.2098, 0, \dots, 0, \dots]^T$ and $x_1 = [-0.8924, 1.3521, 0, \dots, 0, \dots]^T$.

We plot the graphs of errors against the number of iterations in each case. The numerical results can be found in Figure 2.

Figure 2: Example 4.2, top left: Case (1); top right: Case (2) and bottom: Case (3).

Example 4.3. Let $E_1 = E_2 = X = Y = \mathbb{R}$ and $L : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $L(x) = \frac{2x}{5}$ for every $x \in \mathbb{R}$. Then, $L^*(y) = \frac{2y}{5}$ for each $y \in E_1$. Let $C = \{x \in \mathbb{R} : |x| \leq 4\}$, define the mapping $T : C \rightarrow C$ by $T(x) = \frac{2x}{7}$ for all $x \in \mathbb{R}$. Define the function $F : C \rightarrow \mathbb{R}$ by $F(x) = (6 - |x|)x$. Then, F is semistrictly quasi-monotone and weakly sequentially continuous on C , see [25].

Also, define the mappings $f : X \times X \rightarrow Y$ by $f(x) = -x^2 + y^2$, $\forall x, y \in \mathbb{R}$, $A(x) = \frac{x}{5}$, $\forall x \in \mathbb{R}$ and $\psi(x) = 0$, $\forall x \in \mathbb{R}$. It is easy to see that the mappings f , A and ψ satisfy conditions (R1)–(R7) of Theorem 3.6 and $0 \in \text{GVMEP}(f, A, \psi, X)$. Let $\{r_k\} = 1$ for all k , by Lemma 2.10, we can easily find $z \in \mathbb{R}$ such that

$$f(z, y) + \langle Az, y - z \rangle \psi(y) - \psi(x) + \frac{e}{r_k} \langle y - z, Jz - Jx \rangle \in P, \quad \forall y \in X.$$

It can be easily checked that

$$(K_{r_k}(I - r_k A) - I)Lu_k = \frac{5Lu_k}{16}.$$

It follows therefore that $\Gamma = \{0\} \neq \emptyset$. Choose the sequences $\alpha_k = \frac{3k}{4k^2+7}$, $\beta_k = \frac{1}{1000(k+13)}$, $\eta = \frac{0.17}{13}$, $\theta_k = \frac{1}{3} - \frac{2k}{k^2+3k}$ and $\rho = 0.07$. Take $\varepsilon = 1 \times 10^{-6}$ and choose the following initial values:

- (Case 1) $x_0 = 1.5$, $x_1 = 0.5$, $u = -0.5$ and $\lambda_k = \rho$;
 (Case 2) $x_0 = 2.5$, $x_1 = -0.5$, $u = -0.5$ and $\lambda_k = \rho^4$;
 (Case 3) $x_0 = 10$, $x_1 = 15$, $u = 1$ and $\lambda_k = \rho^4$.

We plot the graphs of errors against the number of iterations in each case. The numerical results can be found in Figure 3.

Figure 3: Example 4.3, top left: Case (1); top right: Case (2); and bottom: Case (3).

5 Conclusion

In this paper, we introduced an iterative algorithm of inertial form for approximating an element in the solution set of SGVMEP, which is also a fixed point of a quasi- ϕ -nonexpansive mapping and solves a VIP for a weakly sequentially continuous and quasi-monotone mapping in Banach spaces. The result obtained in this sequel extends and unifies the works of Chang et al. [25], Kazmi and Farid [11], Shan and Huang [31] and others in the literature. Using numerical example, we showed the efficacy of our method for arriving at an element in the solution set.

Remark 5.1.

- (i) If we take $F = 0$, then Theorem 3.6 reduces to the theorems for finding a common element in the solution of SGVMEP and fixed point of quasi- ϕ -nonexpansive mapping.
- (ii) Let $Y = \mathbb{R}$ and $P = [0, +\infty)$, then the result presented in this paper reduces to finding a common element in the solution set of SGMEP considered in [12,55] and fixed point of quasi- ϕ -nonexpansive mapping which is also a solution of a VIP.
- (iii) If we let $E_1 = E_2 = H$ a real Hilbert space, the result presented in this sequel is a unification of the result presented in [25] and [31].

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