Research Article

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Robust numerical method for singularly perturbed differential equations with large delay

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Abstract: In this paper, a singularly perturbed differential equation with a large delay is considered. The considered problem contains a large delay parameter on the reaction term. The solution of the problem exhibits the interior layer due to the delay parameter and the strong right boundary layer due to the small perturbation parameter ε . The resulting singularly perturbed problem is solved using the fitted non-polynomial spline method. The stability and parameter uniform convergence of the proposed method is proved. To validate the applicability of the scheme, two model problems of the variable coefficient are considered for numerical experimentation.

Keywords: singularly perturbed, large delay, fitted spline, varepsilon-uniformly convergent

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1 Introduction

A differential equation is said to be a singularly perturbed delay differential equation, if it includes at least one delay term, involving unknown functions occurring with different arguments, and also, the highest derivative term is multiplied by a small parameter. Such type of delay differential equations plays a very important role in the mathematical models of science and engineering, such as the human pupil light reflex with mixed delay type [1], variational problems in control theory with small state problem [2], models of HIV infection [3], and signal transition [4].

Any system involving a feedback control almost involves time delay. The delay occurs because a finite time is required to sense the information and then react to it. Finding the solution of singularly perturbed delay differential equations, whose application mentioned above, is a challenging problem. In response to these, in recent years, there has been a growing interest in numerical methods on singularly perturbed delay differential equations. The authors of [5–7] have developed various numerical schemes on uniform meshes for singularly perturbed second-order differential equations having a small delay on convection term. The authors of [8–12] have presented second-order differential equations having large delay.

In the present paper, motivated by the works of [8–12], we developed a fitted non-polynomial spline method for the numerical solution of second-order singularly perturbed convection–diffusion equations with large delay. Therefore, the main objective of this study is to develop a more accurate, stable and uniformly convergent scheme using an exponentially fitted non-polynomial spline method for the problem under consideration.

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Throughout our analysis, C is a generic positive constant that is independent of the parameter ε and number of mesh points N . We assume that $\bar{\Omega} = [0, 2]$, $\Omega = (0, 2)$, $\Omega_1 = (0, 1]$, $\Omega_2 = (1, 2)$, and $\Omega^* = \Omega_1 \cup \Omega_2$.

2 Definition of the problem

Consider the following singularly perturbed problem

$$Ly(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), \quad x \in \Omega, \quad (1)$$

$$y(x) = \phi(x), \quad x \in [-1, 0], \quad y(2) = l, \quad (2)$$

where ε , $0 < \varepsilon \ll 1$ is a small parameter, $\phi(x)$ is sufficiently smooth on $[-1, 0]$. For all $x \in \Omega$, it is assumed that the sufficient smooth functions $a(x)$, $b(x)$, and $c(x)$ satisfy at $a(x) > a > 0$, $b(x) > b \geq 0$, $c(x) \leq \gamma < 0$, and $2a + 5b + 5\gamma > 0$, $a + 2\gamma > 0$. The above assumptions ensure that $y \in X = C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^*)$.

The boundary value problem of equations (1)–(2) exhibits strong boundary layer at $x = 2$ and interior layer at $x = 1$.

As we observed from equations (1) and (2), the values of $y(x-1)$ are known for the domain Ω_1 and unknown for the domain Ω_2 due to the large delay at $x = 1$. So, it impossible to treat the problem throughout the domain $(\bar{\Omega})$. Thus, we have to treat the problem at Ω_1 and Ω_2 separately.

Equations (1) and (2) are equivalent to

$$Ly(x) = R(x), \quad (3)$$

where

$$Ly(x) = \begin{cases} L_1 y(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x), & x \in \Omega_1, \\ L_2 y(x) = -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1), & x \in \Omega_2, \end{cases} \quad (4)$$

$$R(x) = \begin{cases} f(x) - c(x)\phi(x-1), & x \in \Omega_1, \\ f(x), & x \in \Omega_2, \end{cases} \quad (5)$$

with boundary conditions

$$\begin{cases} y(x) = \phi(x), & x \in [-1, 0], \\ y(1^-) = y(1^+), & y'(1^-) = y'(1^+), \\ y(2) = l, \end{cases} \quad (6)$$

where $y(1^-)$ and $y(1^+)$ denote the left and right limits of y at $x = 1$, respectively.

3 Properties of continuous solution

Lemma 3.1. (Maximum principle) *Let $\psi(x)$ be any function in X such that $\psi(0) \geq 0$, $\psi(2) \geq 0$, $L_1\psi(x) \geq 0$, $\forall x \in \Omega_1$, $L_2\psi(x) \geq 0$, $\forall x \in \Omega_2$, and $[\psi'](1) \leq 0$. Then, $\psi(x) \geq 0$, $\forall x \in \bar{\Omega}$.*

Proof. For the proof, refer [8]. □

Lemma 3.2. (Stability result) *The solution $y(x)$ of the problem in equations (1 and 2), satisfies the bound*

$$|y(x)| \leq C \max \left\{ |y(0)|, |y(2)|, \sup_{x \in \Omega^*} |Ly(x)| \right\}, \quad x \in \bar{\Omega}.$$

Proof. For the proof, refer [8]. □

Lemma 3.3. Let $y(x)$ be the solution of equations (1 and 2). Then, we have the following bounds:

$$|y^{(k)}(x)|_{\Omega^*} \leq C\varepsilon^{-k}, \quad k = 1, 2, 3.$$

Proof. For the proof, refer [8]. □

4 Numerical scheme formulation

We divide the interval $[0, 2]$ into $2N$ equal parts with constant mesh length h . Let $0 = x_0, x_1, \dots, x_N = 1, x_{N+1}, x_{N+2}, \dots, x_{2N} = 2$ be the mesh points. Then, we have $x_i = ih$, $i = 0, 1, 2, \dots, 2N$.

Consider a uniform mesh with an interval $[0, 1]$ in which $0 = x_0 < x_1 < \dots < x_N = 1$, where $h = \frac{1}{N}$ and $x_i = ih$, $i = 0, 1, 2, \dots, N$.

For the domain Ω_1 , from equations (4) and (5), we have

$$-\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = Q(x), \quad x \in \Omega_1, \quad (7)$$

with interval and boundary conditions of

$$y(x) = \phi(x), \quad x \in [-1, 0], y(2) = l, \quad (8)$$

where $Q(x) = f(x) - c(x)\phi(x - 1)$.

For each segment $[x_i, x_{i+1}]$, $i = 1, 2, \dots, N - 1$, the non-polynomial cubic spline $S_\Delta(x)$ has the following form:

$$S_\Delta(x) = a_i + b_i(x - x_i) + c_i(e^{w(x-x_i)} + e^{-w(x-x_i)}) + d_i(e^{w(x-x_i)} - e^{-w(x-x_i)}), \quad (9)$$

where a_i, b_i, c_i , and d_i are unknown coefficients, and $w \neq 0$ arbitrary parameter which will be used to increase the accuracy of the method.

To determine the unknown coefficients in equation (9) in terms of y_i, y_{i+1}, M_i , and M_{i+1} , first we define:

$$\begin{cases} S_\Delta(x_i) = y_i, & S_\Delta(x_{i+1}) = y_{i+1}, \\ S''_\Delta(x_i) = M_i, & S''_\Delta(x_{i+1}) = M_{i+1}. \end{cases} \quad (10)$$

The coefficients in equation (9) are determined as

$$\begin{cases} a_i = y_i - \frac{M_i}{w^2}, \\ b_i = \frac{y_{i+1} - y_i}{h} + \frac{M_i - M_{i+1}}{w\theta}, \\ c_i = \frac{M_{i+1}}{w^2(e^\theta - e^{-\theta})} - \frac{M_i(e^\theta + e^{-\theta})}{2w^2(e^\theta - e^{-\theta})}, \\ d_i = \frac{M_i}{2w^2}, \end{cases} \quad (11)$$

where $\theta = wh$.

Using the continuity condition of the first derivative at x_i , $S'_{\Delta-1}(x_i) = S'_\Delta(x_i)$, we have

$$b_{i-1} + wc_{i-1}(e^\theta + e^{-\theta}) + wd_{i-1}(e^\theta - e^{-\theta}) = b_i + 2wc_i. \quad (12)$$

Reducing indices of equation (11) by one and substituting into equation (12), we obtain

$$\begin{aligned} \frac{y_i - y_{i-1}}{h} + \frac{M_i - M_{i+1}}{w\theta} + w \left(\frac{2M_i - (e^\theta + e^{-\theta})M_{i-1}}{2w^2(e^\theta + e^{-\theta})} \right) &= \frac{y_{i+1} - y_i}{h} + \frac{M_i - M_{i+1}}{w\theta} + 2w \left(\frac{M_{i+1}}{w^2(e^\theta - e^{-\theta})} - \frac{M_i(e^\theta + e^{-\theta})}{2w^2(e^\theta - e^{-\theta})} \right), \\ \Rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} &= \alpha M_{i-1} + 2\beta M_i + \alpha M_{i+1}, \end{aligned} \quad (13)$$

where $\alpha = \frac{1}{\theta^2} \left(1 - \frac{2\theta}{(e^\theta - e^{-\theta})} \right)$, $\beta = \frac{1}{\theta^2} \left(\frac{\theta(e^\theta + e^{-\theta})}{(e^\theta - e^{-\theta})} - 1 \right)$.

If $h \rightarrow 0$, then $\theta = hk \rightarrow 0$. Thus, using L'Hopitals rule, we have $\lim_{h \rightarrow 0} \alpha = \frac{1}{6}$ and $\lim_{h \rightarrow 0} \beta = \frac{1}{3}$.

Using $S''_\Delta(x_i) = y''_i = M_i$, equation (7) can be re-written as

$$\begin{cases} -\varepsilon M_i = Q_i - a_i y'_i - b_i y_i, \\ -\varepsilon M_{i-1} = Q_{i-1} - a_{i-1} y'_{i-1} - b_{i-1} y_{i-1}, \\ -\varepsilon M_{i+1} = Q_{i+1} - a_{i+1} y'_{i+1} - b_{i+1} y_{i+1}. \end{cases} \quad (14)$$

Using Taylor's series expansions of y_{i-1} , y_{i+1} , y'_{i-1} , and y'_{i+1} simplifying, we have

$$\begin{cases} y'_i = \frac{y_{i+1} - y_{i-1}}{2h} + T_1, \\ y'_{i-1} = \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h} + T_2, \\ y'_{i+1} = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h} + T_2, \end{cases} \quad (15)$$

where $T_1 = -\frac{h^2}{6} y'''(\xi)$ and $T_2 = \frac{h^2}{12} y'''(\xi)$, for $\xi \in (x_{i-1}, x_i)$.

Substituting equation (15) into equation (14), we obtain

$$\begin{cases} M_i = -\frac{1}{\varepsilon} \left\{ Q_i - a_i \left(\frac{y_{i+1} - y_{i-1}}{2h} + T_1 \right) - b_i y_i \right\}, \\ M_{i-1} = -\frac{1}{\varepsilon} \left\{ Q_{i-1} - a_{i-1} \left(\frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h} + T_2 \right) - b_{i-1} y_{i-1} \right\}, \\ M_{i+1} = -\frac{1}{\varepsilon} \left\{ Q_{i+1} - a_{i+1} \left(\frac{3y_{i+1} - 4y_i + y_{i-1}}{2h} + T_2 \right) - b_{i+1} y_{i+1} \right\}. \end{cases} \quad (16)$$

Plugging equation (16) into equation (13) and rearranging, we get

$$\begin{aligned} & \frac{-\varepsilon}{h^2} (y_{i-1} - 2y_i + y_{i+1}) + \frac{\alpha a_{i-1}}{2h} (-y_{i+1} - 4y_i - 3y_{i-1}) + \frac{2\beta a_i}{2h} (y_{i+1} - y_{i-1}) + \frac{\alpha a_{i+1}}{2h} (3y_{i+1} - 3y_i + y_{i-1}) \\ & = \alpha (Q_{i-1} - b_{i-1} y_{i-1} + Q_{i+1} - b_{i+1} y_{i+1}) + 2\beta (Q_i - b_i y_i) + T, \end{aligned} \quad (17)$$

where $T = (4\beta a_i - \alpha a_{i-1} - \alpha a_{i+1}) \frac{h^2}{12} y'''(\xi)$ is the local truncation error.

To find the numerical solution of equation (7), we use the theory applied in an asymptotic method for solving singularly perturbed BVPs. In the considered case, the boundary layer is on the right side of the domain, i.e., near $x = 1$. From the theory of singular perturbations given by [12], we get the asymptotic solution up to first-order approximation as

$$y(x) = y_0(x) + \frac{a(1)}{a(x)} (\theta_1 - y_0(1)) \exp \left(- \int_x^1 \left(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)} \right) dx \right) + O(\varepsilon). \quad (18)$$

Using Taylor series approximation for $a(x)$ and $b(x)$ around $x = 1$ and simplifying, we obtain

$$y(x) = y_0(x) + (\theta_1 - y_0(1)) \exp \left(\frac{-a(1)(1-x)}{\varepsilon} \right), \quad (19)$$

where $y_0(x)$ is the solution of the reduced problem (obtained by setting $\varepsilon = 0$ of equation (7) which is given by

$$a(x)y'(x) + b(x)y(x) = Q(x), \quad \text{with } y(x) = \phi(x), \quad -1 \leq x \leq 0, \quad (20)$$

where the reduced problem of (20) is solved by the classical Runge-Kutta method and the value of $y(1) = y_0(1) = \theta_1$ is obtained.

To control the effect of the perturbation parameter in the solution, we introduce a fitting factor $\sigma(\rho)$ in to equation (17) and we get

$$-\frac{\sigma(\rho)\varepsilon}{h^2}(y_{i-1} - 2y_i + y_{i+1}) + \frac{\alpha a_{i-1}}{2h}(-y_{i+1} - 4y_i - 3y_{i-1}) + \frac{2\beta a_i}{2h}(y_{i+1} - y_{i-1}) + \frac{\alpha a_{i+1}}{2h}(3y_{i+1} - 3y_i + y_{i-1}) \\ = \alpha(Q_{i-1} - b_{i-1}y_{i-1} + Q_{i+1} - b_{i+1}y_{i+1}) + 2\beta(Q_i - b_i y_i) + T. \quad (21)$$

Multiplying equation (21) by h and taking a limit as $h \rightarrow 0$, we obtain

$$-\frac{\sigma(\rho)}{\rho} \lim_{h \rightarrow 0} (y_{i-1} - 2y_i + y_{i+1}) + \frac{\alpha a(1)}{h} \lim_{h \rightarrow 0} (-y_{i+1} - 4y_i - 3y_{i-1}) + \beta a(1) \lim_{h \rightarrow 0} (y_{i+1} - y_{i-1}) \\ + \frac{\alpha a(1)}{2} \lim_{h \rightarrow 0} (3y_{i+1} - 3y_i + y_{i-1}) = 0. \quad (22)$$

Since the problem has the right boundary layer for Ω_1 , we have

$$\begin{cases} \lim_{h \rightarrow 0} (y_{i-1} - 2y_i + y_{i+1}) = (\theta_1 - y_0(1))e^{-a(1)\rho}(e^{a(1)\rho} + e^{-a(1)\rho} - 2), \\ \lim_{h \rightarrow 0} (-y_{i+1} - 4y_i - 3y_{i-1}) = (\theta_1 - y_0(1))e^{-a(1)\rho}(-3e^{a(1)\rho} - e^{-a(1)\rho} + 4), \\ \lim_{h \rightarrow 0} (y_{i+1} - y_{i-1}) = (\theta_1 - y_0(1))e^{-a(1)\rho}(e^{a(1)\rho} + 3e^{-a(1)\rho} - 4), \\ \lim_{h \rightarrow 0} (3y_{i+1} - 3y_i + y_{i-1}) = (\theta_1 - y_0(1))e^{-a(1)\rho}(e^{-a(1)\rho} - e^{a(1)\rho}). \end{cases} \quad (23)$$

Substituting equation (23) into equation (22) and simplifying, we get

$$\sigma_1 = \rho a(1)(\alpha + \beta) \coth\left(\frac{a(1)\rho}{2}\right), \quad (24)$$

where $\rho = \frac{h}{\varepsilon}$.

Thus, equation (21) can be written as

$$\left\{ -\frac{\varepsilon\sigma_1}{h^2} - \frac{3\alpha a_{i-1}}{2h} + \alpha b_{i-1} - \frac{\beta a_i}{h} + \frac{\alpha a_{i+1}}{2h} \right\} y_{i-1} - \left\{ -\frac{2\varepsilon\sigma_1}{h^2} - \frac{2\alpha a_{i-1}}{h} - 2\beta b_i + \frac{2\alpha a_{i+1}}{h} \right\} y_i \\ + \left\{ -\frac{\varepsilon\sigma_1}{h^2} - \frac{\alpha a_{i-1}}{2h} + \alpha b_{i+1} + \frac{\beta a_i}{h} + \frac{3\alpha a_{i+1}}{2h} \right\} y_{i+1} = \alpha(Q_{i-1} + Q_{i+1}) + 2\beta Q_i. \quad (25)$$

Simplifying equation (25), for the domain $\Omega_1 = (0, 1)$, we get the tri-diagonal system of the equation of the form

$$L^N \equiv E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, \dots, N, \quad (26)$$

where

$$\begin{cases} E_i = -\frac{\varepsilon\sigma_1}{h^2} - \frac{3\alpha a_{i-1}}{2h} + \alpha b_{i-1} - \frac{\beta a_i}{h} + \frac{\alpha a_{i+1}}{2h}, \\ F_i = -\frac{2\varepsilon\sigma_1}{h^2} - \frac{2\alpha a_{i-1}}{h} - 2\beta b_i + \frac{2\alpha a_{i+1}}{h}, \\ G_i = -\frac{\varepsilon\sigma_1}{h^2} - \frac{\alpha a_{i-1}}{2h} + \alpha b_{i+1} + \frac{\beta a_i}{h} + \frac{3\alpha a_{i+1}}{2h}, \\ H_i = \alpha(Q_{i-1} + Q_{i+1}) + 2\beta Q_i. \end{cases}$$

Similarly, for the domain Ω_2 , from equations (4) and (5), we have

$$\begin{cases} -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) + c(x)y(x-1) = f(x), & x \in \Omega_2, \\ y(1) = \theta_1, & y(2) = l. \end{cases} \quad (27)$$

From the theory of singular perturbations given by [13], we get the asymptotic solution for the domain Ω_2 up to first-order approximation as

$$y(x) = y_0(x) + \frac{a(2)}{a(x)}(l - y_0(2)) \exp\left(-\int_x^2 \left(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)}\right) dx\right) + O(\varepsilon). \quad (28)$$

Using Taylor series approximation for $a(x)$ and $b(x)$ around $x = 2$ and simplifying, we obtain

$$y(x) = y_0(x) + (l - y_0(2)) \exp\left(\frac{-a(2)}{\varepsilon}(2 - x)\right), \quad (29)$$

Consider a uniform mesh with the interval $[1, 2]$ in which $1 < x_N < x_{N+1} < \dots < x_{2N} < 2$. where $h = \frac{1}{N}$ and $x_i = ih$, $i = 0, 1, 2, \dots, N$. Using $S''_h(x_i) = y''_i = M_i$, equation (27) can be re-written as

$$\begin{cases} -\varepsilon M_i = f_i - a_i y'_i - b_i y_i - c_i y(x_i - 1), \\ -\varepsilon M_{i-1} = f_{i-1} - a_{i-1} y'_{i-1} - b_{i-1} y_{i-1} - c_{i-1} y(x_{i-1} - 1), \\ -\varepsilon M_{i+1} = f_{i+1} - a_{i+1} y'_{i+1} - b_{i+1} y_{i+1} - c_{i+1} y(x_{i+1} - 1). \end{cases} \quad (30)$$

Substituting equation (15) in to equation (30), we get

$$\begin{cases} M_i = -\frac{1}{\varepsilon} \left\{ f_i - a_i \left(\frac{y_{i+1} - y_{i-1}}{2h} + T_1 \right) - b_i y_i - c_i y(x_i - 1) \right\}, \\ M_{i-1} = -\frac{1}{\varepsilon} \left\{ f_{i-1} - a_{i-1} \left(\frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h} + T_2 \right) - b_{i-1} y_{i-1} - c_{i-1} y(x_{i-1} - 1) \right\}, \\ M_{i+1} = -\frac{1}{\varepsilon} \left\{ f_{i+1} - a_{i+1} \left(\frac{3y_{i+1} - 4y_i + y_{i-1}}{2h} + T_2 \right) - b_{i+1} y_{i+1} - c_{i+1} y(x_{i+1} - 1) \right\}. \end{cases} \quad (31)$$

Substituting equations (31) into (13) and rearranging, we get

$$\begin{aligned} & -\frac{\varepsilon}{h^2}(y_{i-1} - 2y_i + y_{i+1}) + \frac{\alpha a_{i-1}}{2h}(-y_{i+1} - 4y_i - 3y_{i-1}) + \frac{2\beta a_i}{2h}(y_{i+1} - y_{i-1}) + \frac{\alpha a_{i+1}}{2h}(3y_{i+1} - 3y_i + y_{i-1}) \\ & = \alpha(f_{i-1} - b_{i-1}y_{i-1} - c_{i-1}y_{x_{i-1}-1}) + f_{i+1} - b_{i+1}y_{i+1} - c_{i+1}y_{x_{i+1}-1}) + 2\beta(f_i - b_i y_i - c_i y_{x_i-1}) + T^*, \end{aligned} \quad (32)$$

where $T^* = (4\beta a_i - \alpha a_{i-1} - \alpha a_{i+1})\frac{h^2}{12}y'''(\eta)$ is the local truncation error.

To get the fitting factor for the second domain, we introduce a fitting factor $\sigma(\rho)$ in to equation (32) and we get

$$\begin{aligned} & -\frac{\sigma(\rho)\varepsilon}{h^2}(y_{i-1} - 2y_i + y_{i+1}) + \frac{\alpha a_{i-1}}{2h}(-y_{i+1} - 4y_i - 3y_{i-1}) + \frac{2\beta a_i}{2h}(y_{i+1} - y_{i-1}) + \frac{\alpha a_{i+1}}{2h}(3y_{i+1} - 3y_i + y_{i-1}) \\ & = \alpha(f_{i-1} - b_{i-1}y_{i-1} - c_{i-1}y_{x_{i-1}-1}) + f_{i+1} - b_{i+1}y_{i+1} - c_{i+1}y_{x_{i+1}-1}) + 2\beta(f_i - b_i y_i - c_i y_{x_i-1}) + T^*, \end{aligned} \quad (33)$$

Multiplying equation (33) by h and taking a limit as $h \rightarrow 0$, we get

$$\begin{aligned} & -\frac{\sigma(\rho)}{\rho} \lim_{h \rightarrow 0} (y_{i-1} - 2y_i + y_{i+1}) + \frac{\alpha a(2)}{h} \lim_{h \rightarrow 0} (-y_{i+1} - 4y_i - 3y_{i-1}) + \beta a(2) \lim_{h \rightarrow 0} (y_{i+1} - y_{i-1}) \\ & + \frac{\alpha a(2)}{2} \lim_{h \rightarrow 0} (3y_{i+1} - 3y_i + y_{i-1}) = 0. \end{aligned} \quad (34)$$

Since the problem has the right boundary layer for Ω_2 , we have

$$\begin{cases} \lim_{h \rightarrow 0} (y_{i-1} - 2y_i + y_{i+1}) = (l - y_0(2))e^{-a(2)\rho}(e^{a(2)\rho} + e^{-a(2)\rho} - 2), \\ \lim_{h \rightarrow 0} (-y_{i+1} - 4y_i - 3y_{i-1}) = (l - y_0(2))e^{-a(2)\rho}(-3e^{a(2)\rho} - e^{-a(2)\rho} + 4), \\ \lim_{h \rightarrow 0} (y_{i+1} - y_{i-1}) = (l - y_0(2))e^{-a(2)\rho}(e^{a(2)\rho} + 3e^{-a(2)\rho} - 4), \\ \lim_{h \rightarrow 0} (3y_{i+1} - 3y_i + y_{i-1}) = (l - y_0(2))e^{-a(2)\rho}(e^{-a(2)\rho} - e^{a(2)\rho}). \end{cases} \quad (35)$$

Substituting equation (35) into equation (34) and simplifying, we get

$$\sigma_2 = \rho a(2)(\alpha + \beta) \coth\left(\frac{a(2)\rho}{2}\right). \quad (36)$$

where $\rho = \frac{h}{\varepsilon}$.

Thus, equation (33) can be written as

$$\begin{aligned} & \left\{ -\frac{\varepsilon\sigma_2}{h^2} - \frac{3\alpha a_{i-1}}{2h} + \alpha b_{i-1} - \frac{\beta a_i}{h} + \frac{\alpha a_{i+1}}{2h} \right\} y_{i-1} - \left\{ -\frac{2\varepsilon\sigma_2}{h^2} - \frac{2\alpha a_{i-1}}{h} - 2\beta b_i + \frac{2\alpha a_{i+1}}{h} \right\} y_i \\ & + \left\{ -\frac{\varepsilon\sigma_2}{h^2} - \frac{\alpha a_{i-1}}{2h} + \alpha b_{i+1} + \frac{\beta a_i}{h} + \frac{3\alpha a_{i+1}}{2h} \right\} y_{i+1} \\ & = \alpha(f_{i-1} + f_{i+1}) + 2\beta f_i + \alpha\{c_{i-1}y(x_{i-1} - 1) + c_{i+1}y(x_{i+1} - 1) + 2\beta c_i y(x_i - 1)\}. \end{aligned} \quad (37)$$

Simplifying equation (37) for the domain $\Omega_2 = (1, 2)$, we get the bi tri-diagonal system of the equation of the form

$$L^N \equiv E_i y_{i-1} - F_i y_i + G_i y_{i+1} + T_i = H_i, \quad i = N + 1, N + 2, \dots, 2N - 1, \quad (38)$$

where

$$\begin{cases} E_i = -\frac{\varepsilon\sigma_2}{h^2} - \frac{3\alpha a_{i-1}}{2h} + \alpha b_{i-1} - \frac{\beta p_i}{h} + \frac{\alpha a_{i+1}}{2h}, \\ F_i = -\frac{2\varepsilon\sigma_2}{h^2} - \frac{2\alpha a_{i-1}}{h} - 2\beta b_i + \frac{2\alpha a_{i+1}}{h}, \\ G_i = -\frac{\varepsilon\sigma_2}{h^2} - \frac{\alpha a_{i-1}}{2h} + \alpha b_{i+1} + \frac{\beta p_i}{h} + \frac{3\alpha a_{i+1}}{2h}, \\ H_i = \alpha(f_{i-1} + f_{i+1}) + 2\beta f_i, \\ T_i = \alpha\{c_{i-1}y(x_{i-1} - 1) + c_{i+1}y(x_{i+1} - 1) + 2\beta c_i y(x_i - 1)\}. \end{cases}$$

Therefore, on the whole domain $\bar{\Omega} = [0, 2]$, the basic schemes to solve equations (1 and 2) are given in equations (26) and (38).

5 Stability and convergence analysis

Theorem 5.1. (Stability) *Let B be a coefficient matrix of the tri-diagonal system, equation (26). Then, for all $\varepsilon > 0$ and sufficiently small h , the matrix B is an irreducible and diagonally dominant matrix, and hence, the scheme is stable.*

Proof. Substituting equations (24) in (25) and multiplying both sides of the equation by h , we get the equivalent tri-diagonal scheme

$$\begin{aligned} & \left\{ \frac{a_i}{2} \coth\left(\frac{a_i \rho}{2}\right) - \frac{3\alpha a_{i-1}}{2} + h\alpha b_{i-1} - \beta a_i + \frac{\alpha a_{i+1}}{2} \right\} y_{i-1} - \left\{ a_i \coth\left(\frac{a_i \rho}{2}\right) - 2\alpha a_{i-1} - 2h\beta b_i + 2\alpha a_{i+1} \right\} y_i \\ & + \left\{ \frac{a_i}{2} \coth\left(\frac{a_i \rho}{2}\right) + \frac{3\alpha a_{i+1}}{2} + h\alpha b_{i+1} + \beta a_i - \frac{\alpha a_{i-1}}{2} \right\} y_{i+1} = h(\alpha(Q_{i-1} + Q_{i+1}) + 2\beta Q_i). \end{aligned} \quad (39)$$

This can be written as

$$E_i^* y_{i-1} - F_i^* y_i + G_i^* y_{i+1} = H_i^*, \quad (40)$$

where

$$\begin{cases} E_i^* = \frac{a_i}{2} \coth\left(\frac{a_i \rho}{2}\right) - \frac{3\alpha a_{i-1}}{2} + h a b_{i-1} - \beta a_i + \frac{\alpha a_{i+1}}{2}, \\ F_i^* = a_i \coth\left(\frac{a_i \rho}{2}\right) - 2\alpha a_{i-1} - 2h\beta b_i + 2\alpha p_{i+1}, \\ G_i^* = \frac{a_i}{2} \coth\left(\frac{p_i \rho}{2}\right) + \frac{3\alpha a_{i+1}}{2} + h a b_{i+1} + \beta a_i - \frac{\alpha a_{i-1}}{2}, \\ H_i^* = h(\alpha(Q_{i-1} + Q_{i+1}) + 2\beta Q_i). \end{cases}$$

Rewriting equation (40) in a matrix-vector form, we obtain

$$BY = C, \quad (41)$$

where B is a coefficient matrix, $Y = (y_1, y_2, \dots, y_{N-1})^T$, and $C = (H_1^* - E_1^* \phi_0, H_2^* - E_2^* \phi_1, \dots, H_{N-1}^* - E_{N-1}^* \varphi)^T$.

The matrix B is a tri-diagonal matrix, and its off-diagonal elements are E_i^* and G_i^* .

Now,

$$|E_i^* + G_i^*| = \left| a_i \coth\left(\frac{a_i \rho}{2}\right) + 2\alpha a_{i+1} - 2\alpha a_{i-1} \right| = \left| a_i \coth\left(\frac{a_i \rho}{2}\right) + 2\alpha(a_{i+1} - a_{i-1}) \right| \leq |F_i^*|.$$

This implies that for each row of B , the sum of the two off-diagonal elements is less than the modulus of the diagonal element. Therefore, B is diagonally dominant.

Further, for sufficiently small h (i.e., $h \rightarrow 0$), we have, $E_i^* \neq 0$ and $G_i^* \neq 0 \forall i = 1, 2, \dots, N-1$.

Hence, B is irreducible [14]. Therefore, from these two conditions, the scheme in equation (26) is stable [15]. $\square$

Theorem 5.2. (Convergence) *Let $y(x)$ be the analytical solution of the problem in equations (1) and (2), and y^N be the numerical solution of the discretized problem of equation (26). Then, $\|y - y^N\| \leq Ch^2$ for sufficiently small h and C is positive constant.*

Proof. Multiplying both sides of equation (25) by $\frac{-h^2}{\varepsilon \sigma_i}$ and simplifying, we obtain

$$(-1 + u_i)y_{i-1} + (2 + v_i)y_i + (-1 + w_i)y_{i+1} + g_i + T_i = 0, \quad (42)$$

where

$$\begin{cases} u_i = \frac{1}{\varepsilon \sigma_i} \left(\frac{3\alpha h a_{i-1}}{2} - \alpha h^2 b_{i-1} + \beta a_i h - \frac{\alpha h a_{i+1}}{2} \right), \\ v_i = \frac{2}{\varepsilon \sigma_i} (\alpha h a_{i+1} - \alpha h a_{i-1} - \beta h^2 b_i), \\ w_i = \frac{1}{\varepsilon \sigma_i} \left(\frac{\alpha h a_{i-1}}{2} - \beta a_i h - \frac{3\alpha h a_{i+1}}{2} - \alpha h^2 b_{i+1} \right), \\ g_i = \frac{-h^2}{\varepsilon \sigma_i} \{ \alpha(Q_{i-1} + Q_{i+1}) + 2\beta Q_i \}, \end{cases}$$

and $T_i = \frac{\alpha(a_{i-1} + a_{i+1}) - 4\beta a_i}{12\varepsilon \sigma_i} h^4 y'''(\xi)$ is a local truncation error for $i = 1, 2, \dots, N-1$.

Incorporating the boundary condition $y_0 = \phi(x_0) = \phi_0$, $y_N = \phi(1) = \varphi$ in equation (42), we get the system of equation of the form

$$(D + P)y + M + T(h) = \bar{0}, \quad (43)$$

where

$$D = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & - & - & & 0 \\ \vdots & & & & -1 \\ 0 & - & - & -1 & 2 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} v_1 & w_1 & 0 & \dots & 0 \\ u_2 & v_2 & w_2 & \dots & 0 \\ 0 & - & - & & 0 \\ \vdots & & & & w_{N-2} \\ 0 & - & - & u_{N-1} & u_{N-1} \end{pmatrix}$$

are tri-diagonal matrices of order $N - 1$, $M = [g_1 + (-1 + u_1)\phi_0, g_2, g_3, \dots, g_{N-1} + (-1 + w_{N-1})\varphi]^T$, $T(h) = O(h^4)$ and $y = [y_1, y_2, \dots, y_{N-1}]^T$, $T(h) = [T_1, T_2, \dots, T_{N-1}]^T$, $\bar{0} = [0, 0, \dots, 0]^T$ are associated vectors of equation (43).

Let $y^N = [y_1^N, y_2^N, \dots, y_{N-1}^N]^T \equiv y$ be the solution which satisfies equation (43), we have

$$(D + P)y^N + M = \bar{0}. \quad (44)$$

Let $e_i = y_i - y_i^N$ for $i = 1, 2, \dots, N - 1$ be the discretization error then $y - y^N = [e_1, e_2, \dots, e_{N-1}]^T$.

Subtracting equation (43) from equation (44), we get

$$(D + P)(y^N - y) = T(h). \quad (45)$$

Let $|a_{i-1}| \leq c_1$, $|a_i| \leq c_2$, $|a_{i+1}| \leq c_3$, $|b_{i-1}| \leq k_1$, $|b_i| \leq k_2$, $|b_{i+1}| \leq k_3$ and $t_{i,j}$ be (i, j) th of the matrix P , then

$$\begin{cases} |t_{i,i+1}| = |w_i| \leq \frac{h}{\varepsilon\sigma_i} \left(\frac{3\alpha c_1}{2} + \alpha h c_2 + \frac{3\alpha c_3}{2} + \alpha k_3 \right), & i = 1, 2, \dots, N - 2, \\ |t_{i,i-1}| = |u_i| \leq \frac{h}{\varepsilon\sigma_i} \left(\frac{3\alpha c_1}{2} + \alpha h k_2 + \beta c_2 + \frac{\alpha c_3}{2} \right), & i = 2, 3, \dots, N - 1. \end{cases}$$

Thus, for sufficiently small h , we have

$$\begin{cases} -1 + |t_{i,i+1}| \neq 0, & i = 1, 2, \dots, N - 2, \\ -1 + |t_{i,i-1}| \neq 0, & i = 2, 3, \dots, N - 1. \end{cases}$$

Hence, the matrix $(D + P)$ is irreducible [14].

Let A_i be the sum of the elements of the i th row of the matrix $(D + P)$, then

$$\begin{cases} A_i = 1 + v_i + w_i = 1 + \frac{2h}{\varepsilon\sigma_i} \left(\alpha a_{i+1} - \alpha a_{i-1} + \frac{\alpha a_{i-1}}{4} - \frac{\beta a_i}{2} - \frac{3\alpha a_{i+1}}{4} \right) + O(h^2), & i = 1, \\ A_i = u_i + v_i + w_i = \frac{h^2}{\varepsilon\sigma_i} (-\alpha b_{i-1} - \beta b_i - \alpha b_{i+1}), & i = 2, 3, \dots, N - 2, \\ A_i = 1 + u_i + v_i = 1 + \frac{2h}{\varepsilon\sigma_i} \left(\frac{3\alpha a_{i+1}}{4} - \frac{\alpha a_{i-1}}{4} + \frac{\beta a_i}{2} \right) + O(h^2), & i = N - 1. \end{cases}$$

Let

$$\begin{cases} d_1 = \min_{1 \leq i \leq N-1} \frac{1}{\varepsilon\sigma_i} (-\alpha b_{i-1} - 2\beta b_i - \alpha b_{i+1}), \\ d_2 = \max_{1 \leq i \leq N-1} \frac{1}{\varepsilon\sigma_i} (-\alpha b_{i-1} - 2\beta b_i - \alpha b_{i+1}), \end{cases}$$

then, $0 \leq d_1 \leq d_2$.

For sufficiently small h , $(D + P)$ is monotone [14] and [16]. Hence, $(D + P)^{-1}$ exists and $(D + P)^{-1} \geq 0$.

From the error in equation (45), we have

$$\|y - y^N\| \leq \|(D + P)^{-1}\| \|T(h)\|. \quad (46)$$

For sufficiently small h , we have $A_i > h^2 d_1$ for $i = 1, \dots, N - 1$, where $d_1 = \min_{1 \leq i \leq N-1} \left(\frac{1}{\varepsilon\sigma_i} (-\alpha b_{i-1} - 2\beta b_i - \alpha b_{i+1}) \right)$.

Let $(D + P)_{i,k}^{-1}$ be the (i, k) th elements of $(D + P)^{-1}$ and we define,

$$\|(D + P)^{-1}\| = \max_{1 \leq i \leq N-1} \sum_{k=1}^{N-1} (D + P)^{-1}, \quad \|T(h)\| = \max_{1 \leq i \leq N-1} |T_i| \quad (47)$$

Since $(D + P)_{i,k}^{-1} \geq 0$ from the theory of matrices, we have $\sum_{k=1}^{N-1} (D + P)_{i,k}^{-1} A_k = 1, i = 1, 2, \dots, N - 1$.

Hence,

$$\sum_{k=1}^{N-1} (D + P)_{i,k}^{-1} A_k \leq \frac{1}{\min_{1 \leq i \leq N-1} A_k} \leq \frac{1}{h^2 d_1}. \quad (48)$$

Now, from equations (46)–(48), we get

$$\|y - y^N\| \leq \frac{1}{h^2 d_1} \left| \left(\frac{\alpha(a_{i-1} + a_{i+1} - 4\beta a_i)}{\varepsilon \sigma_i} \right) \frac{1}{12} h^4 y'''(\xi) \right| \leq \left(\frac{y'''(\xi)(4\beta a_i + \alpha(a_{i-1} + a_{i+1}))}{12 d_1 \varepsilon \sigma_i} \right) h^2 = Ch^2,$$

where $C = \frac{y'''(\xi)(4\beta a_i + \alpha(a_{i-1} + a_{i+1}))}{12 d_1 \varepsilon \sigma_i}$ which is independent of mesh size h and perturbation parameter ε . This establishes that the method is of second-order convergent.

Remark: A similar analysis for convergence may be carried out for finite difference scheme, equation (38). $\square$

6 Numerical examples and results

In this section, two examples are given to illustrate the numerical method discussed above. The exact solutions of the test problem are not known. Therefore, we use the double mesh principle to estimate the error and compute the experiment rate of convergence to the computed solution. For this, we put

$$E_\varepsilon^N = \max_{0 \leq i \leq 2N} |Y_i^N - Y_{2i}^{2N}|, \quad (49)$$

where Y_i^N and Y_{2i}^{2N} are the i th components of the numerical solutions on meshes of N and $2N$, respectively. We compute the uniform error and the rate of convergence as

$$E^N = \max_\varepsilon E_\varepsilon^N, \quad R^N = \log_2 \left(\frac{E^N}{E^{2N}} \right). \quad (50)$$

The numerical results are presented for the values of the perturbation parameter $\varepsilon \in \{10^{-4}, 10^{-8}, \dots, 10^{-20}\}$.

Example 6.1. Consider the model singularly perturbed boundary value problem

$$-\varepsilon y''(x) + (x + 10)y'(x) - y(x - 1) = x, \quad x \in (0, 1) \cup (1, 2),$$

subject to the boundary conditions

$$y(x) = x, \quad x \in [-1, 0], \quad y(2) = 2.$$

Example 6.2. Consider the model singularly perturbed boundary value problem

$$-\varepsilon y''(x) + (3 + x^2)y'(x) - y(x - 1) = \exp(x) \quad x \in (0, 1) \cup (1, 2),$$

subject to the boundary conditions

$$y(x) = \exp(x), \quad x \in [-1, 0], \quad y(2) = 2.$$

Table 1: Maximum absolute errors and rate of convergence for Example 6.1 at different number of mesh points N

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
10^{-4}	7.2887×10^{-4}	3.7587×10^{-4}	1.9081×10^{-4}	9.6127×10^{-5}	4.8244×10^{-5}
10^{-8}	7.2887×10^{-4}	3.7587×10^{-4}	1.9081×10^{-4}	9.6127×10^{-5}	4.8244×10^{-5}
10^{-12}	7.2887×10^{-4}	3.7587×10^{-4}	1.9081×10^{-4}	9.6127×10^{-5}	4.8244×10^{-5}
10^{-16}	7.2887×10^{-4}	3.7587×10^{-4}	1.9081×10^{-4}	9.6127×10^{-5}	4.8244×10^{-5}
10^{-20}	7.2887×10^{-4}	3.7587×10^{-4}	1.9081×10^{-4}	9.6127×10^{-5}	4.8244×10^{-5}
E^N	7.2887×10^{-4}	3.7587×10^{-4}	1.9081×10^{-4}	9.6127×10^{-5}	4.8244×10^{-5}
R^N	0.9554	0.9781	0.9891	0.9946	

Table 2: Comparisons of maximum absolute errors and rate of convergence for Example 6.1 at different perturbation parameters and N

	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
Present method					
E^N	3.7587×10^{-4}	1.9081×10^{-4}	9.6127×10^{-5}	4.8244×10^{-5}	2.4167×10^{-5}
R^N	0.9611	0.9809	0.9906	0.9953	
Ref. [10]					
E^N	—	—	1.3646×10^{-1}	3.1663×10^{-2}	7.7456×10^{-3}
R^N	—	—	2.0340	2.0258	
Ref. [11]					
E^N	1.4425×10^{-3}	7.3349×10^{-4}	3.6981×10^{-4}	1.8567×10^{-4}	9.3026×10^{-5}
R^N	0.9757	0.9880	0.9940	0.9970	
Ref. [17]					
E^N	2.766×10^{-3}	1.402×10^{-3}	7.056×10^{-4}	3.540×10^{-4}	1.772×10^{-4}
R^N	0.9806	0.9904	0.9952	0.9979	

Table 3: Maximum absolute errors and rate of convergence for Example 6.2 at different number of mesh points N

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
10^{-4}	2.0864×10^{-3}	1.1120×10^{-3}	5.7327×10^{-4}	2.9096×10^{-4}	1.4656×10^{-4}
10^{-8}	2.0864×10^{-3}	1.1120×10^{-3}	5.7327×10^{-4}	2.9096×10^{-4}	1.4656×10^{-4}
10^{-12}	2.0864×10^{-3}	1.1120×10^{-3}	5.7327×10^{-4}	2.9096×10^{-4}	1.4656×10^{-4}
10^{-16}	2.0864×10^{-3}	1.1120×10^{-3}	5.7327×10^{-4}	2.9096×10^{-4}	1.4656×10^{-4}
10^{-20}	2.0864×10^{-3}	1.1120×10^{-3}	5.7327×10^{-4}	2.9096×10^{-4}	1.4656×10^{-4}
E^N	2.0864×10^{-3}	1.1120×10^{-3}	5.7327×10^{-4}	2.9096×10^{-4}	1.4656×10^{-4}
R^N	0.9079	0.9559	0.9784	0.9893	

Table 4: Comparisons of maximum absolute errors and rate of convergence for Example 6.2 at different perturbation parameters and $N = 23$

$\varepsilon \rightarrow$	2^{-5}	2^{-9}	2^{-13}	2^{-17}	2^{-20}
Present method					
E^N	2.4989×10^{-3}	1.5082×10^{-3}	1.5096×10^{-3}	1.5096×10^{-3}	1.5096×10^{-3}
R^N	0.9769	0.9886	0.9943	0.9972	
Ref. [18]					
E^N	0.0257	0.0250	0.0250	0.0250	0.0250
R^N	2.2519	2.2453	2.3100	2.3133	2.3134

Figure 1: The behavior of the numerical solution for Examples 6.1 and 6.2 at $\varepsilon = 10^{-12}$ and $N = 32$ respectively.

Figure 2: Point wise absolute error of Examples 6.1 and 6.2 at $\varepsilon = 10^{-12}$ with different mesh point N respectively.

Figure 3: ε -uniform convergence with fitted operator in log-log scale for Examples 6.1 and 6.2 respectively.

7 Discussion and conclusion

This study introduces a fitted non-polynomial cubic spline method for singularly perturbed differential equations with a large delay. The numerical scheme is developed on the uniform mesh using a fitted operator in the given differential equation. The stability of the developed numerical method is established, and its uniform convergence is proved. To validate the applicability of the method, two model problems of variable coefficient are considered for numerical experimentation for different values of the perturbation parameter and mesh points. The numerical results are tabulated in terms of maximum absolute errors, numerical rate of convergence and uniform errors (see Tables 1–4), and compared with the results of the previously developed numerical methods existing in the literature (see Tables 2 and 4). Further, behavior of the numerical solution (Figure 1), point-wise absolute error (Figure 2), and the ε -uniform convergence of the method is shown by the log–log plot (Figure 3). The proposed method gives more accurate, stable, and uniform convergent numerical results.

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