Research Article

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Global existence and dynamic structure of solutions for damped wave equation involving the fractional Laplacian

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Abstract: We consider strong damped wave equation involving the fractional Laplacian with nonlinear source. The results of global solution under necessary conditions on the critical exponent are established. The existence is proved by using the Galerkin approximations combined with the potential well theory. Moreover, we showed new decay estimates of global solution.

Keywords: fractional Laplacian, global existence, fractional Sobolev spaces, potential well, hyperbolic problem

MSC 2020: 35R11, 35L20, 35L70, 47G20, 35Q91

1 Introduction, function spaces and auxiliary results

Let $\Omega_1 \subset \mathbb{R}^n$, $n \geq 1$ with Lipschitz boundary $\partial\Omega_1$ and $\Omega_2 = \mathbb{R}^n \setminus \Omega_1$, $w = w(x, t)$. In this article, we consider the hyperbolic initial boundary value problem involving the fractional Laplacian with power nonlinearity

$$\begin{cases} \partial_t^2 w + (-\Delta)^r w + (-\Delta)^r \partial_t w = w |w|^{p-2}, & x \in \Omega_1, \quad t > 0, \\ w = 0, & x \in \Omega_2, \quad t > 0, \\ w(x, 0) = w_0(x), \partial_t w(x, 0) = w_1(x), & x \in \Omega_1, \end{cases} \quad (1.1)$$

where the exponent p satisfies

$$2 < p \leq \frac{2n}{n-2r} = 2_r^*, \quad n > 2r. \quad (1.2)$$

Here, $(-\Delta)^r$, $r \in (0, 1)$ is the fractional Laplacian. The fractional Laplacian of the function w is a singular integral operator defined by

$$(-\Delta)^r w(x) = C \int_{\mathbb{R}^n} \frac{w(x) - w(z)}{|x - z|^{n+2r}} dz, \quad \forall x \in \mathbb{R}^n, \quad (1.3)$$

where $C^{-1} = \int_{\mathbb{R}^n} \frac{1 - \cos(\zeta_1)}{|\zeta|^{n+2r}} d\zeta$.

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Similar problems were studied, we refer for example to the pioneer works of MacCamy et al. [1,2] and the books of Zuazua [3] and other authors [4–11] and references therein, for a complete analysis and review on this topic.

Motivated by the aforementioned works, we complete the study of weak solutions for problem (1.1) in the setting of fractional Laplacian by potential well theory and Galerkin approximations. More precisely, we shall prove the existence of global solutions for problem (1.1). Furthermore, we show anew decay estimates of global solutions.

It is very important to note that our model involving fractional Laplacian is surely well studied in recent years. This type of problem arises much more in many different applications, such as for example image processing, finance, population dynamics, fluid dynamics, minimal surfaces and game theory and especially in physics.

The rest of the paper is organized as follows. In Section 1, we introduce our problem and recall some necessary definitions and properties of the fractional Sobolev spaces. In Section 3, we study the global existence of weak solutions for problem (1.1). In Section 4, we show the decay rate of global solutions of (1.1).

Some necessary definitions and properties regarding the fractional Sobolev spaces are stated here, see [12] for further details.

We define the fractional-order Sobolev space by

$$W^{r,2}(\Omega_1) = \left\{ v \in L^2(\Omega_1) : \int_{\Omega_1} \int_{\Omega_1} \frac{|v(x) - v(z)|^2}{|x - z|^{n+2r}} dx dz < \infty \right\}, \quad (1.4)$$

equipped with the norm

$$\|w\|_{W^{r,2}(\Omega_1)} = \left(\int_{\Omega_1} |w|^2 dx + \int_{\Omega_1} \int_{\Omega_1} \frac{|w(x) - w(z)|^2}{|x - z|^{n+2r}} dx dz \right)^{\frac{1}{2}}, \quad (1.5)$$

and

$$W_0^{r,2}(\Omega_1) = \{w \in W^{r,2}(\Omega_1) : w = 0 \text{ a.e. in } \Omega_2\}, \quad (1.6)$$

is a closed linear subspace of $W^{r,2}(\Omega_1)$, and its norm is given by

$$\|w\|_{W_0^{r,2}(\Omega_1)} = \left(\int_{\Omega_1} \int_{\Omega_1} \frac{|w(x) - w(z)|^2}{|x - z|^{n+2r}} dx dz \right)^{\frac{1}{2}}. \quad (1.7)$$

The space $W_0^{r,2}(\Omega_1)$ is a Hilbert space with inner product

$$\langle w, u \rangle_{W_0^{r,2}(\Omega_1)} = \int_{\Omega_1} \int_{\Omega_1} \frac{(w(x) - w(z))(u(x) - u(z))}{|x - z|^{n+2r}} dx dz. \quad (1.8)$$

2 The potential well

We define

$$\mathcal{J}(w) = \frac{1}{2} \|w\|_{W_0^{r,2}(\Omega_1)}^2 - \frac{1}{p} \|w\|_p^p \quad (2.1)$$

and

$$\mathcal{I}(w) = \|w\|_{W_0^{r,2}(\Omega_1)}^2 - \|w\|_p^p. \quad (2.2)$$

We define then the stable set as follows:

$$\mathcal{W} = \{w \in W_0^{r,2}(\Omega_1) : I(w) > 0, \mathcal{J}(w) < d\} \cup \{0\}, \quad (2.3)$$

where the mountain pass level d is defined by

$$d = \inf_{w \in W_0^{r,2}(\Omega_1) \setminus \{0\}} \left\{ \sup_{\mu \geq 0} \mathcal{J}(\mu w) \right\}. \quad (2.4)$$

We introduce the so-called “Nehari manifold”

$$\mathcal{N} = \{w \in W_0^{r,2}(\Omega_1) \setminus \{0\} : I(w) = 0\}, \quad (2.5)$$

then potential depth d is characterized by

$$d = \inf_{w \in \mathcal{N}} \mathcal{J}(w), \quad (2.6)$$

which implies that

$$\text{dist}(0, \mathcal{N}) = \min_{w \in \mathcal{N}} \|w\|_{W_0^{r,2}(\Omega_1)}. \quad (2.7)$$

We will prove the invariance of the set $\mathcal{W}$.

Lemma 2.1.

1. d is a positive constant.
2. $\mathcal{J}(\mu w)$ attains maximum, with respect to μ , at

$$\mu^* = \left(\frac{\|w\|_{W_0^{r,2}(\Omega_1)}^2}{\|w(t)\|_p^p} \right)^{1/(p-2)}. \quad (2.8)$$

Proof. We have

$$\mathcal{J}(\mu w) = \frac{\mu^2}{2} \|w\|_{W_0^{r,2}(\Omega_1)}^2 - \frac{\mu^p}{p} \|w\|_p^p. \quad (2.9)$$

Differentiating with respect to μ to get

$$\frac{d}{d\mu} \mathcal{J}(\mu w) = \mu \|w\|_{W_0^{r,2}(\Omega_1)}^2 - \mu^{p-1} \|w\|_p^p. \quad (2.10)$$

For $\mu_1 = 0$ and $\mu_2 = \left(\frac{\|w\|_{W_0^{r,2}(\Omega_1)}^2}{\|w(t)\|_p^p} \right)^{1/(p-2)}$, we have

$$\frac{d}{d\mu} \mathcal{J}(\mu w) = 0. \quad (2.11)$$

As $\mathcal{J}(\mu_1) = 0$, we have

$$\begin{aligned} \mathcal{J}(\mu_2 w) &= \frac{1}{2} \left(\frac{\|w\|_{W_0^{r,2}(\Omega_1)}^2}{\|w(t)\|_p^p} \right)^{2/(p-2)} \|w\|_{W_0^{r,2}(\Omega_1)}^2 - \frac{1}{p} \left(\frac{\|w\|_{W_0^{r,2}(\Omega_1)}^2}{\|w(t)\|_p^p} \right)^{p/(p-2)} \|w(t)\|_p^p \\ &= \frac{1}{2} (\|w(t)\|_p^p)^{-2/(p-2)} (\|w\|_{W_0^{r,2}(\Omega_1)}^2)^{p/(p-2)} - \frac{1}{p} (\|w(t)\|_p^p)^{p/(p-2)} \left(\|w\|_{W_0^{r,2}(\Omega_1)}^2 \right)^{p/(p-2)} \\ &= \left(\frac{1}{2} - \frac{1}{p} \right) (\|w(t)\|_p^p)^{-2/(p-2)} (\|w\|_{W_0^{r,2}(\Omega_1)}^2)^{p/(p-2)}. \end{aligned}$$

Then

$$\sup \{ \mathcal{J}(\mu w) \} > 0. \quad (2.12)$$

So, by the definition of d , we conclude that $d > 0$. $\square$

Lemma 2.2. $\mathcal{W}$ is a bounded neighborhood of 0 in $W_0^{r,2}(\Omega_1)$.

Proof. For $w \in W_0^{r,2}(\Omega_1)$, $w \neq 0$, we have by (2.1)–(2.3)

$$\mathcal{J}(w) = \frac{1}{2} \|w\|_{W_0^{r,2}(\Omega_1)}^2 - \frac{1}{p} \|w\|_p^p = \left(\frac{p-2}{2p} \right) (\|w\|_{W_0^{r,2}(\Omega_1)}^2) + \frac{1}{p} \mathcal{I}(w) \geq \left(\frac{p-2}{2p} \right) \|w\|_{W_0^{r,2}(\Omega_1)}^2, \quad (2.13)$$

then

$$\|w\|_{W_0^{r,2}(\Omega_1)}^2 \leq \left(\frac{2p}{p-2} \right) \mathcal{J}(w) < \left(\frac{2p}{p-2} \right) d = R. \quad (2.14)$$

Consequently, for all $w \in \mathcal{W}$ we have $w \in \mathcal{B}$, where

$$\mathcal{B} = \{w \in W_0^{r,2}(\Omega_1) : \|w\|_{W_0^{r,2}(\Omega_1)}^2 < R\}. \quad (2.15)$$

The proof of Lemma 2.2 is completed. $\square$

Denote by $\mu_1 < \mu_2 < \mu_3 < \dots$ the distinct eigenvalues and e_k the eigenfunction corresponding to μ_k of the elliptic eigenvalue problem

$$\begin{cases} (-\Delta)^r w = \mu w |w|^{p-2}, & x \in \Omega_1, \quad t > 0, \\ w = 0, & x \in \Omega_2, \quad t > 0. \end{cases} \quad (2.16)$$

More precisely, the following weak formulation of (2.16) is discussed: there is a function $w \in W_0^{r,2}(\Omega_1)$ such that

$$(w, \varphi)_{W_0^{r,2}(\Omega_1)} = \int_{\Omega_1} \mu w |w|^{p-2} \varphi dx. \quad (2.17)$$

Let $\{u_{ij}\}$ denote the eigenfunctions corresponding to $\{\mu_{ij}\}$ of problem (2.16). Then $\|u_i\| = 1$, $\{u_{ij}\}$ is an orthonormal basis in $L^2(\Omega_1)$ and an orthogonal basis of $W_0^{r,2}(\Omega_1)$. Set $V_n = \text{span}\{u_1, \dots, u_n\}$. Then $\{V_n\}_n$ is a dense subset of $W_0^{r,2}(\Omega_1)$. Furthermore, we have the following property:

- For $w_0 \in W_0^{r,2}(\Omega_1)$, there exists a sequence $\{w_{0,n}\}_n$ with $w_{0,n} \in V_n$, such that $w_{0,n} \rightarrow w_0$ in $W_0^{r,2}(\Omega_1)$ as $n \rightarrow \infty$.

Lemma 2.3. [12] Let Ω_1 be bounded domain, then we have

- (1) The embedding $W_0^{r,2}(\Omega_1) \hookrightarrow L^p(\Omega_1)$ is compact for any $p \in [1, 2_r^*)$.
- (2) The embedding $W_0^{r,2}(\Omega_1) \hookrightarrow L^{2_r^*}(\Omega_1)$ is continuous.

Lemma 2.4. [12]

- (1) For any $s \in [1, 2_r^*]$, there exists a positive constant $C_0 = C_0(n, s, r)$ such that for any $u \in W_0^{r,2}(\Omega_1)$

$$\|u\|_{L^s(\Omega_1)} \leq C_0 \int \int_{\Omega_1 \times \Omega_1} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2r}} dx dy. \quad (2.18)$$

- (2) $\forall s \in [1, 2_r^*]$ and for any bounded sequence $(w_j)_j \in W_0^{r,2}(\Omega_1)$, there exists w in $L^s(\mathbb{R}^n)$, with $w = 0$ a.e. in $\mathbb{R}^n - \Omega$, such that up to a subsequence, still denoted by $(w_j)_j$,

$$w_j \rightarrow w \text{ strongly in } L^s(\Omega_1) \text{ as } j \rightarrow \infty. \quad (2.19)$$

Definition 2.5. Let $w(t)$ be a weak solution of problem (1.1). We define the maximal existence time T of $w(t)$ as follows:

- If $w(t)$ exists for $0 \leq t < \infty$, then $T = +\infty$.
- If there exists a $t_0 \in (0, \infty)$ such that $w(t)$ exists for $0 \leq t < t_0$, but does not exist at $t = t_0$, then $T = t_0$.

For problem (1.1) and $\delta \in (0, 1)$, we define

$$\mathcal{J}_\delta(w) = \frac{\delta}{2} \|w\|_{W_0^{r,2}(\Omega_1)}^2 - \frac{1}{p} \|w\|_p^p, \quad (2.20)$$

$$d(\delta) = \frac{1-\delta}{2} \left(\frac{p}{2K^p} \delta \right)^{\frac{2}{p-2}}, \quad (2.21)$$

$$r(\delta) = \left(\frac{p}{2K^p} \delta \right)^{\frac{1}{p-2}}, \quad (2.22)$$

where K is the best imbedding constant of the embedding $W_0^{r,2}(\Omega_1)$ into $L^p(\Omega_1)$.

We have

$$\mathcal{J}(w) = \mathcal{J}_\delta(w) + \frac{1-\delta}{2} \|w\|_{W_0^{r,2}(\Omega_1)}^2, \quad \forall \delta \in [0, 1]. \quad (2.23)$$

Lemma 2.6. *If $w \in W_0^{r,2}(\Omega_1)$ and $\mathcal{J}(w) \leq d(\delta)$, then*

- (1) $\mathcal{J}_\delta(w) > 0$ if and only if $\|w\|_{W_0^{r,2}(\Omega_1)}^2 < r(\delta)$;
- (2) $\mathcal{J}_\delta(w) < 0$ if and only if $\|w\|_{W_0^{r,2}(\Omega_1)}^2 > r(\delta)$;
- (3) $\mathcal{J}_\delta(w) = 0$ if and only if $\|w\|_{W_0^{r,2}(\Omega_1)}^2 = r(\delta)$.

Proof. If $w \in W_0^{r,2}(\Omega_1)$ and $\mathcal{J}(w) \leq d(\delta)$,

- (1) Assume that $\mathcal{J}_\delta(w) > 0$, we have,

$$\mathcal{J}(w) = \mathcal{J}_\delta(w) + \frac{1-\delta}{2} \|w\|_{W_0^{r,2}(\Omega_1)}^2 < d(\delta),$$

so that

$$\frac{1-\delta}{2} \|w\|_{W_0^{r,2}(\Omega_1)}^2 < d(\delta) = \frac{1-\delta}{2} \left(\frac{p}{2K^p} \delta \right)^{\frac{2}{p-2}},$$

then,

$$\|w\|_{W_0^{r,2}(\Omega_1)}^2 < d(\delta) = \left(\frac{p}{2K^p} \delta \right)^{\frac{1}{p-2}} = r(\delta).$$

On the other hand, if $0 \leq \|w\|_{W_0^{r,2}(\Omega_1)}^2 < r(\delta)$, then

$$\|w\|_p^p \leq K^p \|w\|_{W_0^{r,2}(\Omega_1)}^p = K^p \|w\|_{W_0^{r,2}(\Omega_1)}^{p-2} \|w\|_{W_0^{r,2}(\Omega_1)}^2 < \frac{p}{2} \delta \|w\|_{W_0^{r,2}(\Omega_1)}^2.$$

Thus, $\mathcal{J}_\delta(w) > 0$.

- (2) $\mathcal{J}_\delta(w) > 0$, we have

$$\frac{p}{2} \delta \|w\|_{W_0^{r,2}(\Omega_1)}^2 < \|w\|_p^p \leq K^p \|w\|_{W_0^{r,2}(\Omega_1)}^{p-2} \|w\|_{W_0^{r,2}(\Omega_1)}^2.$$

So that,

$$\|w\|_{W_0^{r,2}(\Omega_1)}^2 > r(\delta).$$

On the other hand, if $\|w\|_{W_0^{r,2}(\Omega_1)}^2 > r(\delta)$, then

$$\frac{1-\delta}{2} \|w\|_{W_0^{r,2}(\Omega_1)}^2 > \frac{1-\delta}{2} r(\delta)^2 = d(\delta),$$

so,

$$\mathcal{J}(w) = \mathcal{J}_\delta(w) + \frac{1-\delta}{2} \|w\|_{W_0^{r,2}(\Omega_1)}^2 < d(\delta).$$

We get $\mathcal{J}_\delta(w) < 0$. □

Lemma 2.7. *The function $d : [0, 1] \rightarrow \mathbb{R}$ has the following properties.*

- $d(0) = d(1) = 0$.
- d takes the maximum value at $\delta_0 = \frac{2}{p}$ and $d(\delta_0) = e$.
- d is strictly increasing in $[0, \delta_0]$ and is strictly decreasing in $[\delta_0, 1]$.
- For any $e \in [0, d(\delta_0)]$, the equation $d(\delta) = e$ has exactly two roots $\delta_1 \in [0, \delta_0]$ and $\delta_2 \in [\delta_0, 1]$.

Proof. We have $d(\delta) = \frac{1-\delta}{2} \left(\frac{p}{2K^p} \delta \right)^{\frac{2}{p-2}}$. By differentiation, we get,

$$\begin{aligned} d'(\delta) &= -\frac{1}{2} \left(\frac{p}{2K^p} \delta \right)^{\frac{2}{p-2}} + \frac{1-\delta}{2} \left(\frac{2}{p-2} \times \frac{p}{2K^p} \right) \left(\frac{p}{2K^p} \delta \right)^{\frac{2}{p-2}-1}, \\ d'(\delta) &= \frac{1}{2} \left(\frac{p}{2K^p} \delta \right)^{\frac{2}{p-2}} \left(-1 + \frac{1-\delta}{\delta} \frac{2}{p-2} \right). \end{aligned}$$

If $\delta_0 = \frac{2}{p}$, then, $d'(\delta_0) = 0$, and we have $d(\delta) > 0$ for all $\delta \in [0, \delta_0]$ and $d(\delta) < 0$ for any $\delta \in [\delta_0, 1]$. $\square$

Lemma 2.8.

(1) $\forall \delta \in (0, 1)$; we have

$$d(\delta) = \inf \{ \mathcal{J}(w); w \in W_0^{r,2}(\Omega_1) \neq 0, \|w\|_{W_0^{r,2}(\Omega_1)}^2 = 0, \mathcal{J}_\delta(w) = 0 \}.$$

(2) $d = d(\delta_0) = \inf \{ \mathcal{J}(w); w \in W_0^{r,2}(\Omega_1) \neq 0, \|w\|_{W_0^{r,2}(\Omega_1)}^2 = 0, \mathcal{J}_{\delta_0}(w) = 0 \}.$

Proof.

(1) Let $\delta \in (0, 1)$ and $\mathcal{J}_\delta(w) = 0, \|w\|_{W_0^{r,2}(\Omega_1)}^2 = 0$, then

$$\frac{p}{2} \delta \|w\|_{W_0^{r,2}(\Omega_1)}^2 = \|w\|_p^p \leq K^p \|w\|_{W_0^{r,2}(\Omega_1)}^{p-2} \|w\|_{W_0^{r,2}(\Omega_1)}^2.$$

Hence,

$$\|w\|_{W_0^{r,2}(\Omega_1)} > \left(\frac{p}{2K^p} \delta \right)^{\frac{1}{p-2}},$$

so,

$$\mathcal{J}(w) = \mathcal{J}_\delta(w) + \frac{1-\delta}{2} \|w\|_{W_0^{r,2}(\Omega_1)}^2 = \frac{1-\delta}{2} \|w\|_{W_0^{r,2}(\Omega_1)}^2 \geq d(\delta).$$

The proof is completed.

(2) We put $\delta_0 = \frac{2}{p}$, we have $\mathcal{J}_{\delta_0}(w) = \frac{1}{p} I(w)$, which completes the proof. $\square$

Lemma 2.9. Let $\delta \in (0, 1)$ and $w \in W_0^{r,0}(\Omega_1)$, we define $a(\delta) = \frac{1-\delta}{2}$, we have

(1) If $\mathcal{J}(w) \leq d(\delta)$ and $\mathcal{J}_\delta(w) > 0$, then

$$0 \leq \|w\|_{W_0^{r,2}(\Omega_1)}^2 < \frac{d(\delta)}{a(\delta)}. \quad (2.24)$$

In particular, if $\mathcal{J}_\delta(w) \leq d$ and $I(w) > 0$, then

$$0 \leq \|w\|_{W_0^{r,2}(\Omega_1)}^2 < \frac{2p}{p-1} d. \quad (2.25)$$

(2) If $\mathcal{J}(w) \leq d(\delta)$ and $\|w\|_{W_0^{r,2}(\Omega_1)}^2 > \frac{d(\delta)}{a(\delta)}$, then $\mathcal{J}_\delta(w) < 0$. Furthermore, if $\mathcal{J}(w) \leq d$ and $\|w\|_{W_0^{r,2}(\Omega_1)}^2 > \frac{2p}{p-1} d$, then $I(w) < 0$.

Proof. Let $\delta \in (0, 1)$ and $w \in W_0^{r,2}(\Omega_1)$, we have

$$a(\delta) \|w\|_{W_0^{r,2}(\Omega_1)}^2 < a(\delta) \|w\|_{W_0^{r,2}(\Omega_1)}^2 + \mathcal{J}_\delta(w) = \mathcal{J}(w) < d(\delta),$$

so that,

$$0 \leq \|w\|_{W_0^{r,2}(\Omega_1)}^2 < \frac{d(\delta)}{a(\delta)}.$$

In particular, if $\mathcal{J}_\delta(w) \leq d$ and $\mathcal{I}(w) > 0$, then

$$\frac{p-2}{2p} \|w\|_{W_0^{r,2}(\Omega_1)}^2 < \left(\frac{1}{2} - \frac{1}{p} \right) \|w\|_{W_0^{r,2}(\Omega_1)}^2 < \mathcal{J}(w) < d.$$

The proof is now completed. $\square$

Let us define the following family of potential wells for all $\delta \in (0, 1)$

$$\begin{aligned} F &= \{w \in W_0^{r,2}(\Omega_1) : \mathcal{J}(w) < 0, \mathcal{J}(w) < d\}; \\ \mathcal{W}_\delta &= \{w \in W_0^{r,2}(\Omega_1) : \mathcal{J}_\delta(w) > 0, \mathcal{J}(w) < d(\delta)\} \cup \{0\}; \\ \overline{\mathcal{W}}_\delta &= \mathcal{W}_\delta \cup \partial \mathcal{W}_\delta = \{w \in W_0^{r,2}(\Omega_1) : \mathcal{J}_\delta(w) \geq 0, \mathcal{J}(w) \leq d(\delta)\}; \\ F_\delta &= \{w \in W_0^{r,2}(\Omega_1) : \mathcal{J}_\delta(w) < 0, \mathcal{J}(w) < d(\delta)\}; \\ \overline{F}_\delta &= F_\delta \cup \partial F_\delta = \{w \in W_0^{r,2}(\Omega_1) : \mathcal{J}_\delta(w) \leq 0, \mathcal{J}(w) \leq d(\delta)\}; \\ G_\delta &= \{w \in W_0^{r,2}(\Omega_1) : \|w\|_{W_0^{r,2}(\Omega_1)}^2 < r(\delta)\}; \\ \overline{G}_\delta &= G_\delta \cup \partial G_\delta = \{w \in W_0^{r,2}(\Omega_1) : \|w\|_{W_0^{r,2}(\Omega_1)}^2 \leq r(\delta)\}; \\ G_\delta^c &= \{w \in W_0^{r,2}(\Omega_1) : \|w\|_{W_0^{r,2}(\Omega_1)}^2 > r(\delta)\}. \end{aligned}$$

The following result is a consequence of Lemma 2.6.

Lemma 2.10. *If $\mathcal{J}_\delta(w) > 0$, $\delta \in (0, 1)$, then*

- $F_{\delta_0} = F$ and $\mathcal{W}_{\delta_0} = \mathcal{W}$.
- $G_{\bar{\delta}} \subset \mathcal{W}_\delta \subset G_\delta$ and $F_{\bar{\delta}} \subset G_\delta^c$, where $\bar{\delta}$ is such that $r(\bar{\delta}) = (1 - \delta)^{\frac{1}{2}} r(\delta)$.

The following Lemmas are a consequence of Lemmas 2.7 and 2.10.

Lemma 2.11.

- If $\delta_0 > \delta'' > \delta' > 0$, then $\mathcal{W}_{\delta'} \subset \mathcal{W}_{\delta''}$.
- If $1 > \delta'' > \delta' > \delta_0$, then $F_{\delta''} \subset F_{\delta'}$.

Lemma 2.12. *Let $0 < \mathcal{J}(w) < d$ for some $w \in W_0^{r,2}(\Omega_1)$, and that $\delta_2 > \delta_1$ are the two solutions of the equation $\mathcal{J}(w) = d(\delta)$. Then $\mathcal{J}_\delta(w)$ does not change sign for $\delta \in (\delta_1, \delta_2)$.*

3 Existence of global solutions

As in [13], we are now ready to apply the Galerkin method by constructing finite-dimensional Galerkin approximations for (1.1) and then present a priori estimates, which allow us to pass to the limit to obtain the desired weak solution w of (1.1). Indeed, w verifies the conditions of initial data and belongs to the family of potential wells.

Definition 3.1. A function $w = w(x, t)$ is said to be a global (weak) solution of problem (1.1), if

$$\begin{aligned} w &\in L^\infty(0, \infty, W_0^{r,2}(\Omega_1)), & w_t &\in L^\infty(0, \infty, L^2(\Omega_1)); \\ w_0 &\in L^\infty(0, \infty, W_0^{r,2}(\Omega_1)), & w_1 &\in L^\infty(0, \infty, L^2(\Omega_1)), \end{aligned}$$

and for any $\phi \in L^\infty(0, \infty, W_0^{r,2}(\Omega_1))$, $t \in \mathbb{R}_+^*$,

$$\begin{aligned} & (\partial_t w(\cdot, t), \phi(\cdot, t)) + \frac{1}{2} \int_0^t (w(\cdot, \tau), \phi(\cdot, \tau))_{W_0^{r,2}(\Omega_1)} d\tau + \int_0^t (w_t(\cdot, \tau), \phi(\cdot, \tau))_{W_0^{r,2}(\Omega_1)} d\tau \\ &= (w_1, \phi(\cdot, 0)) + \int_0^t (w(\cdot, \tau) |w(\cdot, \tau)|^{p-2}, \phi(\cdot, \tau)) d\tau. \end{aligned}$$

Remark 3.2. If $w \in C(0, \infty; W_0^{r,2}(\Omega_1))$, we say that w is a strong global solution of problem (1.1).

We introduce the energy $\mathcal{E}$ of solution at time t as

$$\mathcal{E}(t) = \frac{1}{2} \|\partial_t w(t)\|_2^2 + \mathcal{J}(w). \quad (3.1)$$

Theorem 3.3. Let $w_0 \in W_0^{r,2}(\Omega_1)$ and $w_1 \in L^2(\Omega_1)$, suppose that $d > \mathcal{E}(0) > 0$, δ_1 and δ_2 , with $0 < \delta_1 < \delta_2$, are the two solutions of the equation $d(\delta) = \mathcal{E}(0)$, that either $\mathcal{J}(w) > 0$ or $\|w_0\|_{W_0^{r,2}(\Omega_1)} = 0$. Then problem (1.1) admits a global solution w , with $w \in L^\infty(0, \infty, W_0^{r,2}(\Omega_1))$, $w_t \in L^\infty(0, \infty, L^2(\Omega_1))$ and $w \in \mathcal{W}_\delta$ for all $\delta \in (\delta_1, \delta_2)$ and for all $t \in \mathbb{R}_+^*$.

Proof. By (2.16) there exists a sequence $(u_j)_j \subset C_0^\infty(\Omega_1)$ of eigenfunctions of $(-\Delta)^r$, which is an orthonormal basis of $L^2(\Omega_1)$ and an orthogonal basis of $W_0^{r,2}(\Omega_1)$.

We use the Galerkin method to prove the existence of weak solutions of (1.1), by finding the approximate solutions

$$w_n(x, t) = \sum_{j=1}^n g_j^n(t) u_j(x); \quad n = 1, 2, 3, \dots$$

be the Galerkin approximate solutions of problem (1.1) satisfying

$$\begin{cases} (\partial_t^2 w(\cdot, t), u_j) + (w(\cdot, t), u_j)_{W_0^{r,2}(\Omega_1)} + (\partial_t w(\cdot, t), u_j)_{W_0^{r,2}(\Omega_1)} = (w(\cdot, t) |w(\cdot, t)|^{p-2}, u_j), & j = 1, \dots, n, \\ w_n(\cdot, 0) = \sum_{j=1}^n A_j u_j \rightarrow w_0, & n \rightarrow \infty \text{ in } W_0^{r,2}(\Omega_1), \\ w_{nt}(\cdot, 0) = \sum_{j=1}^n B_j u_j \rightarrow w_1, & n \rightarrow \infty \text{ in } L^2(\Omega_1). \end{cases}$$

Substituting w_n into (1.1), we get

$$\begin{cases} g_j^{n''} + \mu_j g_j^n + \mu_j g_j^{n'} = \sum_{l=1}^m g_l^l |g_j^l|^{p-2} \int_{\Omega_1} u_l |u_l|^{p-2} u_j dx, \\ g_j^n(0) = a_j, & j = 1, \dots, m, \\ g_j^{n'}(0) = b_j, & j = 1, \dots, m. \end{cases}$$

According to the standard ordinary differential equation theory, problem admits a solution g_{jm} of class $C^1([0, T])$ for each n .

Multiplying problem (1.1) by $g_j^{n'}$, summing for j , we have

$$\begin{aligned} & \int_{\Omega_1} \partial_t^2 w_n(x, \tau) \partial_t w_n(x, \tau) dx + \int_{\Omega_1} \int_{\Omega_1} \frac{w_n(x, t) - w_n(z, t) (\partial_t w_n(x, t) - \partial_t w_n(z, t))}{|x - z|^{n+2r}} dx dz \\ &+ \int_{\Omega_1} \int_{\Omega_1} \frac{(\partial_t w_n(x, t) - \partial_t w_n(z, t))^2}{|x - z|^{n+2r}} dx dz = \int_{\Omega_1} w_n(x, \tau) |w_n(x, \tau)|^{p-2} \partial_t w_n(x, \tau) dx. \end{aligned}$$

Integrating with respect to τ , we get for all $t \in \mathbb{R}_+^*$

$$\begin{aligned} & \frac{1}{2} \int_0^t \frac{d}{d\tau} \left(\int_{\Omega_1} |\partial_t w_n(x, \tau)|^2 dx \right) d\tau + \frac{1}{2} \int_0^t \frac{d}{d\tau} \left(\int_{\Omega_1} \int_{\Omega_1} \frac{(w_n(x, \tau) - w_n(z, \tau))^2}{|x - z|^{n+2\tau}} dx dz \right) d\tau + \int_0^t \|\partial_t w_n(z, \tau)\|_{W_0^{r,2}(\Omega_1)}^2 d\tau \\ &= \frac{1}{p} \int_0^t \frac{d}{d\tau} \left(\int_{\Omega_1} |w_n(x, \tau)|^p dx \right) d\tau. \end{aligned}$$

For all $t \in \mathbb{R}_+^*$, we obtain

$$\begin{aligned} & \frac{1}{2} \|\partial_t w_n(\cdot, t)\|_2^2 - \frac{1}{2} \|\partial_t w_n(\cdot, 0)\|_2^2 + \frac{1}{2} \|w_n(\cdot, t)\|_{W_0^{r,2}(\Omega_1)}^2 - \frac{1}{2} \|w_n(\cdot, 0)\|_{W_0^{r,2}(\Omega_1)}^2 + \int_0^t \|\partial_t w_n(z, \tau)\|_{W_0^{r,2}(\Omega_1)}^2 d\tau \\ &= \frac{1}{p} \|w_n(\cdot, t)\|_p^p - \frac{1}{p} \|w_n(\cdot, 0)\|_p^p. \end{aligned}$$

So,

$$\begin{aligned} & \frac{1}{2} \|\partial_t w_n(\cdot, t)\|_2^2 + \frac{1}{2} \|w_n(\cdot, t)\|_{W_0^{r,2}(\Omega_1)}^2 + \int_0^t \|\partial_t w_n(z, \tau)\|_{W_0^{r,2}(\Omega_1)}^2 d\tau - \frac{1}{p} \|w_n(\cdot, t)\|_p^p \\ &= \frac{1}{2} \|\partial_t w_n(\cdot, 0)\|_2^2 + \frac{C}{2} \|w_n(\cdot, 0)\|_{W_0^{r,2}(\Omega_1)}^2 - \frac{1}{p} \|w_n(\cdot, 0)\|_p^p = \mathcal{E}_n(0). \end{aligned}$$

Hence, we obtain

$$\mathcal{E}_n(0) = \frac{1}{2} \|\partial_t w_n(\cdot, \tau)\|_2^2 + \mathcal{J}(w_n(\cdot, t)) + \int_0^t \|\partial_t w_n(z, \tau)\|_{W_0^{r,2}(\Omega_1)}^2 d\tau.$$

If $\|w_0\|_{W_0^{r,2}(\Omega_1)}^2 = 0$, then $w_0 \in \mathcal{W}_\delta$; therefore, $w_n(\cdot, 0) \in G_\delta$ for sufficient large n , so that $w_n(\cdot, 0) \in \mathcal{W}_\delta$ for all $\delta \in (\delta_1, \delta_2)$ by $(G_\delta \subset \mathcal{W}_\delta)$.

Assume that $\mathcal{J}_{\delta_2}(w_0) > 0$, by Lemma 2.12.

If $\|w_0\|_{W_0^{r,2}(\Omega_1)} \neq 0$ and $\mathcal{J}_\delta(w_0) > 0$ for all $\delta \in (\delta_1, \delta_2)$

$w_0 \in \mathcal{W}_\delta$ for all $\delta \in (\delta_1, \delta_2)$, since $\mathcal{J}(w_0) \leq \mathcal{E}(0) = d(\delta_1) = d(\delta_2)$ for fixed $\delta \in (\delta_1, \delta_2)$, the inequality $\mathcal{J}_{\delta_2}(w_0) > 0$ implies that $\mathcal{J}_{\delta_2}(w_n(\cdot, 0)) > 0$ and $\mathcal{E}(0) < d(\delta)$ provided that n is sufficiently large.

Thus, for all $\delta \in (\delta_1, \delta_2)$, $w_n(\cdot, 0) \in \mathcal{W}_\delta$ and n is sufficiently large.

Next we claim $\forall t \in \mathbb{R}_+^*$ that $w_n(\cdot, t) \in \mathcal{W}_\delta$ for sufficiently large n . Suppose that $w_n(\cdot, t)$ is not contained in $\mathcal{W}_\delta$, and let T be the smallest time t for which $w_n(\cdot, t)$ is not contained in $\mathcal{W}_\delta$. Then, $w_n(\cdot, T) \in \partial \mathcal{W}_\delta$ by the continuity of $w_n(\cdot, t)$. Hence, either $\mathcal{A}(w_n(\cdot, T)) = d(\delta)$ or $\mathcal{J}_\delta(w_n(\cdot, T)) = 0$.

Therefore, for n large enough, we have

$$\mathcal{J}_\delta(w_n(\cdot, t)) \leq \mathcal{E}_n(0) < d(\delta), \quad (3.2)$$

which contradicts the fact that $\mathcal{J}_\delta(w_n(\cdot, t)) = d(\delta)$.

If $\mathcal{J}_\delta(w_n(\cdot, T)) = 0$ and $\|w_n(\cdot, T)\|_2 \neq 0$, then $\mathcal{J}_\delta(w_n(\cdot, T)) \geq 0$, which is impossible, since it contradicts with $\mathcal{J}(w_n(\cdot, T)) < \mathcal{E}_n(0) < d(\delta)$.

In conclusion, $w_n(\cdot, t) \in \mathcal{W}_\delta$, $\forall t \in \mathbb{R}_+^*$ and sufficiently large n , so that $\mathcal{J}_\delta(w_n(\cdot, t)) > 0$. Thus, $\forall t \in \mathbb{R}_+^*$ and sufficiently large n

$$\mathcal{J}(w_n(\cdot, t)) = \frac{1-\delta}{2} \|w_n(\cdot, t)\|_{W_0^{r,2}(\Omega_1)}^2 + \mathcal{J}_\delta(w_n(\cdot, t)) > \frac{1-\delta}{2} \|w_n(\cdot, t)\|_{W_0^{r,2}(\Omega_1)}^2,$$

yields,

$$\frac{1}{2} \|\partial_t w_n(\cdot, t)\|_2^2 + \mathcal{J}(w_n(\cdot, t)) + \int_0^t \|\partial_t w_n(z, \tau)\|_{W_0^{r,2}(\Omega_1)}^2 d\tau = \mathcal{E}_n(0) \leq d(\delta), \quad (3.3)$$

by (3.3), we have

$$\begin{cases} \|w_n(z, t)\|_{W_0^{r,2}(\Omega_1)}^2 < 2 \frac{d(\delta)}{1-\delta} = r(\delta) \leq C_1; \\ \|\partial_t w_n(z, t)\|_2^2 \leq 2d(\delta); \\ \|w_n(z, t)\|_2^2 \leq \frac{2}{1-\delta} d(\delta); \\ \int_0^t \|\partial_t w_n(z, \tau)\|_{W_0^{r,2}(\Omega_1)}^2 d\tau \leq d(\delta), \end{cases}$$

which gives

$$\|w_n(z, t)\|_p^p \leq K^p \|w_n(z, t)\|_{W_0^{r,2}(\Omega_1)}^p < K^p r(\delta)^{\frac{p}{2}}. \quad (3.4)$$

Hence, there exist ξ , w and a subsequence of $(w_n)_n$, such that

$$\begin{aligned} w_n &\rightharpoonup^* w \text{ in } L^\infty(0, \infty, W_0^{r,2}(\Omega_1)) \quad \text{and} \quad w_n \rightarrow w \text{ in } \Omega_1 \times \mathbb{R}_+^*, \\ \partial_t w_n &\rightharpoonup^* \partial_t w \text{ in } L^\infty(0, \infty, W_0^{r,2}(\Omega_1)) \quad \text{and} \quad \partial_t w_n \rightarrow \partial_t w \text{ in } \Omega_1 \times \mathbb{R}_+^*, \\ w_n |w_n|^{p-2} &\rightarrow \xi \text{ in } L^\infty(0, \infty, L^{p'}(\Omega_1)) \quad \text{and} \quad w_n |w_n|^{p-2} \rightarrow \xi \text{ in } \Omega_1 \times \mathbb{R}_+^*, \\ \partial_t w_n &\rightharpoonup^* \partial_t w \text{ in } L^\infty(0, \infty, L^2(\Omega_1)) \quad \text{and} \quad \partial_t w_n \rightarrow \partial_t w \text{ in } L^\infty(0, \infty, L^2(\Omega_1)), \end{aligned}$$

as $n \rightarrow \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Integrating with respect to τ from 0 to t , we have

$$(\partial_t w(\cdot, t), u_j) + \frac{1}{2} \int_0^t (w(\cdot, \tau), u_j)_{W_0^{r,2}(\Omega_1)} d\tau + \int_0^t (\partial_t w(\cdot, \tau), u_j)_{W_0^{r,2}(\Omega_1)} d\tau = (w_1, u_j) + \int_0^t (\xi, u_j) d\tau.$$

Therefore, since $C_0^\infty(\Omega_1)$ is dense in $W_0^{r,2}(\Omega_1)$, the fact that $(u_j)_j \in C_0^\infty(\Omega_1)$ is an orthonormal basis of $L^2(\Omega_1)$, we obtain for all $v \in W_0^{r,2}(\Omega_1)$

$$(\partial_t w(\cdot, t), v(x)) + \frac{1}{2} \int_0^t (w(\cdot, \tau), v(x))_{W_0^{r,2}(\Omega_1)} d\tau + \int_0^t (\partial_t w(\cdot, \tau), v(x))_{W_0^{r,2}(\Omega_1)} d\tau = (w_1, v(x)) + \int_0^t (\xi, v(x)) d\tau,$$

for any $\phi \in L^1(0, \infty; W_0^{r,2}(\Omega_1))$, putting $v(x) = \phi(x, t)$, with t fixed, and integration with respect to t , we conclude that w is a global solution of the problem. Finally, $w_n(\cdot, t) \in \mathcal{W}_\delta$ for any $\delta \in (\delta_1, \delta_2)$, for any n and for $t \in \mathbb{R}_+^*$, so that $w(\cdot, t) \in \mathcal{W}_\delta$ for any $\delta \in (\delta_1, \delta_2)$ and $t \in \mathbb{R}_+^*$. $\square$

Theorem 3.4. Let $w_0 \in W_0^{r,2}(\Omega_1)$ and $w_1 \in L^2(\Omega_1)$, suppose that δ_1 and δ_2 , with $0 < \delta_1 < \delta_2$, are the two solutions of the equation $d(\delta) = 0$, that either $\mathcal{J}(w) > 0$ or $\|w_0\|_{W_0^{r,2}(\Omega_1)} = 0$. Then problem (1.1) admits a unique global solution $w(x, t) = 0$.

Proof. $\mathcal{J}(w_0) = 0$ since $\|w_0\|_{W_0^{r,2}(\Omega_1)} = 0$, hence $0 = \mathcal{E}(0) = \frac{1}{2} \|w_1\|_2^2 + \mathcal{J}(w_0)$, gives $w_1 \equiv 0$. Thus, $w(x, t) = 0$ is a global solution $w(x, t) = 0$ of problem (1.1). $\square$

Theorem 3.5. If $\mathcal{J}_{\delta_2}(w_0) > 0$ replaced by $\mathcal{I}_{\delta_2}(w_0) > 0$ in Theorem 3.3, then problem (1.1) admits a global solution w , with $w \in L^\infty(0, \infty, W_0^{r,2}(\Omega_1))$, $w_t \in L^\infty(0, \infty, L^2(\Omega_1))$ and $w \in \mathcal{W}_\delta$ for all $\delta \in (\delta_1, \delta_2)$ and for all $t \in \mathbb{R}_+^*$.

Proof. If $\mathcal{I}_{\delta_2}(w_0) > 0$ ($\delta_0 = \frac{2}{p}$), then

$$\mathcal{J}_{\delta_0}(w_0) = \frac{\delta_0}{2} \|w_0\|_{W_0^{r,2}(\Omega_1)}^2 - \frac{1}{p} \|w_0\|_2^2 = \frac{1}{p} (\|w_0\|_{W_0^{r,2}(\Omega_1)}^2 - \|w_0\|_2^2) = \frac{1}{p} \mathcal{I}(w_0) > 0,$$

since $\delta_2 \in [\delta_0, 1]$ by Lemma 2.7 a.e. $\delta_2 \geq \delta_0$, we get that $\mathcal{J}_{\delta_2}(w_0) > \mathcal{J}_{\delta_0}(w_0) > 0$. $\square$

Theorem 3.6. Let $w_0 \in W_0^{r,2}(\Omega_1)$ and $w_1 \in L^2(\Omega_1)$, suppose that $0 < \mathcal{E}(0) < d$, δ_1 and δ_2 , with $0 < \delta_1 < \delta_2$, are the two solutions of equation $d(\delta) = \mathcal{E}(0)$, that either $\mathcal{J}(w) > 0$ or $\|w_0\|_{W_0^{r,2}(\Omega_1)} = 0$. Then w the global solution of problem (1.1) belongs to $\overline{\mathcal{W}}_{\delta_1}$ for all $t \in \mathbb{R}_+^*$.

Proof. The global solution w obtained in Theorem 3.3 satisfies

$$\begin{cases} \mathcal{E}(0) < d(\delta), & \forall \delta \in (\delta_1, \delta_2), & t \in \mathbb{R}_+^*, \\ \mathcal{J}(w(\cdot, t)) < \mathcal{E}(0) = d(\delta_1) = d(\delta_2) & \text{fix } t \in \mathbb{R}_+^*. \end{cases}$$

Since $\mathcal{J}_{\delta_2}(w) > 0$ yields $\mathcal{J}_{\delta}(w) > 0$ for all $\delta \in (\delta_1, \delta_2)$ letting $\delta \rightarrow \delta_1$, we get $\mathcal{J}_{\delta_1}(w) > 0$. Thus, $w(\cdot, t) \in \overline{\mathcal{W}}_{\delta_1}$ for all $t \in \mathbb{R}_+^*$. $\square$

Theorem 3.7. Suppose that (1.2) holds. Let $w_0 \in W_0^{r,2}(\Omega_1)$, $w_1 \in L^2(\Omega_1)$, then problem (1.1) admits a global weak solution $w \in \overline{\mathcal{W}}$ satisfying

$$w \in L^\infty(0, \infty; W_0^{r,2}(\Omega_1)), \quad \partial_t w \in L^2(0, \infty; L^2(\Omega_1)). \quad (3.5)$$

We will prove that if the initial data (or for some $t_0 > 0$) are in the set $\mathcal{W}$, then the solution stays there forever.

Lemma 3.8. Suppose that (1.2) holds. If $w_0 \in \mathcal{W}$, $w_1 \in L^2(\Omega_1)$, then the solution $w \in \mathcal{W}$, $\forall t \geq 0$.

Proof. Since $w_0 \in \mathcal{W}$, we have

$$\mathcal{I}(w_0) = \|w_0\|_{W_0^{r,2}(\Omega_1)}^2 - \|w_0\|_p^p > 0.$$

This gives

$$\mathcal{J}(w) = \frac{1}{2} \|w\|_{W_0^{r,2}(\Omega_1)}^2 - \frac{1}{p} \|w\|_p^p = \left(\frac{p-2}{2p} \right) (\|w\|_{W_0^{r,2}(\Omega_1)}^2) + \frac{1}{p} \mathcal{I}(w) \geq \left(\frac{p-2}{2p} \right) \|w\|_{W_0^{r,2}(\Omega_1)}^2, \quad (3.6)$$

then by (3.1), (2.1) and (2.2), we have

$$\|w\|_{W_0^{r,2}(\Omega_1)}^2 \leq \left(\frac{2p}{p-2} \right) \mathcal{J}(w) \leq \left(\frac{2p}{p-2} \right) \mathcal{E}(t) \leq \left(\frac{2p}{p-2} \right) \mathcal{E}(0). \quad \square$$

4 Asymptotic behavior

Lemma 4.1. [14] Let φ be a bounded positive (nonnegative nonincreasing) function on $\mathbb{R}_+$ satisfying, for some constant k ,

- If $\alpha > 0$,

$$k\varphi(t)^{\alpha+1} \leq (\varphi(t) - \varphi(t+1)), \quad t \geq 0.$$

Then we have

$$\varphi(t) \leq (\alpha k(t-1) + M^{-\alpha})^{-\frac{1}{\alpha}}, \quad \forall t \geq 1, \quad (4.1)$$

where

$$M = \max_{t \in [0,1]} \varphi(t).$$

• If $\alpha = 0$, then we have

$$\varphi(t) \leq M \exp(-ht), \quad \forall t \geq 1, \quad (4.2)$$

where $h = -\log(1-k) > 0$.

Theorem 4.2. Let $w \in \mathcal{W}$, $w \in L^2(\Omega_1)$. If $\mathcal{E}(0) < d$, any global (weak) solution w of problem (1.1) has the following decay estimate:

$$\mathcal{E}(t) \leq \mathcal{E}(0) \exp(-ht), \quad \forall t \geq 1,$$

where $h = -\log(1-k) > 0$.

Proof. Let w be a global weak solution of (1.1), since $w_0 \in \mathcal{W}$ by Lemma 2.6, then $w \in \mathcal{W}$ and we have,

$$\mathcal{E}(0) \geq \mathcal{E}(t) \geq \mathcal{J}(w) \geq \left(\frac{1}{2} - \frac{1}{p} \right) \|w\|_{W_0^{r,2}(\Omega_1)}^p. \quad (4.3)$$

Next, multiplying equation (1.1) by $w_t(x, t)$ and integrating in $[t, t+1]$, we get

$$-\int_t^{t+1} \frac{d\mathcal{E}(\tau)}{d\tau} d\tau = \int_t^{t+1} \|\partial_t w(\cdot, \tau)\|_{W_0^{r,2}(\Omega_1)}^2 d\tau = \mathcal{E}(t) - \mathcal{E}(t+1) \geq 0. \quad (4.4)$$

Thus, $\mathcal{E}$ is nonincreasing in $\mathbb{R}_+^*$.

It follows from (4.4) and the embedding $W_0^{r,2}(\Omega_1)$ into $L^p(\Omega_1)$ that

$$\int_t^{t+1} \|w_t(\cdot, t)\|_2^2 dt \leq k \int_t^{t+1} \|w_t(\cdot, t)\|_{W_0^{r,2}(\Omega_1)}^2 dt \leq k[\mathcal{E}(t) - \mathcal{E}(t+1)]. \quad (4.5)$$

Applying the mean value theorem to the left-hand side of (4.5), there exist numbers $t_1 \in [t, t + \frac{1}{4}]$ and $t_2 \in [t + \frac{3}{4}, t+1]$ such that

$$\|w_t(\cdot, t_i)\|_2 \leq k^{\frac{1}{2}} [\mathcal{E}(t) - \mathcal{E}(t+1)]^{\frac{1}{2}} \quad (i = 1, 2). \quad (4.6)$$

Next, multiplying equation (1.1) by $w(x, t)$ and integrating in $\Omega_1 \times [t_1, t_2]$, we can see that

$$\int_{t_1}^{t_2} \int_{\Omega_1} w_{tt}(\cdot, t) w(\cdot, t) dx dt + \int_{t_1}^{t_2} \|w(\cdot, t)\|_{W_0^{r,2}(\Omega_1)}^2 dt + \int_{t_1}^{t_2} (w_t(\cdot, t), w(\cdot, t))_{W_0^{r,2}(\Omega_1)} dt = \int_{t_1}^{t_2} \|w(\cdot, t)\|_p^p dt, \quad (4.7)$$

by (4.7), we have

$$\int_{t_1}^{t_2} \|w(\cdot, t)\|_{W_0^{r,2}(\Omega_1)}^2 dt - \int_{t_1}^{t_2} \|w(\cdot, t)\|_p^p dt = - \int_{t_1}^{t_2} \int_{\Omega_1} w_{tt}(\cdot, t) w(\cdot, t) dx dt - \int_{t_1}^{t_2} (w_t(\cdot, t), w(\cdot, t))_{W_0^{r,2}(\Omega_1)} dt, \quad (4.8)$$

so that,

$$\begin{aligned} & \int_{t_1}^{t_2} \|w(\cdot, t)\|_{W_0^{r,2}(\Omega_1)}^2 dt - \int_{t_1}^{t_2} \|w(\cdot, t)\|_p^p dt \\ &= - \int_{t_1}^{t_2} \int_{\Omega_1} w_{tt}(\cdot, t) w(\cdot, t) dx dt - \int_{t_1}^{t_2} (w_t(\cdot, t), w(\cdot, t))_{W_0^{r,2}(\Omega_1)} dt \end{aligned}$$

$$\begin{aligned}
&\leq \left[\int_{\Omega_1} -w_{tt}(., t)w(., t)dx \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \|w_t(., t)\|_2^2 dt + \int_{t_1}^{t_2} \| -w_t(., t)\|_{W_0^{r,2}(\Omega_1)} \|w(., t)\|_{W_0^{r,2}(\Omega_1)} dt \\
&\leq \sum_{i=1}^2 \|w_t(., t_i)\|_2 \|w(., t_i)\|_2 + \int_{t_1}^{t_2} \|w_t(., t)\|_2^2 dt + \int_{t_1}^{t_2} \|w_t(., t)\|_{W_0^{r,2}(\Omega_1)} \|w(., t)\|_{W_0^{r,2}(\Omega_1)} dt.
\end{aligned}$$

On the other hand,

$$\|w_t(., t_i)\|_2 \|w(., t_i)\|_2 \leq k^{\frac{1}{2}} [\mathcal{E}(t) - \mathcal{E}(t+1)]^{\frac{1}{2}} \sup_{t, t+1} \mathcal{E}(s)^{\frac{1}{2}} \leq k(\varepsilon) [\mathcal{E}(t) - \mathcal{E}(t+1)] + \varepsilon \sup_{[t, t+1]} \mathcal{E}(s), \quad (4.9)$$

and

$$\begin{aligned}
\int_{t_1}^{t_2} \|w_t(., t)\|_{W_0^{r,2}(\Omega_1)} \|w(., t)\|_{W_0^{r,2}(\Omega_1)} dt &\leq \left(\int_{t_1}^{t_2} \|w_t(., t)\|_{W_0^{r,2}(\Omega_1)}^2 dt \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \|w(., t)\|_{W_0^{r,2}(\Omega_1)}^2 dt \right)^{\frac{1}{2}} \\
&\leq k^{\frac{1}{2}} [\mathcal{E}(t) - \mathcal{E}(t+1)]^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \mathcal{E}(t) dt \right)^{\frac{1}{2}}.
\end{aligned}$$

So,

$$\int_{t_1}^{t_2} \|w_t(., t)\|_{W_0^{r,2}(\Omega_1)} \|w(., t)\|_{W_0^{r,2}(\Omega_1)} dt \leq k(\varepsilon) [\mathcal{E}(t) - \mathcal{E}(t+1)] + \varepsilon \int_{t_1}^{t_2} \mathcal{E}(t) dt. \quad (4.10)$$

In view of (4.5), (4.9), (4.10), letting $\varepsilon \rightarrow 0^+$, we get

$$\mathcal{E}(t) \leq \sup_{[t, t+1]} \mathcal{E}(s) \leq k[\mathcal{E}(t) - \mathcal{E}(t+1)]. \quad (4.11)$$

The application of Nakao's inequality to (4.2) yields global (weak) solution w of problem (1.1), which has the following decay estimate:

$$\mathcal{E}(t) \leq \mathcal{E}(0) \exp(\log(1-k)t), \quad \forall t \geq 1.$$

This completes the proof. $\square$

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