Research Article

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Dynamical study of Lyapunov exponents for Hide's coupled dynamo model

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Abstract: In this paper, we introduced the Lyapunov exponents (LEs) as a significant tool that is used to study the numerical solution behavior of the dynamical systems. Moreover, Hide's coupled dynamo model presents a valuable dynamical study. We simulate the convergence of the LEs of the model in three cases by means of periodic flow, regular flow, and chaos flow. In addition, we compared these cases in logic connections and proved them in a mathematical way.

Keywords: Lyapunov exponents, dynamical system, dynamo model, periodic flow, chaotic flow, regular flow

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1 Introduction

The convergence or divergence average of exponential rates of nearby trajectories in the dynamical system phase space is given by the Lyapunov exponents (LEs) [1]. It is named after Aleksandr Lyapunov who has developed many major methods in 1899, which can be used to state the set of stability of the ordinary differential equations (ODEs). There are two types of LEs. First, when the LEs are positive, the average exponential of two nearby trajectories is diverge. Second, when the LEs are negative, the average exponential of two close trajectories is converge [2]. Recently, the Lyapunov characteristic exponent (LCE) becomes a hot research topic, which can be used to characterize the dynamical systems quantitatively and their stochasticity properties including the nearby orbits, exponential divergence [3]. The estimation of LEs is considered as one of the most important tasks in the dynamical system studies, which can be shown in theoretical studies conducted by Oseledets et al. [4] and numerical algorithms conducted by Benettin et al. [5].

Furthermore, there are many methods that are used to measure the LE numerical values. Wolf's algorithm is one of the simplest methods among other methods [6]. Nowadays, the idea of the algorithm is to affect the initial sphere evolution that has perturbation to a nominal orbit. Moreover, the orthogonalization numerically plays a significant role in measuring the LEs. If we do not use the orthogonalization in the major solution, vectors turn to the largest growth direction. For the continuous dynamical system, we used the continuous orthogonalization methods that are more applicable than Wolf's algorithm [7,8].

The study of dynamics and dynamical systems has a great role in our life, which can be applied it in several fields such as mathematics, biology, history, economics, and physics [9–16]. For example, we can describe the dynamical system as an ensemble of particles or a particle whose change depends on time and thus particle obeys time derivative equations [17]. Moreover, the dynamical systems are the main part of different theories such as bifurcation theory, dynamics of the logistic map, the theory of chaos, self-

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assembly processes, and self-organization [10]. In this paper, we use LEs as a significant dynamical to study the behavior of three different cases in Hide's coupled dynamo model. These cases represent the most common flows that demonstrate how the flow acts when the parameters are changed.

2 Methodology and preliminary results

In this study, we used the LEs as a significant dynamical tool for studying the stability of non-stationary solutions of ODEs. Moreover, we introduced a novel set of some nonlinear ODEs that have been investigated by Hide et al. [18], who explained the system by using many differential equations:

$$\begin{aligned}\dot{x} &= x(y - 1) - \beta z, \\ \dot{y} &= \alpha(1 - x^2) - \kappa y, \\ \dot{z} &= x - \lambda z,\end{aligned}\tag{1}$$

where $\dot{x}$, $\dot{y}$, and $\dot{z}$ are dependent variables on time. $\dot{x}$ is the electric current of the dynamical system, $\dot{y}$ is the disk angular rotation rate, and $\dot{z}$ is the motor angular speed. Also, as shown in (1), there are four parameters denoted as β , α , κ , and λ , which are used to make control for a dynamical system. β is used to measure the armature inverse moment, α is used to measure the applied couple, and λ and κ are used to measure the mechanical friction occurring in the motor and disk [19].

Moroz [20] and Hide et al. [18] studied the equilibrium point stability by determining the system steady equilibrium. Suppose that $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} = 0$, then (1) becomes

$$x(y - 1) - \beta z = 0,\tag{2}$$

$$\alpha(1 - x^2) - \kappa y = 0,\tag{3}$$

$$x - \lambda z = 0.\tag{4}$$

From (4), we get $x = \lambda z$. By substituting $x = \lambda z$ into (2), we obtain

$$\lambda z(y - 1) - \beta z = 0,$$

$$z[\lambda(y - 1) - \beta] = 0.$$

Thus, we get either $z = 0$ or $\lambda(y - 1) = \beta$. Consequently, we get $y = 1 + \beta/\lambda$.

From these equations, we can see that the first steady state that satisfies all parameters values of $\lambda\beta$, α , and κ has the form:

$$(x, y, z) = \left(0, \frac{\alpha}{\kappa}, 0\right).\tag{5}$$

In system (1), the current cannot flow as $x = 0$; the result shows that the motor is stationary $z = 0$ while the disk spins. The slow torque is given by the mechanical friction to make a balance between the applied couples [18]. To get the second steady state, we substitute $y = 1 + \frac{\beta}{\lambda}$ into (3). Hence, we get

$$\alpha(1 - x^2) - \kappa \left(1 + \frac{\beta}{\lambda}\right) = 0,$$

$$\left(\alpha - \alpha x^2\right) - \kappa - \frac{\kappa\beta}{\lambda} = 0,$$

$$(\alpha x^2) = \alpha - \kappa \left(1 + \frac{\beta}{\lambda}\right) = 0,$$

$$(x^2) = 1 - \left(\frac{\kappa}{\alpha} \left(1 + \frac{\beta}{\lambda}\right)\right) = 0,$$

$$x = \pm \sqrt{1 - \left(\frac{\kappa}{\alpha} \left(1 + \frac{\beta}{\lambda}\right)\right)}.$$

Also, we substitute $y = 1 + \frac{\beta}{\lambda}$ into (2). Hence, we get

$$x\left(1 + \frac{\beta}{\lambda} - 1\right) - \beta z = 0, \quad \left(x\frac{\beta}{\lambda}\right) = \beta z, \quad \left(\frac{x}{\lambda}\right) = z.$$

Thus, the second steady-state is in the form:

$$(x, y, z) = \left(x = \pm \sqrt{1 - \left(\frac{\kappa}{\alpha}\left(1 + \frac{\beta}{\lambda}\right)\right)}, 1 + \frac{\beta}{\lambda}, \frac{x}{\lambda}\right) = E_{\pm}. \quad (6)$$

2.1 Linear stability of the first steady state

As we have seen that the first steady state is in the form:

$$(x, y, z) = \left(0, \frac{\alpha}{\kappa}, 0\right). \quad (7)$$

To study the stability of the first steady state from (5) and (6), the linearization method is applied. Then, the Jacobi matrix that ends with eigenvalue is as follows:

$$\sigma_1 = -\kappa, \quad \sigma_{2,3} = \frac{1}{2} \left\{ \frac{\alpha}{\kappa} - 1 - \lambda \pm \sqrt{\left(\frac{\alpha}{\kappa} - 1 - \lambda\right)^2 - (4\beta)} \right\}. \quad (8)$$

The study of the first steady-state stability analysis of (5) stated two cases. First, the solution becomes stable when all the three eigenvalues have a real part with negative signs. Second, the change occurs in the stability when at least one of the real parts of the eigenvalue changes its sign.

2.2 Linear stability of the second steady state

The second steady state can be written as follows:

$$(x, y, z) = \left(x = \pm \sqrt{1 - \left(\frac{\kappa}{\alpha}\left(1 + \frac{\beta}{\lambda}\right)\right)}, 1 + \frac{\beta}{\lambda}, \frac{x}{\lambda}\right) = E_{\pm}. \quad (9)$$

The linear stability of (1) applied in the second equilibrium point (5) shows that the Jacobian matrix ends with cubic eigenvalues equation:

$$\sigma^3 + a\sigma^2 + b\sigma + c = 0,$$

where a, b, c are equal to

$$a = \kappa + \lambda - \frac{\beta}{\lambda}, \quad b = 2(\alpha - \kappa) + \kappa\lambda - 3\frac{\beta\kappa}{\lambda}, \quad c = 2(\alpha\lambda - \kappa\lambda + \kappa\beta).$$

The Hopf bifurcation in (5) occurs, when $ab = c$, and $b > 0$.

3 Numerical results and discussion

In this study, we introduced the LE as a dynamical tool to demonstrate the behavior of the dynamical systems. We used the change in the choice of parameters to study the change in the behavior of systems,

$\beta - \frac{\alpha}{\kappa}$ space. Our numerical results represent the convergence of the LEs for Hide's model of periodic flow, regular flow, and chaotic flow. Also, we can see that the LEs give the convergence of trajectories in each dimension of the attractor depending on changing some selected parameters. In order to integrate the equation of system (1), we applied the most popular numerical method, i.e., the fourth-order Runge-Kutta method with an effective time step. Next, the results for a certain choice of parameter values are shown.

3.1 Convergence plot of LEs for Hide's model of periodic flow

The Hide flow is shown in equation (1). In all calculations that have been shown in this section, we used Hide's parameters $\lambda = 0$, $\kappa = 0.1$, $\beta = 1$, and $\alpha = 0.10$. Our provided numerical results all the time approaches to the solution given by (1) for all the values of parameters λ , β , α , and κ . We introduced all output simulated results based on the Mathematica software. In this section, we start with the case $\lambda = 0$, which is physically unrealistic; we analyzed the convergence plot of the LEs for Hide's model as shown in Figure 1. There are two isolated periodic orbit examples that are given for the parameter values as represented in Figure 1, the parameter values are $\lambda = 0$, $\kappa = 0.1$, $\beta = 1$, $\alpha = 0.10$ and initial conditions are $x(0) = 0$, $y(0) = 1$, $z(0) = 0.81$. In Figure 1, when parameters $\kappa = 0.1$ and α has a value more than 0.1, the periodic solution occurs. Also, we can see that the convergence of LEs as a periodic flow in the following coordinates x , y , and z (red, orange, green). Moreover, when α increases, the periodic solution also increases [18]. In addition, from our results, we can see that the flow wraps around the x -axis, and the initial conditions have a significant effect on the convergence of the LEs in the dynamical systems [22].

3.2 Convergence of the LEs for Hide's model of regular flow

In this section, we examined Hide's parameter values $\beta = 1$, $\alpha = 50$, $\lambda = 0$, $\kappa = 0.1$, and the initial conditions are $x(0) = 0$, $y(0) = 4.9$, and $z(0) = 5.64$. The convergence for the above initial condition warps around the y -axis as shown in Figure 2. As we know that with an increases of α to 50, the periodic flow expands in the size. Hence, the several solution states are possible, and these solutions depend on the initial conditions. The LE convergence and the periodic solution number raised as the value of α is increased [22] as shown in Figure 2. The flow is regular and expands in the size until bends out of $y = 2$, as the α increased and the values of convergence tend to be positive as observed in Figure 2. Moreover, as shown in Figure 2, the convergence at this initial condition seems to be mostly regular flow.

Figure 1: The convergence plot of the LEs for Hide's model where initial conditions and parameter values $x_0 = \{0, 1, 0.81\}$, $\beta = 1$, $\alpha = 0.10$, $\lambda = 0$, $\kappa = 0.1$.

Figure 2: The convergence plot of the LEs for Hide's model where initial conditions and parameter values $x_0 = \{0, 4.9, 5.641\}$, $\beta = 1$, $\alpha = 50$, $\lambda = 0$, $\kappa = 0.1$.

Also, we can show that Figure 2 exhibited more and more periodic solutions while there is an increase in the value of α .

3.3 Convergence of the LEs for Hide's model of chaos flow

Now, in this section, we introduced two isolated cases of Hide's parameter values. The first case has parameters given as $\beta = 2$, $\alpha = 20$, $\lambda = 1.2$, $\kappa = 1$, with initial conditions $x(0) = -0.1$, $y(0) = 5.1$, $z(0) = -0.34715835$. Also, the second case has parameters given as $\beta = 1.01$, $\alpha = 100$, $\lambda = 1$, $\kappa = 1$, with initial conditions $x(0) = 0.2$, $y(0) = 0.1$, $z(0) = 0.59$. It is clear that the flow in both cases bends around the y -axis as shown in Figures 3 and 4, which in turn indicate the behavior of chaotic. The chaotic behavior has appeared if $\lambda \neq 0$ for all of these numerical results. Also, the system of the LEs converges to negative, positive, and zero values, which is a clear indication of chaotic behavior [23]. In addition, the chaos presence depends on the negative value of the LEs [24]. Furthermore, the common characteristic of driven systems is the long time needed for the convergence of the LEs [25]. Moreover, from the figures, we can see that when the value of α increased from 20 to 100, the convergence of the LEs becomes more and more regular and periodic [22]. The two cases are considered as two examples of chaotic behavior as shown in Figures 3 and 4. Furthermore, the chaotic solutions are shown

Figure 3: The convergence plot of the LEs for Hide's model with initial conditions and parameter values $x_0 = \{-0.1, 5.1, -0.34715835\}$, $\beta = 2$, $\alpha = 20$, $\lambda = 1.2$, $\kappa = 1$.

Figure 4: The convergence plot of the LEs for Hide's model with initial conditions and parameter values $x_0 = \{0.2, 0.1, 0.59\}$, $\beta = 1.01$, $\alpha = 100$, $\lambda = 1$, $\kappa = 1$.

by increasing α to 100 and $\lambda = 1$; these solutions go and back between the two unstable periodic cycles which introduce chaotic attractor similar to the well-known Lorenz attractor [22].

4 Conclusion

A novel set of nonlinear ODEs developed by Hide et al. has been studied by using a significant dynamical tool that so-called LEs. Due to the significance of these equations and its rich behavior of these equations, numerous studies that have an interest in this method have been published. One of the key objectives of this study is to return to Hide et al. and to put their result in perspective as well as to provide an interpretation and numerical description of these findings. In this study, we introduced the use of the LEs, which is a great tool to measure the nearby trajectories in the dynamical system. Also, from the theoretical results that have been found and with the aid of computational methods, we used Hide's coupled dynamo model and code up the LEs in three cases of periodic flow, regular flow, and chaos flow. In addition, we have compared between these systems in logic connections and proved it in this study. An analysis and description of the dynamics of the system in the figure including the periodic, regular, and chaos flows are obtained. Moreover, in this study, we showed the convergence of the LEs for the Hide dynamo model depends on the change in the initial conditions and some parameter values. In addition, we showed that the system behavior is based on its sensitivity to the four parameters (β , α , κ , λ) and the initial conditions. Three cases such as the periodic flow, regular flow, and chaotic flow are presented here. Further research in the future should include the determination of various forms of periodic orbit bifurcation that can be found numerically. Another idea is to detect the chaotic behavior by adding the alien attractors and the LE. In addition, further experiments with various parameter values and initial conditions can be investigated.

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