Research Article

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Range-Kernel orthogonality and elementary operators on certain Banach spaces

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Abstract: The characterization of the points in $C_{p,1 \leq p < \infty}(\mathcal{H})$, the Von Neuman-Schatten p -classes, that are orthogonal to the range of elementary operators has been done for certain kinds of elementary operators. In this paper, we shall study this problem of characterization on an abstract reflexive, smooth and strictly convex Banach space for arbitrary operator. As an application, we consider other kinds of elementary operators defined on the spaces $C_{p,1 \leq p < \infty}(\mathcal{H})$, and finally, we give a counterexample to Mecheri's result given in this context.

Keywords: Range-Kernel orthogonality, elementary operator, Schatten p -classes, trace class operators

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1 Introduction

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators acting on a complex separable Hilbert space $\mathcal{H}$ and $X \in B(\mathcal{H})$ be a compact operator, and let $s_1(X) \geq s_2(X) \geq \dots \geq 0$ denote the eigenvalues of $|X| = (X^*X)^{\frac{1}{2}}$ arranged in their decreasing order. $X \in C_{p,1 \leq p < \infty}(\mathcal{H})$, the Von Neumann-Schatten p -classes, if

$$\|X\|_p = \left[\sum_{i=1}^{\infty} s_i(X)^p \right]^{\frac{1}{p}} = \text{tr}(|X|^p)^{\frac{1}{p}} < +\infty, \quad (1)$$

where tr denotes the trace functional. Hence, $C_1(\mathcal{H})$ is the trace class, $C_2(\mathcal{H})$ is the Hilbert-Schmidt class and the case $p = \infty$ corresponds to the class of compact operators $C_{\infty}(\mathcal{H})$ equipped with the norm $\|X\|_{\infty} = s_1(X)$. For more details, the reader is referred to [10].

We recall the definition of Birkhoff-James's orthogonality in Banach spaces [2,12].

Definition 1. If X is a complex Banach space, then for any elements $x, y \in X$, we say that x is orthogonal to y , noted by $x \perp y$, iff for all $\alpha, \beta \in \mathbb{C}$ there holds

$$\|\alpha y + \beta x\| \geq \|\beta x\|. \quad (2)$$

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If M and N are linear subspaces in X , we say that M is orthogonal to N , noted by $M \perp N$, if $\|x + y\| \geq \|x\|$ for all $x \in M$ and all $y \in N$. If $M = \{x\}$, we simply write $x \perp N$.

- (i) The orthogonality in this sense is not symmetric;
- (ii) If X is a Hilbert space with its inner product $\langle \cdot, \cdot \rangle$, then it follows from (2) that $\langle x, y \rangle = 0$ which means that Birkhoff-James's orthogonality generalizes the usual sense in Hilbert space.

We also recall the definition of the Range-Kernel orthogonality for a pair of operators (E, T) on Banach spaces introduced by R. Harte [11].

Definition 2. If $E : X \rightarrow Y$ and $T : Y \rightarrow Z$ are bounded linear operators between Banach spaces. T is called orthogonal to E provided

$$s \in \ker T \Rightarrow \|s + E(x)\| \geq \|s\|, \text{ for all } x \in X. \quad (3)$$

If $T = E$, we shall call that E is orthogonal.

The elementary operator is an operator E defined on Banach $(\mathcal{A}, \mathcal{B})$ -bimodule $\mathcal{M}$ with its representation $E(x) = \sum_{i=1}^n a_i x b_i$, where $a = (a_i)_i \in \mathcal{A}^n$, $b = (b_i)_i \in \mathcal{B}^n$ are n -tuples of algebra elements. The length of E is defined to be the smallest number of multiplication terms required for any representation $\sum_j a_j x b_j$ for E . The case of the elementary operator is restricted to the operator $\delta_{A,B} \in B[B(\mathcal{H})]$, which is well-known as generalized derivation induced by fixed operators A, B in $B(\mathcal{H})$ and defined by $\delta_{A,B}(X) = AX - XB$.

Anderson [5] proved that if A and B are normal operators, then

$$\text{for all } X, S \in B(\mathcal{H}) : S \in \ker \delta_{A,B} \Rightarrow \|\delta_{A,B}(X) + S\| \geq \|S\|, \quad (4)$$

where $\ker \delta_{A,B}$ denotes the kernel of $\delta_{A,B}$.

This means that the kernel of $\delta_{A,B}$ is orthogonal to its range. This result has been generalized in different directions, to nonnormal operators (see [6,7]), to $C_{p,1 \leq p < \infty}(\mathcal{H})$ and to some elementary operators (see [8,9]).

The main purpose of this paper is to give necessary conditions and characterize the elements that are orthogonal to the range of an operator defined on an abstract reflexive and strictly convex Banach space. As an application, we consider certain classes of elementary operators defined on the spaces $C_{p,1 \leq p < \infty}(\mathcal{H})$.

In addition to the notations and the definitions already introduced, we set, if X is a normed linear space over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we denote by $B(X)$ the space of all linear bounded operators on X , the closure of the range of an operator $T \in B(X)$ will be denoted by $\overline{\text{ran}(T)}$, the restriction of T to an invariant subspace M will be denoted by $T|_M$ and the commutator $AB - BA$ of the operators A, B will be denoted by $[A, B]$.

2 Main results

Let X be a normed linear space over the field $\mathbb{K}$ and $X^\dagger$ its topological dual. Let D be the (multivalued) mapping defined from X to $X^\dagger$ as:

$$\forall x \in X : D(x) = \{\varphi \in X^\dagger : \varphi(x) = \|x\|^2; \|\varphi\| = \|x\|\}, \quad (5)$$

called the normalized duality mapping. Hahn-Banach's theorem ensures that there always exists at least one support functional (a support functional ψ at $x \in X$ is a norm-one linear functional in $X^\dagger$ such that $\psi(x) = \|x\|$ at each vector $x \in X$) and therefore $D(x)$ is non-empty for every $x \in X$. Moreover, it is well known that $D(x)$ is convex and weak*-compact subset of $X^\dagger$. D is not linear in general but it is homogeneous, i.e., $\forall \alpha \in \mathbb{R} : D(\alpha x) = \alpha D(x)$.

Theorem 3. Let K be a closed linear subset of X and $x \notin K$, then

$$x \perp K \Leftrightarrow \exists \tilde{\varphi} \in D(x) : K \subseteq \ker \tilde{\varphi}.$$

Proof. Let $\varphi \in D(x)$ such that $\langle \varphi, y \rangle = 0$, for all $y \in K$. Then,

$$\varphi(x + y) = \varphi(x) = \|x\|^2$$

and

$$\|x\|^2 \leq \|\varphi\| \|x + y\| \leq \|x\| \|x + y\|,$$

that is, $\|x\| \leq \|x + y\|$, for all $y \in K$. Hence, $x \perp K$.

For the converse, let $x \notin K$ such that $x \perp K$. Then, for all $y \in K$, x and y are linearly independent vectors. Let L be the closed subspace spanned by K and $\{x\}$, $L = [K, \{x\}]$, and define the function φ on L by $\varphi(\alpha x + \beta y) = \alpha \|x\|^2$ for all $y \in K$ and all $\alpha, \beta \in \mathbb{C}$. Clearly φ is linear (by the assumption that K is a linear subset of X). To prove the continuity of φ , let $z \in L$, then $z = \alpha x + \beta y$ and $\varphi(z) = \alpha \|x\|^2$. By the definition of $\perp$ and the assumption that $x \perp K$, we derive that $\|z\| \geq \|\alpha x\|$. If $\alpha \neq 0$, we have

$$|\varphi(z)| = |\alpha| \|x\|^2 \cdot \frac{\|z\|}{\|z\|} \leq \frac{|\alpha| \|x\|^2}{\|\alpha x\|} \|z\| = \|x\| \|z\|.$$

If $\alpha = 0$, then $|\varphi(z)| = |\varphi(\beta y)| = 0$. Hence,

$$|\varphi(z)| \leq \|x\| \|z\|, \quad \text{for all } z \in L.$$

Therefore, φ is continuous on L and $\|\varphi\| = \|x\|$. By Hahn-Banach's theorem there is a continuous linear functional $\tilde{\varphi}$ on X such that $\tilde{\varphi}|_L = \varphi$ and $\|\tilde{\varphi}\| = \|\varphi\|$, where $\tilde{\varphi}|_L$ is the restriction of $\tilde{\varphi}$ on L . It follows, by the definition of φ and $\tilde{\varphi}|_L = \varphi$, that $K \subseteq \ker \tilde{\varphi}$ and $\tilde{\varphi} \in D(x)$. $\square$

Corollary 4.

- (i) $\forall x, y \in X \quad x \perp y \Leftrightarrow \exists \varphi \in D(x) : \varphi(y) = 0$.
- (ii) $\forall x \in X, \forall \varphi \in D(x) : x \perp \ker \varphi$.

Notation. Let K be a nonempty subset of X and $T \in B(X)$, we denote the duality adjoint of T by $T^\perp$ and set $K^\perp = \{x \in X : x \perp y; \forall y \in K\}$.

Remark 5. It is clear that if $\{x_n\}_n$ is a sequence in a subset K converging to y and $x \perp x_n$, for all n , then $x \perp y$. Hence, $x \perp K \Rightarrow x \perp \overline{K}$.

Lemma 6. Let X be a Banach space.

- 1. If K and L are closed subspaces of X and $K \subseteq L^\perp$, then $K \oplus L$ is closed.
- 2. Let $T \in B(X)$ and $s \in X$. Then
 - (i) $s \perp \text{ran } T \Leftrightarrow \exists \varphi \in D(s) : \varphi \in \ker T^\perp$;
 - (ii) If T is orthogonal, then $\ker T \oplus \overline{\text{ran}(T)}$ is a closed subspace of X .

Proof.

- 1. Let $z \in X : z = \lim_n (x_n + y_n)$ and $z_n = (x_n + y_n)$; for all $n \geq 1$, where $x_n \in K$ and $y_n \in L$. From $K \subseteq L^\perp$, we get

$$\|z_n - z_{n+p}\| = \|y_n - y_{n+p} + x_n - x_{n+p}\| \geq \|x_n - x_{n+p}\|, \quad \text{for all } n, p.$$

Then, $\{x_n\}_n$ is a Cauchy sequence, hence $\lim_n x_n$ exists in K . Setting $x = \lim_n x_n$, we get $\lim_n y_n = z - x \in L$, and therefore, $z \in K \oplus L$.

- 2. (i) By Theorem 3 and Remark 5, $s \perp \text{ran } T \Leftrightarrow \exists \varphi \in D(s), \text{ran}(T) \subseteq \ker \varphi$, that is, $\varphi(Tx) = (T^\perp \varphi)x = 0$, for all $x \in X$. Hence, $s \perp \text{ran } T \Leftrightarrow \exists \varphi \in D(s) : \varphi \in \ker T^\perp$.
- (ii) It is a direct consequence of assertions (1) and (2)(i). $\square$

Let $f : X \rightarrow X$ be a map on X , not necessarily linear or additive, and $F_f : X \rightarrow \mathbb{R}^+$ be a map defined by

$$F_f(x) = \|f(x)\|, \quad \text{for all } x \in X. \quad (6)$$

We say that F_f has a global minimum at $a \in X$ if

$$\|f(a)\| \leq \|f(x)\|, \quad \text{for all } x \in X. \quad (7)$$

As an application of the previous results, the following theorem gives us necessary and sufficient conditions in terms of Birkhoff-James orthogonality for minimizing the map F_f .

Theorem 7. *Let T and $f : X \rightarrow X$ be bounded maps not necessarily linear or additive, and $a \in X$. Then the following assertions hold:*

- (i) *Suppose that the relation $f(x) + T(y) = f(y) + T(x)$ holds for all $x, y \in X$; if $f(a) \perp T(x)$, for all $x \in X$, then the map F_f defined by (6) has a global minimum at a ;*
- (ii) *If we suppose that T is linear and $f(x) = T(x) + f(0)$, for all $x \in X$, then F_f defined by (6) has a global minimum at a if and only if $f(a) \perp \text{ran } T$ if and only if there is $\varphi \in D(f(a))$ such that $\varphi \in \ker T^+$, where T^+ is the duality map of T . Moreover, if $f(a)$ is a smooth point (or the space X is smooth), then the existence of φ is unique.*

Proof.

- (i) It follows from Corollary 1 that for all $x \in X$, $f(a) \perp T(x) \Leftrightarrow \exists \varphi \in X^+ : \varphi(f(a)) = \|f(a)\|^2 = \|\varphi\|^2$ and $\varphi(T(x)) = 0$. Then, by the relation defined in (i), we get $\varphi(f(a)) = \varphi(f(a) + T(x)) = \varphi(f(x) + T(a)) = \varphi(f(x))$. So that $\|\varphi(f(a))\| = \|f(a)\| \|\varphi\| \leq \|\varphi\| \|f(x)\|$. Hence, $\|f(a)\| \leq \|f(x)\|$, for all $x \in X$.
- (ii) Since the maps f, T satisfy the relation cited in (i), the sufficient condition follows from (i). By linearity of T , we get $f(a) + \lambda T(x) = f(a + \lambda x)$ for all $x \in X, \lambda \in \mathbb{C}$. Hence, F_f has a global minimum at a implies $\|f(a) + \lambda g(x)\| = \|f(a + \lambda x)\| \geq \|f(a)\|$. The other equivalence follows from Lemma 1.
If $f(a)$ is a smooth point (or the space X is smooth), then $f(a)$ has only one functional support (for the functional support of smooth points, see [10]) and therefore $D(f(a))$ has one element (the map D is a single-valued function if X is smooth). $\square$

Lemma 8. [8] *If $\mathcal{I}$ is a separable ideal of compact operators in $B(\mathcal{H})$ equipped with unitary invariant norm, then its dual $\mathcal{I}^+$ is isometrically isomorphic to an ideal of compact operators $\mathcal{J}$ not necessarily separable as follows:*

$$\phi : \mathcal{J} \rightarrow \mathcal{I}^+, \quad R \mapsto \phi_R : \phi_R(X) = \text{tr}(XR). \quad (8)$$

Corollary 9. *Let $A \in C_p(\mathcal{H})$; ($1 \leq p < \infty$), $T \in B(C_p(\mathcal{H}))$ and f, F_f are defined as in Theorem 7(ii) and where $f(A)$ is given by its polar decomposition $f(A) = u|f(A)|$. Then, the following assertions hold*

- (i) *If $A \in C_p(\mathcal{H})$; ($1 < p < \infty$), then F_f has a global minimizer at A if and only if $f(A) \perp \text{ran } T$ if and only if $\|f(A)\|^{p-1}u^* \in \ker T^+$;*
- (ii) *If $A \in C_1(\mathcal{H})$ and $f(A)$ is a smooth point, then F_f has a global minimizer at A if and only if $f(A) \perp \text{ran } T$ if and only if $u^* \in \ker T^+$ when $f(A)$ is injective (or $u \in \ker T^+$ when $f(A)^*$ is injective).*

Proof. From Theorem 7, we have that F_f has a global minimizer at A if and only if there exists $\varphi \in D(f(A))$ such that $\varphi \in \ker T^+$,

- (i) If $A \in C_p(\mathcal{H})$; ($1 < p < \infty$), then by the isomorphism in (8), it follows that

$$\begin{aligned} \varphi \in C_p(\mathcal{H})^+ &\Leftrightarrow \exists R \in C_{q, 1 < q < \infty}(\mathcal{H}); \quad \frac{1}{p} + \frac{1}{q} = 1 : \phi_R = \varphi; \\ \|\varphi\| &= \|R\| \quad \text{and} \quad \varphi(X) = \text{tr } RX \quad \text{for all } X \in C_p(\mathcal{H}). \end{aligned}$$

Hence, by the smoothness of $C_{p, 1 < p < \infty}(\mathcal{H})$, F_f has a global minimizer at A if there is a unique operator R such that $\varphi(f(A)) = \text{tr}(f(A)R) = \|f(A)\|_p^2 = \|R\|_q^2$ and $\text{tr}(T^+(R)X) = 0$, for all $X \in C_{p, 1 < p < \infty}(\mathcal{H})$. To finish the proof, it suffices to take $R = \|f(A)\|_p^{2-p}|f(A)|^{p-1}u^*$ and the rank one operator $X = e \oplus e$, $e \in \mathcal{H}$. So that with simple calculation, we get $\langle T^*(R)e, e \rangle = 0$, for all $e \in \mathcal{H}$ and thus $T^+(R) = T^+((|f(A)|^{p-1}u^*)) = 0$. Conversely, $T^+((|f(A)|^{p-1}u^*)) = 0 \Rightarrow T^+(R) = 0$ and then $\text{tr}(T^+(R)X) = 0$, for all $X \in C_{p, 1 < p < \infty}(\mathcal{H})$.

- (ii) It is well known that $C_1(\mathcal{H})$ is not reflexive, even not smooth space and its dual $C_1(\mathcal{H})^\dagger$ is isometrically isomorphic to $B(\mathcal{H})$. This isomorphism is given by

$$\phi : B(\mathcal{H}) \hookrightarrow C_1(\mathcal{H})^\dagger, \quad R \mapsto \phi_R : \phi_R(X) = \text{tr}(XR). \quad (9)$$

Then,

$$\begin{aligned} \varphi \in C_1(\mathcal{H})^\dagger &\Leftrightarrow \exists R \in B(\mathcal{H}) : \phi_R = \varphi; \\ \|\varphi\| &= \|R\| \quad \text{and} \quad \varphi(X) = \text{tr} RX \quad \text{for all } X \in C_1(\mathcal{H}). \end{aligned}$$

So that if $f(A)$ is a smooth point, then F_f has a global minimizer at A iff there is a unique operator R such that

$$\varphi(f(A)) = \text{tr}(f(A)R) = \|f(A)\|_1^2 = \|R\|^2$$

and

$$\text{tr}(T^\dagger(R)X) = 0, \quad \forall X \in C_1(\mathcal{H}).$$

Since $f(A)$ is smooth, then by Holub [11], either $f(A)$ or $f(A)^*$ is injective, thus either u or u^* is an isometry, i.e., $uu^* = I$ or $u^*u = I$. So it suffices to take $R = \|f(A)\|_1 u^*$ or $R = \|f(A)\|_1 u$, which is the unique operator required in both cases $f(A)$ or $f(A)^*$ is injective. $\square$

Lemma 10. Let K be a closed subspace of $\mathcal{X}$. If $\mathcal{X}$ is reflexive and $K^\perp = \{0\}$, then $K = \mathcal{X}$.

Proof. If $K \neq \mathcal{X}$, then there exists $\varphi \in \mathcal{X}^\dagger : K \subseteq \ker \varphi$. Since $D(\varphi)$ is not empty, then there is $f \in \mathcal{X}^{++}$ with $f(\varphi) = \|\varphi\|^2 = \|f\|^2$. Let J be the natural injection between $\mathcal{X}$ and $\mathcal{X}^{++}$, i.e.,

$$J : \mathcal{X} \rightarrow \mathcal{X}^{++}; \quad \forall x \in \mathcal{X}, \quad \forall \psi \in \mathcal{X}^\dagger : J(x)\psi = \psi(x); \quad \|J(x)\| = \|x\|. \quad (10)$$

So by the reflexivity of $\mathcal{X}$, J is a bijection and then there is $0 \neq x \in \mathcal{X}$ such that $J(x) = f$. Hence, $\varphi(x) = \|x\|^2 = \|\varphi\|^2$. Thus, $\varphi \in D(x)$ and by application of Theorem 1, we get $0 \neq x \in K^\perp = \{0\}$, a contradiction. $\square$

Proposition 11. Let $\mathcal{X}$ be a reflexive, smooth and strictly convex Banach space and $T \in B(\mathcal{X})$, if $T^\dagger$ is orthogonal. Then

$$\forall s \in \mathcal{X} : s \perp \text{ran } T \Rightarrow s \in \ker T. \quad (11)$$

Proof. Let $s \in \mathcal{X}$ such that $s \perp \text{ran } T$. Then, by Lemma 6(2.i) and the smoothness of $\mathcal{X}$, there is a unique $\varphi_s \in D(s)$ such that $\varphi_s \in \ker T^\dagger$. Again, by assumptions and Lemma 6(2.i), there is $\psi_{\varphi_s} \in D(\varphi_s)$ such that $\psi_{\varphi_s} \in \ker T^{++}$. Let J be the natural injection between $\mathcal{X}$ and $\mathcal{X}^{++}$ as defined in (10). We see that $J(s)\varphi_s = \|\varphi_s\|^2$ and $\|J(s)\| = \|\varphi_s\|$, which means that $J(s) \in D(\varphi_s)$. By the reflexivity of $\mathcal{X}$, J is a bijection. Hence, there is $c \in \mathcal{X} : \psi_{\varphi_s} = J(c)$ and $\|J(c)\| = \|c\| = \|\varphi_s\|$, $J(c)\varphi_s = \varphi_s(c) = \psi_{\varphi_s}(\varphi_s) = \|\varphi_s\|^2$. Then $\varphi_s \in D(s) \cap D(c)$ and since $\mathcal{X}$ is strictly convex, we get $c = s$. Thus, $J(c) = J(s) \in \ker T^{++}$. Therefore, $(T^{++}J(s))\varphi = J(s)(T^\dagger\varphi) = (T^\dagger\varphi)s = \varphi(Ts) = 0$, for all $\varphi \in \mathcal{X}^\dagger$, that is, $s \in \ker T$. $\square$

Remark 12. If $\mathcal{X}$ is a reflexive separable Banach space and $T^\dagger$ is orthogonal, then the implication (11) holds with respect to suitable norm in $\mathcal{X}$. Indeed, if $\mathcal{X}$ is separable, then there is an equivalent norm, which is smooth and strictly convex in $\mathcal{X}$.

Corollary 13. Let $\mathcal{X}$ be a reflexive, smooth and strictly convex Banach space and $T \in B(\mathcal{X})$. If T and $T^\dagger$ are orthogonal, then

$$\forall s \in \mathcal{X} : s \perp \text{ran } T \Leftrightarrow s \in \ker T \quad (12)$$

$$\mathcal{X} = \ker T \oplus \overline{\text{ran}(T)}. \quad (13)$$

Proof. If T is orthogonal, then, by Definition 2, it follows that

$$\forall s \in \mathcal{X} : s \in \ker T \Rightarrow s \perp \operatorname{ran} T$$

and the reverse implication of (12) follows by Proposition 1. Let us prove the decomposition (13): let $y \in \mathcal{X}$ such that $y \in (\ker T \oplus \overline{\operatorname{ran}(T)})^\perp$, then there is $\varphi_y \in D(y)$ such that $\varphi_y(s \oplus TX) = 0$, for all $s \in \ker T$ and all $x \in \mathcal{X}$. For $s = 0$, it follows, by Lemma 6(2.i), that $Y \perp \operatorname{ran} T$, and by Proposition 1, $\operatorname{ran} T \subseteq \ker T$. So that, we can choose $x = 0$ and $s = y$, this yields $\varphi_y(y) = 0$. This means $y \perp y$, and hence $y = 0$. Finally, the decomposition (13) follows from Lemma 10. $\square$

3 Applications

In this section, we consider the important case, when the operator T , cited in the previous section, is replaced by the elementary operators

$$E : E(X) = \sum_{i=1, \dots, n} A_i X B_i \quad \text{on } C_p(\mathcal{H}) (1 \leq p < \infty),$$

where $A = (A_1, A_2, \dots, A_n)$, $B = (B_1, B_2, \dots, B_n)$ are n -tuples in $(B(\mathcal{H}))^n$.

By the isomorphisms (8) and (9), we can assert that the duality adjoint of E on $C_p(\mathcal{H})$; $(1 \leq p < \infty)$ has the form $E^\dagger(X) = \sum_{i=1, \dots, n} B_i X A_i$. Indeed, let $X \in C_p(\mathcal{H})$; $(1 \leq p < \infty)$ and $R \in B(\mathcal{H})$ (or $R \in C_q(\mathcal{H})$ if $X \in C_p(\mathcal{H})$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p, q < \infty$), then

$$\phi_R(E(X)) = \operatorname{tr} \left(\sum_{i=1, \dots, n} A_i X B_i R \right) = \operatorname{tr} \left(X \sum_{i=1, \dots, n} B_i R A_i \right) = \operatorname{tr}(X E^\dagger(R)) = \phi_{E^\dagger(R)}(X).$$

Notation. We denote the formal adjoint of E by $\tilde{E}$ and define by

$$\tilde{E}(X) = \sum_{i=1, \dots, n} A_i^* X B_i^*,$$

where $(A_1^*, A_2^*, \dots, A_n^*)$ and $(B_1^*, B_2^*, \dots, B_n^*)$ are n -tuples of operators in $(B(\mathcal{H}))^n$.

Lemma 14. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $E \in B(B(\mathcal{K}, \mathcal{H}))$ such that $E(X) = AXB$. If A and B^* are injective operators, then E is injective.

Proof. If $AXB = 0$ with A injective, then $XB = 0 = B^*X^* = 0$ implies $X^* = 0 = X$ since B^* is injective. Thus, E is injective. $\square$

Lemma 15. Let $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ with A_i, B_i be operators in $B(\mathcal{H})$ such that

$$\sum_i A_i A_i^* \leq 1, \quad \sum_i A_i^* A_i \leq 1, \quad \sum_i B_i B_i^* \leq 1 \quad \text{and} \quad \sum_i B_i^* B_i \leq 1.$$

If E is the elementary operator defined on $C_{p, 1 \leq p < \infty}(\mathcal{H})$ by

$$E(X) = \sum_{i=1, \dots, n} A_i X B_i - X,$$

then

- (i) $\ker E = \ker \tilde{E} \Rightarrow \ker E^\dagger = \ker \tilde{E}^\dagger$;
- (ii) If $E(S) = 0 = \tilde{E}(S)$ for some compact operator S , then $[|S|, B_i] = 0$ for all $1 \leq i \leq n$.

Proof.

- (i) $E^+(S) = 0 \Leftrightarrow \widetilde{E}(S^*) = 0$. From the equality, $\ker E = \ker \widetilde{E}$, it follows that $E^+(S) = 0 \Leftrightarrow E(S^*) = 0 \Leftrightarrow \widetilde{E}^+(S) = 0$.
(ii) See [12]. $\square$

As an application of the previous section, we shall give certain necessary conditions and characterization of the operators in $C_{p,1 < p < \infty}(\mathcal{H})$ that are orthogonal to the range of certain kinds of elementary operators.

Proposition 16. *Let E be an elementary operator defined on $C_{p,1 < p < \infty}(\mathcal{H})$, then*

$$\forall S, X \in C_{p,1 < p < \infty}(\mathcal{H}) : \|E(X) + S\|_p \geq \|S\|_p \Rightarrow S \in \ker E$$

in one of the following cases:

- (i) $E(X) = AXB$ with A^* and B injective operators;
(ii) $E(X) = AXB - CXD$, where A, B normal operators, D, C^* hyponormal operators with $[A, C] = [B, D] = 0$ and $\ker A^* \cap \ker C^* = \{0\} = \ker B \cap \ker D$.

Proof.

- (i) The duality adjoint E^+ is defined by $E^+(X) = BXA$ and using Lemma 6, we get E^+ is injective and then is orthogonal. So the result follows by Proposition 1.
(ii) We have $E^+(X) = BXA - DXC$ and applying Duggal' result [6], we get E^+ is orthogonal, and then by Proposition 1, the proof is complete. $\square$

Theorem 17. *Let E be an elementary operator defined on $C_{p,1 < p < \infty}(\mathcal{H})$ and the map $F_E : C_{p,1 < p < \infty}(\mathcal{H}) \rightarrow \mathbb{R}^+$ defined by $F_E(X) = \|E(X) + S\|_p$. Then the following assertions:*

- (i) F_E has a global minimizer at $S \in C_{p,1 < p < \infty}(\mathcal{H})$;
(ii) $S \perp \text{ran } E$;
(iii) $S \in \ker E$

are equivalent, when the elementary operator E takes one of the following forms:

- (a) $E(X) = AXB$ such that A, A^*, B and B^* are injective operators;
(b) $E(X) = AXB - CXD$; such that (A, C) and (B, D) are 2-tuples of commuting normal operators and $\ker A \cap \ker C = \{0\} = \ker B \cap \ker D$;
(c) $E(X) = \sum_{i=1, \dots, n} A_i X B_i - X$; such that $\sum_{i=1}^n A_i A_i^* \leq 1$, $\sum_{i=1}^n A_i^* A_i \leq 1$, $\sum_{i=1}^n B_i B_i^* \leq 1$, $\sum_{i=1}^n B_i^* B_i \leq 1$ and $\ker E = \ker \widetilde{E}$.

Proof. From Theorem 2, we know that (i) $\Leftrightarrow$ (ii) for any operator E . When the elementary operator E takes the form (a) or (b) we have, by Proposition 16, that (ii) $\Rightarrow$ (iii) and with respect to the hypothesis cited in our theorem, we argue similarly as in Proposition 16 in the reverse sense, to get (iii) $\Rightarrow$ (ii).

The case when E takes the form (3) (as an application of the previous section, we give a simpler and shorter proof of (iii) $\Leftrightarrow$ (ii) than given by Duggal [13]). We start by proving that E is orthogonal.

Let $0 \neq S = u|S| \in \ker E$, then $\sum_{i=1, \dots, n} A_i u|S|B_i = u|S| = \sum_{i=1, \dots, n} A_i^* u|S|B_i^*$ and by Lemma 6(ii), we get $(\sum_{i=1, \dots, n} A_i^* u|S|B_i^*)|S| = u|S|^{p-1}$, multiply both sides of the equation by $|S|^{p-2}$, hence by the commutativity derived from Lemma 6(ii), we obtain $\sum_{i=1, \dots, n} A_i^* u|S|^{p-1}B_i^* = u|S|^{p-1}$. Then, it follows by taking the adjoint that $\sum_{i=1, \dots, n} B_i |S|^{p-1}u^*A_i = |S|^{p-1}u^*$, which means that $|S|^{p-1}u^* \in \ker E^+$ and by Corollary 4(i), we get that E is orthogonal.

From Lemma 5(i), and the symmetry of the forms of E^+ and E we derive that E^+ is orthogonal. So that the equivalence (iii) $\Rightarrow$ (ii) in this case follows immediately by Corollary 4. $\square$

Mecheri showed, in [14, Theorem 2.9], that if A, B are operators in $B(\mathcal{H})$ such that $\ker \delta_{A,B} \subseteq \ker \delta_{A^*, B^*}$. Then $T \in \ker \delta_{A,B} \cap C_{p,1 < p < \infty}(\mathcal{H})$ if and only if

$$\|\delta_{A,B}(X) + T\|_p \geq \|T\|_p,$$

for all $X \in C_{p:1 < p < \infty}(\mathcal{H})$.

In the following, we give a counterexample to Mecheri's result.

Example 18. Consider the ideal $C_2(\mathcal{H})$ with its inner product $\langle X, Y \rangle = \text{tr}(Y^*X)$. Let $\delta_{S,B}$ be the generalized derivation induced by the operators S, B

$$\delta_{S,B}(X) = SX - XB,$$

where S is the unilateral shift operator, i.e., $S(e_n) = e_{n+1}; n \in \mathbb{N}$, for a basis $\{e_n\}_n$ in $\mathcal{H}$ and $B = \frac{1}{2}I$; I is the identity operator on $B(\mathcal{H})$. If P denotes the orthogonal projection on the subspace spanned by the vector $e = \sum_n \frac{1}{2^n} e_n$, then by simple computation, we get that $\delta_{S,B}(P) \neq 0$, $\delta_{S^*,B^*}(P) = 0$ and, for all $X \in C_2(\mathcal{H})$,

$$\|\delta_{S,B}(X) + P\|_2^2 = \|\delta_{S,B}(X)\|_2^2 + \|P\|_2^2.$$

So that $P \perp \text{ran}(\delta_{S,B})$ and $P \notin \ker \delta_{S,B}$. Since S is a quasinormal operator, then $\ker \delta_{S,B} \cap C_2(\mathcal{H}) \subseteq \ker \delta_{S^*,B^*} \cap C_2(\mathcal{H})$, which means that this example is a counterexample to Theorem 2.9 [14]. If we compare this example with respect to our general result in the previous section, we find that, $\delta_{S,B}$ is orthogonal and δ_{S^*,B^*} is not orthogonal,

$$\|\delta_{S^*,B^*}(X) + P\|_2^2 \neq \|\delta_{S^*,B^*}(X)\|_2^2 + \|P\|_2^2.$$

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