Research Article

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More on μ -semi-Lindelöf sets in μ -spaces

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Abstract: Sarsak [On μ -compact sets in μ -spaces, Questions Answers Gen. Topology **31** (2013), no. 1, 49–57] introduced and studied the class of μ -Lindelöf sets in μ -spaces. Mustafa [μ -semi compactness and μ -semi Lindelöfness in generalized topological spaces, Int. J. Pure Appl. Math. **78** (2012), no. 4, 535–541] introduced and studied the class of μ -semi-Lindelöf sets in generalized topological spaces (GTSs); the primary purpose of this paper is to investigate more properties and mapping properties of μ -semi-Lindelöf sets in μ -spaces. The class of μ -semi-Lindelöf sets in μ -spaces is a proper subclass of the class of μ -Lindelöf sets in μ -spaces. It is shown that the property of being μ -semi-Lindelöf is not monotonic, that is, if (X, μ) is a μ -space and $A \subset B \subset X$, where A is μ_B -semi-Lindelöf, then A need not be μ -semi-Lindelöf. We also introduce and study a new type of generalized open sets in GTSs, called ω_μ -semi-open sets, and investigate them to obtain new properties and characterizations of μ -semi-Lindelöf sets in μ -spaces.

Keywords: generalized topology, μ -space, μ -open, μ -semi-open, μ -Lindelöf set, μ -Lindelöf space, μ -semi-Lindelöf set, μ -semi-Lindelöf space

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1 Introduction and preliminaries

A topological space X is said to be Lindelöf [1] if every open cover of X has a countable subcover. Lindelöf spaces play a vital role in the theory of general topology as a natural generalization of compact spaces. Since then, related concepts had been of special interests to several mathematicians. For instance, nearly Lindelöf spaces [2], strongly Lindelöf spaces [3], almost Lindelöf spaces [4], semi-Lindelöf spaces [5], and rc-Lindelöf spaces [6]. For more related studies, one can see [7–12].

The study of generalized topological spaces (GTSs) was first initiated by Császár [13], which in role, motivated a lot of authors to generalize the topological notions including covering properties to the generalized topological surroundings. For instance, Sarsak [14,15], Mustafa [16], Arar [17,18], Abuage et al. [19], and Roy [20] studied several analogous notions via GTSs. The primary purpose of this paper is to continue the study of μ -semi-Lindelöf sets in μ -spaces introduced by Mustafa in [16]. In Section 2, we study the relationship between μ -semi-Lindelöf sets and μ -Lindelöf sets as we also obtain new characterizations of μ -semi-Lindelöf sets; in Section 3, we mainly deduce that if (X, μ) is a μ -space and $A \subset B \subset X$, where A is μ_B -semi-Lindelöf, then A need not be μ -semi-Lindelöf, as we also give corrections to some results in [16]; in Section 4, we introduce and study a new type of generalized open sets in generalized topological spaces, called ω_μ -semi-open sets, and investigate them in Sections 5 and 6 to obtain more characterizations of μ -semi-Lindelöf spaces and to obtain several properties of μ -semi-Lindelöf sets related to sums, products, images, and preimages.

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A generalized topology (GT) [13] μ on a nonempty set X is a collection of subsets of X such that $\emptyset \in \mu$ and μ is closed under arbitrary unions. Elements of μ will be called μ -open sets, and a subset A of (X, μ) will be called μ -closed if $X \setminus A$ is μ -open. Clearly, a subset A of (X, μ) is μ -open if and only if for each $x \in A$, there exists $U_x \in \mu$ such that $x \in U_x \subset A$, or equivalently, A is the union of μ -open sets. The pair (X, μ) will be called GTS. A space X or (X, μ) will always mean a GTS. A space (X, μ) is called a μ -space [21] if $X \in \mu$. (X, μ) is called a quasi-topological space [22] if μ is closed under finite intersections. Clearly, every topological space is a quasi-topological space, every quasi-topological space is a GTS, and a space (X, μ) is a topological space if and only if (X, μ) is both μ -space and quasi-topological space.

If A is a subset of a space (X, μ) , then the μ -closure of A [23], $c_\mu(A)$, is the intersection of all μ -closed sets containing A and the μ -interior of A [23], $i_\mu(A)$, is the union of all μ -open sets contained in A . It was pointed out in [23] that each of the operators c_μ and i_μ are monotonic [24], i.e., if $A \subset B \subset X$, then $c_\mu(A) \subset c_\mu(B)$ and $i_\mu(A) \subset i_\mu(B)$, idempotent [24], i.e., if $A \subset X$, then $c_\mu(c_\mu(A)) = c_\mu(A)$ and $i_\mu(i_\mu(A)) = i_\mu(A)$, c_μ is enlarging [24], i.e., if $A \subset X$, then $c_\mu(A) \supset A$, i_μ is restricting [24], i.e., if $A \subset X$, then $i_\mu(A) \subset A$, A is μ -open if and only if $A = i_\mu(A)$, and $c_\mu(A) = X \setminus i_\mu(X \setminus A)$.

Clearly, A is μ -closed if and only if $A = c_\mu(A)$, $c_\mu(A)$ is the smallest μ -closed set containing A and $i_\mu(A)$ is the largest μ -open set contained in A , and $x \in c_\mu(A)$ if and only if any μ -open set containing x intersects A .

A function $f: (X, \mu) \rightarrow (Y, \kappa)$ is called (μ, κ) -continuous [25] if the inverse image of each κ -open set is μ -open, and called (μ, κ) -closed [25] if the image of every μ -closed set is κ -closed.

If (X, τ) is a topological space and $A \subset X$, then $\bar{A}$ and $\text{Int } A$ will stand, respectively, for the closure of A in X and the interior of A in X .

A subset A of a topological space (X, τ) is called semi-open [26] if $A \subset \overline{\text{Int } A}$; and A is called semi-closed if $X \setminus A$ is semi-open. A is called preopen [27] (resp. α -open [28]) if $A \subset \text{Int } \bar{A}$ (resp. $A \subset \text{Int } \overline{\text{Int } A}$). It is known that the arbitrary union of semi-open (resp. preopen, α -open) sets is semi-open (resp. preopen, α -open).

The families of semi-open (resp. preopen, α -open) subsets of a topological space (X, τ) will be denoted by $SO(X)$ (resp. $PO(X)$, $\alpha(X)$). Clearly, if $\mu = SO(X)$ (resp. $PO(X)$, $\alpha(X)$), then (X, μ) is a μ -space.

For the concepts and terminology not defined here, the reader is referred to [29]. In concluding this section, we recall the following definitions and facts for their importance in the material of our paper.

Definition 1.1. [23] Let A be a subset of a space (X, μ) . Then A is called

- (i) μ -semi-open if $A \subset c_\mu(i_\mu(A))$;
- (ii) μ -preopen if $A \subset i_\mu(c_\mu(A))$;
- (iii) μ - α -open if $A \subset i_\mu(c_\mu(i_\mu(A)))$.

As in [23], for a space (X, μ) , we will denote the class of μ -semi-open sets by $\sigma(\mu)$ or σ , the class of μ -preopen sets by $\pi(\mu)$ or π , and the class of μ - α -open sets by $\alpha(\mu)$ or α .

Proposition 1.2. [23] Let (X, μ) be a space. Then each of the families σ , π , and α is a GT.

Proposition 1.3. [23] Let (X, μ) be a space. Then

- (i) $\mu \subset \alpha$;
- (ii) $\alpha = \sigma \cap \pi$;
- (iii) $\sigma(\sigma) = \sigma$;
- (iv) $\sigma(\alpha) = \sigma$.

Remark 1.4. If (X, μ) is a μ -space, then it is easy to see that

- (i) $(X, \sigma(\mu))$ is a $\sigma(\mu)$ -space;
- (ii) $(X, \alpha(\mu))$ is an $\alpha(\mu)$ -space.

Definition 1.5. [30] Let A be a subset of a space (X, μ) . Then A is called μ -semi-closed if $X \setminus A$ is μ -semi-open.

Definition 1.6. [31]

- (i) A subset A of a μ -space (X, μ) is called μ -Lindelöf if any cover of A by μ -open subsets of X has a countable subcover.
- (ii) A μ -space (X, μ) is called μ -Lindelöf if any cover of X by μ -open sets has a countable subcover.

2 μ -semi-Lindelöf sets

This section is mainly devoted to investigate more properties of μ -semi-Lindelöf sets in μ -spaces.

Definition 2.1.

- (i) A subset A of a topological space (X, τ) is called semi-compact relative to X [32] (resp. semi-Lindelöf in X [33]) if any cover of A by semi-open subsets of X has a finite (resp. countable) subcover. We will usually use the term “in X ” to mean “relative to X ”.
- (ii) A topological space (X, τ) is called semi-compact [34] (resp. semi-Lindelöf [5]) if any cover of X by semi-open subsets of X has a finite (resp. countable) subcover.

Definition 2.2. [16]

- (i) A subset A of a μ -space (X, μ) is called μ -semi-compact (resp. μ -semi-Lindelöf) relative to X if any cover of A by μ -semi-open subsets of X has a finite (resp. countable) subcover. We will say A is μ -semi-compact (resp. μ -semi-Lindelöf) to mean A is μ -semi-compact (resp. μ -semi-Lindelöf) relative to X .
- (ii) A μ -space (X, μ) is called μ -semi-compact (resp. μ -semi-Lindelöf) if any cover of X by μ -semi-open sets has a finite (resp. countable) subcover.

Remark 2.3. Let A be a subset of a μ -space (X, μ) . Then it is clear from Proposition 1.3(iii) and (iv) that

- (i) A is μ -semi-Lindelöf if and only if A is $\sigma(\mu)$ -semi-Lindelöf.
- (ii) A is μ -semi-Lindelöf if and only if A is $\alpha(\mu)$ -semi-Lindelöf.

Remark 2.4. If A is a subset of a topological space (X, τ) and $\mu = \tau$ (resp. $SO(X)$, $\alpha(X)$), then

$$A \text{ is } \mu\text{-semi-Lindelöf} \Leftrightarrow A \text{ is semi-Lindelöf in } (X, \tau).$$

In particular,

$$(X, \mu) \text{ is } \mu\text{-semi-Lindelöf} \Leftrightarrow (X, \tau) \text{ is semi-Lindelöf}.$$

Remark 2.5. If a subset A of a μ -space (X, μ) is μ -semi-Lindelöf, then A is μ -Lindelöf; in particular, if a μ -space (X, μ) is μ -semi-Lindelöf, then (X, μ) is μ -Lindelöf. However, the converse need not be true even for topological spaces as the following example tells.

Example 2.6. Let X be an uncountable set and consider the topology $\tau = \{X, \emptyset, \{p\}\}$ on X , where $p \in X$. Now, if $x \in A = X \setminus \{p\}$, then $A_x = \{p, x\}$ is τ -semi-open because $\text{Int } A_x = \{p\}$, and thus, $\overline{\text{Int } A_x} = \overline{\{p\}} = X$. Therefore, $A_x \subset \overline{\text{Int } A_x}$. Observe that the collection $\mathcal{A} = \{A_x : x \in A\}$ is a cover of X by τ -semi-open sets, but $\mathcal{A}$ has no countable subcover, so, (X, τ) is not semi-Lindelöf. However, (X, τ) is Lindelöf (even compact).

Remark 2.7.

- (i) A subset A of a μ -space (X, μ) is μ -semi-Lindelöf if and only if A is $\sigma(\mu)$ -Lindelöf.
- (ii) A μ -space (X, μ) is μ -semi-Lindelöf if and only if $(X, \sigma(\mu))$ is $\sigma(\mu)$ -Lindelöf.

The proofs of the following three propositions are either straightforward or from [35] and thus omitted.

Proposition 2.8.

- (i) A subset A of a μ -space (X, μ) is μ -semi-Lindelöf if and only if for every family $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ of μ -semi-closed sets having the property that for every countable subfamily $\mathcal{F}^i$ of $\mathcal{F}$, $(\bigcap \mathcal{F}^i) \cap A \neq \emptyset$, then $(\bigcap \mathcal{F}) \cap A \neq \emptyset$.
- (ii) A μ -space (X, μ) is μ -semi-Lindelöf if and only if for every family $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ of μ -semi-closed sets having the property that for every countable subfamily $\mathcal{F}^i$ of $\mathcal{F}$, $(\bigcap \mathcal{F}^i) \neq \emptyset$, then $(\bigcap \mathcal{F}) \neq \emptyset$.

Definition 2.9. [35] A filter base $\mathcal{F}$ on a nonempty set X is called a strong filter base on X if whenever $\mathcal{F}^i$ is a countable subcollection of $\mathcal{F}$, there exists $F \in \mathcal{F}$ such that $F \subset \bigcap \mathcal{F}^i$.

Definition 2.10. [35] A strong filter base $\mathcal{F}$ on a nonempty set X is called a maximal strong filter base on X if whenever $\mathcal{H}$ is a strong filter base on X with $\mathcal{F} \subset \mathcal{H}$, then $\mathcal{F} = \mathcal{H}$.

Proposition 2.11. [35] Every strong filter base $\mathcal{F}$ on a nonempty set X is contained in a maximal strong filter base on X .

Definition 2.12. A filter base $\mathcal{F}$ on a μ -space (X, μ) is said to μ_σ -converge to a point $x \in X$ if for each μ -semi-open subset U of X such that $x \in U$, there exists $F \in \mathcal{F}$ such that $F \subset U$. $\mathcal{F}$ is said to μ_σ -accumulate at $x \in X$ if $U \cap F \neq \emptyset$ for every $F \in \mathcal{F}$ and for every μ -semi-open subset U of X such that $x \in U$.

Proposition 2.13. Let $\mathcal{F}$ be a filter base on a μ -space (X, μ) and $x \in X$. Then

- (i) If $\mathcal{F}$ μ_σ -converges to x , then $\mathcal{F}$ μ_σ -accumulates at x ;
- (ii) If $\mathcal{F}$ is a maximal filter base (maximal strong filter base), then $\mathcal{F}$ μ_σ -converges to x if and only if $\mathcal{F}$ μ_σ -accumulates at x .

Proposition 2.14. For a subset A of a μ -space (X, μ) , the following are equivalent:

- (i) A is μ -semi-Lindelöf;
- (ii) Every maximal strong filter base on X , each of whose members meets A , μ_σ -converges to some point of A ;
- (iii) Every strong filter base on X , each of whose members meets A , μ_σ -accumulates at some point of A .

Proof. (i) $\Rightarrow$ (ii): Let $\mathcal{F}$ be a maximal strong filter base on X , each of whose members meets A , such that $\mathcal{F}$ does not μ_σ -converge to any point of A . Since $\mathcal{F}$ is maximal, it follows from Proposition 2.13(ii) that $\mathcal{F}$ does not μ_σ -accumulate at any point of A . Thus, for each $x \in A$, there exist $F_x \in \mathcal{F}$ and a μ -semi-open subset U_x of X such that $x \in U_x$ and $U_x \cap F_x = \emptyset$. But A is μ -semi-Lindelöf, so there exist $x_1, x_2, x_3, \dots \in X$ such that $A \subset \bigcup_{i=1}^{\infty} U_{x_i}$. Since $\mathcal{F}$ is a strong filter base on X , there exists $F \in \mathcal{F}$ such that $F \subset \bigcap_{i=1}^{\infty} F_{x_i}$, but $U_{x_i} \cap F_{x_i} = \emptyset$ for each $i \in \{1, 2, 3, \dots\}$, so $U_{x_i} \cap F = \emptyset$ for each $i \in \{1, 2, 3, \dots\}$, i.e., $\emptyset = (\bigcup_{i=1}^{\infty} U_{x_i}) \cap F \supset A \cap F$, a contradiction.

(ii) $\Rightarrow$ (iii): Let $\mathcal{F}$ be a strong filter base on X , each of whose members meets A . Then $\mathcal{F}^A = \{F \cap A : F \in \mathcal{F}\}$ is a strong filter base on X . Thus by Proposition 2.11, $\mathcal{F}^A$ is contained in a maximal strong filter base $\mathcal{H}$ on X , each of whose members meets A . By (ii), $\mathcal{H}$ μ_σ -converges to some point x of A , thus by Proposition 2.13(i), $\mathcal{H}$ μ_σ -accumulates at x , but $\mathcal{F}^A \subset \mathcal{H}$, so $\mathcal{F}^A$ μ_σ -accumulates at x . Hence, $\mathcal{F}$ μ_σ -accumulates at x .

(iii) $\Rightarrow$ (i): Suppose that A is not μ -semi-Lindelöf. Then by Proposition 2.8(i), there exists a cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of A by μ -semi-open subsets of X such that for any countable subset Λ_0 of Λ ,

$$[\bigcap \{(X \setminus U_\alpha) : \alpha \in \Lambda_0\}] \cap A \neq \emptyset.$$

For each countable subset Λ_0 of Λ , let

$$F_{\Lambda_0} = [\bigcap \{(X \setminus U_\alpha) : \alpha \in \Lambda_0\}] \cap A.$$

Then $\mathcal{F} = \{F_{\Lambda_0} : \Lambda_0 \text{ is a countable subset of } \Lambda\}$ is a strong filter base on X , each of whose members meets A . Thus by (iii), $\mathcal{F}$ μ_σ -accumulates at some point x of A . Since $\mathcal{U}$ is a cover of A , there exists $\alpha_0 \in \Lambda$ such that

$x \in U_{\alpha_0}$, but $\mathcal{F}$ μ_σ -accumulates at x and U_{α_0} is μ -semi-open, so $U_{\alpha_0} \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. Let $F = (X \setminus U_{\alpha_0}) \cap A$. Then $F \in \mathcal{F}$ and thus, $U_{\alpha_0} \cap (X \setminus U_{\alpha_0}) \cap A \neq \emptyset$, a contradiction. $\square$

Corollary 2.15. *For a μ -space (X, μ) , the following are equivalent:*

- (i) X is μ -semi-Lindelöf;
- (ii) Every maximal strong filter base on X μ_σ -converges to some point of X ;
- (iii) Every strong filter base on X μ_σ -accumulates at some point of X .

3 Subspaces

This section is mainly devoted to discuss some statements in [16] and [33] concerning subspaces.

Definition 3.1. [31] Let A be a nonempty subset of a space (X, μ) . The generalized subspace topology on A is the collection $\{U \cap A : U \in \mu\}$ and will be denoted by μ_A . The generalized subspace A is the GTS (A, μ_A) .

Remark 3.2. [31] Let A be a nonempty subset of a space (X, μ) . Then it is easy to see that

- (i) If (X, μ) is a μ -space, then (A, μ_A) is a μ_A -space;
- (ii) A subset B of A is μ_A -closed if and only if $B = F \cap A$, for some μ -closed set F .

Proposition 3.3. [33] Let (X, τ) be a topological space, and $A \subset B \subset X$, where B is preopen. Then A is semi-open (resp. semi-closed) in B if and only if $A = S \cap B$, for some semi-open (resp. semi-closed) set S .

Proposition 3.4. ([33, Corollary 2.6], sufficiency of [33, Corollary 2.8]) Let A be a preopen subset of a topological space X . If A is semi-compact (resp. semi-Lindelöf) in X , then A is a semi-compact (resp. semi-Lindelöf) subspace.

Proposition 3.5. [16, Theorem 2.7] Let A, B be subsets of a μ -space (X, μ) with $A \subset B$. If A is μ -semi-compact (resp. μ -semi-Lindelöf), then A is μ_B -semi-compact (resp. μ_B -semi-Lindelöf).

In the proof of Proposition 3.5 [16, Theorem 2.7], the author used without proof the following: If (X, μ) is a space, and $A \subset B \subset X$, where A is μ_B -semi-open, then $A = S \cap B$, for some μ -semi-open set S . For a matter of convenience and importance, we will prove this fact, to proceed, we introduce the following two lemmas.

Lemma 3.6. [36, Proposition 3.8] Let (X, μ) be a space, and $A \subset X$. Then A is μ -semi-open if and only if $U \subset A \subset c_\mu(U)$, for some μ -open set U .

Lemma 3.7. [30, Lemma 3.11] Let (X, μ) be a space, and $A \subset B \subset X$. Then $c_{\mu_B}(A) = c_\mu(A) \cap B$.

Proposition 3.8. Let (X, μ) be a space, and $A \subset B \subset X$. Then

- (i) If A is μ_B -semi-open (resp. μ_B -semi-closed), then $A = S \cap B$, for some μ -semi-open (resp. μ -semi-closed) set S ;
- (ii) If A is μ -open (resp. μ -closed), then A is μ_B -open (resp. μ_B -closed);
- (iii) If A is μ -semi-open, then A is μ_B -semi-open.

Proof.

- (i) Let A be μ_B -semi-open. Then it follows from Lemmas 3.6 and 3.7 that there exists a μ -open set U such that

$$U \cap B \subset A \subset c_\mu(U \cap B) \cap B \subset c_\mu(U).$$

Let $S = U \cup A$. Since $A \subset c_\mu(U)$, then clearly,

$$U \subset S \subset c_\mu(U).$$

Thus by Lemma 3.6, S is μ -semi-open. Since $U \cap B \subset A$,

$$S \cap B = (U \cap B) \cup A = A.$$

Now if A be μ_B -semi-closed, then $B \setminus A$ is μ_B -semi-open, and thus, $B \setminus A = S \cap B$, for some μ -semi-open set S . Hence, $A = (X \setminus S) \cap B$, and $X \setminus S$ is μ -semi-closed.

- (ii) This is clear from Definition 3.1 and Remark 3.2(ii) since $A = A \cap B$.
- (iii) Since A is μ -semi-open, it follows from Lemma 3.6 that there exists a μ -open U such that

$$U \subset A \subset c_\mu(U).$$

But $A \subset B$, so

$$U \subset A \subset c_\mu(U) \cap B.$$

Since U is μ -open, and $U \subset B$, so by (ii), U is μ_B -open. Also by Lemma 3.7, $c_\mu(U) \cap B = c_{\mu_B}(U)$. Thus by Lemma 3.6, A is μ_B -semi-open. $\square$

Remark 3.9. From Proposition 3.8(i), we observe that the condition “preopen” is not essential for the necessity of Proposition 3.3.

Corollary 3.10. [16, Corollary 2.8] *Let A be a subset of a μ -space (X, μ) . If A is μ -semi-compact (resp. μ -semi-Lindelöf), then A is μ_A -semi-compact (resp. μ_A -semi-Lindelöf).*

Corollary 3.11. *Let A be a subset of a topological space X . If A is semi-compact (resp. semi-Lindelöf) in X , then A is a semi-compact (resp. semi-Lindelöf) subspace.*

Remark 3.12. From Corollary 3.11, we observe that the condition “preopen” of Proposition 3.4 is not essential.

Proposition 3.13. (necessity of [33, Corollary 2.8]) *Let A be a preopen subset of a topological space X . If A is a semi-compact (resp. semi-Lindelöf) subspace, then A is semi-compact (resp. semi-Lindelöf) in X .*

The following example shows that the condition “preopen” in Proposition 3.13 is essential.

Example 3.14. Consider the space of Example 2.6, that is, the space (X, τ) , where X is an uncountable set, and $\tau = \{X, \emptyset, \{p\}\}$, where $p \in X$. Then $A = X \setminus \{p\}$ is not τ -preopen because $\overline{A} = A$, and thus, $\text{Int } \overline{A} = \text{Int } A = \emptyset$. Therefore, $A \not\subset \text{Int } \overline{A}$. Now, if $x \in A$, then $A_x = \{p, x\}$ is τ -semi-open. Thus, the collection $\mathcal{A} = \{A_x : x \in A\}$ is a cover of A by τ -semi-open sets and $\mathcal{A}$ has no countable subcover of A . Hence, A is not semi-compact (semi-Lindelöf) in X . Now, the subspace topology $\tau_A = \{A, \emptyset\}$, therefore, the only τ_A -semi-open sets are $A, \emptyset$. Hence, A is a semi-compact (semi-Lindelöf) subspace.

(Sufficiency of [16, Theorem 2.9]) Let A, B be subsets of a μ -space (X, μ) with $A \subset B$. If A is μ_B -semi-compact (resp. μ_B -semi-Lindelöf), then A is μ -semi-compact (resp. μ -semi-Lindelöf).

We point out here to an error in the proof of the sufficiency of [16, Theorem 2.9], that is, the use of that if (X, μ) is a space, and $S \subset X$, where S is μ -semi-open, then $S \cap B$ is μ_B -semi-open, where $B \subset X$, actually, this need not be true (even for topological spaces) as the following two easy examples tell.

Example 3.15. Let $X = \{1, 2, 3, 4\}$ and $\mu = \{X, \emptyset, \{1, 2, 3\}, \{3, 4\}\}$. Then μ is a GT on X . If $S = \{1, 3, 4\}$, then S is μ -semi-open. Let $B = \{1, 2\}$. Then $\mu_B = \{B, \emptyset\}$, and $S \cap B = \{1\}$ is not μ_B -semi-open.

Example 3.16. Let $X = \{1, 2, 3\}$ and $\tau = \{X, \emptyset, \{1\}\}$. Then $S = \{1, 2\}$ is τ -semi-open. Let $B = \{2, 3\}$. Then $\tau_B = \{B, \emptyset\}$, and $S \cap B = \{2\}$ is not τ_B -semi-open.

Remark 3.17. We observe from Example 3.16 that the condition “preopen” is essential for the sufficiency of Proposition 3.3 (note that B is not preopen as $B \not\subset \text{Int } \overline{B} = \text{Int } B = \emptyset$).

Concerning the sufficiency of Proposition 3.3, it might be convenient to raise the following question.

Question. Let (X, μ) be a GTS, where S is μ -semi-open and B is μ -preopen. Is $S \cap B$ necessarily μ_B -semi-open?

(**Necessity of [16, Corollary 2.10]**) Let A be a subset of a μ -space (X, μ) . If A is μ_A -semi-compact (resp. μ_A -semi-Lindelöf), then A is μ -semi-compact (resp. μ -semi-Lindelöf).

We observe that since the condition “preopen” in Proposition 3.13 is essential, the sufficiency of [16, Theorem 2.9] and the necessity of [16, Corollary 2.10] are not correct.

The following proposition includes a correction of the sufficiency of [16, Theorem 2.9].

Proposition 3.18. Let B be a nonempty subset of a μ -space (X, μ) and $A \subset B$. Then A is μ -semi-compact (resp. μ -semi-Lindelöf) if and only if A is $(\sigma(\mu))_B$ -compact (resp. $(\sigma(\mu))_B$ -Lindelöf).

Proof. We will see the case of μ -semi-Lindelöf, the other case is similar.

Necessity. Observe first by Remark 1.4(i) that since (X, μ) is a μ -space, $(X, \sigma(\mu))$ is a $\sigma(\mu)$ -space, thus by Remark 3.2(i), $(B, (\sigma(\mu))_B)$ is a $(\sigma(\mu))_B$ -space. Suppose that $\mathcal{A} = \{A_\alpha : \alpha \in \Lambda\}$ is a cover of A by $(\sigma(\mu))_B$ -open sets. Then $A_\alpha = S_\alpha \cap B$, where S_α is $\sigma(\mu)$ -open for each $\alpha \in \Lambda$. Thus, $\mathcal{S} = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of A by μ -semi-open sets, but A is μ -semi-Lindelöf, so there exist $\alpha_1, \alpha_2, \alpha_3, \dots \in \Lambda$ such that $A \subset \bigcup_{i=1}^{\infty} S_{\alpha_i}$, and thus $A \subset \bigcup_{i=1}^{\infty} (S_{\alpha_i} \cap B) = \bigcup_{i=1}^{\infty} A_{\alpha_i}$. Hence, A is $(\sigma(\mu))_B$ -Lindelöf.

Sufficiency. Suppose that $\mathcal{S} = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of A by μ -semi-open sets. Then $\mathcal{A} = \{S_\alpha \cap B : \alpha \in \Lambda\}$ is a $(\sigma(\mu))_B$ -open cover of A , but A is $(\sigma(\mu))_B$ -Lindelöf, so there exist $\alpha_1, \alpha_2, \alpha_3, \dots \in \Lambda$ such that $A \subset \bigcup_{i=1}^{\infty} (S_{\alpha_i} \cap B) \subset \bigcup_{i=1}^{\infty} S_{\alpha_i}$. Hence, A is μ -semi-Lindelöf. $\square$

The following corollary includes a correction of the necessity of [16, Corollary 2.10].

Corollary 3.19. Let A be a nonempty subset of a μ -space (X, μ) . Then A is μ -semi-compact (resp. μ -semi-Lindelöf) if and only if A is $(\sigma(\mu))_A$ -compact (resp. $(\sigma(\mu))_A$ -Lindelöf).

4 ω_μ -semi-open sets

Definition 4.1. Let (X, μ) be a space and A be a subset of X . Then A is called ω_μ -semi-open if whenever $x \in A$, there exists a μ -semi-open set U_x containing x such that $U_x \setminus A$ is countable. A is called ω_μ -semi-closed if $X \setminus A$ is ω_μ -semi-open. The collection of all ω_μ -semi-open subsets of X will be denoted by $\omega_{\sigma(\mu)}$.

Proposition 4.2. Let (X, μ) be a space and A be a subset of X . Then A is ω_μ -semi-open if and only if whenever $x \in A$, there exists a μ -semi-open set U_x containing x and a countable subset C_x of X such that $U_x \setminus C_x \subset A$.

Proof. Necessity. Let A be ω_μ -semi-open and $x \in A$. Then there exists a μ -semi-open set U_x containing x such that $U_x \setminus A$ is countable. Let $C_x = U_x \setminus A$. Then $U_x \setminus C_x \subset A$.

Sufficiency. Let $x \in A$. Then by assumption, there exists a μ -semi-open set U_x containing x and a countable subset C_x of X such that $U_x \setminus C_x \subset A$. Thus, $U_x \setminus A \subset C_x$, and therefore, $U_x \setminus A$ is countable. Hence, A is ω_μ -semi-open. $\square$

Corollary 4.3. Let (X, μ) be a space and A be an ω_μ -semi-closed subset of X . Then $A \subset B \cup C$ for some μ -semi-closed subset B of X and some countable subset C of X .

Proof. Let A be an ω_μ -semi-closed subset of X . Then $X \setminus A$ is ω_μ -semi-open. If $X \setminus A = \emptyset$, choose $B = X$ and $C = \emptyset$. If $X \setminus A \neq \emptyset$, choose $x \in X \setminus A$. Thus by Proposition 4.2, there exists a μ -semi-open set U_x containing x and a countable subset C_x of X such that $U_x \setminus C_x \subset X \setminus A$. Therefore, $A \subset (X \setminus U_x) \cup C_x$. Let $B = X \setminus U_x$ and $C = C_x$. Then B is μ -semi-closed, and $A \subset B \cup C$. $\square$

Proposition 4.4.

- (i) If (X, μ) is a space, then $(X, \omega_{\sigma(\mu)})$ is a space.
- (ii) If (X, μ) is a μ -space, then $(X, \omega_{\sigma(\mu)})$ is an $\omega_{\sigma(\mu)}$ -space.
- (iii) If A is a μ -semi-open subset of a space (X, μ) , then A is ω_μ -semi-open, that is, $\sigma(\mu) \subset \omega_{\sigma(\mu)}$.
- (iv) Let A be a subset of a space (X, μ) . Then A is ω_μ -semi-open if and only if A is $\omega_{\sigma(\mu)}$ -semi-open, that is, $\omega_{\sigma(\mu)} = \omega_{\sigma(\sigma(\mu))}$.
- (v) Let A be a subset of a space (X, μ) . Then A is ω_μ -semi-open if and only if A is $\omega_{\omega_{\sigma(\mu)}}$ -semi-open.
- (vi) Let A be a countable subset of a μ -space (X, μ) . Then A is ω_μ -semi-closed.

Proof.

- (i) Clearly, $\emptyset$ is ω_μ -semi-open, that is, $\emptyset \in \omega_{\sigma(\mu)}$. Now let U_α be ω_μ -semi-open for each $\alpha \in \Lambda$ and let $x \in \bigcup_{\alpha \in \Lambda} U_\alpha$. Then $x \in U_{\alpha_0}$ for some $\alpha_0 \in \Lambda$. Since U_{α_0} is ω_μ -semi-open, there exists a μ -semi-open set V_{α_0} containing x such that $V_{\alpha_0} \setminus U_{\alpha_0}$ is countable. Thus, $V_{\alpha_0} \setminus (\bigcup_{\alpha \in \Lambda} U_\alpha)$ is countable. Hence, $\bigcup_{\alpha \in \Lambda} U_\alpha$ is ω_μ -semi-open, that is, $\bigcup_{\alpha \in \Lambda} U_\alpha \in \omega_{\sigma(\mu)}$. Hence, $(X, \omega_{\sigma(\mu)})$ is a space.
- (ii) As (X, μ) is a μ -space, $X \in \mu$. Thus, X is μ -semi-open. Let $x \in X$. Then $X \setminus X = \emptyset$ is countable. Thus, X is ω_μ -semi-open, that is, $X \in \omega_{\sigma(\mu)}$. Hence, $(X, \omega_{\sigma(\mu)})$ is an $\omega_{\sigma(\mu)}$ -space.
- (iii) Let A be μ -semi-open and $x \in A$. Then $A \setminus A = \emptyset$ is countable. Thus, A is ω_μ -semi-open.
- (iv) This follows from Proposition 1.3(iii).
- (v) Suppose that A is ω_μ -semi-open and let $x \in A$. Then there exists a μ -semi-open set U containing x such that $U \setminus A$ is countable. By (iii), U is ω_μ -semi-open. By (iv), U is $\omega_{\sigma(\mu)}$ -semi-open. Thus, A is $\omega_{\omega_{\sigma(\mu)}}$ -semi-open. Conversely, suppose that A is $\omega_{\omega_{\sigma(\mu)}}$ -semi-open and let $x \in A$. Then there exists an $\omega_{\sigma(\mu)}$ -semi-open set U containing x such that $U \setminus A$ is countable. By (iv), U is ω_μ -semi-open. Thus, there exists a μ -semi-open set V containing x such that $V \setminus U$ is countable. Now $V \setminus A = ((V \setminus U) \setminus A) \cup ((U \cap V) \setminus A)$. As $U \setminus A$ and $V \setminus U$ are both countable, $(V \setminus U) \setminus A$ and $(U \cap V) \setminus A$ are both countable, and thus, $V \setminus A$ is countable. Hence, A is ω_μ -semi-open.
- (vi) Let $x \in X \setminus A$. As (X, μ) is a μ -space, $X \in \mu$, and thus, X is μ -semi-open. Now $X \setminus (X \setminus A) = A$ is countable by assumption. Thus, $X \setminus A$ is ω_μ -semi-open, that is, A is ω_μ -semi-closed. $\square$

The following easy example shows that the converse of Proposition 4.4(iii) need not be correct in general even for topological spaces.

Example 4.5. Let X be an uncountable set and μ be the indiscrete topology on X . If A is a nonempty countable subset of X , then by Proposition 4.4(vi), $X \setminus A$ is ω_μ -semi-open; however, $X \setminus A$ is not μ -semi-open.

5 Applications

Proposition 5.1. Let A be an ω_μ -semi-open subset of a μ -space (X, μ) . If A is μ -semi-Lindelöf, then $A = B \setminus C$ for some μ -semi-open subset B of X and some countable subset C of X .

Proof. Since A is ω_μ -semi-open, it follows that for each $x \in A$ there exists a μ -semi-open set U_x containing x such that $U_x \setminus A$ is countable, but A is μ -semi-Lindelöf, so there exist $x_1, x_2, x_3, \dots \in A$ such that $A \subset \bigcup_{i=1}^\infty U_{x_i}$. Thus, $A = (\bigcup_{i=1}^\infty U_{x_i}) \setminus (\bigcup_{i=1}^\infty (U_{x_i} \setminus A))$. Let $B = \bigcup_{i=1}^\infty U_{x_i}$ and $C = \bigcup_{i=1}^\infty (U_{x_i} \setminus A)$. Then B is μ -semi-open and C is countable. $\square$

Proposition 5.2. *Let A be a subset of a μ -space (X, μ) . Then A is μ -semi-Lindelöf if and only if A is $\omega_{\sigma(\mu)}$ -Lindelöf.*

Proof. Necessity. Suppose that $\mathcal{S} = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of A by $\omega_{\sigma(\mu)}$ -open sets. Then for each $x \in A$, $x \in S_{\alpha(x)}$, for some $\alpha(x) \in \Lambda$. Thus, there exists a μ -semi-open set U_x containing x such that $U_x \setminus S_{\alpha(x)}$ is countable. Now $\mathcal{A} = \{U_x : x \in A\}$ is a cover of A by μ -semi-open sets, but A is μ -semi-Lindelöf, so there exist $x_1, x_2, x_3, \dots \in A$ such that $A \subset \bigcup_{i=1}^\infty (U_{x_i} \cap A) \subset (\bigcup_{i=1}^\infty ((U_{x_i} \setminus S_{\alpha(x_i)}) \cap A)) \cup (\bigcup_{i=1}^\infty S_{\alpha(x_i)})$. Since $(\bigcup_{i=1}^\infty ((U_{x_i} \setminus S_{\alpha(x_i)}) \cap A))$ is a countable subset of A , it is covered by a countable subcollection $\mathcal{B}$ of $\mathcal{S}$. Thus, $\mathcal{S}$ has $\mathcal{B} \cup \{S_{\alpha(x_i)} : i \in \mathbb{N}\}$ as a countable subcover. Hence, A is $\omega_{\sigma(\mu)}$ -Lindelöf.

Sufficiency. Follows from Proposition 4.4(iii). $\square$

Proposition 5.3. *Let A be a μ -semi-Lindelöf subset of a μ -space (X, μ) and B be an ω_μ -semi-closed subset of X . Then $A \cap B$ is μ -semi-Lindelöf. In particular, an ω_μ -semi-closed subset A of a μ -semi-Lindelöf μ -space (X, μ) is μ -semi-Lindelöf.*

Proof. Suppose that $\mathcal{S} = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of $A \cap B$ by μ -semi-open sets. Then $\mathcal{A} = \{S_\alpha : \alpha \in \Lambda\} \cup \{X \setminus B\}$ is a cover of A by ω_μ -semi-open sets, but A is μ -semi-Lindelöf, so it follows from Proposition 5.2 that there exist $\alpha_1, \alpha_2, \alpha_3, \dots \in \Lambda$ such that $A \subset (\bigcup_{i=1}^\infty S_{\alpha_i}) \cup (X \setminus B)$. Thus, $A \cap B \subset \bigcup_{i=1}^\infty (S_{\alpha_i} \cap B) \subset \bigcup_{i=1}^\infty S_{\alpha_i}$. Hence, $A \cap B$ is μ -semi-Lindelöf. $\square$

Corollary 5.4. [16, Theorem 2.11] *Let A be a μ -semi-Lindelöf subset of a μ -space (X, μ) and B be a μ -semi-closed subset of X . Then $A \cap B$ is μ -semi-Lindelöf. In particular, a μ -semi-closed subset A of a μ -semi-Lindelöf μ -space (X, μ) is μ -semi-Lindelöf.*

Corollary 5.5. *For a μ -space (X, μ) , the following are equivalent:*

- (i) X is μ -semi-Lindelöf;
- (ii) Every proper ω_μ -semi-closed subset of X is μ -semi-Lindelöf;
- (iii) Every proper μ -semi-closed subset of X is μ -semi-Lindelöf.

Proof. (i) $\Rightarrow$ (ii): Follows from Proposition 5.3.

(ii) $\Rightarrow$ (iii): Follows from Proposition 4.4(iii).

(iii) $\Rightarrow$ (i): Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a cover of X by μ -semi-open subsets of X . Choose $\alpha_0 \in \Lambda$ such that $U_{\alpha_0} \neq \emptyset$. Then $X \setminus U_{\alpha_0}$ is a proper μ -semi-closed subset of X , thus by (iii), $X \setminus U_{\alpha_0}$ is μ -semi-Lindelöf, so there exist $\alpha_1, \alpha_2, \alpha_3, \dots \in \Lambda$ such that $X \setminus U_{\alpha_0} \subset \bigcup_{i=1}^\infty U_{\alpha_i}$ and thus $X = \bigcup_{i=0}^\infty U_{\alpha_i}$. Hence, X is μ -semi-Lindelöf. $\square$

Definition 5.6. [31] Let (X_α, μ_α) be a GTS for each $\alpha \in \Lambda$, where $\{X_\alpha : \alpha \in \Lambda\}$ is a disjoint family of sets. The collection μ of subsets of $\bigcup X_\alpha$ is defined as follows:

$$\mu = \{U \subset \bigcup X_\alpha : U \cap X_\alpha \in \mu_\alpha, \forall \alpha \in \Lambda\}.$$

Proposition 5.7. [31] *Let (X_α, μ_α) be a GTS for each $\alpha \in \Lambda$, where $\{X_\alpha : \alpha \in \Lambda\}$ is a disjoint family of sets, and let μ be as in Definition 5.6. Then μ is a GT on $\bigcup X_\alpha$. The GTS $(\bigcup X_\alpha, \mu)$ will be called the generalized topological sum of $X_\alpha, \alpha \in \Lambda$ and will be denoted by $\oplus X_\alpha$.*

Corollary 5.8. [31] Let $(\oplus X_\alpha, \mu)$ be the generalized topological sum of (X_α, μ_α) , $\alpha \in \Lambda$. Then a subset A of $\oplus X_\alpha$ is μ -closed if and only if $A \cap X_\alpha$ is μ_α -closed for each α .

Remark 5.9. [31] Let (X_α, μ_α) be a GTS for each $\alpha \in \Lambda$, and let $(\oplus X_\alpha, \mu)$ be the generalized topological sum of (X_α, μ_α) , $\alpha \in \Lambda$. Then it is easy to see the following:

- (i) $\bigcup \mu_\alpha \subset \mu$;
- (ii) $\mu_{X_\alpha} = \mu_\alpha$ for each $\alpha \in \Lambda$.

Remark 5.10. [31] Let (X_α, μ_α) be a μ_α -space for each $\alpha \in \Lambda$, and let $(\oplus X_\alpha, \mu)$ be the generalized topological sum of (X_α, μ_α) , $\alpha \in \Lambda$. Then it is easy to see the following:

- (i) $(\oplus X_\alpha, \mu)$ is a μ -space;
- (ii) X_α is μ -open (μ -closed) for each $\alpha \in \Lambda$.

Remark 5.11. We observe from Remark 5.10(ii) that if (X_α, μ_α) is a μ_α -space for each $\alpha \in \Lambda$, and if $(\oplus X_\alpha, \mu)$ is the generalized topological sum of (X_α, μ_α) , $\alpha \in \Lambda$, then X_α is μ -semi-open (μ -semi-closed) for each $\alpha \in \Lambda$.

Proposition 5.12. [16] Let (X, μ) be a μ -space. Then the countable union of subsets of X , each of which μ -semi-Lindelöf, is μ -semi-Lindelöf.

Proposition 5.13. Let (X_α, μ_α) be a μ_α -space for each $\alpha \in \Lambda$, and let $(\oplus X_\alpha, \mu)$ be the generalized topological sum of (X_α, μ_α) , $\alpha \in \Lambda$. Then $\oplus X_\alpha$ is μ -semi-Lindelöf if and only if X_α is μ -semi-Lindelöf for each $\alpha \in \Lambda$ and Λ is countable.

Proof. Necessity. By Remark 5.11, X_α is μ -semi-closed for each $\alpha \in \Lambda$. Since $\oplus X_\alpha$ is μ -semi-Lindelöf, so by Corollary 5.4, X_α is μ -semi-Lindelöf for each $\alpha \in \Lambda$. Also by Remark 5.11, X_α is μ -semi-open for each $\alpha \in \Lambda$, thus, $\mathcal{A} = \{X_\alpha : \alpha \in \Lambda\}$ is a cover of $\oplus X_\alpha$ by μ -semi-open sets, but $\oplus X_\alpha$ is μ -semi-Lindelöf, so, Λ is countable.

Sufficiency. Follows from Proposition 5.12. $\square$

6 Mapping properties

This section is mainly devoted to study several mapping properties of μ -semi-Lindelöf sets in μ -spaces.

Proposition 6.1. Let $f : (X, \mu) \rightarrow (Y, \kappa)$ be a $(\omega_{\sigma(\mu)}, \sigma(\kappa))$ -continuous function, where (X, μ) is a μ -space and (Y, κ) is a κ -space. If A is μ -semi-Lindelöf, then $f(A)$ is κ -semi-Lindelöf.

Proof. Suppose that $\mathcal{S} = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of $f(A)$ by κ -semi-open sets. Then $\mathcal{A} = \{f^{-1}(S_\alpha) : \alpha \in \Lambda\}$ is a cover of A , but f is $(\omega_{\sigma(\mu)}, \sigma(\kappa))$ -continuous, so $f^{-1}(S_\alpha)$ is ω_μ -semi-open for each $\alpha \in \Lambda$. Since A is μ -semi-Lindelöf, it follows from Proposition 5.2 that there exist $\alpha_1, \alpha_2, \alpha_3, \dots \in \Lambda$ such that $A \subset \bigcup_{i=1}^{\infty} f^{-1}(S_{\alpha_i})$. Thus, $f(A) \subset \bigcup_{i=1}^{\infty} f(f^{-1}(S_{\alpha_i})) \subset \bigcup_{i=1}^{\infty} S_{\alpha_i}$. Hence, $f(A)$ is κ -semi-Lindelöf. $\square$

Corollary 6.2. Let $f : (X, \mu) \rightarrow (Y, \kappa)$ be a $(\sigma(\mu), \sigma(\kappa))$ -continuous function, where (X, μ) is a μ -space and (Y, κ) is a κ -space. If A is μ -semi-Lindelöf, then $f(A)$ is κ -semi-Lindelöf.

Corollary 6.3. Let $f : (X, \mu) \rightarrow (Y, \kappa)$ be a $(\sigma(\mu), \sigma(\kappa))$ -continuous surjection, where (X, μ) is a μ -space and (Y, κ) is a κ -space. If X is μ -semi-Lindelöf, then Y is κ -semi-Lindelöf.

Proposition 6.4. Let $f : (X, \mu) \rightarrow (Y, \kappa)$ be a $(\sigma(\mu), \omega_{\sigma(\kappa)})$ -closed function, where (X, μ) is a μ -space and (Y, κ) is a κ -space. If for each $y \in Y$, $f^{-1}(y)$ is μ -semi-Lindelöf, then $f^{-1}(A)$ is μ -semi-Lindelöf whenever A is κ -semi-Lindelöf.

Proof. Suppose that $S = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of $f^{-1}(A)$ by μ -semi-open sets. Then it follows by assumption that for each $y \in A$, there exists a countable subcollection S^y of S such that $f^{-1}(y) \subset \bigcup S^y$. Let $V_y = \bigcup S^y$. Then V_y is μ -semi-open. Let $H_y = Y \setminus f(X \setminus V_y)$. Then H_y is ω_κ -semi-open as f is $(\sigma(\mu), \omega_{\sigma(\kappa)})$ -closed, also $y \in H_y$ for each $y \in A$ as $f^{-1}(y) \subset V_y$. Thus, $\mathcal{H} = \{H_y : y \in A\}$ is a cover of A by ω_κ -semi-open sets, but A is κ -semi-Lindelöf, so it follows from Proposition 5.2 that there exist $y_1, y_2, y_3, \dots \in A$ such that $A \subset \bigcup_{i=1}^{\infty} H_{y_i}$. Thus, $f^{-1}(A) \subset \bigcup_{i=1}^{\infty} f^{-1}(H_{y_i}) \subset \bigcup_{i=1}^{\infty} V_{y_i}$. Since S^i is a countable subcollection of S for each $i \in \mathbb{N}$, it follows that $\bigcup_{i=1}^{\infty} S^i$ is a countable subcollection of S . Hence, $f^{-1}(A)$ is μ -semi-Lindelöf. $\square$

Corollary 6.5. [16, Theorem 2.18] *Let $f : (X, \mu) \rightarrow (Y, \kappa)$ be a $(\sigma(\mu), \sigma(\kappa))$ -closed function, where (X, μ) is a μ -space and (Y, κ) is a κ -space. If for each $y \in Y$, $f^{-1}(y)$ is μ -semi-Lindelöf, then $f^{-1}(A)$ is μ -semi-Lindelöf whenever A is κ -semi-Lindelöf.*

Corollary 6.6. *Let $f : (X, \mu) \rightarrow (Y, \kappa)$ be a $(\sigma(\mu), \sigma(\kappa))$ -closed function, where (X, μ) is a μ -space and (Y, κ) is a κ -space. If for each $y \in Y$, $f^{-1}(y)$ is μ -semi-Lindelöf, then (X, μ) is μ -semi-Lindelöf whenever (Y, κ) is κ -semi-Lindelöf.*

Proposition 6.7. [31] *Let (X, μ) and (Y, κ) be GTSS, and let $\mathcal{U} = \{U \times V : U \in \mu, V \in \kappa\}$. Then $\mathcal{U}$ generates a GT λ on $X \times Y$, called the generalized product topology on $X \times Y$, that is,*

$$\lambda = \{\text{all possible unions of members of } \mathcal{U}\}.$$

λ will sometimes be denoted by $\mu \times \kappa$.

Remark 6.8. [31] Let (X, μ) and (Y, κ) be GTSS, σ be the generalized product topology on $X \times Y$, $A \subset X$, $B \subset Y$, and $K \subset X \times Y$. Then it is easy to see the following:

- (i) K is λ -open if and only if for each $(x, y) \in K$, there exist $U_x \in \mu$ and $V_y \in \kappa$ such that $(x, y) \in U_x \times V_y \subset K$;
- (ii) $c_\lambda(A \times B) = c_\mu(A) \times c_\kappa(B)$;
- (iii) $i_\lambda(A \times B) = i_\mu(A) \times i_\kappa(B)$;
- (iv) $\lambda_{A \times B} = \mu_A \times \kappa_B$.

Remark 6.9. [31] Let (X, μ) be a μ -space, (Y, κ) be a κ -space, and λ be the generalized product topology on $X \times Y$. Then it is clear that $(X \times Y, \lambda)$ is a λ -space.

Lemma 6.10. *Let (X, μ) be a μ -space, (Y, κ) be a κ -space, and λ be the generalized product topology on $X \times Y$. Then the projection $P_X : (X \times Y, \lambda) \rightarrow (X, \mu)$ (resp. $P_Y : (X \times Y, \lambda) \rightarrow (Y, \kappa)$) is $(\sigma(\lambda), \sigma(\mu))$ -continuous (resp. $(\sigma(\lambda), \sigma(\kappa))$ -continuous).*

Proof. We will show that the projection $P_X : (X \times Y, \lambda) \rightarrow (X, \mu)$ is $(\sigma(\lambda), \sigma(\mu))$ -continuous, the other case is similar. Let A be a μ -semi-open subset of X . Then $(P_X)^{-1}(A) = A \times Y$. We want to show that $A \times Y$ is λ -semi-open. Now by Remark 6.8, $c_\lambda(i_\lambda(A \times Y)) = c_\mu(i_\mu(A)) \times Y \supset A \times Y$. Thus, $A \times Y$ is λ -semi-open. $\square$

Corollary 6.11. *Let (X, μ) be a μ -space, (Y, κ) be a κ -space, and λ be the generalized product topology on $X \times Y$. If $X \times Y$ is λ -semi-Lindelöf, then (X, μ) is μ -semi-Lindelöf and (Y, κ) is κ -semi-Lindelöf.*

Proof. Observe first by Remark 6.9 that since (X, μ) is a μ -space and (Y, κ) is a κ -space, $(X \times Y, \lambda)$ is a λ -space. The result follows from Corollary 6.3 and Lemma 6.10. $\square$

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