Research Article

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A new iteration method for the solution of third-order BVP via Green's function

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Abstract: In this study, a new iterative method for third-order boundary value problems based on embedding Green's function is introduced. The existence and uniqueness theorems are established, and necessary conditions are derived for convergence. The accuracy, efficiency and applicability of the results are demonstrated by comparing with the exact results and existing methods. The results of this paper extend and generalize the corresponding results in the literature.

Keywords: boundary value problem, fixed point iteration method, Green's function, integral operator, rate of convergence

MSC 2020: 47H10, 47J05

1 Introduction

The iterative methods are used to solve initial and boundary value problems (BVPs) in image and restoration problems, variational inequality problems and etc. Successive approximation method was introduced by Liouville in 1837. Then, Picard [1] developed his classical and well-known proof of the existence and uniqueness of the solution of initial value problems for ordinary differential equations in 1890. The iterative methods of Picard [1] and Mann [2] are generated by an arbitrary point $p_0 \in Y$ and defined as follows:

$$\begin{aligned} p_{(n+1)} &= Tp_n = T^n p_0, \quad n \in \mathbb{Z}_+, \\ p_{(n+1)} &= (1 - r_n)p_n + r_n Tp_n, \quad n \in \mathbb{Z}_+. \end{aligned}$$

Afterward, several notable researchers introduced many fixed point iterative methods to approximate the solution of a given problem for better approximation with a minimum error (see e.g. [3–5]). In particular, third-order BVPs have received much attention in many scientific and engineering applications and many branches of pure and applied mathematics in the last decade. Thus, finding the solution of nonlinear initial and BVPs, particularly, second- or third-order differential equations, has become a very interesting problem. [1–24] and references therein are some of these studies. In recent years, Abushammala et al. [25] and Khuri and Sayfy [4] designed the methods based on Green's function and fixed-point iterative methods, e.g. Picard-Green's and Krasnoselski-Mann's iterative methods, to approximate the solution of nonlinear initial and BVPs. Recently, Khuri and Louhichi [26] have developed a novel Ishikawa-Green's fixed point method to approximate the solution of second-order BVPs. The authors have also shown that the proposed method has a better approximation with a minimum error. Ali et al. [3] introduced Khan-Green's fixed point

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iterative method for the approximate solution of second-order BVPs and showed that the proposed method has a better approximation with a minimum error than the Ishikawa-Green's method.

The strategy of this paper is motivated by the work of Khan-Green's iterative method. Khan's iterative method is defined in the subsequent form:

$$\begin{aligned} q_n &= (1 - r_n)p_n + r_n T(p_n), \\ p_{(n+1)} &= T(q_n), \quad n \in \mathbb{Z}_+, \end{aligned} \quad (1)$$

where $p_0 \in Y$, $T : Y \rightarrow Y$ is a mapping on a non-empty and convex subset Y of a Banach space X and $\{r_n\}$ is a parametric sequence in $(0, 1)$. This iterative method was established by combining Picard and Mann's methods. The existence of one parametric sequence makes the method easier than Ishikawa's method and besides, the convergence rate is better than the mentioned methods for the third-order BVPs. More specifically, the following third-order BVP:

$$u'''(t) = f(t, u(t), u'(t), u''(t)) \quad (2)$$

is subject to the boundary conditions:

$$u(0) = 0, \quad u'(0) = 0, \quad u'(1) = 0, \quad (3)$$

where $t \in (0, 1)$ is considered. Here,

- (1) $f(t, x, y, z)$ is a continuous function in $(0, 1)$;
- (2) $f(t, x, y, z)$ and $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are bounded;
- (3) $f(t, x, y, z) > 0$ on $[0, 1] \times \mathbb{R}^3$.

In this paper, Khan-Green's fixed point iterative method is generalized and extended for the approximate solution of third-order BVPs. The existence and uniqueness theorems for generalized method are established, and necessary conditions are derived for convergence. The new method is implemented on several numerical examples including linear and nonlinear third-order BVPs. Effectiveness is established with better approximation with minimum error when compared to exact solutions and Picard-Green's solutions.

2 Green's function and methodology

Consider the following third-order BVP,

$$L[u] = u''' = f(t, u(t), u'(t), u''(t)) \quad (4)$$

with the boundary conditions

$$\begin{aligned} B_a[u] &= \alpha_1 u(a) + \alpha_2 u'(a) + \alpha_3 u''(a) = \alpha, \\ B_b[u] &= \beta_1 u(b) + \beta_2 u'(b) + \beta_3 u''(b) = \beta, \\ B_c[u] &= \gamma_1 u(c) + \gamma_2 u'(c) + \gamma_3 u''(c) = \gamma, \end{aligned} \quad (5)$$

where $L[u]$ and $f(t, u(t), u'(t), u''(t))$ are linear or non-linear terms, $t \in [a, b]$, α, β, γ are constants and either $c = a$ or $c = b$. The existence and uniqueness results for the solution of the problem equations (4) and (5) are given in [19,26,27].

Green's function $G(t, s)$ corresponding to linear term $L[u] = u''' = f(t)$ is then given by

$$G(t, s) = \begin{cases} a_1 u_1 + b_1 u_2 + c_1 u_3, & a < t < s, \\ a_2 u_1 + b_2 u_2 + c_2 u_3, & s < t < b, \end{cases} \quad (6)$$

where $t \neq s$, u_1 , u_2 and u_3 are linearly independent solutions of $L[u]$ and a_i , b_i and c_i , ($i = 1, 2$) are constants. To find the constants and the final version of Green's function of third-order BVP, the following four properties should be followed.

1. G satisfies the homogeneous boundary conditions:

$$B_a[G(t, s)] = B_b[G(t, s)] = B_c[G(t, s)] = 0; \quad (7)$$

2. G is continuous at $t = s$:

$$a_1 u_1(s) + b_1 u_2(s) + c_1 u_3(s) = a_2 u_1(s) + b_2 u_2(s) + c_2 u_3(s); \quad (8)$$

3. G' is continuous at $t = s$:

$$a_1 u_1'(s) + b_1 u_2'(s) + c_1 u_3'(s) = a_2 u_1'(s) + b_2 u_2'(s) + c_2 u_3'(s); \quad (9)$$

4. G'' has jumping discontinuous at $t = s$:

$$a_1 u_1''(s) + b_1 u_2''(s) + c_1 u_3''(s) + 1/p(s) = a_2 u_1''(s) + b_2 u_2''(s) + c_2 u_3''(s). \quad (10)$$

As a consequence of these calculations, Green's function for the problems equations (4) and (5) can be written in the following form:

$$u_p = \int_a^b G(t, s) f(s, u_p, u_p', u_p'') ds, \quad (11)$$

where u_p is the particular solution of equation (4).

3 Khan-Green's fixed point iterative method

This method is based on a non-linear differential function

$$L(u) + N(u) = f(t, u, u', u''), \quad (12)$$

where $L(u)$ and $N(u)$ are linear and nonlinear operators and $f(t, u, u', u'')$ is a linear or nonlinear function. Consider the following integral operator:

$$M(u_p) = \int_a^b G(t, s) L(u_p) ds, \quad (13)$$

where u_p is the particular solution of equation (12) and G is Green's function of a linear operator $L(u_p)$. For easiness, we set $u_p = u$. From equations (11) and (12), the following operator can be obtained.

$$\begin{aligned} M(u) &= \int_a^b G(t, s) (L(u) + N(u) - f(s, u, u', u'') - N(u) + f(s, u, u', u'')) ds \\ &= \int_a^b G(t, s) (L(u) + N(u) - f(s, u, u', u'')) ds + \int_a^b G(t, s) (f(s, u, u', u'') - N(u)) ds \\ &= u + \int_a^b G(t, s) (L(u) + N(u) - f(s, u, u', u'')) ds. \end{aligned} \quad (14)$$

By using the above operator and Khan's fixed point iterative method defined in equation (1),

$$\begin{aligned} q_n &= (1 - r_n) p_n + r_n M(p_n), \\ p_{n+1} &= M(q_n), \quad n \in \mathbb{Z}_+ \end{aligned} \quad (15)$$

are obtained. Afterwards, using the results in equations (14) and (15), the reduced form:

$$\begin{aligned} q_n &= p_n + r_n \int_a^b G(t, s)(L(p_n) + N(p_n) - f(s, p_n, p'_n, p''_n))ds, \\ p_{n+1} &= q_n + \int_a^b G(t, s)(L(q_n) + N(q_n) - f(s, q_n, q'_n, q''_n))ds \end{aligned} \quad (16)$$

is obtained. Equation (16) is the general form of Khan's iterative method. Moreover, as the parametric sequence $\{r_n\}$ is within the interval $(0, 1)$, this function is independent of Picard-Green's, Mann-Green's and Ishikawa-Green's iterative methods. The initial function p_0 is chosen to be the exact solution of the homogeneous equation $L[u] = 0$ under given boundary conditions.

4 Convergence analysis and rate of convergence

In this section, the convergence analysis and convergence rates will be introduced. Moreover, the existence of better and faster convergence rate than Picard Green's, Krasnoselskii-Mann's and Ishikawa-Green's will be proved. The proof of convergence will be based on a nonlinear differential equation with certain boundary conditions. For other sets of boundary conditions, the proof follows in an analogous way.

Consider, the third-order BVP

$$u'''(t) = f(t, u(t), u'(t), u''(t)) \quad (17)$$

complimented with the boundary conditions:

$$u(0) = u'(0) = u(1) = 0. \quad (18)$$

Green's function $G(t, s)$ of this BVP is

$$G(t, s) = \begin{cases} \left(\frac{-s^2}{2} + s - \frac{1}{2}\right)t^2, & 0 < t < s, \\ \frac{s^2}{2} - st + \left(\frac{-s^2}{2} + s\right)t^2, & s < t < 1. \end{cases} \quad (19)$$

The adjoint of the above function is

$$G^*(t, s) = - \begin{cases} \frac{s^2}{2} - st + \left(\frac{-s^2}{2} + s\right)t^2, & 0 < s < t, \\ \left(\frac{-s^2}{2} + s - \frac{1}{2}\right)t^2, & t < s < 1. \end{cases} \quad (20)$$

By applying Khan-Green's iterative method,

$$\begin{aligned} q_n &= p_n + r_n \int_0^1 G^*(t, s)(L(p_n) + N(p_n) - f(s, p_n, p'_n, p''_n))ds, \\ p_{n+1} &= q_n + \int_0^1 G^*(t, s)(L(q_n) + N(q_n) - f(s, q_n, q'_n, q''_n))ds, \quad n \in \mathbb{Z}_+, \end{aligned} \quad (21)$$

and more precisely,

$$\begin{aligned}
q_n &= p_n - r_n \int_0^t \left(\frac{s^2}{2} - st + \left(\frac{-s^2}{2} + s \right) t^2 \right) (p_n'''(s) - f(s, p_n(s), p_n'(s), p_n''(s))) ds \\
&\quad - r_n \int_t^1 \left(\left(\frac{-s^2}{2} + s - \frac{1}{2} \right) t^2 \right) (p_n'''(s) - f(s, p_n(s), p_n'(s), p_n''(s))) ds, \\
p_{n+1} &= q_n - \int_0^t \left(\frac{s^2}{2} - st + \left(\frac{-s^2}{2} + s \right) t^2 \right) (q_n'''(s) - f(s, q_n(s), q_n'(s), q_n''(s))) ds \\
&\quad - \int_t^1 \left(\left(\frac{-s^2}{2} + s - \frac{1}{2} \right) t^2 \right) (q_n'''(s) - f(s, q_n(s), q_n'(s), q_n''(s))) ds
\end{aligned} \tag{22}$$

are obtained. To show the rate of convergence for equations (17) and (18), the following integral operator should be considered.

$$T(u) = u + \int_0^1 G^*(t, s)(u'''(s) - f(s, u(s), u'(s), u''(s))) ds. \tag{23}$$

The integral operator $T(u)$ defined in equation (23) is a Banach's contraction with respect to the supnorm under the following hypothesis on the function f . Let

$$\delta = \frac{1}{20\sqrt{3}} \sup_{([0,1] \times \mathbb{R}^3)} \left| \frac{\partial f}{\partial u} \right| < 1. \tag{24}$$

Consider

$$\begin{aligned}
\|T(u) - T(z)\| &= \left\| \int_0^1 G^*(t, s)f(s, z, z', z'') ds - \int_0^1 G^*(t, s)f(s, u, u', u'') ds \right\| \\
&\leq \left(\int_0^1 |G^*(t, s)|^2 ds \right)^{\left(\frac{1}{2}\right)} \left(\int_0^1 |f(s, z, z', z'') - f(s, u, u', u'')|^2 ds \right)^{\left(\frac{1}{2}\right)} \\
&= \frac{\sqrt{-t^7 + 5t^6 - 7t^5 + 3t^4}}{2\sqrt{15}} \left(\int_0^1 |f(s, z, z', z'') - f(s, u, u', u'')|^2 ds \right)^{\left(\frac{1}{2}\right)} \\
&\leq \frac{1}{20\sqrt{3}} \left(\int_0^1 |f(s, z, z', z'') - f(s, u, u', u'')|^2 ds \right)^{\left(\frac{1}{2}\right)}.
\end{aligned} \tag{25}$$

By applying the mean value theorem

$$\|T(u) - T(z)\| \leq \frac{1}{20\sqrt{3}} \sup_{([0,1] \times \mathbb{R}^3)} \left| \frac{\partial f}{\partial u} \right| \sup_{[0,1]} |z(t) - u(t)| \leq \delta \sup_{[0,1]} |z(t) - u(t)| = \delta \|u - z\| \tag{26}$$

is obtained. Thus, T is a contraction.

Now, the convergence of the new method will be introduced.

4.1 Theorem

Assume that the condition equation (24) holds. Then the sequence $\{p_n\}$ defined by Khan-Green's method converges strongly to the solution of the problem equations (17) and (18). Furthermore, if Picard-Green's, Mann-Green's, Ishikawa-Green's and Khan-Green's iterative methods converge to the same point, then Khan-Green's method converges faster than defined iterative methods.

4.2 Proof

Let x^* be the solution of the problem equations (17) and (18), then $T(x^*) = x^*$.

Let $p_0 \rightarrow x^*$ as $n \rightarrow \infty$. Then,

$$\begin{aligned}\|p_{n+1} - x^*\| &= \|T(q_n) - x^*\| \leq \delta \|q_n - x^*\| \\ &\leq \delta \|(1 - r_n)p_n + r_n T(p_n) - x^*\| \\ &\leq \delta((1 - r_n)\|p_n - x^*\| + r_n\|T(p_n) - x^*\|) \\ &\leq \delta((1 - r_n)\|p_n - x^*\| + \delta r_n\|p_n - x^*\|) \\ &= \delta(1 - r_n + \delta r_n)\|p_n - x^*\| \\ &= \delta(1 - (1 - \delta)r_n)\|p_n - x^*\|.\end{aligned}\tag{27}$$

By using the fact $(1 - (1 - \delta)r_n) < 1$ where $0 < \delta < 1$ and $r_n \in (0, 1)$, the following inequality is obtained.

$$\|p_{n+1} - x^*\| \leq \delta \|p_n - x^*\|.\tag{28}$$

From equation (28), the following expression is true

$$\|p_{n+1} - x^*\| \leq \delta^{n+1} \|p_0 - x^*\|.\tag{29}$$

Since $0 < \delta < 1$, it concludes that $\{p_n\}$ converges strongly to x^* . The rest of the proof can be completed from the proof of Proposition 1 in [17].

5 Numerical examples

In this section, numerical examples of both linear and non-linear differential equations solved by Khan-Green's fixed point iterative method are shown as proof. In addition, the examples were also computed by Picard-Green's method to show comparisons of the outcomes for both methods to reveal the high accuracy of Khan-Green's method. Computations are carried out by MATLAB.

Example 1: Consider the nonlinear BVP

$$u'''(t) + u(t)u''(t) - (u'(t))^2 + 1 = 0\tag{30}$$

with boundary conditions

$$u(0) = u(1) = u'(0) = 0.\tag{31}$$

First, it is worth to mention that the exact solution for the problem equations (30) and (31) is unknown. Second, Green's function of BVP is defined for equations (30) and (31).

$$G(t, s) = \begin{cases} \left(\frac{-s^2}{2} + s - \frac{1}{2}\right)t^2, & 0 < t < s, \\ \left(\frac{s^2}{2} - st + \left(\frac{-s^2}{2} + s\right)t^2\right), & s < t < 1. \end{cases}$$

Therefore, Khan-Green's iteration method can be presented as follows:

$$\begin{aligned}q_n &= p_n - r_n \int_0^t \left(\frac{s^2}{2} - st + \left(\frac{-s^2}{2} + s\right)t^2\right) (p_n'''(s) + p_n(s)p_n''(s) - (p_n')^2(s) + 1) ds \\ &\quad - r_n \int_t^1 \left(\left(\frac{-s^2}{2} + s - \frac{1}{2}\right)t^2\right) (p_n'''(s) + p_n(s)p_n''(s) - (p_n')^2(s) + 1) ds,\end{aligned}\tag{32}$$

$$p_{n+1} = q_n - \int_0^t \left(\frac{s^2}{2} - st + \left(\frac{-s^2}{2} + s \right) t^2 \right) (q_n'''(s) + q_n(s)q_n''(s) - (q_n')^2(s) + 1) ds \\ - \int_t^1 \left(\left(\frac{-s^2}{2} + s - \frac{1}{2} \right) t^2 \right) (q_n'''(s) + q_n(s)q_n''(s) - (q_n')^2(s) + 1) ds,$$

where the parametric sequence is chosen to be $r_n = 0.99$. The starting point $p_0 = 0$ is the homogeneous solution of $L[u] = u''' = 0$ with the corresponding boundary conditions. The absolute error will be found by the formula

$$\text{Error}(n) = |u_{n+1} - u_n|. \quad (33)$$

The maximum errors of the problem equations (30) and (31) are given in Table 1, whereas Table 2 shows the numerical solutions of Example 1 and its absolute errors obtained by applying Khan-Green's method and comparing to the results found by Picard-Green's method, the relative absolute errors $\left| \frac{u_3 - u_4}{u_4} \right|$ computed at various values of t and are presented together with the relative absolute errors done by Picard-Green's method in Table 3. Considering the numerical results obtained in both tables, both the relative error and the absolute error converge to zero faster in all iteration steps than the known methods in our method for third-order BVPs. Thus, we get a better approximation.

Table 1: Maximum errors of Example 1

No. of iterations	1	2	3
Max error (n)	2.44×10^{-2}	1.36×10^{-6}	2.61×10^{-11}

Table 2: Absolute errors (n) of Example 1

t	Numerical solution	Khan-Green's error (3)	Picard-Green's error (3)
0.0	0.0	0.0	0.0
0.1	0.00149606946549956	8.85422×10^{-13}	2.95707×10^{-8}
0.2	0.00531781872516839	3.50394×10^{-12}	1.1718×10^{-7}
0.3	0.01046620292094920	7.66851×10^{-12}	2.57374×10^{-7}
0.4	0.01594328097748140	1.29396×10^{-11}	4.37316×10^{-7}
0.5	0.02075232464561610	1.85634×10^{-11}	6.34702×10^{-7}
0.6	0.02389786114270560	2.3442×10^{-11}	8.15945×10^{-7}
0.7	0.02438584789719030	2.61×10^{-11}	9.33111×10^{-7}
0.8	0.02122417680320920	2.46645×10^{-11}	9.17006×10^{-7}
0.9	0.01342370402248610	1.68034×10^{-11}	6.62401×10^{-7}
1.0	0.0	0.0	0.0

Table 3: Relative absolute errors of Example 1

t	Khan-Green's	Picard-Green's
0.1	5.91832×10^{-10}	1.97656×10^{-5}
0.3	7.32692×10^{-10}	2.4591×10^{-5}
0.5	8.94521×10^{-10}	3.05846×10^{-5}
0.7	1.07051×10^{-9}	3.82644×10^{-5}
0.9	1.25177×10^{-9}	4.93456×10^{-5}

Figure 1: The relative absolute errors of Example 1.

Figure 1 shows the relative errors of the problem equations (30) and (31). With reference to the graph, it can be said that when applying Khan-Green's fixed point iteration method, the relative errors approach zero faster.

Example 2: Consider the linear BVP

$$u'''(t) = -tu''(t) - 2t^2 + t - 2 \quad (34)$$

with the boundary conditions

$$u(0) = u'(0) = u'(1) = 0. \quad (35)$$

The exact solution of the problem equations (23) and (24) is given as $u(t) = \frac{t^2}{2} - \frac{t^3}{3}$. Green's function of the given problem is

$$G(t, s) = \begin{cases} \left(\frac{(s-1)}{2}\right)t^2, & 0 < t < s, \\ \frac{(s(1-t)^2)}{2} + \frac{(s^2-s)}{2}, & s < t < 1 \end{cases}$$

and applying Khan-Green's fixed point iteration method

$$\begin{aligned} q_n &= p_n - r_n \int_0^t \left(\frac{(s(1-t)^2)}{2} + \frac{(s^2-s)}{2} \right) (p_n'''(s) + sp_n''(s) + 2s^2 - s + 2) ds \\ &\quad - r_n \int_t^1 \left(\left(\frac{(s-1)}{2} \right) t^2 \right) (p_n'''(s) + sp_n''(s) + 2s^2 - s + 2) ds, \\ p_{n+1} &= q_n - \int_0^t \left(\frac{(s(1-t)^2)}{2} + \frac{(s^2-s)}{2} \right) (q_n'''(s) + sq_n''(s) + 2s^2 - s + 2) ds \\ &\quad - \int_t^1 \left(\left(\frac{(s-1)}{2} \right) t^2 \right) (q_n'''(s) + sq_n''(s) + 2s^2 - s + 2) ds, \end{aligned} \quad (36)$$

where the parametric sequence is chosen to be $r_n = 0.99$. The starting point $p_0 = 0$ is the homogeneous solution of $L[u] = u''' = 0$ with the corresponding boundary conditions. For various number of iterations,

the maximum absolute errors are reported in Table 4. According to Table 4, it is obvious that with an increase in the number of iterations, the high accuracy of the numerical values will be approached. The formula to estimate the maximum absolute errors is given as:

$$|u_n(t) - u(t)|. \tag{37}$$

Simultaneously, Table 5 shows the numerical results of the problem equations (34) and (35) at 16th iteration, namely the 16th iteration, and the maximum errors of Khan-Green’s iteration method in comparison with Picard Green’s Method (PGEM), respectively. At the same time, from Figure 2 visualizing the 16th iteration for both Khan-Green’s and PGEM, it can be seen that the errors obtained via Khan-Green’s iteration method are much closer to zero than Picard-Green’s.

Table 4: Maximum absolute errors of Example 2

No. of iteration	Max error (<i>n</i>)
5	2.33268×10^{-12}
10	1.90790×10^{-23}
15	1.60579×10^{-33}
20	3.08010×10^{-44}
25	4.99665×10^{-55}
30	2.77113×10^{-65}

Table 5: Numerical results of Example 2 and comparison of Khan-Green’s and Picard-Green’s

<i>t</i>	Numerical solution	Khan-Green’s error (16)	Picard-Green’s error (16)
0.0	0.0	0.0	0.0
0.2	0.01733333333333330	8.25017×10^{-37}	6.47066×10^{-20}
0.4	0.05866666666666660	3.32753×10^{-36}	2.27313×10^{-19}
0.6	0.10800000000000000	7.03846×10^{-36}	3.50808×10^{-19}
0.8	0.14933333333333300	9.67542×10^{-36}	2.02491×10^{-19}
1.0	0.16666666666666600	9.46172×10^{-36}	3.9911×10^{-20}

Figure 2: Comparison of errors for Example 2.

6 Conclusion

This study was motivated by Khan-Green's iterative method for the second-order BVP. The new results were generalized and new theorems proved for third-order BVP. It was shown numerically that the values approach fixed point faster than existing methods. It was also revealed that the proposed method has a better approximation with a minimum error. On the other hand, many results have been obtained for classical non-negative solutions of nonlinear three-dimensional wave equations for initial value problems [28]. Our method offers a novel approach that can be developed to these results.

Conflict of interest: Authors state no conflict of interest.

References

- [1] E. Picard, *Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives*, Journal de Mathématiques Pures et Appliquées **6** (1890), 145–210.
- [2] W. R. Mann, *Mean value methods in iteration*, Proc. Am. Math. Soc. **4** (1953), 506–510, DOI: <https://doi.org/10.1090/S0002-9939-1953-0054846-3>.
- [3] F. Ali, J. Ali, and I. Uddin, *A novel approach for the solution of BVPs via Green's function and fixed point iterative method*, J. Korean Math. Soc. **66** (2021), 167–181, DOI: <https://doi.org/10.1007/s12190-020-01431-7>.
- [4] S. A. Khuri and A. Sayfy, *Variational iteration method: Green's functions and fixed point iterations perspective*, Appl. Math. Lett. **32** (2014), 24–34, DOI: <https://doi.org/10.1016/j.aml.2014.01.006>.
- [5] M. I. Mohamed and M. A. Ragusa, *Solvability of Langevin equations with two Hadamard fractional derivatives via Mittag-Leffler functions*, Appl. Anal. **99** (2021), 1–15, DOI: <https://doi.org/10.1080/00036811.2020.1839645>.
- [6] D. R. Anderson, *Green's function for a third-order generalized right focal problem*, J. Math. Anal. Appl. **288** (2003), no. 1, 1–14, DOI: [https://doi.org/10.1016/S0022-247X\(03\)00132-X](https://doi.org/10.1016/S0022-247X(03)00132-X).
- [7] D. R. Anderson, *Multiple positive solutions for a three-point boundary value problem*, Math. Comput. Model. **27** (1998), no. 6, 49–57, DOI: [https://doi.org/10.1016/S0895-7177\(98\)00028-4](https://doi.org/10.1016/S0895-7177(98)00028-4).
- [8] D. R. Anderson and J. M. Davis, *Multiple solutions and eigenvalues for third-order right focal boundary value problems*, J. Math. Anal. Appl. **267** (2002), no. 1, 135–157, DOI: <https://doi.org/10.1006/jmaa.2001.7756>.
- [9] F. A. Abd El-Salam, A. A. El-Sabbagh, and Z. A. Zaki, *The numerical solution of linear third-order boundary value problems using nonpolynomial spline technique*, J. Amer. Sci. **6** (2010), no. 12, 303–309.
- [10] A. Benramdane, N. Mezouar, M. S. Alqawba, S. M. Boulaaras, and B. B. Cherif, *Blow-up for a stochastic viscoelastic Lamé equation with logarithmic nonlinearity*, J. Funct. Spaces **2021** (2021), 9943969, DOI: <https://doi.org/10.1155/2021/9943969>.
- [11] A. Boucherif and N. Al-Malki, *Nonlinear three-point third-order boundary value problems*, Appl. Math. Comput. **190** (2007), no. 2, 1168–1177, DOI: <https://doi.org/10.1016/j.amc.2007.02.039>.
- [12] Z. Bai and X. Fei, *Existence of triple positive solutions for a third-order generalized right focal problem*, Math. Inequal. Appl. **9** (2006), 437–444, DOI: <https://doi.org/10.7153/mia-09-42>.
- [13] H. Chen, *Positive solutions for the nonhomogeneous three-point boundary value problem of second-order differential equations*, Math. Comput. Model. **45** (2007), no. 7–8, 844–852, DOI: <https://doi.org/10.1016/j.mcm.2006.08.004>.
- [14] E. Dulacska, *Soil Settlement Effects on Buildings, Developments in Geotechnical Engineering*, vol. 69, Elsevier, Amsterdam, 1992.
- [15] J. R. Graef and B. Yang, *Multiple positive solutions to a three point third-order boundary value problem*, Discrete Contin. Dyn. Syst. **2005** (2005), 337–344.
- [16] M. Do Rosário Grossinho and F. M. Minhos, *Existence result for some third-order separated boundary value problems*, Nonlinear Anal. **47** (2001), no. 4, 2407–2418, DOI: [https://doi.org/10.1016/S0362-546X\(01\)00364-9](https://doi.org/10.1016/S0362-546X(01)00364-9).
- [17] S. H. Khan, *A Picard-Mann hybrid iterative process*, Fixed Point Theory Appl. **2013** (2013), 69, DOI: <https://doi.org/10.1186/1687-1812-2013-69>.
- [18] J. Prescott, *Applied Elasticity*, Green and Company, Longmans, 1924.
- [19] W. Soedel, *Vibrations of Shells and Plates*, CRC Press, Indiana, USA, 2004.
- [20] Y. Sun, *Positive solutions of singular third-order three-point boundary value problems*, J. Math. Anal. Appl. **306** (2005), no. 2, 589–603, DOI: <https://doi.org/10.1016/j.jmaa.2004.10.029>.
- [21] S. P. Timoshenko and M. G. James, *Theory of Elastic Stability*, Courier Corporation, New York, USA, 2009.
- [22] W. Zhao, *Existence and uniqueness of solutions for third-order nonlinear boundary value problems*, Tohoku Math. J. **44** (1992), no. 4, 545–555, DOI: <https://doi.org/10.2748/tmj/1178227249>.

- [23] H. Yu, L. Haiyan, and Y. Liu, *Multiple positive solutions to third-order three-point singular semipositone boundary value problem*, Proc. Indian Acad. Sci. Math. Sci. **114** (2004), 409–422, DOI: <https://doi.org/10.1007/BF02829445>.
- [24] Q.-L. Yao, *The existence and multiplicity of positive solutions for a third-order three-point boundary value problem*, Acta Math. Appl. Sin. **19** (2003), 117–122, DOI: <https://doi.org/10.1007/s10255-003-0087-1>.
- [25] M. Abushammala, S. A. Khuri, and A. Sayfy, *A novel fixed point iteration method for the solution of third-order boundary value problems*, Appl. Math. Comput. **271** (2015), 131–141, DOI: <https://doi.org/10.1016/j.amc.2015.08.129>.
- [26] S. A. Khuri and I. Louhichi, *A novel Ishikawa-Green's fixed point scheme for the solution of BVPs*, Appl. Math. Lett. **82** (2018), 50–57, DOI: <https://doi.org/10.1016/j.aml.2018.02.016>.
- [27] R. P. Agarwal, *Existence-uniqueness and iterative methods for third-order boundary value problems*, J. Comput. Appl. Math. **17** (1987), no. 3, 271–289, DOI: [https://doi.org/10.1016/0377-0427\(87\)90105-1](https://doi.org/10.1016/0377-0427(87)90105-1).
- [28] S. G. Georgiev and K. Zennir, *Classical solutions for a class of IVP for nonlinear two-dimensional wave equations via new fixed point approach*, Partial Differ. Equ. Appl. Math. **2** (2020), 100014, DOI: <https://doi.org/10.1016/j.padiff.2020.100014>.