Research Article

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New class of operators where the distance between the identity operator and the generalized Jordan $*$ -derivation range is maximal

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Abstract: A new class of operators, larger than $*$ -finite operators, named generalized $*$ -finite operators and noted by $\mathcal{GF}^*(\mathcal{H})$ is introduced, where:

$$\mathcal{GF}^*(\mathcal{H}) = \{(A, B) \in \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) : \|TA - BT^* - \lambda I\| \geq |\lambda|, \forall \lambda \in \mathbb{C}, \forall T \in \mathcal{B}(\mathcal{H})\}.$$

Basic properties are given. Some examples are also presented.

Keywords: $*$ -finite operator, numerical range, generalized Jordan $*$ -derivation, finite operator, paranormal operator

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1 Introduction

Let $\mathcal{A}$ be a real or complex Banach $*$ -algebra. For $a, b \in \mathcal{A}$, a linear mapping $d : \mathcal{A} \rightarrow \mathcal{A}$ is called Derivation if we have: $d(ab) = ad(b) + d(a)b$. It is called Jordan derivation if it satisfies: $d(a^2) = ad(a) + d(a)a$, for $a \in \mathcal{A}$.

Recently, in 1990, Šemrl [1] introduced a derivation called Jordan $*$ -derivation. An additive mapping $J : \mathcal{A} \rightarrow \mathcal{A}$ is said to be Jordan $*$ -derivation if: $J(a^2) = aJ(a) + J(a)a^*$, for all $a \in \mathcal{A}$, where a^* stands for the adjoint of a . A Jordan $*$ -derivation J on $\mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ denotes the Banach algebra of all bounded linear operators acting on a complex and infinite dimensional separable Hilbert space $\mathcal{H}$, is called inner if there exists an operator A that satisfies:

$$J(T) \equiv J_A(T) = TA - AT^*, \quad \text{for each } T \in \mathcal{B}(\mathcal{H}).$$

Šemrl [1] showed that every Jordan $*$ -derivation on $\mathcal{B}(\mathcal{H})$ is inner.

The motivation for the study of Jordan $*$ -derivations is the well-known mapping called Derivation map. The properties of derivations on $\mathcal{B}(\mathcal{H})$, their spectra, norms and ranges have been studied extensively by Williams [2], Stampfli [3] and others. In a similar way, some results were obtained for Jordan $*$ -derivations, for instance, Molnár [4] proved that, just as derivations, the range of Jordan $*$ -derivations cannot be dense in $\mathcal{B}(\mathcal{H})$ in the operator norm topology.

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Another purpose of this study is the problem of representability of quadratic forms by sesquilinear ones. Šemrl showed that the structure of Jordan $*$ -derivations arises as a “measure” of this representation problem.

This kind of mapping was studied by many authors like Molnár, Brešar, Battyanyi, Zalar and others. Some of them studied the structure of these mappings. They showed that a Jordan $*$ -derivation defined on certain algebras is inner, see for example [5–8]. Others were interested in studying the range of Jordan $*$ -derivations [4,9].

In [1] Šemrl treated a question in regard to generalized Jordan $*$ -derivation as the concept of Jordan $*$ -derivation pairs, which was introduced by Zalar [10]. It is shown that on a complex $*$ -algebra Jordan $*$ -derivation pairs are of the form:

$$J_{A,B}(T) = TA - BT^*, \quad \text{for all } A, B, T \in \mathcal{B}(\mathcal{H}),$$

and its range is defined by: $R_{A,B} = \{TA - BT^*; T \in \mathcal{B}(\mathcal{H})\}$. We note $R_{A,A} = R_A$.

The class of operators A in $\mathcal{B}(\mathcal{H})$, where the distance between the inner derivation range $R(\delta_A)$ (where $\delta_A(X) = AX - XA$) and the identity operator I is maximal, is called finite operator class and noted by $\mathcal{F}(\mathcal{H})$. In other words, $A \in \mathcal{B}(\mathcal{H})$ is a finite operator if:

$$\|AT - TA - I\| \geq 1; \quad \forall T \in \mathcal{B}(\mathcal{H}).$$

This class was first introduced by Williams [11]. Also, the author in [11] proved that $\mathcal{F}(\mathcal{H})$ contains every normal and hyponormal operators. Many authors extended these results to non-normal operators (see [12–17]).

Based on the study of finite operators and inner Jordan $*$ -derivation, Hamada [18] introduced a new class called the class of $*$ -finite operators defined by:

$$\mathcal{F}^*(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : \|TA - AT^* - I\| \geq 1; \quad \forall T \in \mathcal{B}(\mathcal{H})\}.$$

In [19], the author presented some properties of $*$ -finite operators and proved that a paranormal operator under certain scalar perturbation is $*$ -finite operator.

Depending on the researches of Williams [11] and Hamada [19] on finite and $*$ -finite operators, we will introduce in this paper, a new class of operators named generalized $*$ -finite operators denoted by $\mathcal{GF}^*(\mathcal{H})$. It is the class of operators $A, B \in \mathcal{B}(\mathcal{H})$ where the distance between $R_{A,B}$ and the identity operator I is maximal, i.e.,

$$\mathcal{GF}^*(\mathcal{H}) = \{(A, B) \in \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) : \|TA - BT^* - \lambda I\| \geq |\lambda|; \quad \forall \lambda \in \mathbb{C}, \forall T \in \mathcal{B}(\mathcal{H})\}.$$

The aim of this paper is, first, to investigate $\mathcal{GF}^*(\mathcal{H})$ and give some basic algebraic properties of this class. The last part of the paper focuses on presenting some pairs of operators $(A, B) \in \mathcal{GF}^*(\mathcal{H})$.

2 Preliminaries

Definition 2.1. Let $A \in \mathcal{B}(\mathcal{H})$. A is called normal if: $AA^* = A^*A$, hyponormal if: $AA^* \leq A^*A$, p -hyponormal ($0 < p \leq 1$) if: $(AA^*)^p \leq (A^*A)^p$, paranormal if: $\|Ax\|^2 \leq \|A^2x\|\|x\|$, for all $x \in \mathcal{H}$, normaloid if: $r(A) = \|A\|$ (where $r(A)$ denotes the spectral radius of A) and log-hyponormal if: A is invertible and satisfies $\log(A^*A) \geq \log(AA^*)$.

It is known that:

$$\{\text{invertible } p\text{-hyponormal operators}\} \Rightarrow \{\text{log-hyponormal operators}\},$$

but the converse is not true [20].

$A \in \mathcal{B}(\mathcal{H})$ is a class $\mathcal{A}$ operator if: $|A^2| - |A|^2 \geq 0$ (where $|A|^2 = A^*A$). Class $\mathcal{A}$ is a subclass of paranormal operators (see [21]).

We have:

$$\begin{array}{ccc} \{p\text{-hyponormal operators}\} & \longrightarrow & \{\text{class } \mathcal{A} \text{ operators}\}, \\ & \nearrow & \\ \{\text{log-hyponormal operators}\} & & \end{array}$$

and

$$\text{normal} \subset \text{hyponormal} \subset p\text{-hyponormal} \subset \text{class } \mathcal{A} \subset \text{paranormal} \subset \text{normaloid}.$$

Definition 2.2. [22] Let $\lambda \in \mathbb{C}$. We say that $\lambda \in \sigma_{ar}(A)$ (the approximate reduced spectrum of A), if there is a normed sequence $\{x_n\} \in \mathcal{H}$ such that: $\lim_n (A - \lambda I)x_n = 0$ and $\lim_n (A - \lambda I)^*x_n = 0$.

Definition 2.3. The numerical range of an operator A is defined by:

$$W(A) = \{\langle Ax, x \rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}.$$

We have $\sigma(A) \subseteq \overline{W(A)}$, for all $A \in \mathcal{B}(\mathcal{H})$ (where $\sigma(A)$ is the spectrum of A and $\overline{W(A)}$ denotes the closure of the numerical range of A).

Definition 2.4. [18] An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be *-finite if:

$$0 \in \overline{W(AT - TA)}, \quad \text{for all } T \in \mathcal{B}(\mathcal{H}).$$

Proposition 2.5. [18] These are equivalent conditions on an operator A .

- (1) A is *-finite.
- (2) $\|TA - AT^* - I\| \geq 1$; $\forall T \in \mathcal{B}(\mathcal{H})$.

3 Main results

Definition 3.1. Let $A, B \in \mathcal{B}(\mathcal{H})$. The pair (A, B) is said to be a pair of generalized *-finite operators if:

$$\|TA - BT^* - \lambda I\| \geq |\lambda|; \quad \forall \lambda \in \mathbb{C}, \quad \forall T \in \mathcal{B}(\mathcal{H}).$$

We note by $\mathcal{GF}^*(\mathcal{H})$ the class of generalized *-finite operators, i.e.:

$$\mathcal{GF}^*(\mathcal{H}) = \{(A, B) \in \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) : \|TA - BT^* - \lambda I\| \geq |\lambda|; \quad \forall \lambda \in \mathbb{C}, \quad \forall T \in \mathcal{B}(\mathcal{H})\}.$$

Theorem 3.2. Let $A, B \in \mathcal{B}(\mathcal{H})$. These are equivalent conditions on A, B .

- (i) $0 \in \overline{W(TA - BT^*)}$; $\forall T \in \mathcal{B}(\mathcal{H})$;
- (ii) $\|TA - BT^* - \lambda I\| \geq |\lambda|$; $\forall \lambda \in \mathbb{C}, \quad \forall T \in \mathcal{B}(\mathcal{H})$.

Proof. In [11] the author has shown that for any operator A , we have: $0 \in \overline{W(A)}$ iff: $\|A - \lambda I\| \geq |\lambda|$, for all $\lambda \in \mathbb{C}$. Replacing A by $TA - BT^*$ we get:

$$0 \in \overline{W(TA - BT^*)}, \quad \text{iff: } \|TA - BT^* - \lambda I\| \geq |\lambda|; \quad \forall \lambda \in \mathbb{C}, \quad \forall T \in \mathcal{B}(\mathcal{H}). \quad \square$$

Remark 3.3. $(0, I)$ is not a pair of generalized *-finite operators. Otherwise $0 \in \overline{W(-T^*)}$, for all $T \in \mathcal{B}(\mathcal{H})$. Moreover, if $T = iI$, we get: $0 \in \overline{W(iI)} = \{i\}$, which is a contradiction.

Remark 3.4. The condition (ii) above means that $\inf_T \|TA - BT^* - \lambda I\| = |\lambda|$, for all $\lambda \in \mathbb{C}$.

In the following, we give some basic properties of generalized *-finite operator class.

Proposition 3.5. Let $(A, B) \in \mathcal{GF}^*(\mathcal{H})$. Then $(\alpha A, \alpha B) \in \mathcal{GF}^*(\mathcal{H})$, for all $\alpha \in \mathbb{C}$.

Proof. Clearly, the pair of null operators is a generalized pair of $*$ -finite operators. Let $\alpha \in \mathbb{C}^*$, given $\varepsilon > 0$, then we have:

$$|\langle (T(\alpha A) - (\alpha B)T^*)x, x \rangle| \leq |\alpha| |\langle (TA - BT^*)x, x \rangle| < |\alpha| \frac{\varepsilon}{|\alpha|},$$

for all $T \in \mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$. Thus,

$$|\langle (T(\alpha A) - (\alpha B)T^*)x, x \rangle| < \varepsilon,$$

i.e., $0 \in \overline{W(T(\alpha A) - (\alpha B)T^*)}$. □

Proposition 3.6. If $(B, A) \in \mathcal{GF}^*(\mathcal{H})$, then $(A^*, B^*) \in \mathcal{GF}^*(\mathcal{H})$.

Proof. Suppose that $(B, A) \in \mathcal{GF}^*(\mathcal{H})$. Hence,

$$\|TB - AT^* - \lambda I\| \geq |\lambda|; \quad \forall \lambda \in \mathbb{C}, \quad \forall T \in \mathcal{B}(\mathcal{H}).$$

Since the map $\mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) : T \rightarrow T^*$ is surjective, then:

$$\|(TB - AT^* - \lambda I)^*\| \geq |\lambda|; \quad \forall \lambda \in \mathbb{C}, \quad \forall T \in \mathcal{B}(\mathcal{H}).$$

So,

$$\|TA^* - B^*T^* - \bar{\lambda}I\| \geq |\lambda|; \quad \forall \lambda \in \mathbb{C}, \quad \forall T \in \mathcal{B}(\mathcal{H}),$$

i.e., (A^*, B^*) is generalized $*$ -finite. □

Proposition 3.7. If $(A, B), (C, D)$ are pairs of generalized $*$ -finite operators, then $(A + C, B + D)$ is generalized $*$ -finite.

Proof. Let $(A, B), (C, D) \in \mathcal{GF}^*(\mathcal{H})$. For $\varepsilon > 0$, we have:

$$|\langle (TA - BT^*)x, x \rangle| < \frac{\varepsilon}{2} \quad \text{and} \quad |\langle (TC - DT^*)x, x \rangle| < \frac{\varepsilon}{2},$$

for all $T \in \mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$. Hence,

$$|\langle (T(A + C) - (B + D)T^*)x, x \rangle| \leq |\langle (TA - BT^*)x, x \rangle| + |\langle (TC - DT^*)x, x \rangle| < \varepsilon. \quad \square$$

Proposition 3.8. There exists $(A, B) \in \mathcal{GF}^*(\mathcal{H})$, such that:

$$(A - \lambda I, B - \lambda I) \notin \mathcal{GF}^*(\mathcal{H}), \quad \text{for all } \lambda \in \mathbb{C}^*.$$

Proof. Suppose that $(A - \lambda I, B - \lambda I) \in \mathcal{GF}^*(\mathcal{H})$. Then $0 \in \overline{W(T(A - \lambda I) - (B - \lambda I)T^*)}$ for all $T \in \mathcal{B}(\mathcal{H})$. Put $(A, B) = (0, 0)$ and $T = iI$, so $0 \in \overline{W(-2\lambda iI)} = \{-2\lambda i\}$, for all $\lambda \in \mathbb{C}^*$, which is a contradiction. □

Theorem 3.9. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and let $A, B \in \mathcal{B}(\mathcal{H})$ be operators of the form:

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}.$$

If for all $i; i = \overline{1, 2} : (A_{ii}, B_{ii}) \in \mathcal{GF}^*(\mathcal{H}_i)$, then $(A, B) \in \mathcal{GF}^*(\mathcal{H})$.

Proof. We have for all $X \in \mathcal{B}(\mathcal{H})$

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} I_{11} & 0 \\ 0 & I_{22} \end{pmatrix}.$$

Then, for all $\lambda \in \mathbb{C}$, we have:

$$\begin{aligned} \|XA - BX^* - \lambda I\| &= \left\| \begin{pmatrix} X_{11}A_{11} - B_{11}X_{11}^* - \lambda I_{11} & X_{12}A_{22} + B_{11}X_{21}^* \\ X_{21}A_{11} + B_{22}X_{12}^* & X_{22}A_{22} - B_{22}X_{22}^* - \lambda I_{22} \end{pmatrix} \right\| \\ &\geq \max_{i=1,2} \|X_{ii}A_{ii} - B_{ii}X_{ii}^* - \lambda I_i\| \geq |\lambda|. \end{aligned} \quad \square$$

Theorem 3.10. $\mathcal{GF}^*(\mathcal{H})$ is closed in $\mathcal{B}(\mathcal{H})$ in the operator norm topology.

Proof. Let $(A_n, B_n)_{n \in \mathbb{N}^*}$ be a sequence in $\mathcal{GF}^*(\mathcal{H})$ such that $A_n \rightarrow A$ and $B_n \rightarrow B$. Then,

$$|\lambda| \leq \|TA_n - B_nT^* - \lambda I\| \leq \|TA - BT^* - \lambda I\| + \|A_n - A\| \|T\| + \|B_n - B\| \|T\|$$

for all $T \in \mathcal{B}(\mathcal{H})$ and all $\lambda \in \mathbb{C}$. By letting $n \rightarrow +\infty$, we obtain $\|TA - BT^* - \lambda I\| \geq |\lambda|$. Hence, $(A, B) \in \mathcal{GF}^*(\mathcal{H})$. $\square$

Proposition 3.11. $\mathcal{GF}^*(\mathcal{H})$ is not dense in $\mathcal{B}(\mathcal{H})$ in the operator norm topology.

Proof. Theorem 3.10 and Remark 3.3 show that $\mathcal{GF}^*(\mathcal{H})$ cannot be dense in $\mathcal{B}(\mathcal{H})$ in the operator norm topology since $\overline{\mathcal{GF}^*(\mathcal{H})} \neq \mathcal{B}(\mathcal{H})$. $\square$

If (A, B) is generalized *-finite, what about the unitarily equivalent operators?

Proposition 3.12. $\mathcal{GF}^*(\mathcal{H})$ is invariant under unitary equivalence.

Proof. Let $(A, B) \in \mathcal{GF}^*(\mathcal{H})$ and U is a unitary operator. Then by Theorem 3.2 we have:

$$\begin{aligned} (A, B) \in \mathcal{GF}^*(\mathcal{H}) &\Leftrightarrow 0 \in \overline{W(TA - BT^*)}, \quad \forall T \in \mathcal{B}(\mathcal{H}) \\ &\Leftrightarrow 0 \in \overline{W(U^*(TA - BT^*)U)}, \quad \forall T \in \mathcal{B}(\mathcal{H}) \\ &\Leftrightarrow 0 \in \overline{W(U^*(TUU^*A - BUU^*T^*)U)}, \quad \forall T \in \mathcal{B}(\mathcal{H}) \\ &\Leftrightarrow 0 \in \overline{W(S(U^*AU) - (U^*BU)S^*)}, \quad \forall S \in \mathcal{B}(\mathcal{H}) \\ &\Leftrightarrow ((U^*AU), (U^*BU)) \in \mathcal{GF}^*(\mathcal{H}). \end{aligned} \quad \square$$

In what follows, we will present some pairs of generalized *-finite operators.

Lemma 3.13. Let A, B be finite rank operators, then (A, B) is generalized *-finite.

Proof. For a finite rank operator $T \in \mathcal{B}(\mathcal{H})$, $TA - BT^*$ is also of finite rank. Since 0 belongs to the spectrum of each finite rank operator, then $0 \in \sigma(TA - BT^*) \subseteq \overline{W(TA - BT^*)}$, for all $T \in \mathcal{B}(\mathcal{H})$. Consequently, (A, B) is generalized *-finite. $\square$

Theorem 3.14. Let $A, B \in \mathcal{B}(\mathcal{H})$. If there exist a normed sequence $(x_n)_{n \geq 1} \subset \mathcal{H}$ and some scalar λ verifying:

$$\|(A - \lambda I)x_n\| \rightarrow 0 \quad \text{and} \quad \|(B - \lambda I)^*x_n\| \rightarrow 0,$$

then $(A - \lambda I, B - \lambda I)$ is generalized *-finite.

Proof. We have for all $T \in \mathcal{B}(\mathcal{H})$:

$$\begin{aligned} \|T(A - \lambda I) - (B - \lambda I)T^* - \lambda I\| &\geq |\langle [T(A - \lambda I) - (B - \lambda I)T^* - \lambda I]x_n, x_n \rangle| \\ &= |\langle (A - \lambda I)x_n, T^*x_n \rangle - \langle T^*x_n, (B - \lambda I)^*x_n \rangle - \lambda|. \end{aligned}$$

Letting $n \rightarrow \infty$, we get $\|T(A - \lambda I) - (B - \lambda I)T^* - \lambda I\| \geq |\lambda|$, for all $T \in \mathcal{B}(\mathcal{H})$ and all $\lambda \in \mathbb{C}$. $\square$

Corollary 3.15. *If $A \in \mathcal{B}(\mathcal{H})$, then for all $\lambda \in \sigma_a(A)$ and for all $C \in \mathcal{B}(\mathcal{H})$,*

$$(C(A - \lambda I), (A - \lambda I)^*) \in \mathcal{GF}^*(\mathcal{H}).$$

Proof. Let $\lambda \in \sigma_a(A)$. Then there exists a normed sequence $(x_n)_{n \geq 1}$ in $\mathcal{H}$ verifying: $\|(A - \lambda I)x_n\| \rightarrow 0$. If $T = A - \lambda I$ and $S = CT$ with $C \in \mathcal{B}(\mathcal{H})$, then:

$$\|((T - 0)^*)^*x_n\| = \|(A - \lambda I)x_n\| \rightarrow 0$$

and

$$\|(S - 0)x_n\| = \|(C(A - \lambda I)x_n)\| \rightarrow 0.$$

Thus,

$$(C(A - \lambda I), (A - \lambda I)^*) = (S, T^*) \in \mathcal{GF}^*(\mathcal{H}). \quad \square$$

Lemma 3.16. [23] *If A is paranormal, then $\sigma_{ar}(A) \neq \emptyset$.*

Proposition 3.17. *If $A, B \in \mathcal{B}(\mathcal{H})$ are paranormal operators, then $(A - \lambda I, B - \lambda I) \in \mathcal{GF}^*(\mathcal{H})$, for all $\lambda \in \sigma_{ar}(A) \cap \sigma_{ar}(B)$.*

Proof. Let $A, B \in \mathcal{B}(\mathcal{H})$ be paranormal operators. Then by Lemma 3.16 there exist $\lambda \in \sigma_{ar}(A) \cap \sigma_{ar}(B)$ and a normed sequence $(x_n)_{n \geq 1}$ in $\mathcal{H}$ satisfying:

$$\|(A - \lambda I)x_n\| \rightarrow 0, \|(A - \lambda I)^*x_n\| \rightarrow 0 \quad \text{and} \quad \|(B - \lambda I)x_n\| \rightarrow 0, \|(B - \lambda I)^*x_n\| \rightarrow 0,$$

Hence, by Theorem 3.14 we get $(A - \lambda I, B - \lambda I) \in \mathcal{GF}^*(\mathcal{H})$. $\square$

Corollary 3.18. *In all the next cases, the pair $(A - \lambda I, B - \lambda I)$ is generalized $*$ -finite for all $\lambda \in \sigma_{ar}(A) \cap \sigma_{ar}(B)$.*

- (i) *A and B are hyponormal operators,*
- (ii) *A and B are p -hyponormal operators,*
- (iii) *A and B are class $\mathcal{A}$ operators,*
- (iv) *A and B are log-hyponormal operators.*

In the rest of this paper, we will study the range $R_{A,B}$ when (A, B) is generalized $*$ -finite.

Proposition 3.19. *Let $A, B \in \mathcal{B}(\mathcal{H})$. If $R_{A,B}$ contains no invertible operator, then $(A, B) \in \mathcal{GF}^*(\mathcal{H})$.*

Proof. Suppose that $C \in R_{A,B}$ (i.e., $C = TA - BT^*$) is a noninvertible operator. Thus by [24, p. 162]: $\|C - \lambda I\| \geq |\lambda|$, $\forall \lambda \in \mathbb{C}$. Then $(A, B) \in \mathcal{GF}^*(\mathcal{H})$. $\square$

Proposition 3.20. *If $(A, B) \in \mathcal{GF}^*(\mathcal{H})$, then $\overline{R_{A,B}}$ (the closure of $R_{A,B}$) contains no nonzero scalar operator.*

Proof. Suppose that $\lambda I \in \overline{R_{A,B}}$ for some $\lambda \in \mathbb{C}^*$, then there exists a sequence (T_n) in $\mathcal{B}(\mathcal{H})$ satisfying

$$\|T_n A - B T_n^* - \lambda I\| \rightarrow 0.$$

Since $(A, B) \in \mathcal{GF}^*(\mathcal{H})$, $|\lambda| \leq \|T_n A - B T_n^* - \lambda I\| \rightarrow 0$, a contradiction. $\square$

4 Conclusion

For $A, B \in \mathcal{B}(\mathcal{H})$, the pair (A, B) is said to be generalized $*$ -finite operators if it is an element of $\mathcal{GF}^*(\mathcal{H})$, where:

$$\mathcal{GF}^*(\mathcal{H}) = \{(A, B) \in \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) : \|TA - BT^* - \lambda I\| \geq |\lambda|; \quad \forall \lambda \in \mathbb{C}, \quad \forall T \in \mathcal{B}(\mathcal{H})\}.$$

The class of generalized $\ast$ -finite operators is a generalization of $\ast$ -finite operators class, and this work provides basic tools for further research on this subject.

First, a necessary and sufficient condition for belonging to $\mathcal{GF}^*(\mathcal{H})$ is given and some algebraic properties of $\mathcal{GF}^*(\mathcal{H})$ are presented and proved that the class of generalized $\ast$ -finite operators is closed for uniform topology.

Second, we proved that $\mathcal{GF}^*(\mathcal{H})$ is invariant under unitary equivalence.

Finally, we presented some pairs of operators $(A, B) \in \mathcal{GF}^*(\mathcal{H})$.

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