Research Article

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Some aspects of generalized Zbăganu and James constant in Banach spaces

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Abstract: We shall introduce a new geometric constant $C_Z(\lambda, \mu, X)$ based on a generalization of the parallelogram law, which was proposed by Moslehian and Rassias. First, it is shown that, for a Banach space, $C_Z(\lambda, \mu, X)$ is equal to 1 if and only if the norm is induced by an inner product. Next, a characterization of uniformly non-square is given, that is, X has the fixed point property. Also, a sufficient condition which implies weak normal structure is presented. Moreover, a generalized James constant $J(\lambda, X)$ is also introduced. Finally, some basic properties of this new coefficient are presented.

Keywords: Banach spaces, geometric constants, weak normal structure, Euler-Lagrange identity

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1 Introduction

In recent years, a lot of geometric constants have been defined and studied in the literature, which makes it easier for us to deal with some problems in Banach space, because it can describe the geometric properties of space quantitatively, and these geometric constants have mathematical beauty, and there are countless relationships between different geometric constants. Special attention is paid to the von Neumann-Jordan constant $C_{NJ}(X)$ and the James constant $J(X)$, which are rigorously studied based on their importance. For a Banach space X , several studies on the James constant $J(X)$ and also on the von Neumann-Jordan constant $C_{NJ}(X)$ have been conducted by Gao [1,2], Yang and Wang [3], and Kato et al. [4,5]. For readers interested in this field are advised to see [6–13] and references mentioned therein. It is worth mentioning that geometric constants play a vital role as a tool for solving other problems, such as in the study of Banach-Stone theorem, Bishop-Phelps-Bollobás theorem, and Tingley's problem. These are important research topics in the area of functional analysis and we recommend readers to read the literature [14–16].

If we consider the usual Euclidean space $(\mathbb{R}^n, \|\cdot\|)$, then the identity $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ is called the parallelogram law, and it is well known. This identity can be naturally extended to the more general case. Research on the equivalent characterization of inner product space has been attracting much attention. We refer the readers to [17–19] for more details.

Moslehian and Rassias obtained a new equivalent characterization of inner product space in the literature [20]. For the reader's convenience, we present the theorem as follows:

Theorem 1.1. [20] *A normed space $(X, \|\cdot\|)$ is an inner product space if and only if*

$$\|\lambda x + \mu y\|^2 + \|\mu x - \lambda y\|^2 = (\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)$$

for any non-negative real numbers λ, μ and any $x, y \in X$.

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Inspired by the above theorem, we introduce a new geometric constant $C_Z(\lambda, \mu, X)$ in a Banach space X . It is well known that the constant $C_{NJ}(X)$ is used to describe the parallelogram law, and this new constant is used to describe the generalizations of the parallelogram law.

The paper is organized as follows: we recall some fundamental concepts and introduce some well-known geometric constants related to the ones we introduce in the next section. In Section 3, some basic properties of this new coefficient are investigated. Section 4 is devoted to relationships between the constant $C_Z(\lambda, \mu, X)$ and other well-known constants, emphasized in terms of non-trivial inequalities involving these constants. Furthermore, we establish a new necessary condition for weak normal Banach spaces in the form of $C_Z(\lambda, \mu, X)$. In Section 5, we introduced a generalized James constant $J(\lambda, X)$, some basic properties of this new coefficient are presented. Finally, in Section 6, we establish a new sufficient condition for normal structure of Banach spaces in terms of $J(\lambda, X)$.

2 Preliminaries

We now give some definitions related to geometric constants. Let real Banach space be represented by X with $\dim X \geq 2$ and the dual space by X^* of X , then the unit ball, as well as the unit sphere of X are, respectively, symbolized by B_X and S_X .

Recall that the Banach space X was called uniformly non-square [21] if there exists a $\delta \in (0, 1)$ such that for any $x, y \in S_X$ either $\frac{\|x+y\|}{2} \leq 1 - \delta$ or $\frac{\|x-y\|}{2} \leq 1 - \delta$.

The James constant $J(X)$ of a Banach space X was introduced by Gao and Lau [22] as follows:

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}.$$

In combination with Jordan and von Neumann's [23] brilliant work on the characterization of inner product spaces by the parallelogram law, Clarkson [24] first proposed the von Neumann-Jordan constant $C_{NJ}(X)$ of Banach space. More precisely, the von Neumann-Jordan constant of X is defined by

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, (x, y) \neq (0, 0)\right\}.$$

Moreover, the various axioms of these constants are given in [2,4,25]:

- (1) $\sqrt{2} \leq J(X) \leq 2$.
- (2) $J(X) = \sqrt{2}$ whenever X represents Hilbert space; the converse is not necessarily correct.
- (3) $1 \leq C_{NJ}(X) \leq 2$.
- (4) X is a Hilbert space iff $C_{NJ}(X) = 1$.
- (5) X is uniformly non-square iff $C_{NJ}(X) < 2$.
- (6) $C_{NJ}(X) = C_{NJ}(X^*)$.

The constant $C_Z(X)$ was introduced by Zbăganu [26]:

$$C_Z(X) = \sup\left\{\frac{\|x+y\|\|x-y\|}{\|x\|^2 + \|y\|^2} : x, y \in X, (x, y) \neq (0, 0)\right\}.$$

It is worth mentioning that Zbăganu conjectured that $C_Z(X)$ coincides with the von Neumann-Jordan constant $C_{NJ}(X)$, but Alonso and Martín [17] gave a counterexample that $C_{NJ}(X) \neq C_Z(X)$.

The constant $J(t, X)$

$$J(t, X) = \sup\{\min\{\|x+ty\|, \|x-ty\|\} : x, y \in S_X\}$$

was studied by He and Cui [27] as a generalization of the constant $J(X)$.

Definition 2.1. [28] A non-empty bounded and convex subset K of a Banach space X is said to have normal structure if for every convex subset H of K that contains more than one point, there exists a point $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < \sup\{\|x - y\| : x, y \in H\}.$$

Normal structure is an important concept in fixed point theory [29]. A Banach space X is said to have weak normal structure if each weakly compact convex set K of X that contains more than one point has normal structure. Obviously, for a reflexive Banach space, normal structure and weak normal structure coincide. Furthermore, every reflexive Banach space with normal structure has the fixed point property.

The next definition is extremely important, as it was in [30]. Space X holds the axioms WORTH if

$$\limsup_{n \rightarrow \infty} \|\|x_n + x\| - \|x_n - x\|\| = 0$$

for every x_n in X and all the elements x of X , where x_n is a null weakly sequence. In connection with property WORTH we shall use the following parameter from Sims [30]. That is,

$$w(X) = \sup\left\{\lambda > 0 : \lambda \cdot \liminf_{n \rightarrow \infty} \|x_n + x\| \leq \liminf_{n \rightarrow \infty} \|x_n - x\|\right\}.$$

In the above, by taking the supremum over all x_n in X and every element x of X . It was proved [31] that $\frac{1}{3} \leq w(X) \leq 1$ for Banach space X .

3 The constant $C_Z(\lambda, \mu, X)$

From now on, we will consider only Banach spaces of dimension at least 2. We begin by introducing the following key definition: for $\lambda, \mu > 0$

$$C_Z(\lambda, \mu, X) = \sup\left\{\frac{2\|\lambda x + \mu y\|\|\mu x - \lambda y\|}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} : x, y \in X, (x, y) \neq (0, 0)\right\}.$$

Clearly, $C_Z(1, 1, X) = C_Z(X)$.

Proposition 3.1. Suppose that X is a Banach space. Then

$$1 \leq C_Z(\lambda, \mu, X) \leq 2.$$

Proof. Let $y = \frac{\mu - \lambda}{\lambda + \mu}x$, then we can obtain

$$\frac{2\|\lambda x + \mu y\|\|\mu x - \lambda y\|}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} = 1.$$

For the second part of the proof:

$$\frac{2\|\lambda x + \mu y\|\|\mu x - \lambda y\|}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} \leq \frac{\|\lambda x + \mu y\|^2 + \|\mu x - \lambda y\|^2}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} \leq \frac{(2\lambda^2 + 2\mu^2)(\|x\|^2 + \|y\|^2)}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} = 2.$$

This completes the proof. $\square$

The following great result is known as “the parallelogram law.”

Theorem 3.2. [23] Let $(X, \|\cdot\|)$ be a real normed linear space. Then $\|\cdot\|$ derives from an inner product if and only if the parallelogram law holds, i.e.,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in X$.

Theorem 3.2 has some other versions where the sign of equality is replaced by the sign of inequality, and only $x, y \in S_X$ needs to be considered.

Theorem 3.3. [32] *Let $(X, \|\cdot\|)$ be a real normed linear space. Then $\|\cdot\|$ derives from an inner product if and only if*

$$\|x + y\|^2 + \|x - y\|^2 \sim 4$$

for all $x, y \in S_X$, where $\sim$ stands either for $\leq$ or $\geq$.

We now give the following proposition, which is inspired by Theorem 1.1. We remark that the Zbăganu constant $C_Z(X)$ can be in the following form:

$$C_Z(X) = \sup \left\{ \frac{4\|x\|\|y\|}{\|x + y\|^2 + \|x - y\|^2} : x, y \in X, (x, y) \neq (0, 0) \right\}.$$

Theorem 3.4. *Let X be a Banach space. Then $C_Z(\lambda, \mu, X) = 1$ if and only if X is a Hilbert space.*

Proof. Suppose $C_Z(\lambda, \mu, X) = 1$, it follows from setting $\lambda = \mu = 1$ that

$$\frac{4\|x\|\|y\|}{\|x + y\|^2 + \|x - y\|^2} \leq 1,$$

then we have

$$\|x + y\|^2 + \|x - y\|^2 \geq 4$$

for any $x, y \in S_X$. Thus, X is a Hilbert space directly from Theorem 3.3.

For the second part of the proof, assume that X is a Hilbert space, using Theorem 1.1 we have

$$\frac{\|\lambda x + \mu y\|^2 + \|\mu x - \lambda y\|^2}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} = 1$$

and hence

$$\frac{2\|\lambda x + \mu y\|\|\mu x - \lambda y\|}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} \leq 1$$

for any non-negative real numbers λ, μ and any $x, y \in X$. Then $C_Z(\lambda, \mu, X) = 1$. □

4 Properties of $C_Z(\lambda, \mu, X)$

In this section, we will show that the constant $C_Z(\lambda, \mu, X)$ is closely related to other important constants and geometric properties. First, we use the following equivalent definition of $C_{NJ}(X)$ to present a relation between $C_Z(\lambda, \mu, X)$ and $C_{NJ}(X)$.

Definition 4.1. [33] Let X be a Banach space. Then

$$C_{NJ}(X) = \sup \left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{4} : \|x\|^2 + \|y\|^2 = 2 \right\}.$$

Proposition 4.2. *Let X be a Banach space. Then,*

$$C_Z(\lambda, \mu, X) \leq \frac{2\lambda^2}{\lambda^2 + \mu^2} C_{NJ}(X) + \frac{2\sqrt{2}\lambda|\lambda - \mu|}{\lambda^2 + \mu^2} \sqrt{C_{NJ}(X)} + \frac{|\lambda - \mu|^2}{\lambda^2 + \mu^2}.$$

Proof. Using the above equivalent definition of $C_{NJ}(X)$ and the elementary inequality, we can obtain the following estimates:

$$\begin{aligned}
 & \frac{2\|\lambda x + \mu y\|\|\mu x - \lambda y\|}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} \\
 & \leq \frac{\|\lambda x + \mu y\|^2 + \|\mu x - \lambda y\|^2}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} \\
 & \leq \frac{(\lambda(\|x + y\|) + |\lambda - \mu|\|y\|)^2 + (\lambda(\|x - y\|) + |\lambda - \mu|\|x\|)^2}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} \\
 & = \frac{\lambda^2(\|x + y\|^2 + \|x - y\|^2) + |\lambda - \mu|^2(\|x\|^2 + \|y\|^2) + 2\lambda|\lambda - \mu|(\|x + y\|\|y\| + \|x - y\|\|x\|)}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} \\
 & \leq \frac{\lambda^2(\|x + y\|^2 + \|x - y\|^2) + |\lambda - \mu|^2(\|x\|^2 + \|y\|^2) + 2\lambda|\lambda - \mu|\sqrt{\|x\|^2 + \|y\|^2}\sqrt{\|x + y\|^2 + \|x - y\|^2}}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} \\
 & \leq \frac{2\lambda^2}{\lambda^2 + \mu^2}C_{NJ}(X) + \frac{2\sqrt{2}\lambda|\lambda - \mu|}{\lambda^2 + \mu^2}\sqrt{C_{NJ}(X)} + \frac{|\lambda - \mu|^2}{\lambda^2 + \mu^2},
 \end{aligned}$$

as desired. $\square$

Example 4.3. Let $\lambda = 2, \mu = 3$, then

$$C_Z(2, 3, X) = \sup \left\{ \frac{2\|2x + 3y\| \cdot \|3x - 2y\|}{13(\|x\|^2 + \|y\|^2)} : x, y \in X, (x, y) \neq (0, 0) \right\}.$$

Thus from Proposition 4.2, we have

$$C_Z(2, 3, X) \leq \frac{8}{13}C_{NJ}(X) + \frac{4\sqrt{2}}{13}\sqrt{C_{NJ}(X)} + \frac{1}{13}.$$

Assume $C_{NJ}(X) = 2$ and hence $C_Z(2, 3, X) \leq \frac{25}{13}$. Furthermore, there exist $x, y \in S_X$ such that

$$\|x + y\| = 2, \|x - y\| = 2.$$

This means that there exist $x, y \in S_X$ such that

$$5 \geq \|2x + 3y\| = \|3x + 3y - x\| \geq 3\|x + y\| - \|x\| = 5$$

and

$$5 \geq \|3x - 2y\| = \|3x - 3y + y\| \geq 3\|x - y\| - \|y\| = 5.$$

So we can deduce that there exist $x, y \in S_X$ such that

$$\|2x + 3y\| = \|3x - 2y\| = 5,$$

which implies that

$$C_Z(2, 3, X) = \frac{25}{13}.$$

Next, combining $C_{NJ}(X)$ and $C_Z(X)$ we give a relationship to $C_Z(\lambda, \mu, X)$.

Proposition 4.4. Let X be a Banach space. Then,

$$C_Z(\lambda, \mu, X) \leq \frac{(\lambda + \mu)|\lambda - \mu|}{\lambda^2 + \mu^2}C_{NJ}(X) + C_Z(X).$$

Proof. According to the following elementary identity:

$$\lambda x + \mu y = \frac{(\lambda + \mu)}{2}(x + y) + \frac{\lambda - \mu}{2}(x - y)$$

and

$$\mu x - \lambda y = \frac{(\mu - \lambda)}{2}(x + y) + \frac{\mu + \lambda}{2}(x - y),$$

we obtain

$$\begin{aligned} & \frac{\|\lambda x + \mu y\| \|\mu x - \lambda y\|}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} \\ &= \frac{\left\| \frac{\lambda + \mu}{2}(x + y) + \frac{\lambda - \mu}{2}(x - y) \right\| \cdot \left\| \frac{\mu - \lambda}{2}(x + y) + \frac{\mu + \lambda}{2}(x - y) \right\|}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} \\ &\leq \frac{\left(\left\| \frac{\lambda + \mu}{2}(x + y) \right\| + \left\| \frac{\lambda - \mu}{2}(x - y) \right\| \right) \cdot \left(\left\| \frac{\lambda - \mu}{2}(x + y) \right\| + \left\| \frac{\lambda + \mu}{2}(x - y) \right\| \right)}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} \\ &= \frac{\frac{(\lambda + \mu)}{4} \|\lambda - \mu\| \|x + y\|^2 + \frac{(\lambda + \mu)}{4} \|\lambda - \mu\| \|x - y\|^2 + \frac{(\lambda + \mu)^2}{4} \|x + y\| \|x - y\| + \frac{|\lambda - \mu| \cdot |\lambda - \mu|}{4} \|x + y\| \|x - y\|}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} \\ &= \frac{\frac{(\lambda + \mu)}{4} \|\lambda - \mu\| (\|x + y\|^2 + \|x - y\|^2) + \frac{(\lambda + \mu)^2 + |\lambda - \mu|^2}{4} \|x + y\| \|x - y\|}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} \\ &\leq \frac{(\lambda + \mu) \|\lambda - \mu\|}{2(\lambda^2 + \mu^2)} C_{\text{NJ}}(X) + \frac{1}{2} C_Z(X). \end{aligned}$$

This completes the proof. $\square$

Example 4.5. Consider $X = \mathbb{R}^2$ be equipped with the norm defined by

$$\|(x_1, x_2)\| = \max\{|x_1| + (\sqrt{2} - 1)|x_2|, |x_2| + (\sqrt{2} - 1)|x_1|\}.$$

It is known that $C_{\text{NJ}}(X) = C_Z(X) = 4 - 2\sqrt{2}$ (see [6]). Without loss of generality, let $\lambda \neq \mu$. Thus from Proposition 4.4, we have

$$C_Z(\lambda, \mu, X) \leq (4 - 2\sqrt{2}) \left(\frac{2 \max\{\lambda^2, \mu^2\}}{\lambda^2 + \mu^2} \right).$$

Proposition 4.6. For a Banach space X the following assertions are equivalent:

- (i) $C_{\text{NJ}}(X) = 2$.
- (ii) $C_Z(\lambda, \mu, X) = \frac{(\lambda + \mu)^2}{\lambda^2 + \mu^2}$ for all λ, μ .
- (iii) $C_Z(\lambda, \mu, X) = \frac{(\lambda_0 + \mu_0)^2}{\lambda_0^2 + \mu_0^2}$ for some λ_0, μ_0 .

Proof. If $\lambda = \mu$, it is obvious (from Proposition 1 in [17]). Without loss of generality, assume $\lambda > \mu$ (the case $\lambda < \mu$ is similar).

(i) $\Rightarrow$ (ii). Since $C_{\text{NJ}}(X) = 2$, we deduce that there exist $x_n, y_n \in S_X$ such that

$$\|x_n + y_n\| \rightarrow 2, \quad \|x_n - y_n\| \rightarrow 2 \quad (n \rightarrow \infty).$$

This means that there exist $x_n, y_n \in S_X$ such that

$$\|\lambda x_n + \mu y_n\| = \|\lambda(x_n + y_n) - (\lambda - \mu)y_n\| \geq \lambda\|x_n + y_n\| - (\lambda - \mu)\|y_n\| = 2\lambda - (\lambda - \mu) = \lambda + \mu$$

and

$$\|\mu x_n - \lambda y_n\| = \|(\lambda + \mu)(x_n - y_n) + \mu y_n - \lambda x_n\| \geq (\lambda + \mu)\|x_n - y_n\| - \mu\|y_n\| - \lambda\|x_n\| = \lambda + \mu.$$

So we can deduce that there exist $x_n, y_n \in S_X$ such that

$$\|\lambda x_n + \mu y_n\| \rightarrow \lambda + \mu, \quad \|\mu x_n - \lambda y_n\| \rightarrow \lambda + \mu \quad (n \rightarrow \infty).$$

On the other hand, by Proposition 4.2, we can deduce that

$$C_Z(\lambda, \mu, X) \leq \frac{4\lambda^2}{\lambda^2 + \mu^2} + \frac{4\lambda|\lambda - \mu|}{\lambda^2 + \mu^2} + \frac{|\lambda - \mu|^2}{\lambda^2 + \mu^2} = \frac{(\lambda + \mu)^2}{\lambda^2 + \mu^2}.$$

This implies that $C_Z(\lambda, \mu, X) = \frac{(\lambda + \mu)^2}{\lambda^2 + \mu^2}$, as desired.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). If $C_{NJ}(X) < 2$, applying Proposition 4.2 twice, then we have

$$C_Z(\lambda, \mu, X) < \frac{4\lambda^2}{\lambda^2 + \mu^2} + \frac{4|\lambda - \mu|}{\lambda^2 + \mu^2} + \frac{|\lambda - \mu|^2}{\lambda^2 + \mu^2} = \frac{(\lambda + \mu)^2}{\lambda^2 + \mu^2}.$$

This contradiction implies the thesis. $\square$

Corollary 4.7. X is uniformly non-square if and only if $C_Z(\lambda, \mu, X) < \frac{(\lambda + \mu)^2}{\lambda^2 + \mu^2}$.

Proof. It can be directly concluded from Proposition 4.6 and the fact X is uniformly non-square if and only if $C_{NJ}(X) < 2$. $\square$

Remark 4.8. It is worth mentioning that every uniformly non-square Banach space has the fixed point property (see [34]). So we can deduce that if X is a Banach space with $C_Z(\lambda, \mu, X) < \frac{(\lambda + \mu)^2}{\lambda^2 + \mu^2}$, then X has the fixed point property.

In the next portion, we will establish a connection between the constant $C_Z(\lambda, \mu, X)$ and the weak normal structure. We begin by starting the following lemma which will be our main tool.

Lemma 4.9. [2] Let X be a Banach space without weak normal structure, then for any $0 < \delta < 1$, there exist x_1, x_2, x_3 in S_X satisfying

- (i) $x_2 - x_3 = ax_1$ with $|a - 1| < \delta$;
- (ii) $\|x_1 - x_2\| - 1, \|x_3 - (-x_1)\| - 1 < \delta$; and
- (iii) $\|\frac{x_1 + x_2}{2}\|, \|\frac{x_3 + (-x_1)}{2}\| > 1 - \delta$.

Theorem 4.10. A Banach space X with $C_Z(\lambda, \mu, X) < \frac{(\lambda + \mu) \cdot (2\mu - \lambda)}{\lambda^2 + \mu^2}$ for some $\lambda, \mu > 0$ has weak normal structure.

Proof. Assume X does not have weak normal structure. For each $\delta > 0$, let x_1, x_2 , and x_3 in S_X satisfy the conditions in Lemma 4.9. Without loss of generality, let $\lambda \geq \mu$ (the case $\lambda \leq \mu$ is similar).

Let $t = \frac{\mu}{\lambda}$, then

$$\|x_1 + tx_2\| = \|(x_1 + x_2) - (1 - t)x_2\| \geq \|x_1 + x_2\| - \|(1 - t)x_2\| \geq 2 - 2\delta - (1 - t) = 1 + t - 2\delta.$$

So we have

$$\|\lambda x_1 + \mu x_2\| = \lambda \|x_1 + tx_2\| \geq \lambda + \mu - 2\delta\lambda.$$

On the other hand, we obtain

$$\begin{aligned} \|x_2 - tx_1\| &= \|x_2 + tx_2 - tx_2 - tx_1\| = \|x_2 + t(ax_1 + x_3) - tx_2 - tx_1\| = \|t(x_3 - x_1) + (1 - t)x_2 + tax_1\| \\ &\geq (2 - 2\delta)t - 1 + t(1 - a) \geq (2 - 2\delta)t - \delta - 1. \end{aligned}$$

Thus,

$$\|\mu x_1 - \lambda x_2\| = \lambda \|x_2 - tx_1\| \geq \lambda((2 - 2\delta)t - \delta - 1).$$

Since δ can be arbitrarily small, we have

$$C_Z(\lambda, \mu, X) \geq \frac{(\lambda + \mu)(2\mu - \lambda)}{\lambda^2 + \mu^2}. \quad \square$$

Remark 4.11. If λ and μ satisfy this condition $\mu^2 - 2\lambda^2 - \lambda\mu \leq 0$, we obtain

$$\frac{(\lambda + \mu)(2\mu - \lambda)}{\lambda^2 + \mu^2} \leq \frac{(\lambda + \mu)^2}{\lambda^2 + \mu^2},$$

which together with Corollary 4.7 shows that X is uniformly non-square, and hence reflexive [21]. As we mentioned earlier, we can deduce that normal structure and weak normal structure coincide, this implies that X also has normal structure.

5 Constant $J(\lambda, X)$

We begin by introducing the following key definition: for $0 < \lambda < 1$.

$$J(\lambda, X) = \sup\{\min\{\|\lambda x + (1 - \lambda)y\|, \|\lambda x - (1 - \lambda)y\|\} : x, y \in S_X\}.$$

Clearly, $J(\frac{1}{2}, X) = \frac{1}{2}J(X)$.

Proposition 5.1. Let X be a Banach space. Then,

$$\sqrt{2\lambda^2 - 2\lambda + 1} \leq J(\lambda, X) \leq 1.$$

Proof. Since $\|x \pm y\| \leq \|x\| + \|y\|$, we have

$$\|\lambda x + (1 - \lambda)y\| \leq \|\lambda x\| + \|(1 - \lambda)y\|$$

and

$$\|\lambda x - (1 - \lambda)y\| \leq \|\lambda x\| + \|(1 - \lambda)y\|,$$

which yields $J(\lambda, X) \leq 1$.

On the other hand, there exist $x, y \in S_X$ such that

$$\|\lambda x + (1 - \lambda)y\| = \|\lambda x - (1 - \lambda)y\| = \sqrt{2\lambda^2 - 2\lambda + 1}$$

and hence

$$J(\lambda, X) \geq \sqrt{2\lambda^2 - 2\lambda + 1}. \quad \square$$

Theorem 5.2. Let X be a Banach space. Then,

$$\max\{\lambda, 1 - \lambda\}J(X) - |2\lambda - 1| \leq J(\lambda, X) \leq \lambda J(X) + |2\lambda - 1|.$$

Proof. For $x, y \in S_X$ and $0 < \lambda < 1$, we have

$$\begin{aligned} \lambda \min\{\|x + y\|, \|x - y\|\} &= \min\{\|\lambda x + \lambda y\|, \|\lambda x - \lambda y\|\} \\ &\leq \min\{\|\lambda x + (1 - \lambda)y\| + |2\lambda - 1|, \|\lambda x - (1 - \lambda)y\| + |2\lambda - 1|\} \\ &\leq J(\lambda, X) + |2\lambda - 1| \end{aligned}$$

and

$$\begin{aligned}
(1 - \lambda) \min\{\|x + y\|, \|x - y\|\} &= \min\{(1 - \lambda)\|x + y\|, (1 - \lambda)\|x - y\|\} \\
&\leq \min\{\|\lambda x + (1 - \lambda)y\| + |2\lambda - 1|, \|\lambda x - (1 - \lambda)y\| + |2\lambda - 1|\} \\
&\leq J(\lambda, X) + |2\lambda - 1|.
\end{aligned}$$

On the other hand,

$$\min\{\|\lambda x + (1 - \lambda)y\|, \|\lambda x - (1 - \lambda)y\|\} \leq \min\{\lambda\|x + y\| + |2\lambda - 1|, \lambda\|x - y\| + |2\lambda - 1|\} \leq \lambda J(X) + |2\lambda - 1|.$$

This completes the proof. $\square$

Corollary 5.3. *If X is not uniformly non-square, then $J(\lambda, X) = 1$.*

Proof. Since X is not uniformly non-square, it gives that $J(X) = 2$. Applying Theorem 5.2, we obtain $J(\lambda, X) \geq 1$. Thus, $J(\lambda, X) = 1$ directly from the estimate of $J(\lambda, X) \leq 1$. $\square$

Lemma 5.4. [35] *Let X be a normed space of $\dim X \geq 3$. Then X is an inner product space iff $x, y \in S_X$ and $x \perp_I y$ (i.e., $\|x + y\| = \|x - y\|$) implies that $\|x + y\| = \sqrt{2}$.*

Using this, we can prove the following proposition. The proof is based on ideas due to Gao [2] and [12].

Proposition 5.5. *Let X be a Banach space of $\dim X \geq 3$. Then $J(\lambda, X) = \sqrt{2\lambda^2 - 2\lambda + 1}$ if and only if X is a Hilbert space.*

Proof. Let $x, y \in S_X$. Suppose that $x \perp_I y$. Then for $\lambda = \frac{1}{2}$, we have

$$\frac{1}{2}\|x + y\| = \frac{1}{2}\|x - y\| \leq J\left(\frac{1}{2}, X\right) = \frac{\sqrt{2}}{2}.$$

To show that $\|x + y\| = \sqrt{2}$. Arguing by contradiction, we suppose that $\|x + y\| < \sqrt{2}$. Set $u = \frac{x+y}{\|x+y\|}$ and $v = \frac{x-y}{\|x-y\|}$. Then we obtain $\|u + v\| = \frac{2}{\|x+y\|} = \|u - v\|$, which implies that

$$\|u + v\| = \|u - v\| \leq J\left(\frac{1}{2}, X\right) = \frac{\sqrt{2}}{2}.$$

However, we now have $\|u + v\| = \frac{2}{\|x+y\|} > \sqrt{2}$, and hence

$$\|u + v\| = \|u - v\| > \sqrt{2},$$

which contradicted to our assumption. So we derive $\|x + y\| = \sqrt{2}$, along with the result as stated by Lemma 5.4 verify that the space X is a Hilbert.

For the converse, assume that X is a Hilbert space, applying parallelogram law for $x, y \in S_X$ we have the property

$$\|\lambda x + (1 - \lambda)y\|^2 + \|\lambda x - (1 - \lambda)y\|^2 = 2\lambda^2 + 2(1 - \lambda)^2.$$

Clearly,

$$\begin{aligned}
\min\{\|\lambda x + (1 - \lambda)y\|, \|\lambda x - (1 - \lambda)y\|\} &\leq \frac{1}{2}(\|\lambda x + (1 - \lambda)y\| + \|\lambda x - (1 - \lambda)y\|) \\
&= \frac{1}{2}\sqrt{(\|\lambda x + (1 - \lambda)y\| + \|\lambda x - (1 - \lambda)y\|)^2} \\
&\leq \frac{1}{2}\sqrt{2\|\lambda x + (1 - \lambda)y\|^2 + 2\|\lambda x - (1 - \lambda)y\|^2} \\
&= \sqrt{2\lambda^2 - 2\lambda + 1}.
\end{aligned}$$

Thus, $J(\lambda, X) = \sqrt{2\lambda^2 - 2\lambda + 1}$ directly from the estimate of $J(\lambda, X)$. $\square$

6 Weak orthogonality coefficient

In the last portion, we will see that the constant $J(\lambda, X)$ and the weak orthogonality coefficient have a nice relationship.

We begin by stating two lemmas which will be our main tools.

Lemma 6.1. [36] Assume that X is a Banach space but does not have the condition of weak normal structure, then for every $0 < \varepsilon < 1$, there exists $\{x_n\} \subseteq S_X$ with the condition that $x_n \xrightarrow{w} 0$, and

$$1 - \varepsilon < \|x_{n+1} - x\| < 1 + \varepsilon,$$

where n is sufficiently large, and $x \in \text{co}\{x_k\}_{k=1}^n$.

From Lemma 6.1, we get the following:

Lemma 6.2. Let X be a Banach space without weak normal structure, then for all $0 < \varepsilon < w(X)$, there is an x_n in S_X satisfying

- (i) $1 - \varepsilon \leq \|x_n - x\| \leq 1 + \varepsilon$;
- (ii) $\|x_n - x_1\| \leq 1 + \varepsilon$;
- (iii) $\|x_n + x_1\| \leq \frac{1 + \varepsilon}{w(X) - \varepsilon}$,

where $x \in \text{co}\{x_k\}_{k=1}^n$.

Proof. The proof is a direct consequence of Lemma 6.1 and definition of $w(X)$, so we omit the proof. $\square$

Here we give a sufficient condition that a Banach space has normal structure using the coefficient $w(X)$ and $J(\lambda, X)$.

Theorem 6.3. For a Banach space X , if

$$J(\lambda, X) < 2(1 - \lambda)w(X)$$

for some $\lambda \in \left[\frac{1}{2}, 1\right)$, then X has normal structure.

Proof. As x_1 and x_n are defined in Lemma 6.2, assume that $x = \frac{x_n - x_1}{\|x_n - x_1\|}$ and $y = \frac{(w(X) - \varepsilon)(x_n + x_1)}{\|(w(X) - \varepsilon)(x_n + x_1)\|}$. Then $\|x\| = 1$ and $\|y\| = 1$. We set

$$A = \frac{1}{\|x_n - x_1\|} + \frac{\frac{1-\lambda}{\lambda}w(X) - \frac{1-\lambda}{\lambda}\varepsilon}{\|(w(X) - \varepsilon)(x_n + x_1)\|}$$

and

$$B = \frac{1}{\|x_n - x_1\|} - \frac{\frac{1-\lambda}{\lambda}w(X) - \frac{1-\lambda}{\lambda}\varepsilon}{\|(w(X) - \varepsilon)(x_n + x_1)\|}.$$

So we obtain

$$\begin{aligned} \|\lambda x + (1 - \lambda)y\| &= \lambda \left\| x + \frac{1 - \lambda}{\lambda}y \right\| \\ &= \lambda \left(A \left\| x_n - \frac{B}{A}x_1 \right\| \right) \\ &= \lambda \left(A \left\| x_n - \left(\frac{B}{A}x_1 + \frac{A - B}{A} \cdot 0 \right) \right\| \right) \\ &\geq \lambda A(1 - \varepsilon) \quad (\text{by Lemma 6.2(i)}) \end{aligned}$$

and

$$\begin{aligned}
\|\lambda x - (1 - \lambda)y\| &= \lambda \left\| x - \frac{1 - \lambda}{\lambda} y \right\| \\
&= \lambda \|Ax_1 - Bx_n\| \\
&\geq \lambda (\|Ax_1\| - \|Bx_n\|) \\
&= \lambda (A - B) \\
&= \lambda \left(\frac{2^{\frac{1-\lambda}{\lambda}} w(X) - 2^{\frac{1-\lambda}{\lambda}} \varepsilon}{\|(w(X) - \varepsilon)(x_n + x_1)\|} \right).
\end{aligned}$$

Note that if $\varepsilon \rightarrow 0$, then $A \geq 1 + \frac{1-\lambda}{\lambda} w(X)$. So, from the definition of $J(\lambda, X)$ and let $\varepsilon \rightarrow 0$ we have

$$J(\lambda, X) \geq \min\{\|\lambda x + (1 - \lambda)y\|, \|\lambda x - (1 - \lambda)y\|\} \geq 2(1 - \lambda)w(X).$$

In the case of $J(\lambda, X) < 2(1 - \lambda)w(X)$, we have $J(\lambda, X) < 1$, and so X is uniformly non-square. Therefore, X is reflexive [21], which implies that normal structure equals weak normal structure, as required. $\square$

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