Research Article

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Generalized split null point of sum of monotone operators in Hilbert spaces

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Abstract: In this paper, we introduce a new type of a generalized split monotone variational inclusion (GSMVI) problem in the framework of real Hilbert spaces. By incorporating an inertial extrapolation method and an Halpern iterative technique, we establish a strong convergence result for approximating a solution of GSMVI and fixed point problems of certain nonlinear mappings in the framework of real Hilbert spaces. Many existing results are derived as corollaries to our main result. Furthermore, we present a numerical example to support our main result and propose an open problem for interested researchers in this area. The result obtained in this paper improves and generalizes many existing results in the literature.

Keywords: generalized split monotone variational inclusion, inertial iterative scheme, firmly nonexpansive, fixed point problem

MSC 2020: 47H06, 47H09, 47J05, 47J25

1 Introduction

The concept of split feasibility problem (SFP) introduced by Censor and Elfving [1] in the framework of finite-dimensional Hilbert spaces is to find

$$x^* \in C \quad \text{such that} \quad \mathcal{B}x^* \in D, \quad (1)$$

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where C and D are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 , respectively, and $\mathcal{B} : H_1 \rightarrow H_2$ is a bounded linear operator. The SFP finds its applications in image recovery, signal processing, control theory, data compression, computer tomography and so on (see [2–5]). Due to this advantage, a lot of researchers have studied the SFP in various abstract spaces (see [6–11]). In 2009, Censor and Segal [12] further extended the notion of SFP by introducing the concept of split common fixed point problem (SCFPP) and found

$$x^* \in F(T) \quad \text{such that} \quad \mathcal{B}x^* \in F(S), \quad (2)$$

where $F(T)$, $F(S)$ denote the set of fixed points of two nonlinear operators $T : C \rightarrow C$ and $S : D \rightarrow D$, respectively, and $\mathcal{B} : H_1 \rightarrow H_2$ is a bounded linear operator. For surveys on methods for approximating the solutions of SFP, [13–15] and references therein.

The variational inclusion problems (VIPs) are being used as mathematical models for the study of several optimization problems arising in finance, economics, network, transportation, science and engineering. For a real Hilbert space H , the VIP consists of finding a point $x^* \in H$ such that

$$0 \in Bx^*, \quad (3)$$

where $B : H \rightarrow 2^H$ is a multivalued (point-to-set) operator. Whenever B is a maximal monotone operator, such element $x^* \in H$ is called the zero of the maximal monotone operator B . The VIP for monotone operators was introduced by Martinet [16]. For approximating the zeroes of (3), the relation generated by the fixed equation $x^* = J_\lambda^B x^*$ has been considered. The single-valued mapping $J_\lambda^B = (I + \lambda B)^{-1}$ with $\lambda > 0$ is called the resolvent of the operator B . Thus, the method for approximating the zeros of B is given by $x_1 \in H$ and

$$x_{n+1} = (I + \lambda_n B)^{-1} x_n.$$

Byrne *et al.* in [17] combined the concept of the VIP and SFP to introduce the split null point problem (SNPP). The SNPP is given as the problem of finding

$$0 \in B_1(x^*) \quad \text{such that} \quad 0 \in B_2(\mathcal{B}x^*), \quad (4)$$

where $B_i : H_i \rightarrow 2^{H_i}$, $i = 1, 2$ are maximal monotone operators and $\mathcal{B} : H_1 \rightarrow H_2$ is a bounded linear operator. They considered the following iterative algorithm: for $\lambda > 0$ and $x_1 \in H_1$

$$x_{n+1} = J_\lambda^{B_1} \left(x_n - \gamma \mathcal{B}^* (I - J_\lambda^{B_2}) \mathcal{B} x_n \right) \quad \forall n \in \mathbb{N}, \quad (5)$$

where $\gamma \in \left(0, \frac{2}{\|\mathcal{B}\|^2}\right)$ and established that $\{x_n\}$ converges weakly to a point x^* in the solution set of (4). The SNPP has been considered by many authors (see [18–21] and references therein).

A generalization of the VIP (3) is a problem of finding an element $x^* \in H$ such that

$$0 \in (A + B)x^*, \quad (6)$$

where $A : H \rightarrow H$ is a single-valued operator and $B : H \rightarrow 2^H$ is a multivalued operator. In the case where A and B are monotone operators, the elements in the solution set of (6) are called the zeros of the sum of monotone operators. We note that the solution of (6) is the fixed points of the operator $J_\lambda^B (I - \lambda A)$, when $\lambda > 0$ (see [22]). Several authors have employed different types of iterative methods for approximating the solution of (6) (see for example [23, 24] and references therein).

In 2011, Moudafi [25] introduced the split monotone variational inclusion problem (SMVI) which generalizes the SNPP. Let H_1 and H_2 be two real Hilbert spaces, $A : H_i \rightarrow H_i$, $i = 1, 2$ be single-valued operator, $B_i : H_i \rightarrow 2^{H_i}$, $i = 1, 2$ be set-valued operators and $\mathcal{B} : H_1 \rightarrow H_2$ be bounded linear operator. The SMVI is to find

$$x^* \in H_1 \quad \text{such that} \quad 0 \in (A_1 + B_1)x^*$$

and

$$\mathcal{B}x^* \in H_2 \quad \text{such that} \quad 0 \in (A_2 + B_2)\mathcal{B}x^*. \quad (7)$$

For solving the SMVI, Moudafi [25] proposed the following iterative algorithm: For any $x_1 \in H_1$ and

$$x_{n+1} = T_1(x_n - \gamma \mathcal{B}^* (I - T_2) \mathcal{B} x_n) \quad \forall n \in \mathbb{N}, \quad (8)$$

where $\gamma \in \left(0, \frac{1}{\|B\|^2}\right)$ and $T_i = J_{\lambda}^{B_i}(I - A_i)$ for $i = 1, 2$. They established that $\{x_n\}$ converges weakly to a point x^* in the solution set of (7).

Question: Can further generalize the SMVI and propose a natural modification of an iterative scheme to obtain a strong convergence result for this type of generalization?

On the other hand, the inertial extrapolation method has proven to be an effective way for accelerating the rate of convergence of iterative algorithms. The technique was introduced in 1964 based on a discrete version of a second-order dissipative dynamical system (see [26,27]). The inertial-type algorithm uses its two previous iterates to obtain its next iterate (see [28,29]). In [30], Moudafi and Oliny proposed the following inertial proximal point algorithm for finding the zero of sum of two maximal monotone operators:

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= (I + \lambda_n B)^{-1}(y_n - \lambda_n A y_n), \end{aligned} \quad (9)$$

where $A : H \rightarrow H$ and $B : H \rightarrow 2^H$ are single and multi-valued operators, respectively, and $\lambda_n > 0$. They established the weak convergence result under the assumption that $\lambda_n < \frac{2}{L}$ with L being the Lipschitz constant of A and $\sum_{n=0}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$. Also, Lorenz and Pock [24] introduced a modified forward-backward splitting algorithm with inertial term. They define the algorithm as follows:

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= (I + \lambda_n M^{-1}B)^{-1}(I - \lambda_n M^{-1}A)x_n, \end{aligned} \quad (10)$$

where $\theta_n \in [0, 1)$, M is a linear self adjoint positive definite map and $\lambda_n > 0$ is a step size parameter. They proved that the sequence $\{x_n\}$ generated by (10) converges weakly to the zero of $A + B$.

Motivated by the results discussed above, we study a generalized split monotone variational inclusion (GSMVI) and then introduce an iterative method for approximating the solution of GSMVI and fixed point problem of the composition of two nonlinear mappings in the framework of real Hilbert spaces. We display a numerical example to show the behavior of our result.

For $i = 1, 2, 3, \dots, N$, let H_i be real Hilbert spaces, $A_i : H_i \rightarrow H_i$ be k_i -ism, $B_i : H_i \rightarrow 2^{H_i}$ be a maximal monotone operator and for $i = 1, 2, 3, \dots, N-1$, $\mathcal{B}_i : H_i \rightarrow H_{i+1}$ be bounded linear operators with dual $B_i^* : H_{i+1} \rightarrow H_i$, such that $\mathcal{B}_i \neq \emptyset$, find point $x^* \in H_1$ such that

$$\begin{cases} 0 \in (A_1 + B_1)x^*, \\ \text{such that } 0 \in (A_2 + B_2)\mathcal{B}_1x^*, \\ \text{such that } 0 \in (A_3 + B_3)\mathcal{B}_1\mathcal{B}_2x^*, \\ \vdots \\ \text{such that } 0 \in (A_N + B_N)\mathcal{B}_1\mathcal{B}_2 \cdots \mathcal{B}_{N-1}x^*. \end{cases} \quad (11)$$

The GSMVI 11 is much more general as it includes SFP, SNPP, SMVI as special cases and thus has more real-life applications. In addition, the GSMVI problem has wide applications in many fields such as machine learning, statistical regression, image processing and signal recovery (see [5,31,32] and references therein).

In this paper, we consider the problem of finding a common element in the solution set of the GSMVI and fixed point of a composition of two mappings T_1 and T_2 , where $T_1 : C \rightarrow C$ is an α -strongly quasi-nonexpansive mapping, $T_2 : C \rightarrow C$ is a firmly nonexpansive mapping and C is a nonempty, closed and convex subset of H_1 . That is, find $x^* \in C$ such that

$$\begin{cases} 0 \in F(T_1 \circ T_2) \cap (A_1 + B_1)x^*, \\ \text{such that } 0 \in (A_2 + B_2)\mathcal{B}_1x^*, \\ \text{such that } 0 \in (A_3 + B_3)\mathcal{B}_1\mathcal{B}_2x^*, \\ \vdots \\ \text{such that } 0 \in (A_N + B_N)\mathcal{B}_1\mathcal{B}_2 \cdots \mathcal{B}_{N-1}x^*. \end{cases} \quad (12)$$

We denote by Γ the solution set of problem (12).

As far as we are concerned, the GSMVI is new and has yet to be considered in any of the recent or previous research papers in this direction. For obtaining the solution of (12), we propose a modified Halpern iterative algorithm together with an inertial term and prove a strong convergence result for approximating the solution of Γ . The result in this paper generalizes, unifies and extends many related results in the literature.

2 Preliminaries

In this section, we begin by recalling some known and useful results which are needed in the sequel.

Let H be a real Hilbert space. The set of fixed point of T will be denoted by $F(T)$, that is $F(T) = \{x \in H : Tx = x\}$. We denote strong and weak convergence by “ $\rightarrow$ ” and “ $\rightharpoonup$,” respectively. For any $x, y \in H$ and $\alpha \in [0, 1]$, it is well known that

$$\langle x, y \rangle = \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x - y\|^2), \quad (13)$$

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2. \quad (14)$$

Definition 2.1. Let $T : H \rightarrow H$ be an operator. Then the operator T is called

(a) L -Lipschitz continuous, if

$$\|Tx - Ty\| \leq L\|x - y\|,$$

where $L > 0$ and $x, y \in H$. If $L = 1$, then T is called nonexpansive. Also, if $y \in F(T)$ and $L = 1$, then T is called quasi-nonexpansive;

(b) monotone, if

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in H;$$

(c) firmly nonexpansive, if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \quad \forall x, y \in H;$$

or equivalently

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H;$$

(d) k -inverse strongly monotone (k -ism), if there exists $k > 0$, such that

$$\langle Tx - Ty, x - y \rangle \geq k\|Tx - Ty\|^2, \quad \forall x, y \in H;$$

(e) α -strongly quasi-nonexpansive mapping, with $\alpha > 0$ if

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \alpha\|x - Tx\|^2, \quad \forall z \in F(T), x \in H. \quad (15)$$

It is well known that for any nonexpansive mapping T , the set of fixed point is closed and convex. Also, T satisfies the following inequality:

$$\langle (x - Tx) - (y - Ty), Ty - Tx \rangle \leq \frac{1}{2}\|(Tx - x) - (Ty - y)\|^2, \quad \forall x, y \in H. \quad (16)$$

Thus, for all $x \in H$ and $x^* \in F(T)$, we have that

$$\langle x - Tx, x^* - Tx \rangle \leq \frac{1}{2}\|Tx - x\|^2, \quad \forall x, y \in H. \quad (17)$$

Lemma 2.2. [33] Let T be a k -ism operator, then

1. T is a $\frac{1}{k}$ -Lipschitz continuous monotone operator.

2. If $\lambda \in (0, 2k]$, then $(I - \lambda T)$ is a nonexpansive mapping, where I is the identity operator on H .

A multi-valued operator $B : H \rightarrow 2^H$ is called monotone, if for all $x, y \in H$, $u \in Bx$ and $v \in By$ implies that $\langle u - v, x - y \rangle \geq 0$. A monotone operator B is maximal monotone, if for all $(x, u) \in H \times H$, $\langle u - v, x - y \rangle \geq 0$ for every $(y, v) \in \text{Graph}(B)$ (the graph of B) implies that $u \in Bx$. Let $B : H \rightarrow 2^H$ be a multi-valued maximal monotone operator. The resolvent operator $J_\lambda^B : H \rightarrow H$ associated with B is defined by

$$J_\lambda^B(x) := (I + \lambda B)^{-1}(x), \quad \forall x \in H,$$

where $\lambda > 0$ and I is the identity operator on H .

Lemma 2.3. [34] Let $B : H \rightarrow 2^H$ be a set-valued maximal monotone mapping and $\lambda > 0$, then J_λ^B is single-valued and a firmly nonexpansive mapping.

Lemma 2.4. [35] Let X be a real Banach space, $B : X \rightarrow 2^X$ be a maximal monotone mapping and $A : X \rightarrow X$ be a k -inverse strongly monotone operator on X . Define $T_\lambda := (I + \lambda B)^{-1}(I - \lambda A)$, $\lambda > 0$. Then, the following hold:

1. for $\lambda > 0$, $F(T_\lambda) = (A + B)^{-1}(0)$;
2. for $0 < r \leq \lambda$ and $x \in X$, $\|x - T_r x\| \leq 2\|x - T_\lambda x\|$.

Lemma 2.5. [36] Let $A : H \rightarrow H$ be a k -ism operator and $B : H \rightarrow 2^H$ be a maximal monotone operator. Then, for all $\lambda, \mu > 0$ and $x \in H$, we have

$$\begin{aligned} J_\lambda^B x &= J_\mu^B \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda} \right) J_\lambda^B x \right), \\ J_\lambda^B (I - \lambda A)x &= J_\mu^B (I - \mu A) \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda} \right) J_\lambda^B (I - \lambda A)x \right). \end{aligned} \quad (18)$$

Lemma 2.6. [37] Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{n=1}^\infty \alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n, \quad n \geq 1.$$

If $\limsup_{k \rightarrow \infty} d_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k \rightarrow \infty} \{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then, $\lim_{n \rightarrow \infty} a_n = 0$.

Let H be a real Hilbert space and C a nonempty, closed and convex subset of H . For any $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C\| \leq \|u - y\|, \quad \forall y \in C.$$

P_C is called the metric projection of H onto C . It is well known that P_C is a nonexpansive mapping and that P_C satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2,$$

for all $x, y \in H$. Furthermore, $P_C x$ is characterized by the properties $P_C x \in C$,

$$\langle x - P_C x, P_C x - y \rangle \geq 0$$

for all $y \in C$ and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$$

for all $x \in H$ and $y \in C$.

3 Main result

Lemma 3.1. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H . Let $T_1 : C \rightarrow C$ be an α -strongly quasi-nonexpansive and $T_2 : C \rightarrow C$ be a firmly nonexpansive mapping, such that $F(T_1) \cap F(T_2) \neq \emptyset$. Then $F(T_1 \circ T_2) = F(T_1) \cap F(T_2)$.*

Proof. It is required that we show that $F(T_1 \circ T_2) \subseteq F(T_1) \cap F(T_2)$ and $F(T_1) \cap F(T_2) \subseteq F(T_1 \circ T_2)$. It is easy to see that $F(T_1) \cap F(T_2) \subseteq F(T_1 \circ T_2)$. We now establish that $F(T_1 \circ T_2) \subseteq F(T_1) \cap F(T_2)$. Let $y \in F(T_1 \circ T_2)$ and $x \in F(T_1) \cap F(T_2)$, we have

$$\|y - x\|^2 = \|T_1(T_2y) - T_1x\|^2 \leq \|T_2y - x\|^2 - \alpha\|T_2y - T_1(T_2y)\|^2 \leq \|T_2y - x\|^2. \quad (19)$$

Also, using (19), we have

$$\begin{aligned} \|T_2y - x\|^2 &= \langle T_2y - x, y - x \rangle \\ &= \frac{1}{2}\|T_2y - x\|^2 + \frac{1}{2}\|y - x\|^2 - \frac{1}{2}\|T_2y - y\|^2 \\ &\leq \frac{1}{2}\|T_2y - x\|^2 + \frac{1}{2}\|T_2y - x\|^2 - \frac{1}{2}\|T_2y - y\|^2, \end{aligned} \quad (20)$$

which implies that $\|T_2y - y\|^2 = 0 \Rightarrow \|T_2y - y\| = 0 \Rightarrow T_2y = y$. Using this fact, we have that

$$y = (T_1 \circ T_2)y = T_1(T_2y) = T_1y \Rightarrow y \in F(T_1) \cap F(T_2). \quad (21)$$

Hence, $F(T_1 \circ T_2) \subseteq F(T_1) \cap F(T_2)$, and so $F(T_1 \circ T_2) = F(T_1) \cap F(T_2)$. $\square$

Lemma 3.2. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H . Let $T_1 : C \rightarrow C$ be an α -strongly quasi-nonexpansive mapping and $T_2 : C \rightarrow C$ be a firmly nonexpansive mapping. Then, $T_1 \circ T_2$ is a quasi-nonexpansive mapping.*

Proof. Let $x \in C$ and $y \in F(T_1 \circ T_2)$, using Lemma 3.1, we have that $y \in F(T_1) \cap F(T_2)$, which implies that $y = T_1y$ and $y = T_2y$. Now, observe that

$$\begin{aligned} \|(T_1 \circ T_2)x - y\|^2 &= \|T_1(T_2x) - T_1y\|^2 \\ &\leq \|T_2x - y\|^2 - \alpha\|T_2x - T_1(T_2x)\|^2 \\ &\leq \|T_2x - T_2y\|^2 \\ &\leq \|x - y\|^2 - \|(x - y) - (T_2x - T_2y)\|^2 \\ &= \|x - y\|^2 - \|x - T_2x\|^2 \\ &\leq \|x - y\|^2. \end{aligned} \quad \square$$

Lemma 3.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $A : H \rightarrow H$ be a k -ism operator, $B : H \rightarrow 2^H$ be a maximal monotone operator. Then, for all $\mu \geq \lambda$, $x, y \in H$ and $p \in (A + B)^{-1}(0)$, we have*

1. $\|x - J_\lambda^B(I - \lambda A)x\| \leq 2\|x - J_\mu^B(I - \mu A)x\|$;
2. $\langle x - y, J_\lambda^B(I - \lambda A)x - J_\lambda^B(I - \lambda A)y \rangle \geq \|J_\lambda^B(I - \lambda A)x - J_\lambda^B(I - \lambda A)y\|^2$;
3. $\langle (I - J_\lambda^B(I - \lambda A))x - (I - J_\lambda^B(I - \lambda A))y, x - y \rangle \geq \|(I - J_\lambda^B(I - \lambda A))x - (I - J_\lambda^B(I - \lambda A))y\|^2$;
4. $\|J_\lambda^B(I - \lambda A)x - p\|^2 \leq \|x - p\|^2 - \|x - J_\lambda^B(I - \lambda A)x\|^2$.

Proof. For all $\mu \geq \lambda$, $x, y \in H$ and $p \in (A + B)^{-1}(0)$, we have established the aforementioned results.

1. Using (18), we have

$$\begin{aligned}
 \|x - J_\lambda^B(I - \lambda A)x\| &\leq \|x - J_\mu^B(I - \mu A)x\| + \|J_\mu^B(I - \mu A)x - J_\lambda^B(I - \lambda A)x\| \\
 &= \|x - J_\mu^B(I - \mu A)x\| + \left\| J_\lambda^B(I - \lambda A) \left(\frac{\lambda}{\mu}x + \left(1 - \frac{\lambda}{\mu}\right)J_\mu^B(I - \mu A)x \right) - J_\lambda^B(I - \lambda A)x \right\| \\
 &\leq \|x - J_\mu^B(I - \mu A)x\| + \left\| \frac{\lambda}{\mu}x + \left(1 - \frac{\lambda}{\mu}\right)J_\mu^B(I - \mu A)x - x \right\| \\
 &\leq \|x - J_\mu B(I - \mu A)x\| + \left(1 - \frac{\lambda}{\mu}\right)\|x - J_\mu^B(I - \mu A)x\| \\
 &\leq \|x - J_\mu^B(I - \mu A)x\| + \|x - J_\mu^B(I - \mu A)x\| \\
 &= 2\|x - J_\mu^B(I - \mu A)x\|.
 \end{aligned}$$

2. Using the property of a monotone operator, we have that

$$\frac{1}{\lambda} \langle J_\lambda^B(I - \lambda A)x - J_\lambda^B(I - \lambda A)y, x - J_\lambda^B(I - \lambda A)x - (y - J_\lambda^B(I - \lambda A)y) \rangle \geq 0,$$

clearly, we have that

$$\langle x - y, J_\lambda^B(I - \lambda A)x - J_\lambda^B(I - \lambda A)y \rangle \geq \|J_\lambda^B(I - \lambda A)x - J_\lambda^B(I - \lambda A)y\|^2.$$

3. Using (2), we have that

$$\begin{aligned}
 &\langle (I - J_\lambda^B(I - \lambda A))x - (I - J_\lambda^B(I - \lambda A))y, x - y \rangle \\
 &= \langle (I - J_\lambda^B(I - \lambda A))x - (I - J_\lambda^B(I - \lambda A))y, (I - J_\lambda^B(I - \lambda A))x - (I - J_\lambda^B(I - \lambda A))y \rangle \\
 &\quad + \langle [J_\lambda^B(I - \lambda A)x - J_\lambda^C(I - \lambda A)y] \rangle \\
 &= \|(I - J_\lambda^B(I - \lambda A))x - (I - J_\lambda^B(I - \lambda A))y\|^2 + \langle (I - J_\lambda^B(I - \lambda A))x - (I - J_\lambda^B(I - \lambda A))y, J_\lambda^B(I - \lambda A)x \\
 &\quad - J_\lambda^B(I - \lambda A)y \rangle \\
 &= \|(I - J_\lambda^B(I - \lambda A))x - (I - J_\lambda^B(I - \lambda A))y\|^2 + \langle x - y, (I - J_\lambda^B(I - \lambda A))x - (I - J_\lambda^B(I - \lambda A))y \rangle \\
 &\quad - \|J_\lambda^B(I - \lambda A)x - J_\lambda^B(I - \lambda A)y\| \\
 &\geq \|(I - J_\lambda^B(I - \lambda A))x - (I - J_\lambda^B(I - \lambda A))y\|^2 + \|J_\lambda^B(I - \lambda A)x - J_\lambda^B(I - \lambda A)y\| \\
 &\quad - \|J_\lambda^B(I - \lambda A)x - J_\lambda^B(I - \lambda A)y\| \\
 &= \|(I - J_\lambda^B(I - \lambda A))x - (I - J_\lambda^B(I - \lambda A))y\|^2.
 \end{aligned}$$

4. It is well known that for any $p \in (A + B)^{-1}(0)$, we have $p \in F(J_\lambda^B(I - \lambda A))$. Using (3), we have that

$$\begin{aligned}
 \|J_\lambda^B(I - \lambda A)x - p\|^2 &= \|x - p + J_\lambda^B(I - \lambda A)x - x\|^2 \\
 &= \|x - p\|^2 + \|x - J_\lambda^B(I - \lambda A)x\|^2 - 2\langle x - J_\lambda^B(I - \lambda A)x, x - p \rangle \\
 &= \|x - p\|^2 + \|x - J_\lambda^B(I - \lambda A)x\|^2 - 2\langle (I - J_\lambda^B(I - \lambda A))x - (I - J_\lambda^B(I - \lambda A))p, x - p \rangle \\
 &\leq \|x - p\|^2 + \|x - J_\lambda^B(I - \lambda A)x\|^2 - 2\|x - J_\lambda^B(I - \lambda A)x\|^2 \\
 &= \|x - p\|^2 - \|x - J_\lambda^B(I - \lambda A)x\|^2.
 \end{aligned}$$

□

Lemma 3.4. Let H be a real Hilbert space and C be a nonempty closed and convex subset of H . Let $T_1 : C \rightarrow C$ be an α -strongly quasi-nonexpansive mapping, $T_2 : C \rightarrow C$ be a firmly nonexpansive mapping and $J_\lambda^B(I - \lambda A)$ the resolvent of $(A + B)^{-1}(0)$. Then, $F(U \circ J_\lambda^B(I - \lambda A)) = F(U) \cap F(J_\lambda^B(I - \lambda A))$, where $U = (T_1 \circ T_2)$.

Proof. It is easy to see that $F(U) \cap F(J_\lambda^B(I - \lambda A)) \subset F(U \circ J_\lambda^B(I - \lambda A))$. We now establish that $F(U \circ J_\lambda^B(I - \lambda A)) \subset F(U) \cap F(J_\lambda^B(I - \lambda A))$. Let $y \in F(U \circ J_\lambda^B(I - \lambda A))$ and $x \in F(U) \cap F(J_\lambda^B(I - \lambda A))$, we have

$$\|y - x\|^2 = \|U(J_\lambda^B(I - \lambda A)y) - Ux\|^2 \leq \|J_\lambda^B(I - \lambda A)y - x\|^2. \quad (22)$$

Using Lemma 3.3(4) and (22), we have

$$\|J_\lambda^B(I - \lambda A)y - x\|^2 \leq \|y - x\|^2 - \|y - J_\lambda^B(I - \lambda A)y\|^2 \leq \|J_\lambda^B(I - \lambda A)y - x\|^2 - \|y - J_\lambda^B(I - \lambda A)y\|^2.$$

This implies that $\|y - J_\lambda^B(I - \lambda A)y\| = 0 \Rightarrow y = J_\lambda^B(I - \lambda A)y$. Since, $y = U(J_\lambda^B(I - \lambda A)y) = Uy$. It follows that $y \in F(U) \cap F(J_\lambda^B(I - \lambda A))$. $\square$

Lemma 3.5. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $A : H \rightarrow H$ be a k -ism operator, $B : H \rightarrow 2^H$ be a maximal monotone operator, $T_1 : C \rightarrow C$ be an α -strongly quasi-nonexpansive mapping and $T_2 : C \rightarrow C$ be a firmly nonexpansive mapping. Then, for all $\mu \geq \lambda$, $x \in H$, $p \in F(T_1 \circ T_2)p$ and $p \in (A + B)^{-1}(0)$, we have $\|(U \circ J_\lambda^B(I - \lambda A))x - p\|^2 \leq \|x - p\|^2 - \|x - (U \circ J_\lambda^B(I - \lambda A))x\|^2$, where $U = T_1 \circ T_2$.

Proof. For any $p \in (A + B)^{-1}(0)$ and $p = F(U)$, we have $p \in F(J_\lambda^B(I - \lambda A))$, using Lemma 3.4, we have that $p \in F(U) \cap F(J_\lambda^B(I - \lambda A)) = F(U \circ J_\lambda^B(I - \lambda A))$. Using a similar approach to that in (4) of Lemma 3.3, we obtain the desired result. $\square$

In the sequel, we suppose that H_i , $i = 1, 2, 3$ be real Hilbert spaces. Let $A_i : H_i \rightarrow H_i$ be a k -ism operator and $B_i : H_i \rightarrow 2^{H_i}$ be a maximal monotone operator. For $i = 1, 2$, let $\mathcal{B}_i : H_i \rightarrow H_{i+1}$ be bounded linear operators with $\mathcal{B}_i^*$ the adjoint of $\mathcal{B}_i$. Let $T_1 : C \rightarrow C$ be an α strongly quasi-nonexpansive mapping and $T_2 : C \rightarrow C$ be a firmly nonexpansive mapping. We assume $\Gamma \neq \emptyset$, where Γ denotes the solution set (12) with $N = 3$.

Algorithm 3

Initialization: Given $\gamma_1, \gamma_2 > 0$ and $\theta_n, \eta_n, \beta_n, \alpha_n \in (0, 1)$, for all $n \in \mathbb{N}$, such that $\eta_n \leq \beta_n \leq \alpha_n$ with $\eta_n + \beta_n + \alpha_n = 1$. Let $x_0, x_1, u \in C$ be arbitrary.

Iterative step:

Step 1: Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min \left\{ \theta, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|} \right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise,} \end{cases} \quad (23)$$

where $\theta > 0$ and $\{\varepsilon_n\}$ is a positive sequence such that $\varepsilon_n = o(\alpha_n)$.

Step 2: Set

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then compute

$$\begin{cases} z_{1,n} = w_n - \gamma_1 \mathcal{B}_1^* \mathcal{B}_2^* (I - J_{\lambda_{3,n}}^{B_3} (I - \lambda_{3,n} A_3)) \mathcal{B}_2 \mathcal{B}_1 w_n, \\ z_{2,n} = z_{1,n} - \gamma_2 \mathcal{B}_1^* (I - J_{\lambda_{2,n}}^{B_2} (I - \lambda_{2,n} A_2)) \mathcal{B}_1 z_{1,n}, \\ z_{3,n} = U J_{\lambda_{1,n}}^{B_1} (I - \lambda_{1,n} A_1) z_{2,n}, \end{cases}$$

where $\gamma_1 \in \left(\varepsilon, \frac{\|(I - J_{\lambda_{3,n}}^{B_3} (I - \lambda_{3,n} A_3)) \mathcal{B}_2 \mathcal{B}_1 w_n\|^2}{\|\mathcal{B}_1^* \mathcal{B}_2^* (I - J_{\lambda_{3,n}}^{B_3} (I - \lambda_{3,n} A_3)) \mathcal{B}_2 \mathcal{B}_1 w_n\|^2} - \varepsilon \right)$, $\gamma_2 \in \left(\varepsilon, \frac{\|(I - J_{\lambda_{2,n}}^{B_2} (I - \lambda_{2,n} A_2)) \mathcal{B}_1 z_{1,n}\|^2}{\|\mathcal{B}_1^* (I - J_{\lambda_{2,n}}^{B_2} (I - \lambda_{2,n} A_2)) \mathcal{B}_1 z_{1,n}\|^2} - \varepsilon \right)$, and $U = (T_1 \circ T_2)$.

Step 3: Compute

$$x_{n+1} = \alpha_n u + \beta_n x_n + \eta_n z_{3,n}.$$

Stopping criterion: If $w_n = z_{1,n} = z_{2,n} = z_{3,n} = x_n$, then stop, otherwise, set $n := n + 1$ and go back to **Step 1**.

Remark 3.6. The following are some of the highlights of our method:

1. The choice of stepsize $\gamma_1 \in \left(\varepsilon, \frac{\|(I - J_{\lambda_{3,n}}^{B_3})(I - \lambda_{3,n}A_3))\mathcal{B}_2\mathcal{B}_1w_n\|^2}{\|\mathcal{B}_1^*\mathcal{B}_2^*(I - J_{\lambda_{3,n}}^{B_3})(I - \lambda_{3,n}A_3))\mathcal{B}_2\mathcal{B}_1w_n\|^2} - \varepsilon \right)$ and $\gamma_2 \in \left(\varepsilon, \frac{\|(I - J_{\lambda_{2,n}}^{B_2})(I - \lambda_{2,n}A_2))\mathcal{B}_1z_{1,n}\|^2}{\|\mathcal{B}_1^*(I - J_{\lambda_{2,n}}^{B_2})(I - \lambda_{2,n}A_2))\mathcal{B}_1z_{1,n}\|^2} - \varepsilon \right)$ used in Algorithm 3 does not require the knowledge of the operator norms $\|\mathcal{B}_1\|\|\mathcal{B}_2\|$ and $\|\mathcal{B}_3\|$. It is known that algorithms with parameters depending on operator norm are not easy to execute in practice.
2. It is easy to see from (23) that $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$.

Since $\{\varepsilon_n\}$ is a positive sequence such that $\varepsilon_n = o(\alpha_n)$, which means that $\lim_{n \rightarrow \infty} \frac{\varepsilon_n}{\alpha_n} = 0$. Clearly, we have that $\theta_n \|x_n - x_{n-1}\| \leq \varepsilon_n$ for all $n \in \mathbb{N}$, which together with $\lim_{n \rightarrow \infty} \frac{\varepsilon_n}{\alpha_n} = 0$, it follows that

$$\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \leq \lim_{n \rightarrow \infty} \frac{\varepsilon_n}{\alpha_n} = 0.$$

Lemma 3.7. Assume that $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$. Then, the sequence $\{x_n\}$ generated by Algorithm 3 is bounded and consequently, $\{z_{1,n}\}$, $\{z_{2,n}\}$, $\{z_{3,n}\}$ and $\{w_n\}$ are bounded.

Proof. Let $p \in \Gamma$ and since $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$, there exists $N_1 > 0$ such that $\frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \leq N_1$. Then from Algorithm 3, we have

$$\begin{aligned} \|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\ &= \|x_n - p\|^2 + 2\theta_n \langle x_n - p, x_n - x_{n-1} \rangle + \theta_n^2 \|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - p\|^2 + 2\theta_n \|x_n - p\| \|x_n - x_{n-1}\| + \theta_n^2 \|x_n - x_{n-1}\|^2 \\ &= \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| [2\|x_n - p\| + \theta_n \|x_n - x_{n-1}\|] \\ &= \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| [2\|x_n - p\| + \alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\|] \\ &\leq \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| [2\|x_n - p\| + \alpha_n N_1] \\ &\leq \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| N_2, \end{aligned} \quad (24)$$

where $N_2 := 2\|x_n - x^*\| + \alpha_n N_1$.

Also, using Algorithm 3, we have

$$\begin{aligned} \|z_{1,n} - p\|^2 &= \|w_n - \gamma_1 \mathcal{B}_1^* \mathcal{B}_2^* (I - J_{\lambda_{3,n}}^{B_3})(I - \lambda_{3,n}A_3)) \mathcal{B}_2 \mathcal{B}_1 w_n - p\|^2 \\ &= \|w_n - p\|^2 - 2\gamma_1 \langle w_n - p, \mathcal{B}_1^* \mathcal{B}_2^* (I - J_{\lambda_{3,n}}^{B_3})(I - \lambda_{3,n}A_3)) \mathcal{B}_2 \mathcal{B}_1 w_n \rangle \\ &\quad + \gamma_1^2 \|\mathcal{B}_1^* \mathcal{B}_2^* (I - J_{\lambda_{3,n}}^{B_3})(I - \lambda_{3,n}A_3)) \mathcal{B}_2 \mathcal{B}_1 w_n\|^2 \\ &= \|w_n - p\|^2 - 2\gamma_1 \langle \mathcal{B}_2 \mathcal{B}_1 (w_n - p), (I - J_{\lambda_{3,n}}^{B_3})(I - \lambda_{3,n}A_3)) \mathcal{B}_2 \mathcal{B}_1 w_n \rangle \\ &\quad + \gamma_1^2 \|\mathcal{B}_1^* \mathcal{B}_2^* (I - J_{\lambda_{3,n}}^{B_3})(I - \lambda_{3,n}A_3)) \mathcal{B}_2 \mathcal{B}_1 w_n\|^2 \\ &\leq \|w_n - p\|^2 - 2\gamma_1 \|(I - J_{\lambda_{3,n}}^{B_3})(I - \lambda_{3,n}A_3)) \mathcal{B}_2 \mathcal{B}_1 w_n\|^2 + \gamma_1^2 \|\mathcal{B}_1^* \mathcal{B}_2^* (I - J_{\lambda_{3,n}}^{B_3})(I - \lambda_{3,n}A_3)) \mathcal{B}_2 \mathcal{B}_1 w_n\|^2 \end{aligned} \quad (25)$$

using $\gamma_1 \in \left(\varepsilon, \frac{\|(I - J_{\lambda_{3,n}}^{B_3})(I - \lambda_{3,n}A_3)) \mathcal{B}_2 \mathcal{B}_1 w_n\|^2}{\|\mathcal{B}_1^* \mathcal{B}_2^* (I - J_{\lambda_{3,n}}^{B_3})(I - \lambda_{3,n}A_3)) \mathcal{B}_2 \mathcal{B}_1 w_n\|^2} - \varepsilon \right)$, and (25) becomes

$$\|z_{1,n} - p\|^2 = \|w_n - p\|^2 - (\gamma_1^2 + 2\varepsilon) \left[\|\mathcal{B}_1^* \mathcal{B}_2^* (I - J_{\lambda_{3,n}}^{B_3})(I - \lambda_{3,n}A_3)) \mathcal{B}_2 \mathcal{B}_1 w_n\|^2 \right] \leq \|w_n - p\|^2. \quad (26)$$

Similarly, using Algorithm 3, we have

$$\begin{aligned} \|z_{2,n} - p\|^2 &= \|z_{1,n} - \gamma_2 \mathcal{B}_1^* (I - J_{\lambda_{2,n}}^{B_2})(I - \lambda_{2,n}A_2)) \mathcal{B}_1 z_{1,n} - p\|^2 \\ &= \|z_{1,n} - p\|^2 - 2\gamma_2 \langle z_{1,n} - p, \mathcal{B}_1^* (I - J_{\lambda_{2,n}}^{B_2})(I - \lambda_{2,n}A_2)) \mathcal{B}_1 z_{1,n} \rangle + \gamma_2^2 \|\mathcal{B}_1^* (I - J_{\lambda_{2,n}}^{B_2})(I - \lambda_{2,n}A_2)) \mathcal{B}_1 z_{1,n}\|^2 \\ &\leq \|z_{1,n} - p\|^2 - 2\gamma_2 \|(I - J_{\lambda_{2,n}}^{B_2})(I - \lambda_{2,n}A_2)) \mathcal{B}_1 z_{1,n}\|^2 + \gamma_2^2 \|\mathcal{B}_1^* (I - J_{\lambda_{2,n}}^{B_2})(I - \lambda_{2,n}A_2)) \mathcal{B}_1 z_{1,n}\|^2, \end{aligned} \quad (27)$$

since $\gamma_2 \in \left(\varepsilon, \frac{\|(I - J_{\lambda_{2,n}}^{B_2})(I - \lambda_{2,n}A_2)\| \mathcal{B}_1 z_{1,n}\|^2}{\|\mathcal{B}_1^*(I - J_{\lambda_{2,n}}^{B_2})(I - \lambda_{2,n}A_2)\| \mathcal{B}_1 z_{1,n}\|^2} - \varepsilon \right)$, and we have from (27) that

$$\|z_{2,n} - p\|^2 = \|z_{1,n} - p\|^2 - (\gamma_2 + 2\varepsilon) \left[\|\mathcal{B}_1^*(I - J_{\lambda_{2,n}}^{B_2})(I - \lambda_{2,n}A_2)\| \mathcal{B}_1 z_{1,n}\|^2 \right] \leq \|z_{1,n} - p\|^2 \leq \|w_n - p\|^2. \quad (28)$$

Furthermore, using Algorithm 3, Lemma 3.2 and (28), we have

$$\begin{aligned} \|z_{3,n} - p\|^2 &= \|UJ_{\lambda_{1,n}}^{B_1}(I - \lambda_{1,n}A_1)z_{2,n} - p\|^2 \leq \|J_{\lambda_{1,n}}^{B_1}(I - \lambda_{1,n}A_1)z_{2,n} - p\|^2 \\ &\leq \|z_{2,n} - p\|^2 \leq \|z_{1,n} - p\|^2 \leq \|w_n - p\|^2. \end{aligned} \quad (29)$$

Finally, using Algorithm 3 and (24), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n u + \beta_n x_n + \eta_n z_{3,n} - p\|^2 \\ &\leq \alpha_n \|u - p\|^2 + \beta_n \|x_n - p\|^2 + \eta_n \|z_{3,n} - p\|^2 \\ &\leq \alpha_n \|u - p\|^2 + (1 - \alpha_n - \eta_n) \|x_n - p\|^2 + \eta_n (\|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| N_2) \\ &\leq \alpha_n \|u - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| N_2 \\ &\leq \max\{\|x_n - p\|^2, \|u - p\|^2\} + \theta_n \|x_n - x_{n-1}\| N_2 \\ &\leq \max\{\max\{\|x_n - p\|^2, \|u - p\|^2\} + \theta_{n-1} \|x_{n-1} - x_{n-2}\| N_2, \|u - p\|^2\} + \theta_n \|x_n - x_{n-1}\| N_2 \\ &= \max\{\|x_n - p\|^2, \|u - p\|^2\} + \alpha_{n-1} \frac{\theta_{n-1}}{\alpha_{n-1}} \|x_{n-1} - x_{n-2}\| N_2 + \alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| N_2, \end{aligned} \quad (30)$$

using the fact that $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$, there exists $N_1 > 0$ such that $\frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \leq N_1$, (30) becomes

$$\|x_{n+1} - p\|^2 \leq \max\{\|x_n - p\|^2, \|u - p\|^2\} + N_4, \quad (31)$$

where $N_4 = N_3 N_2 + N_2 N_1$, thus $\{x_n\}$ generated by Algorithm 3 is bounded and consequently $\{z_{1,n}\}$, $\{z_{2,n}\}$, $\{z_{3,n}\}$ and $\{w_n\}$ are bounded. $\square$

Theorem 3.8. Let $\{x_n\}$ be the sequence generated by Algorithm 3. If $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$, then, $\{x_n\}$ converges strongly to $p \in P_{\Gamma} u$.

Proof. Let $p \in \Gamma$, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n u + \beta_n x_n + \eta_n z_{3,n} - p\|^2 \\ &= \|\alpha_n(u - p) + \beta_n(x_n - p) + \eta_n(z_{3,n} - p)\|^2 \\ &\leq \|\beta_n(x_n - p) + \eta_n(z_{3,n} - p)\|^2 + 2\alpha_n \langle u - p, x_{n+1} - p \rangle \\ &\leq \beta_n^2 \|x_n - p\|^2 + \eta_n^2 \|z_{3,n} - p\|^2 + 2\beta_n \eta_n \|x_n - p\| \|z_{3,n} - p\| + 2\alpha_n \langle u - p, x_{n+1} - p \rangle \\ &\leq \beta_n^2 \|x_n - p\|^2 + \eta_n^2 \|z_{3,n} - p\|^2 + \beta_n \eta_n [\|x_n - p\|^2 + \|z_{3,n} - p\|^2] + 2\alpha_n \langle u - p, x_{n+1} - p \rangle \\ &= \beta_n(\beta_n + \eta_n) \|x_n - p\|^2 + \eta_n(\eta_n + \beta_n) \|z_{3,n} - p\|^2 + 2\alpha_n \langle u - p, x_{n+1} - p \rangle \\ &= \beta_n(1 - \alpha_n) \|x_n - p\|^2 + \eta_n(1 - \alpha_n) \|z_{3,n} - p\|^2 + 2\alpha_n \langle u - p, x_{n+1} - p \rangle \\ &\leq \beta_n(1 - \alpha_n) \|x_n - p\|^2 + \eta_n(1 - \alpha_n) \|z_{3,n} - x_n\|^2 + 2\eta_n(1 - \alpha_n) \|z_{3,n} - x_n\| \|x_n - p\| \\ &\quad + \eta_n(1 - \alpha_n) \|x_n - p\|^2 + 2\alpha_n \langle u - p, x_{n+1} - p \rangle \\ &\leq (\beta_n + \eta_n) \|x_n - p\|^2 + \eta_n(1 - \alpha_n) \|z_{3,n} - x_n\|^2 + 2\eta_n(1 - \alpha_n) \|z_{3,n} - x_n\| \|x_n - p\| \\ &\quad + 2\alpha_n \langle u - p, x_{n+1} - p \rangle \\ &\leq (\beta_n + \eta_n) \|x_n - p\|^2 + \alpha_n(1 - \alpha_n) \|z_{3,n} - x_n\|^2 + 2\alpha_n(1 - \alpha_n) \|z_{3,n} - x_n\| \|x_n - p\| \\ &\quad + 2\alpha_n \langle u - p, x_{n+1} - p \rangle \\ &\leq (1 - \alpha_n) \|x_n - p\|^2 + \alpha_n [\|z_{3,n} - x_n\|^2 + 2\|z_{3,n} - x_n\| \|x_n - p\| + 2\langle u - p, x_{n+1} - p \rangle] \\ &= (1 - \alpha_n) \|x_n - p\|^2 + \alpha_n \delta_n, \end{aligned} \quad (32)$$

where $\delta_n := \|z_{3,n} - x_n\|^2 + 2\|z_{3,n} - x_n\|\|x_n - p\| + 2\langle u - p, x_{n+1} - p \rangle$. According to Lemma 3.1, to conclude our proof, it is sufficient to establish that $\limsup_{k \rightarrow \infty} \delta_{n_k} \leq 0$ for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying the condition:

$$\liminf_{k \rightarrow \infty} \{\|x_{n_{k+1}} - p\| - \|x_{n_k} - p\|\} \geq 0. \quad (33)$$

To establish that $\limsup_{k \rightarrow \infty} \delta_{n_k} \leq 0$, we suppose that for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ such that (33) holds. Then,

$$\liminf_{k \rightarrow \infty} \{\|x_{n_{k+1}} - p\|^2 - \|x_{n_k} - p\|^2\} = \liminf_{k \rightarrow \infty} \{(\|x_{n_{k+1}} - p\| - \|x_{n_k} - p\|)(\|x_{n_{k+1}} - p\| + \|x_{n_k} - p\|)\} \geq 0. \quad (34)$$

Now, using Algorithm 3, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n u + \beta_n x_n + \eta_n z_{3,n} - p\|^2 \\ &= \|\alpha_n(u - p) + \beta_n(x_n - p) + \eta_n(z_{3,n} - p)\|^2 \\ &\leq \|\beta_n(x_n - p) + \eta_n(z_{3,n} - p)\|^2 + \alpha_n^2 \|u - p\|^2 \\ &\leq \beta_n \|x_n - p\|^2 + \eta_n \|z_{3,n} - p\|^2 - \beta_n \eta_n \|z_{3,n} - x_n\|^2 + \alpha_n^2 \|u - p\|^2 \\ &\leq \eta_n [\|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| N_2] + \beta_n \|x_n - p\|^2 - \beta_n \eta_n \|z_{3,n} - x_n\|^2 + \alpha_n^2 \|u - p\|^2 \\ &= (\eta_n + \beta_n) \|x_n - p\|^2 + \eta_n \theta_n \|x_n - x_{n-1}\| N_2 + \alpha_n^2 \|u - p\|^2 - \beta_n \eta_n \|z_{3,n} - x_n\|^2 \\ &\leq \|x_n - p\|^2 + \alpha_n \theta_n \|x_n - x_{n-1}\| N_2 + \alpha_n^2 \|u - p\|^2 - \beta_n \eta_n \|z_{3,n} - x_n\|^2, \end{aligned} \quad (35)$$

this implies from (34)

$$\begin{aligned} \limsup_{k \rightarrow \infty} [\beta_{n_k} \eta_{n_k} \|z_{3,n_k} - x_{n_k}\|^2] &\leq \limsup_{k \rightarrow \infty} [\|x_{n_k} - p\|^2 - \|x_{n_{k+1}} - p\|^2 + \alpha_{n_k} \theta_{n_k} \|x_{n_k} - x_{n_{k-1}}\| N_2 + \alpha_{n_k}^2 \|u - p\|^2] \\ &\leq -\liminf_{k \rightarrow \infty} [\|x_{n_k} - p\|^2 - \|x_{n_{k+1}} - p\|^2] \leq 0, \end{aligned} \quad (36)$$

which gives

$$\lim_{k \rightarrow \infty} \|z_{3,n_k} - x_{n_k}\| = 0. \quad (37)$$

Also, using Algorithm 3 and Lemma 3.5, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n u + \beta_n x_n + \eta_n z_{3,n} - p\|^2 \\ &= \|\alpha_n(u - p) + \beta_n(x_n - p) + \eta_n(z_{3,n} - p)\|^2 \\ &\leq \alpha_n \|u - p\|^2 + \beta_n \|x_n - p\|^2 + \eta_n \|z_{3,n} - p\|^2 \\ &= \alpha_n \|u - p\|^2 + \beta_n \|x_n - p\|^2 + \eta_n \|UJ_{\lambda_{1,n}}^{B_1}(I - \lambda_{1,n} A_3)z_{2,n} - p\|^2 \\ &\leq \alpha_n \|u - p\|^2 + \beta_n \|x_n - p\|^2 + \eta_n [\|z_{2,n} - p\|^2 - \|z_{2,n} - UJ_{\lambda_{1,n}}^{B_1}(I - \lambda_{1,n} A_1)z_{2,n}\|^2] \\ &\leq \alpha_n \|u - p\|^2 + \beta_n \|x_n - p\|^2 + \eta_n \|w_n - p\|^2 - \eta_n \|z_{2,n} - UJ_{\lambda_{1,n}}^{B_1}(I - \lambda_{1,n} A_1)z_{2,n}\|^2 \\ &= \alpha_n \|u - p\|^2 + (\beta_n + \eta_n) \|x_n - p\|^2 + \eta_n \theta_n \|x_n - x_{n-1}\| N_2 - \eta_n \|z_{2,n} - UJ_{\lambda_{1,n}}^{B_1}(I - \lambda_{1,n} A_1)z_{2,n}\|^2 \\ &\leq \alpha_n \|u - p\|^2 + \|x_n - p\|^2 + \alpha_n \theta_n \|x_n - x_{n-1}\| N_2 - \eta_n \|z_{2,n} - UJ_{\lambda_{1,n}}^{B_1}(I - \lambda_{1,n} A_1)z_{2,n}\|^2. \end{aligned} \quad (38)$$

This implies from (34) that

$$\begin{aligned} \limsup_{k \rightarrow \infty} [\eta_{n_k} \|z_{2,n_k} - UJ_{\lambda_{1,n}}^{B_1}(I - \lambda_{1,n} A_1)z_{2,n_k}\|^2] \\ \leq \limsup_{k \rightarrow \infty} [\|x_{n_k} - p\|^2 - \|x_{n_{k+1}} - p\|^2 + \alpha_{n_k} \theta_{n_k} \|x_{n_k} - x_{n_{k-1}}\| N_2 + \alpha_{n_k} \|u - p\|^2] \\ \leq -\liminf_{k \rightarrow \infty} [\|x_{n_k} - p\|^2 - \|x_{n_{k+1}} - p\|^2] \leq 0, \end{aligned} \quad (39)$$

which gives

$$\lim_{k \rightarrow \infty} \|z_{2,n_k} - UJ_{\lambda_{1,n_k}}^{B_1}(I - \lambda_{1,n_k}A_1)z_{2,n_k}\| = 0. \quad (40)$$

By Lemma 3.3 (1), we have

$$\lim_{k \rightarrow \infty} \|z_{2,n_k} - UJ_{\lambda_1}^{B_1}(I - \lambda_1A_1)z_{2,n_k}\| = 0. \quad (41)$$

More so, using Algorithm 3 and (28), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n u + \beta_n x_n + \eta_n z_{3,n} - p\|^2 \\ &= \|\alpha_n(u - p) + \beta_n(x_n - p) + \eta_n(z_{3,n} - p)\|^2 \\ &\leq \alpha_n \|u - p\|^2 + \beta_n \|x_n - p\|^2 + \eta_n \|z_{3,n} - p\|^2 \\ &\leq \alpha_n \|u - p\|^2 + \beta_n \|x_n - p\|^2 + \eta_n \|z_{2,n} - p\|^2 \\ &\leq \alpha_n \|u - p\|^2 + \beta_n \|x_n - p\|^2 + \eta_n \|z_{1,n} - p\|^2 - (\gamma_2 + 2\varepsilon) \left[\|\mathcal{B}_1^*(I - J_{\lambda_{2,n}}^{B_2}(I - \lambda_{2,n}A_2))\mathcal{B}_1 z_{1,n}\|^2 \right] \\ &\leq \alpha_n \|u - p\|^2 + \beta_n \|x_n - p\|^2 + \eta_n \|w_n - p\|^2 - \eta_n (\gamma_2 + 2\varepsilon) \left[\|\mathcal{B}_1^*(I - J_{\lambda_{2,n}}^{B_2}(I - \lambda_{2,n}A_2))\mathcal{B}_1 z_{1,n}\|^2 \right] \\ &= \alpha_n \|u - p\|^2 + (\beta_n + \eta_n) \|x_n - p\|^2 + \eta_n \theta_n \|x_n - x_{n-1}\| \\ &\quad - \eta_n (\gamma_2 + 2\varepsilon) \left[\|\mathcal{B}_1^*(I - J_{\lambda_{2,n}}^{B_2}(I - \lambda_{2,n}A_2))\mathcal{B}_1 z_{1,n}\|^2 \right] \\ &\leq \alpha_n \|u - p\|^2 + \|x_n - p\|^2 + \alpha_n \theta_n \|x_n - x_{n-1}\| - \eta_n (\gamma_2 + 2\varepsilon) \left[\|\mathcal{B}_1^*(I - J_{\lambda_{2,n}}^{B_2}(I - \lambda_{2,n}A_2))\mathcal{B}_1 z_{1,n}\|^2 \right], \end{aligned} \quad (42)$$

this implies from (34)

$$\begin{aligned} &\limsup_{k \rightarrow \infty} \left[\eta_{n_k} (\gamma_2 + 2\varepsilon) \left[\|\mathcal{B}_1^*(I - J_{\lambda_{2,n_k}}^{B_2}(I - \lambda_{2,n_k}A_2))\mathcal{B}_1 z_{1,n_k}\|^2 \right] \right] \\ &\leq \limsup_{k \rightarrow \infty} \left[\|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2 + \alpha_{n_k} \theta_{n_k} \|x_{n_k} - x_{n_k-1}\| N_2 + \alpha_{n_k} \|u - p\|^2 \right] \\ &\leq -\liminf_{k \rightarrow \infty} \left[\|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2 \right] \leq 0, \end{aligned} \quad (43)$$

which gives

$$\lim_{k \rightarrow \infty} \|\mathcal{B}_1^*(I - J_{\lambda_{2,n_k}}^{B_2}(I - \lambda_{2,n_k}A_2))\mathcal{B}_1 z_{1,n_k}\| = 0. \quad (44)$$

More so, using Algorithm 3 and (26), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n u + \beta_n x_n + \eta_n z_{3,n} - p\|^2 \\ &= \|\alpha_n(u - p) + \beta_n(x_n - p) + \eta_n(z_{3,n} - p)\|^2 \\ &\leq \alpha_n \|u - p\|^2 + \beta_n \|x_n - p\|^2 + \eta_n \|z_{3,n} - p\|^2 \\ &\leq \alpha_n \|u - p\|^2 + \beta_n \|x_n - p\|^2 + \eta_n \|z_{1,n} - p\|^2 \\ &\leq \alpha_n \|u - p\|^2 + \beta_n \|x_n - p\|^2 + \eta_n \|w_n - p\|^2 \\ &\quad - \eta_n (\gamma_1^2 + 2\varepsilon) \left[\|\mathcal{B}_1^* \mathcal{B}_2^*(I - J_{\lambda_{3,n}}^{B_3}(I - \lambda_{3,n}A_3))\mathcal{B}_2 \mathcal{B}_1 w_n\|^2 \right] \\ &\leq \alpha_n \|u - p\|^2 + \|x_n - p\|^2 + \alpha_n \theta_n \|x_n - x_{n-1}\| \\ &\quad - \eta_n (\gamma_1^2 + 2\varepsilon) \left[\|\mathcal{B}_1^* \mathcal{B}_2^*(I - J_{\lambda_{3,n}}^{B_3}(I - \lambda_{3,n}A_3))\mathcal{B}_2 \mathcal{B}_1 w_n\|^2 \right], \end{aligned} \quad (45)$$

this implies from (34)

$$\begin{aligned} &\limsup_{k \rightarrow \infty} \left[\eta_{n_k} (\gamma_1^2 + 2\varepsilon) \left[\|\mathcal{B}_1^* \mathcal{B}_2^*(I - J_{\lambda_{3,n_k}}^{B_3}(I - \lambda_{3,n_k}A_3))\mathcal{B}_2 \mathcal{B}_1 w_{n_k}\|^2 \right] \right] \\ &\leq \limsup_{k \rightarrow \infty} \left[\|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2 + \alpha_{n_k} \theta_{n_k} \|x_{n_k} - x_{n_k-1}\| N_2 + \alpha_{n_k} \|u - p\|^2 \right] \\ &\leq -\liminf_{k \rightarrow \infty} \left[\|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2 \right] \leq 0, \end{aligned} \quad (46)$$

which gives

$$\lim_{k \rightarrow \infty} \|\mathcal{B}_1^* \mathcal{B}_2^* (I - J_{\lambda_{3,n_k}}^{B_3} (I - \lambda_{3,n_k} A_3)) \mathcal{B}_2 \mathcal{B}_1 w_{n_k}\| = 0. \quad (47)$$

Using (47), we have that

$$\|z_{1,n_k} - w_{n_k}\| = \|w_{n_k} - \gamma_1 \mathcal{B}_1^* \mathcal{B}_2^* (I - J_{\lambda_{3,n_k}}^{B_3} (I - \lambda_{3,n_k} A_3)) \mathcal{B}_2 \mathcal{B}_1 w_{n_k} - w_{n_k}\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (48)$$

Using this, (25) and (47), we have

$$\begin{aligned} & 2\gamma_1 \|(I - J_{\lambda_{3,n_k}}^{B_3} (I - \lambda_{3,n_k} A_3)) \mathcal{B}_2 \mathcal{B}_1 w_{n_k}\| \\ & \leq \|w_{n_k} - p\|^2 - \|z_{1,n_k} - p\|^2 + \gamma_1^2 \|\mathcal{B}_2^* \mathcal{B}_1^* (I - J_{\lambda_{3,n_k}}^{B_3} (I - \lambda_{3,n_k} A_3)) \mathcal{B}_2 \mathcal{B}_1 w_{n_k}\| \\ & \leq \|w_{n_k} - z_{1,n_k}\| (\|w_{n_k} - p\| + \|z_{1,n_k} - p\|) + \gamma_1^2 \|\mathcal{B}_2^* \mathcal{B}_1^* (I - J_{\lambda_{3,n_k}}^{B_3} (I - \lambda_{3,n_k} A_3)) \mathcal{B}_2 \mathcal{B}_1 w_{n_k}\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence,

$$\lim_{k \rightarrow \infty} \|(I - J_{\lambda_{3,n_k}}^{B_3} (I - \lambda_{3,n_k} A_3)) \mathcal{B}_2 \mathcal{B}_1 w_{n_k}\| = 0,$$

and by Lemma 3.3 (1), we obtain

$$\lim_{k \rightarrow \infty} \|(I - J_{\lambda_3}^{B_3} (I - \lambda_3 A_3)) \mathcal{B}_2 \mathcal{B}_1 w_{n_k}\| = 0. \quad (49)$$

Also, from (40), we obtain that

$$\|z_{2,n_k} - z_{1,n_k}\| = \|z_{1,n_k} - \gamma_2 \mathcal{B}_1^* (I - J_{\lambda_{2,n_k}}^{B_2} (I - \lambda_{2,n_k} A_2)) \mathcal{B}_1 z_{1,n_k} - z_{1,n_k}\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (50)$$

Now from (27), (40) and (50), we have

$$\begin{aligned} & 2\gamma_2 \|(I - J_{\lambda_{2,n_k}}^{B_2} (I - \lambda_{2,n_k} A_2)) \mathcal{B}_1 z_{1,n_k}\|^2 \\ & \leq \|z_{1,n_k} - p\|^2 - \|z_{2,n_k} - p\|^2 + \gamma_2 \|\mathcal{B}_1^* (I - J_{\lambda_{2,n_k}}^{B_2} (I - \lambda_{2,n_k} A_2)) \mathcal{B}_1 z_{1,n_k}\|^2 \\ & \leq \|z_{1,n_k} - z_{2,n_k}\| (\|z_{1,n_k} - p\| + \|z_{2,n_k} - p\|) + \gamma_2 \|\mathcal{B}_1^* (I - J_{\lambda_{2,n_k}}^{B_2} (I - \lambda_{2,n_k} A_2)) \mathcal{B}_1 z_{1,n_k}\|^2 \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus,

$$\lim_{k \rightarrow \infty} \|(I - J_{\lambda_{2,n_k}}^{B_2} (I - \lambda_{2,n_k} A_2)) \mathcal{B}_1 z_{1,n_k}\| = 0, \quad (51)$$

again by Lemma 3.3 (1), we obtain

$$\lim_{k \rightarrow \infty} \|(I - J_{\lambda_2}^{B_2} (I - \lambda_2 A_2)) \mathcal{B}_1 z_{1,n_k}\| = 0. \quad (52)$$

We also have

$$\|z_{2,n_k} - w_{n_k}\| = \|z_{2,n_k} - z_{1,n_k}\| + \|z_{1,n_k} - w_{n_k}\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (53)$$

It is easy to see that, as $k \rightarrow \infty$, we have

$$\lim_{k \rightarrow \infty} \|w_{n_k} - x_{n_k}\| = \theta_{n_k} \|x_{n_k} - x_{n_k-1}\| = \alpha_{n_k} \cdot \frac{\theta_{n_k}}{\alpha_{n_k}} \|x_{n_k} - x_{n_k-1}\| = 0. \quad (54)$$

In addition, we have by triangular inequality that

$$\begin{cases} \|z_{2,n_k} - x_{n_k}\| = \|z_{2,n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \rightarrow 0, & \text{as } n \rightarrow \infty, \\ \|z_{3,n_k} - z_{2,n_k}\| = \|z_{3,n_k} - x_{n_k}\| + \|x_{n_k} - z_{2,n_k}\| \rightarrow 0, & \text{as } n \rightarrow \infty, \\ \|z_{1,n_k} - x_{n_k}\| = \|z_{2,n_k} - x_{n_k}\| + \|z_{1,n_k} - z_{2,n_k}\| \rightarrow 0, & \text{as } n \rightarrow \infty. \end{cases} \quad (55)$$

From the hypothesis and (37), we have that

$$\begin{aligned}\|x_{n_k+1} - z_{3,n_k}\| &= \|\alpha_n u + \beta_n x_n + \eta_n z_{3,n_k} - z_{3,n_k}\| \\ &\leq \alpha_n \|u - z_{3,n_k}\| + \beta_n \|x_{n_k} - z_{3,n_k}\| + \eta_n \|z_{3,n_k} - z_{3,n_k}\| \rightarrow 0, \quad \text{as } k \rightarrow \infty.\end{aligned}\quad (56)$$

Using (56) and (37), it is easy to see that

$$\|x_{n_k+1} - x_{n_k}\| \leq \|x_{n_k+1} - z_{3,n_k}\| + \|z_{3,n_k} - x_{n_k}\| \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad (57)$$

Since $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that converges weakly to x^* such that

$$\limsup_{k \rightarrow \infty} \langle u - p, x_{n_k} - p \rangle = \lim_{j \rightarrow \infty} \langle u - p, x_{n_{k_j}} - p \rangle = \langle u - p, x^* - p \rangle. \quad (58)$$

Thus by $x_{n_k} \rightharpoonup x^*$, (37), (54), we obtain that $\{z_{3,n_{k_j}}\}$, $\{w_{n_{k_j}}\}$, converge weakly to x^* . Also, from (55), we obtain that $\{z_{2,n_{k_j}}\}$ and $\{z_{1,n_{k_j}}\}$ converge weakly to x^* . It follows from this and the fact that $\mathcal{B}_1$ and $\mathcal{B}_2$ are bounded linear operator that $\mathcal{B}_1 z_{1,n_k} \rightharpoonup \mathcal{B}_1 x^*$ and $\mathcal{B}_2 \mathcal{B}_1 w_{n_k} \rightharpoonup \mathcal{B}_2 \mathcal{B}_1 x^*$. Thus, we have by (41) and Lemma 3.4 that $x^* \in F(U) = F(T_1 \circ T_2) \cap (A_1 + B_1)^{-1}(0)$. Also by using Lemma 19(1) with (49) and (52), respectively, we obtain $\mathcal{B}_1 x^* \in (A_2 + B_2)^{-1}(0)$ and $\mathcal{B}_2 \mathcal{B}_1 x^* \in (A_3 + B_3)^{-1}(0)$. Hence, $x^* \in \Gamma$.

Hence, since $p = P_\Gamma u$, we have obtained from (58) that

$$\limsup_{k \rightarrow \infty} \langle u - p, x_{n_k} - p \rangle = \langle u - p, x^* - p \rangle \leq 0, \quad (59)$$

which implies by (57) that

$$\limsup_{k \rightarrow \infty} \langle u - p, x_{n_k+1} - p \rangle \leq 0. \quad (60)$$

Using (37) and (60), we have that $\limsup_{k \rightarrow \infty} \delta_{n_k} := \|z_{3,n_k} - x_{n_k}\|^2 + 2\|z_{3,n_k} - x_{n_k}\| \|x_{n_k} - p\| + 2\langle u - p, x_{n_k+1} - p \rangle \leq 0$. Thus, the last part of Lemma 3.1 is achieved. Hence, we have that $\lim_{n \rightarrow \infty} \|x_n - p\| = 0$. Thus, $\{x_n\}$ converges strongly to $p = P_\Gamma u$. $\square$

4 Numerical example

Example 4.1. Let $H_1 = H_2 = H_3 = C = \mathbb{R}^2$ with the Euclidean norm and inner product defined by $\langle x, y \rangle = x_1 y_1 + x_2 y_2$ for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. Define the bounded linear operators $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, by $\mathcal{B}_1 = (2x_1 + x_2, x_1 - 2x_2)$ and $\mathcal{B}_2 = (x_1 + 2x_2, x_1 + x_2)$. For $i = 1, 2, 3$, let the operators $A_i : H_i \rightarrow H_i$ and $B_i : H_i \rightarrow 2^{H_i}$ be defined as follows:

$$A_1 = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}.$$

Let $U : C \rightarrow C$ be defined by $U(y) = \frac{y}{2}$ for all $y \in \mathbb{R}^2$. Choose $\lambda_{i,n} = \frac{2n}{3in+5}$, $\gamma_1 = 0.36$, $\gamma_2 = 0.10$, $\alpha_n = \frac{2n-2}{7n^3+2n-1}$, $\beta_n = \frac{7n^3}{7n^3+2n-1}$, $\eta_n = \frac{1}{7n^3+2n-1}$ and $\theta_n = \frac{1}{30n+5}$.

For initial values

- (i) $x_0 = (-1.5, 0.1)^T$, $x_1 = (-3, 2.5)^T$ and $u_0 = (-0.5, 1)^T$;
- (ii) $x_0 = (-5, 5)^T$, $x_1 = (-10, 5)^T$ and $u_0 = (-0.5, 1)^T$;
- (iii) $x_0 = (-5, 5)^T$, $x_1 = (-10, 5)^T$ and $u_0 = (10, 10)^T$, see Figure 1.

Figure 1: Top: (i); bottom left: (ii); bottom right: (iii).

Example 4.2. Let $H_1 = H_2 = H_3 = C = L^2([0, 1])$ be the space of function with norm and inner product defined, respectively, by $\|x\| = \left(\int_0^1 |x(t)|^2 dt \right)^{\frac{1}{2}}$, $\forall t \in L^2([0, 1])$ and $\langle x, y \rangle = \int_0^1 x(t)y(t)dt$, $\forall t \in L^2([0, 1])$. Let the operators $\mathcal{B}_1$ and $\mathcal{B}_2$ be defined, respectively, by $\mathcal{B}_1(x) = x$ and $\mathcal{B}_2(x) = \frac{3x}{2}$ for all $x \in L^2([0, 1])$. For $i = 1, 2, 3$, we define the operators A_1, A_2 and A_3 by $A_1(x) = 5x - 1$, $A_2(x) = 3x - 1$ and $A_3(x) = x - 1$, respectively, for every $x \in L^2([0, 1])$. Also, for $i = 1, 2, 3$, we define the mappings B_i by $B_i(x) = ix$, where $x \in L^2([0, 1])$. Let $U : C \rightarrow C$ be defined by $U(y) = \frac{2y}{3}$ for all $y \in L^2([0, 1])$. For this example, we choose $\lambda_{i,n} = \frac{2n}{3in+5}$, $\gamma_1 = 0.36$, $\gamma_2 = 0.10$, $\alpha_n = \frac{2n-2}{7n^3+2n-1}$, $\beta_n = \frac{7n^3}{7n^3+2n-1}$, $\eta_n = \frac{1}{7n^3+2n-1}$ and $\theta_n = \frac{1}{30n+5}$. It is easy to see that the conditions of Theorem 3 are satisfied. The report of this numerical illustration is given in Figure 2 with varying initial values of x_0 and x_1 . Again, we compare the accelerated and unaccelerated algorithms by choosing a tolerance level $\|x_{n+1} - x_n\| < \varepsilon$ with $\varepsilon = 10^{-4}$.

(Case 1) $u = 0.25$, $x_0 = t^2 + 1$ and $x_1 = 3^*t$;

(Case 2) $u = 0.55$, $x_0 = t^{\frac{2}{3}} + 1$ and $x_1 = 3^*t - 2t^2$;

(Case 3) $u = 1.0$, $x_0 = \cos(t + 1)$ and $x_1 = \frac{t}{2}$.

Figure 2: Top: (Case 1); bottom left: (Case 2); bottom right: (Case 3).

5 Conclusion and open problem

In this work, we introduce and study a new type of GSMVI and established a strong convergence result for approximating a solution of GSMVI in the framework of real Hilbert spaces using an inertial extrapolation term and Halpern iterative technique. We have to only consider this problem for $N = 3$ in the framework of Hilbert spaces. It is therefore left open for interested researchers in this area of research to extend the concept to more general spaces and also consider the case when $N \geq 4$. In addition, the authors in [38] introduced the notion of finding the zero of the sum of three monotone operators in the framework of Hilbert spaces. It is natural to ask, if the present results in this work can be extended to three monotone operators.

Abbreviations

SPF	split feasibility problem
SCFPP	split common fixed point problem
VIP	variational inclusion problem
SNPP	split null point problem
SMVI	split monotone variational inclusion problem
GSMVI	generalized split monotone variational inclusion problem

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