Research Article

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On convergence of explicit finite volume scheme for one-dimensional three-component two-phase flow model in porous media

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Abstract: In this work, we develop and analyze an explicit finite volume scheme for a one-dimensional nonlinear, degenerate, convection–diffusion equation having application in petroleum reservoir. The main difficulty is that the solution typically lacks regularity due to the degenerate nonlinear diffusion term. We analyze a numerical scheme corresponding to explicit discretization of the diffusion term and a Godunov scheme for the advection term. L^∞ stability under appropriate CFL conditions and BV estimates are obtained. It is shown that the scheme satisfies a discrete maximum principle. Then we prove convergence of the approximate solution to the weak solution of the problem, and we mount convergence results to a weak solution of the problem in L^1 . Results of numerical experiments are presented to validate the theoretical analysis.

Keywords: finite volume method, degenerate parabolic equation, nonlinear convection–diffusion, porous media

MSC 2020: 76S05, 35Kxx, 74Sxx, 74S10

1 Introduction

In this paper, we develop and analyze finite volume methods for the following nonlinear degenerate convection–diffusion problem (P):

$$(P) \begin{cases} \phi(x) \frac{\partial s}{\partial t} + u \frac{\partial}{\partial x}(f(s, r)) - \frac{\partial}{\partial x} \left(K(x) \alpha(s, r) \frac{\partial s}{\partial x} \right) = 0 & \text{in } I_T = I \times]0, T[, \\ \phi(x) \frac{\partial r}{\partial t} + u \frac{\partial}{\partial x}(g(s, r)) - \frac{\partial}{\partial x} \left(K(x) \alpha(s, r) \frac{\partial r}{\partial x} \right) = 0 & \text{in } I_T, \\ s(x, 0) = s_0(x), \quad r(x, 0) = r_0(x), \quad x \in I =]0, L[, \\ s(0, t) = s_g(t), \quad s(L, t) = s_L(t), \quad t \in [0, T], \\ r(0, t) = r_g(t), \quad r(L, t) = r_L(t), \quad t \in [0, T], \end{cases} \quad (1.1)$$

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where α is the nonlinear function, f, g are the mobility functions, u is the velocity of fluid, and ϕ and K are the porosity and the absolute permeability of the porous medium.

In our problem, (s, r) is the couple of unknowns with $s = s_w$ is the saturation of the wetting phase and $r = cs$ with c is the concentration of polymer component in the wetting phase.

The basic set of equations considered here arises from tertiary oil recovery flow models. It is well known that the main difficulty in approximating (1.1) is that the solution typically lacks regularity due to the degenerate nonlinear diffusion term. For the mathematical analysis of this problem, we refer the reader to [1–8].

Flow simulation in petroleum reservoirs has been extensively studied using finite element methods in the past few years, see [4,9]. Also, a discretization using both finite element and finite volume methods for two-phase flow in porous media is presented in [10] and [11]. However, it appears that there are few results on convergence theory for the degenerate problem [9]. More recently, finite volume methods were developed and analyzed for immiscible two-phase flow in porous media in the case where the diffusion term is neglected, see [11] and [12]. This approach leads to robust schemes applicable for unstructured grids and the approximate solution has various interesting properties which correspond to the properties of the physical solution, see [13] and [14]. Let us also mention that error estimates for finite volume schemes have been obtained by several authors for linear diffusion–convection equations, see [15,16].

In petroleum reservoir simulation, the governing equations that describe fluid flow are usually written in a fractional flow formulation, i.e., in terms of a saturation and a global pressure [4]. This formulation leads to a coupled system of partial differential equations which includes an elliptic pressure–velocity equation and a nonlinear degenerate parabolic saturation equation. The saturation equation is in general convection dominated and thus special care should be taken in the discretization. Also, while the diffusion term is relatively small, it is in general important and cannot be neglected [16]. The main difficulties related to the convergence of the approximate solution of such systems are the coupling between the pressure and the saturation equations and the degeneracy of the saturation equation. In view of these difficulties, we consider the one-dimensional flow of a diphasic incompressible fluid in a porous medium; we assume that the phases are immiscible. This system is decoupled and the pressure equation is solved explicitly. Thus, the saturation equation reduces to (1.1). We are interested in the convection dominated case taking into account the diffusion term. Our goal is to construct explicit finite volume scheme for (1.1) that satisfies a discrete maximum principle and to derive convergence results.

The L^∞ stability under appropriate CFL condition, BV estimate in Section 2, and the convergence of the scheme are derived in Section 3. Finally, in Section 4, numerical results using the explicit finite volume scheme.

1.1 Assumptions

We introduce some relevant assumptions on the coefficients of (1.1). We assume the following:

(H1) $\phi \in L^\infty(I)$ with

$$0 < \phi_- \leq \phi \leq \phi_+ < 1 \quad \text{almost everywhere in } I.$$

(H2) $K \in L^\infty(I)$ with

$$0 < K_- \leq K \leq K_+ < \infty \quad \text{almost everywhere in } I.$$

(H3) $\alpha \in C^0([0, 1] \times [0, 1])$ with

$$\alpha(0, \cdot) = \alpha(1, \cdot) = 0$$

and

$$\alpha(s, r) > 0, \quad \forall r, s \in]0, 1[.$$

(H4) $f, g \in C^1([0, 1] \times [0, 1])$ are monotonic functions such that

$$\frac{\partial}{\partial x} f(s, \cdot) > 0, \quad \frac{\partial}{\partial x} g(\cdot, r) > 0, \quad \forall s \in]0, 1[.$$

(H5) $s^0, r^0 \in L^\infty(I)$ with

$$0 \leq s^0 \leq 1, \quad 0 \leq r^0 \leq 1 \quad \text{almost everywhere in } I.$$

(H6) α^{-1} is a Hölder function¹ of exponent $\theta \in]0, 1[$ with

$$f \circ \alpha^{-1} \quad \text{and} \quad g \circ \alpha^{-1} \quad \text{are Hölderian functions of exponent } \frac{1+\theta}{2}.$$

(H7) $s^0, r^0 \in L^\infty(I) \cap \overline{BV}(I)$.

(H8) $K\alpha(s^0, r^0)s_x^0, K\alpha(s^0, r^0)r_x^0 \in L^\infty(I) \cap \overline{BV}(I)$.

The space $BV(I)$ is a space of the finite total variation functions in I . The basic set of equations considered is not here models of oil recovery flow and groundwater. It is well known that the main difficulty in reporting (1.1) is that the solution is not there. Usually not the regularity due to the non-linear degenerate diffusion term.

In applications in the oil and groundwater reservoirs is the saturation of the wetting phase, ϕ and K are the porosity and absolute permeability of the porous medium, and $\alpha(s, r)$, $f(s, r)$, and $g(s, r)$ are non-linear functions in function of the relative permeability, the viscosity of two phases, and the function of the capillary pressure. For these applications other boundary conditions and the effect of gravity may be specified. The technique developed in this document is applicable to such problems.

In this work, we present a numerical method in one dimension for a three-component two-phase flow model for an incompressible and immiscible flow in porous media. This method has applications in the tertiary oil recovery. This mathematical model is generalization of the mathematical model studied in [10].

Definition 1.1. A couple of functions (s, r) are said a weak solution of (1.1) defined by:

$$\left\{ \begin{array}{ll} s, r :]0, T[\longrightarrow W_0; 0 < s(x, t) \leq 1, 0 \leq r(x, t) \leq 1 \quad \text{in } \Omega_T, & \text{(i)} \\ s_t, r_t \in L^2(0, T; W') \quad \text{and} \quad \alpha(s, r) \in L^2(0, T; W \times W), & \text{(ii)} \\ \int_{\Omega_T} [\phi s v_t + (uf(s, r) - K\alpha(s, r)s_x)v_x] dt dx + \int_I \phi s^0 v(x, 0) dx = 0 \quad \forall v \in V, & \text{(iii)} \\ \int_{\Omega_T} [\phi r v_t + (uf(s, r) - K\alpha(s, r)r_x)v_x] dt dx + \int_I \phi r^0 v(x, 0) dx = 0 \quad \forall v \in V, & \text{(iv)} \end{array} \right. \quad (1.2)$$

where functional spaces W , W_0 , and V are defined by:

$$\begin{aligned} W &= \{w \in H^1(I); w(L) = 0\}, \\ W_0 &= \{(s, r) \in L^2(I) \times L^2(I); \alpha(s, \cdot), \alpha(\cdot, r) \in W\}, \\ V &= \{v \in C^1(0, T; C_0^1(\bar{I})), v(\cdot, T) \equiv 0\}. \end{aligned}$$

The data are assumed to be smooth enough to guarantee the existence and uniqueness of the weak solution. See [4,8,17] for a more detailed analysis.

¹ The function α^{-1} is inverse of function $s \longrightarrow \alpha(s, r)$, r fixed, α^{-1} is the Hölderian function of order θ : i.e., there exists $b > 0$ such that $|\alpha^{-1}(y_2) - \alpha^{-1}(y_1)| \leq b|y_2 - y_1|^\theta$, for all $y_1, y_2 \in [0, \alpha(1, 1)]$.

2 Finite volume discretization

Before describing the explicit finite volume discretization of the problem 1.1 we give some notations.

We divide the interval $[0, T]$ into $(M + 1)$ intervals of length $\Delta t = T/(M + 1)$, the interval $[0, L]$ into $N + 1$ intervals of length $h = L/(N + 1)$. Note that,

1. $x_i = ih$ for $i = 0, \dots, N + 1$ and $t^n = n\Delta t$ for $n = 0, \dots, M$.
2. $x_{i+1/2}$ the center of $I_{i+1/2} = [x_i, x_{i+1}]$, with $x_{-1/2} = 0$, $x_{N+1/2} = L$ for $i = 0, \dots, N$.
3. $I_i =]x_{i-1/2}, x_{i+1/2}[$ said volume of control for $i = 0, \dots, N$.
4. Let $(t^n)_{n=0, \dots, M+1}$ be a partition of $[0, T]$ with a time step $\Delta t = t^{n+1} - t^n$.

We define $\phi_i = \frac{1}{h} \int_{I_i} \phi(x) dx$ we assume that the function $K(x)$ is piecewise constant, and we set $K_{i+1/2} = K|_{I_{i+1/2}}$.

Let s_i^n [resp. $s_{i+1/2}^n$] be an approximation of $s(x_i, t^n)$ [resp. $s(x_{i+1/2}, t^n)$] and let r_i^n [resp. $r_{i+1/2}^n$] be an approximation of $r(x_i, t^n)$ [resp. $r(x_{i+1/2}, t^n)$], which will be defined precisely in the sequel.

We approximate the terms $\frac{\partial r}{\partial x}$ and $\frac{\partial s}{\partial x}$, respectively, definite by

$$\frac{\partial r}{\partial x} \approx \frac{r_{i+1}^n - r_i^n}{h} \quad \text{and} \quad \frac{\partial s}{\partial x} \approx \frac{s_{i+1}^n - s_i^n}{h}.$$

2.1 Explicit scheme of concentration–saturation equation

Using an explicit approximation of concentration–saturation equation, we get

$$\begin{cases} \phi \frac{\partial r}{\partial t} - \frac{\partial}{\partial x} \left(K(x) \alpha(s, r) \frac{\partial r}{\partial x} \right) + u \frac{\partial}{\partial x} (g(s, r)) = 0, & x \in]0, L[, t \in]0, T[, \\ s(x, 0) = s_0(x), r(x, 0) = r_0, & x \in]0, L[, \\ s(0, t) = s_g(t), s(L, t) = s_L(t), r(0, t) = r_g(t), r(L, t) = r_L(t), & t \in [0, T]. \end{cases} \quad (2.1)$$

We integrate over the set $I_i \times [t^n, t^{n+1}]$ the equation (2.1). Using Fubini's theorem, we get

$$\int_{I_i} \int_{t^n}^{t^{n+1}} \left(\phi \frac{\partial r}{\partial t} \right) dt dx - \int_{I_i} \int_{t^n}^{t^{n+1}} \frac{\partial}{\partial x} \left(K(x) \alpha(s, r) \frac{\partial r}{\partial x} \right) dt dx + u \int_{I_i} \int_{t^n}^{t^{n+1}} \frac{\partial}{\partial x} (g(s, r)) dt dx = 0.$$

We obtain the following finite volume scheme, for $i = 0, \dots, N$

$$\begin{aligned} & \phi_i \int_{I_i} (r(x, t^{n+1}) - r(x, t^n)) dx + u \int_{t^n}^{t^{n+1}} (g(s(x_{i+1/2}, t), r(x_{i+1/2}, t)) - g(s(x_{i-1/2}, t), r(x_{i-1/2}, t))) dt \\ &= \alpha(s(x_i, t), r(x_i, t)) K_{i+1/2} \int_{t^n}^{t^{n+1}} \frac{\partial}{\partial x} r(x_{i+1/2}, t) dt - \alpha(s(x_i, t), r(x_i, t)) K_{i-1/2} \int_{t^n}^{t^{n+1}} \frac{\partial}{\partial x} r(x_{i-1/2}, t) dt, \end{aligned}$$

with

$$s_i^n = \frac{1}{h} \int_{I_i} s(x, t^{n+1}) dx \quad \text{and} \quad r_i^n = \frac{1}{h} \int_{I_i} r(x, t^{n+1}) dx,$$

and

$$s_i^0 = \frac{1}{h} \int_{I_i} s^0(x) dx \quad \text{and} \quad r_i^0 = \frac{1}{h} \int_{I_i} r^0(x) dx.$$

The advection term is approximated by an upwind Godunov scheme and since (H4), we can write

$$g(s(x_{i+1/2}, t), r(x_{i+1/2}, t)) = g(s_{i+1/2}(t), r_{i+1/2}(t))$$

and

$$s(x_{i+1/2}, t^n) = s_i^n, \quad r(x_{i+1/2}, t^n) = r_i^n.$$

Then we can write the above formulae in the following form:

$$\begin{cases} \forall n = 0, \dots, M-1, \quad i = 1, \dots, N+1, \\ \phi_i \frac{r_i^{n+1} - r_i^n}{\Delta t} - K_{i+1/2} \alpha(s_i^n, r_i^n) \frac{r_{i+1}^n - r_i^n}{h^2} + K_{i-1/2} \alpha(s_i^n, r_i^n) \frac{r_i^n - r_{i-1}^n}{h^2} + u \frac{g(s_{i+1}^n, r_{i+1}^n) - g(s_i^n, r_i^n)}{h} = 0, \\ s_i^0 = s_0(x_i), r_i^0 = r_0(x_i), \quad i = 0, \dots, N+1, \\ s_0^n = s_g(t^n) = s_{N+1}^0 = s_d(t^n), r_0^n = r_g(t^n), \quad n = 0, \dots, M. \end{cases} \quad (2.2)$$

2.2 Explicit scheme of saturation equation

Using an explicit approximation of saturation equation, we get

$$\begin{cases} \phi \frac{\partial s}{\partial t} - \frac{\partial}{\partial x} \left(K(x) \alpha(s, r) \frac{\partial s}{\partial x} \right) + u \frac{\partial}{\partial x} (f(s, r)) = 0, \quad x \in]0, L[, t \in]0, T[, \\ s(x, 0) = s_0(x), r(x, 0) = r_0, \quad x \in]0, L[, \\ s(0, t) = s_g(t), s(L, t) = s_L(t), r(0, t) = r_g(t), r(L, t) = r_L(t), \quad t \in [0, T]. \end{cases} \quad (2.3)$$

We integrate over the set $I_i \times [t^n, t^{n+1}]$ equation (2.3). Using Fubini's theorem, we obtain

$$\int_{I_i} \int_{t^n}^{t^{n+1}} \left(\phi \frac{\partial s}{\partial t} \right) dt dx - \int_{I_i} \int_{t^n}^{t^{n+1}} \frac{\partial}{\partial x} \left(K(x) \alpha(s, r) \frac{\partial s}{\partial x} \right) dt dx + u \int_{I_i} \int_{t^n}^{t^{n+1}} \frac{\partial}{\partial x} (f(s, r)) dt dx = 0.$$

We obtain the following finite volume scheme, for $i = 0, \dots, N$

$$\begin{aligned} & \phi_i \int_{I_i} (s(x, t^{n+1}) - s(x, t^n)) dx + u \int_{t^n}^{t^{n+1}} (f(s(x_{i+1/2}, t), r(x_{i+1/2}, t)) - f(s(x_{i-1/2}, t), r(x_{i-1/2}, t))) dt \\ &= \alpha(s(x_i, t), r(x_i, t)) K_{i+1/2} \int_{t^n}^{t^{n+1}} \frac{\partial}{\partial x} s(x_{i+1/2}, t) dt - \alpha(s(x_i, t), r(x_i, t)) K_{i-1/2} \int_{t^n}^{t^{n+1}} \frac{\partial}{\partial x} s(x_{i-1/2}, t) dt. \end{aligned}$$

The advection term is approximated by an upwind Godunov scheme and since (H4), we can write

$$f(s(x_{i+1/2}, t), r(x_{i+1/2}, t)) = f(s_{i+1/2}(t), r_{i+1/2}(t)),$$

and

$$s(x_{i+1/2}, t^n) = s_i^n, \quad r(x_{i+1/2}, t^n) = r_i^n.$$

Then we can write the above formulae in the following form:

$$\begin{cases} \forall n = 0, \dots, M-1, \quad i = 1, \dots, N+1 \\ \phi_i \frac{s_i^{n+1} - s_i^n}{\Delta t} - K_{i+1/2} \alpha(s_i^n, r_i^n) \frac{s_{i+1}^n - s_i^n}{h^2} + K_{i-1/2} \alpha(s_i^n, r_i^n) \frac{s_i^n - s_{i-1}^n}{h^2} + u \frac{f(s_{i+1}^n, r_{i+1}^n) - f(s_i^n, r_i^n)}{h} = 0, \\ s_i^0 = s_0(x_i), r_i^0 = r_0(x_i), \quad i = 0, \dots, N+1, \\ s_0^n = s_g(t^n) = s_{N+1}^0 = s_d(t^n), r_0^n = r_g(t^n), \quad n = 0, \dots, M. \end{cases} \quad (2.4)$$

3 Stability and convergence results

We shall analyze the finite volume schemes obtained in (2.2) and (2.4). We present L^∞ stability, BV estimate, and convergence results.

3.1 L^∞ stability and BV estimate

Proposition 3.1. *The explicit scheme obtained in (2.2) and (2.4) is L^∞ stable and BV stable, under the above assumptions and the CFL condition:*

$$\zeta = \zeta_1 + \zeta_2 \leq 1,$$

with

$$\zeta_1 = \max_i \left(\frac{K_{i+1/2} + K_{i-1/2}}{\phi_i} \right) \frac{\Delta t}{h^2} \sup_{s,r} (\alpha_*(s, r)), \quad \zeta_2 = \frac{u \Delta t}{\phi_i h} \sup_{s,r} (g_*(s, r)),$$

and

$$\xi = \xi_1 + \xi_2 \leq 1,$$

with

$$\xi_1 = \max_i \left(\frac{K_{i+1/2} + K_{i-1/2}}{\phi_i} \right) \frac{\Delta t}{h^2} \sup_{s,r} (\alpha_*(s, r)), \quad \xi_2 = \frac{u \Delta t}{\phi_i h} \sup_{s,r} (f_*(s, r)).$$

Definition 3.2. An approximate solution $w_h^n = (w_i^n)_{0 \leq i \leq N+1}$ is L^∞ stable if there exists a constant C such that:

$$\|w_h^n\|_\infty = \sup_{0 \leq i \leq N+1} |w_i^n| \leq C, \quad \forall n = 1, \dots, M$$

and (BV) stable if there exists a constant C such that:

$$\|w_h^n\|_{\text{BV}} = \sum_{1 \leq i \leq N} |w_{i+1}^n - w_i^n| \leq C, \quad \forall n = 1, \dots, M,$$

with $w = r$ or $w = s$.

Proof. We write the finite volume scheme (2.2) in the following form:

$$\begin{aligned} r_i^{n+1} = & \left(1 - \left(\frac{u \Delta t}{\phi_i h} g_{*i}^n + \frac{K_{i+1/2} \Delta t}{\phi_i h^2} \alpha_{*i+1}^n + \frac{K_{i-1/2} \Delta t}{\phi_i h^2} \alpha_{*i}^n \right) \right) r_i^n \\ & + \left(\frac{u \Delta t}{\phi_i h} g_{*i-1}^n + \frac{K_{i-1/2} \Delta t}{\phi_i h^2} \alpha_{*i-1}^n \right) s_{i-1}^n + \frac{K_{i+1/2} \Delta t}{\phi_i h^2} \alpha_{*i+1}^n r_{i+1}^n, \end{aligned} \quad (3.1)$$

with

$$g_{*i}^n = \begin{cases} \frac{g(s_i^n, r_i^n) - g(s_{i-1}^n, r_{i-1}^n)}{r_i^n - r_{i-1}^n}, & \text{if } r_i^n \neq r_{i-1}^n, \\ 0, & \text{if } r_i^n = r_{i-1}^n, \end{cases}$$

and

$$\alpha_{*i}^n = \begin{cases} \frac{\alpha(s_i^n, r_i^n) - \alpha(s_{i-1}^n, r_{i-1}^n)}{r_i^n - r_{i-1}^n}, & \text{if } r_i^n \neq r_{i-1}^n, \\ 0, & \text{if } r_i^n = r_{i-1}^n. \end{cases}$$

Now, by using CFL condition, we get

$$\left(1 - \left(\frac{u\Delta t}{\phi_i h} g_{*i}^n + \alpha_{*i}^n \frac{K_{i+1/2}\Delta t}{\phi_i h^2} + \frac{K_{i-1/2}\Delta t}{\phi_i h^2} \alpha_{*i}^n\right)\right) \geq (1 - \zeta_1 - \zeta_2) \geq (1 - \zeta) \geq 0.$$

So, by recurrence on n , we obtain

$$|r_i^n| \leq C \text{ for } i = 0, \dots, N+1, \Rightarrow |s_i^{n+1}| \leq (1 - 2\zeta_1 - \zeta_1)C + (\zeta_1 + \zeta_2)C + \zeta_1 C = C.$$

Then r_h^n is L^∞ stable.

Now, we prove that r_h^n is (BV) stable, according to (3.2) we obtain

$$\forall n = 1, \dots, M, \quad i = 1, \dots, N+1$$

$$\begin{aligned} (r_{i+1}^{n+1} - r_i^{n+1}) &= (r_{i+1}^n - r_i^n) \left(1 - \left(\frac{u\Delta t}{\phi_i h} g_{*i}^n + \frac{(K_{i+1/2} + K_{i-1/2})\Delta t}{\phi_i h^2} \alpha_{*i+1}^n\right)\right) \\ &\quad + (r_i^n - r_{i-1}^n) \left(\frac{u\Delta t}{\phi_i h} g_{*i}^n + \frac{K_{i-1/2}\Delta t}{\phi_i h^2} \alpha_{*i}^n\right) + (r_{i+2}^n - r_{i+1}^n) \frac{K_{i+3/2}\Delta t}{\phi_i h^2} \alpha_{*i+2}^n. \end{aligned}$$

Using CFL condition and assuming the function r^n is continued by constants outside the interval $[0, L]$ by $r_g(t^n)$ on $]-\infty, 0[$ and $r_d(t^n)$ on $]0, +\infty[$, we get

$$\begin{aligned} \sum_{i \in \mathbb{Z}} |r_{i+1}^{n+1} - r_i^{n+1}| &\leq \sum_{i \in \mathbb{Z}} |r_{i+1}^n - r_i^n| \left(1 - \left(\frac{u\Delta t}{\phi_i h} g_{*i}^n + \frac{(K_{i+1/2} + K_{i-1/2})\Delta t}{\phi_i h^2} \alpha_{*i+1}^n\right)\right) \\ &\quad + \sum_{i \in \mathbb{Z}} |r_i^n - r_{i-1}^n| \left(\frac{u\Delta t}{\phi_i h} g_{*i}^n + \frac{K_{i-1/2}\Delta t}{\phi_i h^2} \alpha_{*i}^n\right) + \sum_{i \in \mathbb{Z}} |r_{i+2}^n - r_{i+1}^n| \frac{K_{i+3/2}\Delta t}{\phi_i h^2} \alpha_{*i+2}^n \\ &\leq \sum_{i \in \mathbb{Z}} |r_{i+1}^n - r_i^n|. \end{aligned}$$

So r_h^n is (BV) stable.

We write finite volume scheme (2.4) in the following form:

$$\begin{aligned} s_i^{n+1} &= \left(1 - \left(\frac{u\Delta t}{\phi_i h} f_{*i}^n + \frac{K_{i+1/2}\Delta t}{\phi_i h^2} \alpha_{*i+1}^n + \frac{K_{i-1/2}\Delta t}{\phi_i h^2} \alpha_{*i}^n\right)\right) s_i^n \\ &\quad + \left(\frac{u\Delta t}{\phi_i h} f_{*i-1}^n + \frac{K_{i-1/2}\Delta t}{\phi_i h^2} \alpha_{*i-1}^n\right) s_{i-1}^n + \frac{K_{i+1/2}\Delta t}{\phi_i h^2} \alpha_{*i+1}^n s_{i+1}^n, \end{aligned} \quad (3.2)$$

with

$$f_{*i}^n = \begin{cases} \frac{f(s_i^n, r_i^n) - f(s_{i-1}^n, r_{i-1}^n)}{s_i^n - s_{i-1}^n}, & \text{if } s_i^n \neq s_{i-1}^n, \\ 0, & \text{if } s_i^n = s_{i-1}^n, \end{cases}$$

and

$$\alpha_{*i}^n = \begin{cases} \frac{\alpha(s_i^n, r_i^n) - \alpha(s_{i-1}^n, r_{i-1}^n)}{s_i^n - s_{i-1}^n}, & \text{if } s_i^n \neq s_{i-1}^n, \\ 0, & \text{if } s_i^n = s_{i-1}^n. \end{cases}$$

Using CFL condition, we get

$$\left(1 - \left(\frac{u\Delta t}{\phi_i h} f_{*i}^n + \alpha_{*i}^n \frac{K_{i+1/2}\Delta t}{\phi_i h^2} + \frac{K_{i-1/2}\Delta t}{\phi_i h^2} \alpha_{*i}^n\right)\right) \geq (1 - \xi_1 - \xi_2) \geq (1 - \xi) \geq 0.$$

By recurrence on n , we obtain

$$|s_i^n| \leq C \text{ for } i = 0, \dots, N+1, \Rightarrow |s_i^{n+1}| \leq (1 - 2\xi_1 - \xi_1)C + (\xi_1 + \xi_2)C + \xi_1 C = C.$$

Then s_h^n is L^∞ stable.

Finally, we prove that s_h^n is (BV) stable, according to (3.2) we obtain

$$\forall n = 1, \dots, M, \quad i = 1, \dots, N+1,$$

$$\begin{aligned} (s_{i+1}^{n+1} - s_i^{n+1}) &= (s_{i+1}^n - s_i^n) \left(1 - \left(\frac{u\Delta t}{\phi_i h} f_{*i}^n + \frac{(K_{i+1/2} + K_{i-1/2})\Delta t}{\phi_i h^2} \alpha_{*i+1}^n \right) \right) \\ &\quad + (s_i^n - s_{i-1}^n) \left(\frac{u\Delta t}{\phi_i h} f_{*i}^n + \frac{K_{i-1/2}\Delta t}{\phi_i h^2} \alpha_{*i}^n \right) + (s_{i+2}^n - s_{i+1}^n) \frac{K_{i+3/2}\Delta t}{\phi_i h^2} \alpha_{*i+2}^n. \end{aligned}$$

Using CFL condition and assuming the function s^n is continued by constants outside the interval $[0, L]$ by $r_g(t^n)$ on $]-\infty, 0[$ and $r_d(t^n)$ on $]0, +\infty[$, we get

$$\begin{aligned} \sum_{i \in \mathbb{Z}} |s_{i+1}^{n+1} - s_i^{n+1}| &\leq \sum_{i \in \mathbb{Z}} |s_{i+1}^n - s_i^n| \left(1 - \left(\frac{u\Delta t}{\phi_i h} f_{*i}^n + \frac{(K_{i+1/2} + K_{i-1/2})\Delta t}{\phi_i h^2} \alpha_{*i+1}^n \right) \right) \\ &\quad + \sum_{i \in \mathbb{Z}} |s_i^n - s_{i-1}^n| \left(\frac{u\Delta t}{\phi_i h} f_{*i}^n + \frac{K_{i-1/2}\Delta t}{\phi_i h^2} \alpha_{*i}^n \right) + \sum_{i \in \mathbb{Z}} |s_{i+2}^n - s_{i+1}^n| \frac{K_{i+3/2}\Delta t}{\phi_i h^2} \alpha_{*i+2}^n \\ &\leq \sum_{i \in \mathbb{Z}} |s_{i+1}^n - s_i^n|. \end{aligned}$$

Then s_h^n is (BV) stable. □

3.2 Convergence results

In this paragraph, we present some results of convergence of the explicit finite volume scheme.

We prove the main result of this subsection, namely Theorem 3.5, we need the following lemma:

Lemma 3.3. *Under hypotheses (H1)–(H8) and CFL condition for the explicit scheme, the following estimate is valid: for $n = 0, \dots, M-1$:*

$$\|s^{n+1} - s^n\|_{L^1(I)} = h \sum_{i=1, \dots, N} \|s_i^{n+1} - s_i^n\| \leq C_1 \Delta t$$

and

$$\|r^{n+1} - r^n\|_{L^1(I)} = h \sum_{i=1, \dots, N} \|r_i^{n+1} - r_i^n\| \leq C_2 \Delta t,$$

where

$$C_1 = (u \sup(f_*)) \|s^0\|_{\text{BV}(I)} + \|K\alpha(s^0, r^0) s_x^0\|_{\text{BV}(I)}$$

and

$$C_2 = (u \sup(g_*)) \|r^0\|_{\text{BV}(I)} + \|K\alpha(s^0, r^0) r_x^0\|_{\text{BV}(I)},$$

where C_1 and C_2 are constants independent of h and Δt .

Proof. We present the proof for the explicit scheme. Similar results hold for the other scheme. Multiply (2.4) and (2.2), respectively, by $\frac{\phi_i h}{\Delta t}$ and sum over $j \leq i$, we obtain, for $i = 1, \dots, N$

$$w_i^n = \sum_{j \leq i} \phi_j h \frac{s_i^{n+1} - s_i^n}{\Delta t} = uf(1, 1) - uf(s_i^n, r_i^n) + \frac{K_{i+1/2}}{h} (\alpha(s_{i+1}^n, r_{i+1}^n) - \alpha(s_i^n, r_i^n))$$

and

$$z_i^n = \sum_{j \leq i} \phi_j h \frac{r_i^{n+1} - r_i^n}{\Delta t} = ug(1, 1) - ug(s_i^n, r_i^n) + \frac{K_{i+1/2}}{h} (\alpha(s_{i+1}^n, r_{i+1}^n) - \alpha(s_i^n, r_i^n)),$$

and we put $w_0^n = 0$, $w_{N+1}^n = f(1, 1)$, $z_0^n = 0$, and $z_{N+1}^n = g(1, 1)$, then we have

$$w_i^n - w_{i-1}^n = \phi_i h \frac{s_i^{n+1} - s_i^n}{\Delta t},$$

hence

$$\begin{aligned} w_i^{n+1} &= w_i^n - uf(s_i^{n+1}, r_i^{n+1}) + uf(s_i^n, r_i^n) + \frac{K_{i+1/2}}{h} (\alpha(s_{i+1}^{n+1}, r_{i+1}^{n+1}) - \alpha(s_i^{n+1}, r_i^{n+1})) - \frac{K_{i+1/2}}{h} (\alpha(s_{i+1}^n, r_{i+1}^n) - \alpha(s_i^n, r_i^n)) \\ &= w_i^n - uf_{i*}^{n+1/2} (s_i^{n+1} - s_i^n) + \frac{K_{i+1/2}}{h} (\alpha_{i+1*}^{n+1/2} (s_{i+1}^{n+1} - s_{i+1}^n) - \alpha_{i*}^{n+1/2} (s_i^{n+1} - s_i^n)), \end{aligned}$$

and

$$z_i^n - z_{i-1}^n = \phi_i h \frac{r_i^{n+1} - r_i^n}{\Delta t},$$

we obtain

$$\begin{aligned} z_i^{n+1} &= z_i^n - ug(s_i^{n+1}, r_i^{n+1}) + ug(s_i^n, r_i^n) + \frac{K_{i+1/2}}{h} (\alpha(s_{i+1}^{n+1}, r_{i+1}^{n+1}) - \alpha(s_i^{n+1}, r_i^{n+1})) - \frac{K_{i+1/2}}{h} (\alpha(s_{i+1}^n, r_{i+1}^n) - \alpha(s_i^n, r_i^n)) \\ &= z_i^n - ug_{i*}^{n+1/2} (r_i^{n+1} - r_i^n) + \frac{K_{i+1/2}}{h} (\alpha_{i+1*}^{n+1/2} (r_{i+1}^{n+1} - r_{i+1}^n) - \alpha_{i*}^{n+1/2} (r_i^{n+1} - r_i^n)). \end{aligned}$$

Then,

$$w_i^{n+1} = w_i^n - \frac{u \Delta t f_{i*}^{n+1/2}}{\phi_i h} (w_i^n - w_{i-1}^n) + \frac{\Delta t K_{i+1/2} \alpha_{i*}^{n+1/2}}{\phi_i h^2} (w_{i+1}^n - w_i^n) - \frac{\Delta t K_{i+1/2} \alpha_{i*}^{n+1/2}}{\phi_i h^2} (w_i^n - w_{i-1}^n)$$

and

$$z_i^{n+1} = z_i^n - \frac{u \Delta t g_{i*}^{n+1/2}}{\phi_i h} (z_i^n - z_{i-1}^n) + \frac{\Delta t K_{i+1/2} \alpha_{i*}^{n+1/2}}{\phi_i h^2} (z_{i+1}^n - z_i^n) - \frac{\Delta t K_{i+1/2} \alpha_{i*}^{n+1/2}}{\phi_i h^2} (z_i^n - z_{i-1}^n).$$

So that,

$$\begin{aligned} w_{i+1}^{n+1} - w_i^{n+1} &= (w_{i+1}^n - w_i^n) \left(1 - \frac{u \Delta t f_{i+1*}^{n+1/2}}{\phi_i h} - \frac{\Delta t K_{i+1/2} \alpha_{i+1*}^{n+1/2}}{\phi_i h^2} - \frac{\Delta t K_{i+3/2} \alpha_{i+1*}^{n+1/2}}{\phi_i h^2} \right) \\ &\quad + (w_i^n - w_{i-1}^n) \left(\frac{u \Delta t f_{i+1*}^{n+1/2}}{\phi_i h} - \frac{\Delta t K_{i+1/2} \alpha_{i*}^{n+1/2}}{\phi_i h^2} \right) + (w_{i+2}^n - w_{i+1}^n) \left(\frac{\Delta t K_{i+3/2} \alpha_{i+1*}^{n+1/2}}{\phi_i h^2} \right) \end{aligned}$$

and

$$\begin{aligned} z_{i+1}^{n+1} - z_i^{n+1} &= (z_{i+1}^n - z_i^n) \left(1 - \frac{u \Delta t g_{i+1*}^{n+1/2}}{\phi_i h} - \frac{\Delta t K_{i+1/2} \alpha_{i+1*}^{n+1/2}}{\phi_i h^2} - \frac{\Delta t K_{i+3/2} \alpha_{i+1*}^{n+1/2}}{\phi_i h^2} \right) \\ &\quad + (z_{i+1}^n - z_i^n) \left(\frac{u \Delta t g_{i+1*}^{n+1/2}}{\phi_i h} - \frac{\Delta t K_{i+1/2} \alpha_{i*}^{n+1/2}}{\phi_i h^2} \right) + (z_{i+2}^n - z_{i+1}^n) \left(\frac{\Delta t K_{i+3/2} \alpha_{i+1*}^{n+1/2}}{\phi_i h^2} \right). \end{aligned}$$

Using CFL condition and assuming the function w^n and z^n are continued by constants outside the interval $[0, L]$, by $w^n|_{-\infty, 0[} = 0$, $w^n|_{L, +\infty[} = f(1, 1)$, $z^n|_{-\infty, 0[} = 0$, and $z^n|_{L, +\infty[} = g(1, 1)$, we obtain:

$$\begin{aligned} \sum_{i \in \mathbb{Z}} |w_{i+1}^{n+1} - w_i^{n+1}| &= \sum_{i \in \mathbb{Z}} |w_{i+1}^n - w_i^n| \left(1 - \frac{u \Delta t f_{i+1*}^{n+1/2}}{\phi_i h} - \frac{\Delta t K_{i+1/2} \alpha_{i+1*}^{n+1/2}}{\phi_i h^2} - \frac{\Delta t K_{i+3/2} \alpha_{i+1*}^{n+1/2}}{\phi_i h^2} \right) \\ &+ \sum_{i \in \mathbb{Z}} |w_i^n - w_{i-1}^n| \left(\frac{u \Delta t f_{i+1*}^{n+1/2}}{\phi_i h} - \frac{\Delta t K_{i+1/2} \alpha_{i*}^{n+1/2}}{\phi_i h^2} \right) + \sum_{i \in \mathbb{Z}} |w_{i+2}^n - w_{i+1}^n| \left(\frac{\Delta t K_{i+3/2} \alpha_{i+1*}^{n+1/2}}{\phi_i h^2} \right) \end{aligned}$$

and

$$\begin{aligned} \sum_{i \in \mathbb{Z}} |z_{i+1}^{n+1} - z_i^{n+1}| &= \sum_{i \in \mathbb{Z}} |z_{i+1}^n - z_i^n| \left(1 - \frac{u \Delta t g_{i+1*}^{n+1/2}}{\phi_i h} - \frac{\Delta t K_{i+1/2} \alpha_{i+1*}^{n+1/2}}{\phi_i h^2} - \frac{\Delta t K_{i+3/2} \alpha_{i+1*}^{n+1/2}}{\phi_i h^2} \right) \\ &+ \sum_{i \in \mathbb{Z}} |z_i^n - z_{i-1}^n| \left(\frac{u \Delta t g_{i+1*}^{n+1/2}}{\phi_i h} - \frac{\Delta t K_{i+1/2} \alpha_{i*}^{n+1/2}}{\phi_i h^2} \right) + \sum_{i \in \mathbb{Z}} |z_{i+2}^n - z_{i+1}^n| \left(\frac{\Delta t K_{i+3/2} \alpha_{i+1*}^{n+1/2}}{\phi_i h^2} \right), \end{aligned}$$

we deduce that

$$\sum_{i \in \mathbb{Z}} \phi_i h \frac{|s_i^{n+1} - s_i^n|}{\Delta t} = \sum_{i \in \mathbb{Z}} |w_{i+1}^{n+1} - w_i^{n+1}| \text{ is decreasing,}$$

and

$$\sum_{i \in \mathbb{Z}} \phi_i h \frac{|r_i^{n+1} - r_i^n|}{\Delta t} = \sum_{i \in \mathbb{Z}} |z_{i+1}^{n+1} - z_i^{n+1}| \text{ is decreasing.}$$

Finally, according to hypotheses (H7) and (H8), we have:

$$\begin{aligned} \sum_{i \in \mathbb{Z}} \phi_i h \frac{s_i^{n+1} - s_i^n}{\Delta t} &\leq \sum_{i \in \mathbb{Z}} \phi_i h \frac{|s_i^1 - s_i^0|}{\Delta t} \\ &= \sum_{i \in \mathbb{Z}} u \left| f(s_{i-1}^0, r_{i-1}^0) - f(s_i^0, r_i^0) + K_{i+1/2} \alpha(s_i^0, r_i^0) \frac{s_{i+1}^0 - s_i^0}{h} - K_{i-1/2} \alpha(s_i^0, r_i^0) \frac{s_i^0 - s_{i-1}^0}{h} \right| \\ &\leq \sup(u f_*) \|s^0\|_{BV(I)} + \|K \alpha(s^0, r^0)(s^0)_x\|_{BV(I)} \end{aligned}$$

and

$$\begin{aligned} \sum_{i \in \mathbb{Z}} \phi_i h \frac{r_i^{n+1} - r_i^n}{\Delta t} &\leq \sum_{i \in \mathbb{Z}} \phi_i h \frac{|r_i^1 - r_i^0|}{\Delta t} \\ &= \sum_{i \in \mathbb{Z}} u \left| g(s_{i-1}^0, r_{i-1}^0) - g(s_i^0, r_i^0) + K_{i+1/2} \alpha(s_i^0, r_i^0) \frac{r_{i+1}^0 - r_i^0}{h} - K_{i-1/2} \alpha(s_i^0, r_i^0) \frac{r_i^0 - r_{i-1}^0}{h} \right| \\ &\leq \sup(u g_*) \|r^0\|_{BV(I)} + \|K \alpha(s^0, r^0)(r^0)_x\|_{BV(I)}. \end{aligned}$$

Then,

$$\sum_{i \in \mathbb{Z}} \phi_i h \frac{|s_i^{n+1} - s_i^n|}{\Delta t} \leq C_1 \quad \text{and} \quad \sum_{i \in \mathbb{Z}} \phi_i h \frac{|r_i^{n+1} - r_i^n|}{\Delta t} \leq C_2,$$

such that,

$$C_1 = (u \sup(f_*)) \|s^0\|_{BV(I)} + \|K \alpha(s^0, r^0) s_x^0\|_{BV(I)} < \infty$$

and

$$C_2 = (u \sup(g_*)) \|r^0\|_{BV(I)} + \|K \alpha(s^0, r^0) r_x^0\|_{BV(I)} < \infty.$$

This completes the proof of Lemma 3.3. $\square$

Corollary 3.4 is an immediate consequence of Lemma 3.3.

Corollary 3.4. *Under the assumptions of Lemma 3.3, we have the following estimates: for $n = 1, \dots, M$*

$$\|K\alpha(s^n, r^n)(s^n)_x\|_{BV(I)} = \sum_{i=1, \dots, N} \left| K_{i+1/2}\alpha(s_i^n, r_i^n) \frac{s_{i+1}^n - s_i^n}{h} - K_{i-1/2}\alpha(s_i^n, r_i^n) \frac{s_i^n - s_{i-1}^n}{h} \right| \leq C$$

and

$$\|K\alpha(s^n, r^n)(s^n)_x\|_{L^\infty(I)} = \sup_{i=1, \dots, N} \left| K_{i+1/2}\alpha(s_i^n, r_i^n) \frac{s_{i+1}^n - s_i^n}{h} \right| \leq C.$$

Furthermore,

$$\|K\alpha(s^n, r^n)(r^n)_x\|_{BV(I)} = \sum_{i=1, \dots, N} \left| K_{i+1/2}\alpha(s_i^n, r_i^n) \frac{r_{i+1}^n - r_i^n}{h} - K_{i-1/2}\alpha(s_i^n, r_i^n) \frac{r_i^n - r_{i-1}^n}{h} \right| \leq C$$

and

$$\|K\alpha(s^n, r^n)(r^n)_x\|_{L^\infty(I)} = \sup_{i=1, \dots, N} \left| K_{i+1/2}\alpha(s_i^n, r_i^n) \frac{r_{i+1}^n - r_i^n}{h} \right| \leq C.$$

Proof. We have

$$\begin{aligned} \|K\alpha(s^n, r^n)(s^n)_x\|_{BV(I)} &= \sum_{i=1, \dots, N} \left| K_{i+1/2}\alpha(s_i^n, r_i^n) \frac{s_{i+1}^n - s_i^n}{h} - K_{i-1/2}\alpha(s_i^n, r_i^n) \frac{s_i^n - s_{i-1}^n}{h} \right| \\ &\leq \sum_{i=1, \dots, N} |w_i^n - w_{i-1}^n| + \sum_{i=1, \dots, N} u |f(s_{i-1}^n, r_{i-1}^n) - f(s_i^n, r_i^n)| \\ &\leq \sum_{i=1, \dots, N} |w_i^0 - w_{i-1}^0| + u \sup(f_*) \|s^n\|_{BV(I)} \\ &\leq 2u \sup(f_*) \|s^0\|_{BV(I)} + \|K\alpha(s^0, r^0)(s^0)_x\|_{BV(I)}, \end{aligned}$$

according to hypothesis (H8), we obtain

$$\|K\alpha(s^n, r^n)(s^n)_x\|_{BV(I)} \leq C,$$

where

$$C = 2u \sup(f_*) \|s^0\|_{BV(I)} + \|K\alpha(s^0, r^0)(s^0)_x\|_{BV(I)} < \infty \quad (3.3)$$

and

$$\begin{aligned} \|K\alpha(s^n, r^n)(s^n)_x\|_{L^\infty(I)} &= \sup_{i=1, \dots, N} \left| K_{i+1/2}\alpha(s_i^n, r_i^n) \frac{s_{i+1}^n - s_i^n}{h} \right| \\ &\leq \sup_{i=1, \dots, N} \left(u |f(s_i^n, r_i^n) - f(1, 1)| + \sum_{j < i} \phi_i h \frac{s_j^{n+1} - s_j^n}{\Delta t} \right) \\ &\leq u \sup(f_*) \|s^0 - 1\|_{L^\infty(I)} + u \sup(f_*) \|s^0\|_{BV(I)} + \|K\alpha(s^0, r^0)(s^0)_x\|_{BV(I)}, \end{aligned}$$

according to hypothesis (H8), we obtain

$$\|K\alpha(s^n, r^n)(s^n)_x\|_{L^\infty(I)} \leq C.$$

In the same way, we prove the other estimates. $\square$

Now we are ready to prove the main result of this subsection. Note that the approximate solution (s_h, r_h) by $s_h(x, t) = s_i^n$ and $r_h(x, t) = r_i^n$ in $I_i \times [t^n, t^{n+1}]$.

Theorem 3.5. *Under the above assumptions (H1)–(H8) and the CFL condition, the approximate solution (s_h, r_h) , given by the explicit scheme, converges to (s, r) in $(L^1(\Omega_T))^2$, where $L^1(\Omega_T) = L^1([0, L] \times [0, T])$ when h and Δt tend to zero.*

Proof. To show the convergence of $s_h = s_h(x, t)$ and $r_h = r_h(x, t)$ in $I_i \times [t^n, t^{n+1}]$ toward the weak solution defined by (1.2), we pass to the limit in the discretized equation.

From Proposition 3.1 and Lemma 3.3, (s_h, r_h) is bounded in $L^\infty(\Omega_T) \cap BV(\Omega_T) \hookrightarrow L^1(\Omega_T)$ with compactness embedding. We can extract, by a diagonal procedure, a subsequence, still denoted (s_h, r_h) , such that

$$\begin{aligned} s_h &\longrightarrow s^* \text{ on } L^1(\Omega_T), \\ r_h &\longrightarrow r^* \text{ on } L^1(\Omega_T), \end{aligned} \quad (3.4)$$

when h and Δt tend to zero where the limit $(s^*, r^*) \in L^\infty(\Omega_T) \cap BV(\Omega_T)$. It is easy to obtain from Corollary 3.4 and hypothesis (H2), the following estimates: for all $t \in [0, T]$

$$\|\alpha_h(s, r)(s_h)_x\|_{L^\infty(I)} \leq C$$

and

$$\|\alpha_h(s, r)(r_h)_x\|_{L^\infty(I)} \leq C,$$

where C is an independent constant of h and Δt . Thus, from the continuous injection of $W^{1,\infty}(I)$ into $H^1(I)$ we get

$$\alpha_h(s, r)s_h \rightharpoonup \alpha^*s^* \text{ on } H^1(I) - \text{weakly}$$

and

$$\alpha_h(s, r)r_h \rightharpoonup \alpha^*r^* \text{ on } H^1(I) - \text{weakly},$$

for all $t \in [0, T]$ and using the compactness embedding $W^{1,\infty}(I) \hookrightarrow C^0(\bar{I})$ we obtain

$$\begin{aligned} \alpha_h(s, r)s_h &\longrightarrow \alpha^*s^* \text{ on } C^0(\bar{I}), \\ \alpha_h(s, r)r_h &\longrightarrow \alpha^*r^* \text{ on } C^0(\bar{I}), \end{aligned} \quad (3.5)$$

where $\alpha^*s^*, \alpha^*r^* \in H^1(I)$ and from (3.5) α^* satisfies the boundary conditions. The next step is to prove that $\alpha^* = \alpha(s^*, r^*)$. Taking into account the assumptions on the data it follows from (3.4) that: for all $t \in [0, T]$

$$\alpha(s_h, r_h) \longrightarrow \alpha(s^*, r^*)^* \text{ on } L^1(I), \quad (3.6)$$

and the fact that $\alpha(s_h, r_h)s_h, \alpha(s_h, r_h)r_h \in BV(I)$ we have

$$\int_I |\alpha(s_h, r_h)s_h - \alpha_h(s, r)s_h| dx \leq \sum_{i=1, \dots, N-1} h |\alpha(s_i^n, r_i^n)s_{i+1}^n - \alpha(s_i^n, r_i^n)s_i^n| \leq h \|\alpha(s^n, r^n)s^n\|_{BV(I)} \longrightarrow 0$$

and

$$\int_I |\alpha(s_h, r_h)r_h - \alpha_h(s, r)r_h| dx \leq \sum_{i=1, \dots, N-1} h |\alpha(s_i^n, r_i^n)r_{i+1}^n - \alpha(s_i^n, r_i^n)r_i^n| \leq h \|\alpha(s^n, r^n)r^n\|_{BV(I)} \longrightarrow 0,$$

and consequently

$$|\alpha(s_h, r_h)s_h - \alpha_h(s, r)s_h| \longrightarrow 0 \text{ on } L^1(I) \quad (3.7)$$

and

$$|\alpha(s_h, r_h)r_h - \alpha_h(s, r)r_h| \longrightarrow 0 \text{ on } L^1(I), \quad (3.8)$$

we now deduce from (3.5)–(3.8) that $\alpha^* = \alpha(s^*, r^*)$.

We conclude that (s^*, r^*) satisfies (1.2). It remains to be shown that (s^*, r^*) satisfies (1.2). For this, let $v \in V$ be a test function and denote by $v_i^n = v(x_{i-1/2}, t^n)$. Multiply by $h\phi_i v_i^n$ the explicit scheme and summing over n and i , we obtain:

$$\begin{aligned} & \sum_{n,i} h\Delta t \, v_i^n \phi_i \frac{r_i^{n+1} - r_i^n}{\Delta t} + \sum_{n,i} u h\Delta t \, v_i^n (g(s_i^n, r_i^n) - g(s_{i-1}^n, r_{i-1}^n)) \\ &= \sum_{n,i} h\Delta t \, v_i^n \left[K_{i+1/2} \alpha(s_i^n, r_i^n) \frac{r_{i+1}^n - r_i^n}{h} - K_{i-1/2} \alpha(s_i^n, r_i^n) \frac{r_i^n - r_{i-1}^n}{h} \right] \end{aligned}$$

and

$$\begin{aligned} & \sum_{n,i} h\Delta t \, v_i^n \phi_i \frac{r_i^{n+1} - r_i^n}{\Delta t} + \sum_{n,i} u h\Delta t \, v_i^n (f(s_i^n, r_i^n) - f(s_{i-1}^n, r_{i-1}^n)) \\ &= \sum_{n,i} h\Delta t \, v_i^n \left[K_{i+1/2} \alpha(s_i^n, r_i^n) \frac{s_{i+1}^n - s_i^n}{h} - K_{i-1/2} \alpha(s_i^n, r_i^n) \frac{s_i^n - s_{i-1}^n}{h} \right]. \end{aligned}$$

These summations are transformed into the following forms:

$$\begin{aligned} -\sum_i h v_i^0 r_i^0 \phi_i + \sum_{n,i} h\Delta t \, r_i^{n+1} \phi_i \frac{v_i^n - v_i^{n+1}}{\Delta t} &= \sum_{n,i} u h\Delta t g(s_i^n, r_i^n) \frac{v_{i+1}^n - v_i^n}{\Delta t} \\ &+ \sum_{n,i} \Delta t K_{i+1/2} \alpha(s_i^n, r_i^n) \frac{r_{i+1}^n - r_i^n}{h} (v_i^n - v_{i-1}^n), \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} -\sum_i h v_i^0 s_i^0 \phi_i + \sum_{n,i} h\Delta t \, s_i^{n+1} \phi_i \frac{v_i^n - v_i^{n+1}}{\Delta t} &= \sum_{n,i} u h\Delta t f(s_i^n, r_i^n) \frac{v_{i+1}^n - v_i^n}{\Delta t} \\ &+ \sum_{n,i} \Delta t K_{i+1/2} \alpha(s_i^n, r_i^n) \frac{s_{i+1}^n - s_i^n}{h} (v_i^n - v_{i-1}^n). \end{aligned} \quad (3.10)$$

Taking into account the assumptions on the data and using the theorem of Lebesgue, it follows as h and Δt go to zero,

$$\begin{aligned} -\sum_i h v_i^0 r_i^0 \phi_i &\longrightarrow -\int_I \phi_i r^0 v(x, 0) dx, \\ \sum_{n,i} h\Delta t \, r_i^{n+1} \phi_i \frac{v_i^n - v_i^{n+1}}{\Delta t} &\longrightarrow -\int \int_{\Omega_T} \phi r^* v_t dt dx, \\ \sum_{n,i} u h\Delta t \, g(s_i^n, r_i^n) \frac{v_{i+1}^n - v_i^n}{h} &\longrightarrow -\int \int_{\Omega_T} u g(s^*, r^*) v_x dt dx, \\ \sum_{n,i} \Delta t K_{i+1/2} \alpha(s_i^n, r_i^n) \frac{r_{i+1}^n - r_i^n}{h} (v_{i+1}^n) &\longrightarrow -\int \int_{\Omega_T} (K(x) \alpha(s^*, r^*) r^* v_x)_x dt dx \end{aligned}$$

and

$$\begin{aligned} -\sum_i h v_i^0 s_i^0 \phi_i &\longrightarrow -\int_I \phi s^0 v(x, 0) dx, \\ \sum_{n,i} h\Delta t \, s_i^{n+1} \phi_i \frac{v_i^n - v_i^{n+1}}{\Delta t} &\longrightarrow -\int \int_{\Omega_T} \phi s^* v_t dt dx, \\ \sum_{n,i} u h\Delta t \, f(s_i^n, r_i^n) \frac{v_{i+1}^n - v_i^n}{h} &\longrightarrow -\int \int_{\Omega_T} u f(s^*, r^*) v_x dt dx, \\ \sum_{n,i} \Delta t K_{i+1/2} \alpha(s_i^n, r_i^n) \frac{s_{i+1}^n - s_i^n}{h} (v_{i+1}^n) &\longrightarrow -\int \int_{\Omega_T} (K(x) \alpha(s^*, r^*) s^* v_x)_x dt dx. \end{aligned}$$

Finally, passing to the limit in (3.9)–(3.10) yields

$$\int_I \phi r^0 v(x, 0) dx + \int \int_{\Omega_T} [\phi r^* v_t + (u g(s^*, r^*) - K \alpha(s^*, r^*) r_x^*) v_x] dt dx = 0$$

and

$$\int_I \phi s^0 v(x, 0) dx + \int \int_{\Omega_T} [\phi s^* v_t + (u f(s^*, r^*) - K \alpha(s^*, r^*) s_x^*) v_x] dt dx = 0.$$

Then, (s^*, r^*) is a weak solution of problem (1.2) which only admits a unique solution (s, r) . Thus, the entire sequence (s_h, r_h) converges to (s, r) . $\square$

4 Numerical simulations

In this section, we present numerical results using the explicit finite volume scheme described in Section 2, for immiscible flow in porous media modeled by problem (1.2).

We deal with the displacement of oil (o) by water (w) “aqueous water and polymer” in a horizontal one-dimensional porous core sample. We study the process for an injection of water with polymer component in the left end of the core sample and before breakthrough time.

The test problem involves simulation with the following data:

Model parameter	Symbol	Value
Length	L	1 m
Porosity	ϕ	0.2
Absolute permeability	K	10^{-13} m^2
Velocity	u	$5 \times 10^{-5} \text{ m s}^{-1}$
Oil viscosity	μ_o	$4 \times 10^{-3} \text{ Pa s}$
Pure water viscosity	μ_a	10^{-3} Pa s
The water saturation in point source	$s_g(t)$	0.8999
The water saturation in point sinks	$s_d(t)$	0.15
The saturation–concentration in point source	$r_g(t)$	0.5
The saturation–concentration in point sinks	$r_d(t)$	0.15

In the numerical simulations presented in this study, the capillary pressure and relative permeability are given by the Van Genuchten model [18]. We put the parameters $\delta = 1$, $n = 2$, $m = 1 - \frac{1}{n} = \frac{1}{2}$, we have

$$p_c(s_w) = p_c(s) = \frac{1}{\delta} (s^{-1/m} - 1)^{1/n} = (s^{-2} - 1)^{1/2},$$

$$k_{rw}(s) = \sqrt{s} [1 - (1 - s^{1/m})^m]^2 = \sqrt{s} [1 - (1 - \sqrt{s})^2]^2,$$

and

$$k_{ro}(s) = s^{1/3} (1 - (1 - s)^{1/m})^{2m} = s^{1/3} [1 - (1 - s)^2],$$

the aqueous phase viscosity μ_w is modeled by a linear function of the polymer concentration

$$\mu_w(c) = \mu_a(1 + \kappa c),$$

where μ_a is the viscosity of pure water and the coefficient κ characterizes the particular polymer, see [19–21].

We obtain

$$\alpha(s, r) = \frac{sk_{rw}(s)k_{ro}(s)}{\mu_o sk_{rw}(s) + \mu_a(1 + \kappa r)k_{ro}(s)} \left(\frac{-1}{s\sqrt{1-s^2}} \right),$$

$$f(s, r) = \frac{\mu_o sk_{rw}(s)}{\mu_o k_{rw}(s) + \mu_a(1 + \kappa r)k_{ro}(s)},$$

and

$$g(s, r) = \frac{\mu_o r sk_{rw}(s)}{\mu_o k_{rw}(s) + \mu_a(1 + \kappa r)k_{ro}(s)}.$$

In the numerical simulations, we vary the viscosity ratio $M_\mu = \mu_o/\mu_a$ for waterflooding and $M_\mu = \mu_o/\mu_w$ for polymer flooding between the values of 1.25 and 10.

4.1 Test 1

In this test, we put the polymer concentration on initial simulation:

$$s(x, t = 0) = \begin{cases} 0.8999 & x = 0, \\ 0.15 & x \neq 0, \end{cases} \quad \text{and} \quad r(x, t = 0) = \begin{cases} 0.5 & x = 0, \\ 0.15 & x \neq 0. \end{cases}$$

This results from right to left for $\kappa = 1.25$, $\kappa = 2$, and $\kappa = 5$, respectively.

The approximate solution of saturation s and concentration–saturation r (Figure 1):

Figure 1: Calculation of the wetting saturation s and saturation–concentration r at instant $T = 10,000$ s by the explicit scheme of saturation equation on 100 Nodes with $\Delta t = 2.12 \times 10^{-4}$ with CFL = 0.9.

Numerical results in one-dimensional tests confirm the stability of the explicit scheme. Remark that if the coefficient κ characterizes the particular polymer as small as possible: oil sweeping by aqueous (water and polymer) is very efficient.

4.2 Test 2

Now, we fix $\kappa = 2$ and we change, respectively, the saturation–concentration “concentration of polymer in injected fluid” on initial simulation:

$$r(x, t = 0) = \begin{cases} 0.3 & x = 0, \\ 0.15 & x \neq 0, \end{cases} \quad r(x, t = 0) = \begin{cases} 0.5 & x = 0, \\ 0.15 & x \neq 0, \end{cases} \quad \text{and} \quad r(x, t = 0) = \begin{cases} 0.6 & x = 0, \\ 0.15 & x \neq 0. \end{cases}$$

See the approximate solution of saturation s and concentration–saturation r (Figure 2):

Figure 2: Calculation of the wetting saturation s and saturation–concentration r at time $T = 10,000$ s by the explicit scheme of the concentration equation on 100 Nodes with $\Delta t = 2.1210^{-4}$ with CFL = 0.9.

Remark that if the greater the initial polymer concentration ratio: oil sweeping by aqueous (water and polymer) is very efficient.

5 Concluding remarks

The purpose of this work was to apply and develop explicit finite volume scheme for a one-dimensional nonlinear, degenerate, convection–diffusion equation. We studied and analyzed the explicit scheme. After proving that the schemes are L^∞ stable and BV stable under CFL conditions and satisfy a discrete maximum principle, we obtained convergence results. Both the schemes were tested numerically for the discretization of the above mentioned equations, in the case of immiscible two-phase flow in porous media. The approximate solution using the present approach was compared to a numerical solution obtained by a standard method which treats the diffusion term with mixed finite elements. The numerical results indicate that the finite volume scheme is particularly well adapted to the discretization of this problem. The extension of the present technique to the two-dimensional problem with uniform rectangular grids is straightforward. In future work, three families of such schemes (explicit, implicit, and semi-implicit) will be studied and analyzed and will concentrate on development of this technique on unstructured grids in the two-dimensional case.

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