Research Article

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An adjusted Grubbs' and generalized extreme studentized deviation

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Abstract: Detecting outlier data is an interesting subject in the statistical field. Grubbs' test is one of the common detection methods of outlier observation at univariate data sets. This approach is based on the mean and standard deviation of univariate data, and hence, these data are highly affected by the presence of outliers. An improvement for Grubbs' test is proposed in this paper to increase the power of detecting data.

Keywords: robust estimation, DetMCD, DetS, DetMM linear discriminant analysis

MSC 2020: 62H30, 62P03, 62P05

1 Introduction

The presence of outliers (extreme points) is the main problem that destroys analysis results. Therefore, studying outlier detection methods is crucial in statistics. These detection methods have a strong impact in estimating data analysis results and might be more interesting than the other data in some cases. Thus, outlier detection is the observation with large deviations from the other observations. This type of observation may be defined as observation generated by different mechanism reasons according to [1]. In the last few decades, many studies have introduced methods to detect the outliers for multivariate and univariate data. Examples of these methods include Z-score, Tukey's method (box-plot) [2], adjusted box-plot [3], minimum covariance determinant [4] and DetS and DetM estimators [5]. One of the interesting subjects is to detect outlier data by using statistical testing. Statistical testing often uses mean and standard deviation to facilitate decision making regarding processing to determine the robustness of evidence to reject the hypothesis of process. Many researchers have studied the robustness of testing statistics for multivariate and univariate data analyses such as [6–9].

Most of the statistical tests for detecting the presence of outliers handle one outlier at once. The outlier is removed from the data set when its measurement is located and the process is repeated until all suspected outliers are detected. One of the common testing methods used for univariate data to detect the presence of outlier data is Grubbs' test, which was introduced in 1950 [10] and extended by the same author in 1969 [11] and 1972 [12]. Grubbs' test is known in different names, such as maximum normed residual test or extreme studentized deviation (ESD) test. The mean, standard deviation, and the tabulated criterion are used to apply Grubbs' test, and the tabulated value for critical value is compared for the level of significance.

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Grubbs' test has been applied in different studies for the last few decades to detect outliers. Previous studies [13–17], applied this test to detect the outlier observation. The advantages and disadvantages of Grubbs' test have been studied over the last few decades, which helps to improve the performance of this test. Rosner [13] compared Grubbs' test and studentized range methods and found that Grubbs' test performed efficiently and was similar to the kurtosis and R-statistic method.

Grubbs' and the generalized ESD tests require mean and standard deviation for their application. Both tests are highly affected by the presence of outlier data, thus yielding accurate results.

Rosner [18] introduced the generalized ESD to test the presence of one or more outliers in a univariate data set that follows an approximately normal distribution. This test requires an upper bound of the suspected number of existing outliers. However, the generalized ESD remained inefficient for sample sizes less than 25. Another improvement was introduced by Brant [19], who reported the combination of ESD rules and box-plot. This combination revealed that the selection of a suitable ESD has comparable performance to box-plot rules in non-Gaussian situations and superior to the pure Gaussian case. Grubbs' test also failed to detect outliers because the sample mean and standard deviation are highly affected by the presence of outliers [20]. The generalized ESD and Grubbs' tests have been applied in different studies, demonstrating a robust equicorrelation effect of observation and a constant significant level [21]. The complete solution of the problem is derived, wherein the characteristics of Grubbs' test are invariant, given a dispersion matrix if and only if it has established Baldessari structure [22]. Grubbs' test is applied to a new transformed data set. The new technique improves the capability of outlier detection, wherein the results show that Grubbs' test can identify the outliers at a significant level of 0.01, after data transformation; notably, the observations are identified at significant level 0.05, before the transformation [23]. Four types of estimation (M , LTS , S , MM , S) are available; Proc Robustreg is performed and compared with sequential application of Grubbs' test, 3-Sigma, and Weisberg t -test. Proc RobustReg may be a useful tool to confirm the results of Grubbs' test and handle signal outliers; dealing with two outliers simultaneously must be performed with care [14]. By contrast, Grubbs' test has been applied with the dependency and independence variables (x_1, x_2, x_3, x_4) of Rosner, where the results were almost similar for detecting the presence of data outliers [24]. Grubbs' test was also applied to the correlation data to obtain a binary decision regarding the presence or absence of the searched watermark. This study showed the efficiency, robustness, and reliability of Grubbs' test [25]. Previous study [26] applied a general extreme studentized residual (GESR) procedure. This approach overcame the type I error and presented a two-phase procedure to detect the outliers with high leverage values. Barbato et al. [27] studies several statistical methods used to detect the presence of outliers and theoretically and experimentally compared their performance. Manoj and Kannan [28] compared different methods for detecting outliers and found that Hampel, Quartile, and generalized ESD are better than Grubbs' and Dixon's tests.

The previous literature showed that all researchers used mean and standard deviation of the data for Grubbs' test. However, modification of the test formula by replacing the test parameters is not attempted due to their substantial sensitivity to outlier observation. One of the most robust parameters for outliers is the median of data. Thus, the mean is replaced by the median and the result showed increased efficacy of the outlier detection observation. Stahel [29] and Donoho [30], revealed that the median is used to adjust box-plots, where the outlyingness of a univariate point x_i is defined as follows.

$$SDO_i = SDO^{(1)}(x_i, x_n) = \frac{|x_i - \text{mad}(x_n)|}{\text{mad}(x_n)}, \quad (1)$$

where x_i refers to the observed sampled and $\text{mad}(x_n)$ is the median of sampled data, where $i = 1, 2, 3, \dots, n$.

This study aims to introduce a method to replace the mean of data by its median for application in Grubbs' test. Furthermore, the replacement of mean by the data median increases the capability of overcoming the presence of outliers due to the substantial sensitivity of the mean to outlier observations. Therefore, the application of the mean in the test may be influenced by the outliers, such as the data presented in Figure 1.

Figure 1: Spread observation example of a group of data.

2 Grubbs' and generalized ESD test of detecting outliers

2.1 Generalized ESD test

Previous study [18] applied ESD test to detect the presence of one or more outliers at univariate data that follows an approximately normal distribution. The generalized ESD test required only a specified upper bound k of the suspected number of outliers. The generalized ESD test is performed k times separately given the k number of outliers. Thus, the test will be performed for one outlier and then repeated for the second outlier, continuing until k time. The generalized ESD test is used for the following hypotheses:

1. H_0 : No outlier is found in the data set.
2. H_1 : Up to k outliers are found in the data set.

Where the test statistic computes:

$$R_i = \frac{\max |x_i - \bar{x}|}{s}. \quad (2)$$

First, the maximum value of $\max |x_i - \bar{x}|$ will remove and then compute the above statistic with $n - 1$ observations. The test will repeat and continues the process until k observations are removed from the data test. The results are obtained in k test statistics $R_1, R_2, R_3, \dots, R_k$.

The significance level and critical value corresponding to the test statistics are calculated K times as follows:

$$\lambda_i = \frac{(n - i)t_{p, n-i-1}}{\sqrt{((n - i - 1) + t_{p, n-i-1}^2)(n - i + 1)}}, \quad (3)$$

where $i = 1, 2, 3, \dots, r$, $t_{p, \nu}$ is the $100p$ percentage point from the distribution t with ν , which is the degrees of freedom and $p = 1 - \frac{\alpha}{3(n - i + 1)}$.

2.2 Grubbs' test

Grubbs' test [10] is used to distinguish and detect the presence of outlier observations, whose distribution is approximately Gaussian. As previously mentioned, Grubbs' test was introduced in 1950 and extended by the same author in 1969 and then 1972. This test is also known as the maximum normed residual test or ESD test and, defined as follows:

$$G = \frac{\max_{i=1,2,3,\dots,N} |x_i - \bar{x}|}{s}, \quad (4)$$

where $\bar{x}$ is the sample mean, and s is the standard deviation of data. Let $x_1, x_2, \dots, x_n$, be the observations sampled, where the order of sample is $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$, which is constructed for the observations. Meanwhile, the tested null hypothesis H_0 indicated that all $x_1, x_2, \dots, x_n$ are not outliers they belong to the data set. The alternative hypothesis H_1 is that $x_1, x_2, \dots, x_{n-1}$ belong to the population and the another x_n is not considered as outlier observation. These hypotheses present a two-sided version of the test. Grubbs' test can be defined as a one-sided test as follows:

1. Test whether the minimum value is an outlier $G = \frac{\bar{x} - x_{\min}}{s}$, with $x_{\min}$ is the minimum value in the data set.
2. Test whether the maximum value is an outlier $G = \frac{x_{\max} - \bar{x}}{s}$, with $x_{\max}$ is the maximum value in the data set.

The critical value of Grubbs' test is

$$G_{\text{crit}} = \frac{(n-1)t_{\text{crit}}}{\sqrt{n(n-2+t_{\text{crit}}^2)}}, \quad (5)$$

where t_{crit} is the critical value of the t distribution T_{n-2} and the significant level is $\alpha/2$. Therefore, if $G > G_{\text{crit}}$, then the null hypothesis is rejected. If the maximum value of G that is related to the pertinent tabulated criterion, then this observation is considered to be an outlier. This process continues until no additional observation is detected at the data set. Grubbs' test is inefficient in detecting outlier observations for data sets of size less than or equal to six, which may be identified as non-outlier in outlier observation [15].

3 Robust tests

The main idea of robust generalized ESD and Grubbs' tests involves replacing the mean of the data set by its median. All previous researchers used the original procedure of Grubbs' test for detecting the outlier observations. However, this study applied both tests as the original ones with the data set median instead of using its mean.

3.1 Generalized ESD

The mean of the data set is replaced by the median to strengthen the generalized ESD test, and the hypothesis test is the same as in 2.1. The test statistic is computed as follows:

$$R_i = \frac{\max |x_i - \text{med}(x)|}{s}. \quad (6)$$

3.2 Grubbs' test

The new Grubbs' test is defined as follows:

$$G = \frac{\max_{i=1,2,3,\dots,N} |X - \text{med}(X)|}{s}, \quad (7)$$

where $\text{med}(X)$ is the median of data set and s is the standard deviation. The remaining step of Grubbs' test procedure is still the same as the original procedure. Let $X_1, X_2, \dots, X_n$ be observation sampled and test steps arranged as follows:

1. Arrange the observations $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$.
2. Calculate the Grubbs' test parameters and the G value.
3. Obtain the G_{crit} value.
4. Decision making.

4 Simulation study

Generalized ESD and Grubbs' test will be applied for univariate generated data from the original and new version. The data should be univariate and the generated data should be normally distributed as follows:

$$\begin{array}{ll} \text{Grubbs' test} & 99N_1(\mu, \sigma) + N_1(\mu, \sigma) \\ \text{Generalized ESD test} & 90N_1(\mu, \sigma) + 25(10N_1(\mu, \sigma)), \end{array}$$

where μ is the mean, and σ is the standard deviation of the data; each set of data contains 100 observations. The group used for Grubbs' test has only one outlier observation, while ten outlier observations are observed for the generalized ESD test data sets.

The data are constructed for single and multiple outliers. The test has also been applied 1,000 times; the data are generated each time and both tests are applied. The tests are constructed in two versions: one version is used with the mean of the data and the second version is applied with the median of data instead of using their mean. However, the two versions have been applied for different levels of alpha values as follows: $\alpha = 0.1$, $\alpha = 0.05$, and $\alpha = 0.01$. The simulation study is conducted to compare the performance of the new version of generalized ESD and Grubbs' test in different levels of significant values of α . The simulations run 1,000 times, and the data are generated each time.

4.1 Grubbs' test result

The univariate data are generated in this simulation study, followed by the normal distribution. Figure 2 presents the generated data set with one unusual observation that is used for this test. Table 1 shows the result of 1,000 replications for Grubbs' test. The test has been applied for original and new versions of Grubbs' test. Table 1 reveals the number of outliers detected, wherein the number of outliers detected by Grubbs' test using the median is more than that of the original test. The difference between the two versions is minimal, the differences are 4, 1, and 18 outliers at $\alpha = 0.01$, 0.05, and 0.01, respectively. However, the difference between the two versions is minimal but still had meaning because the presence of outliers will influence the test result. Table 1 also presents the average of 1,000 values of the maximum value obtained by Grubbs' test. Table 2 shows only ten replications of the Grubbs' test value for each version at all values of α as an example to demonstrate the effect of replacing the data mean by their median. Both versions of the test demonstrate relatively similar performance. However, the same Grubbs' test value is occasionally provided in different data sets, where the outlier value does not have a considerable effect on the

Figure 2: Example of the normal data containing one outlier observation used for Grubbs' test simulation.

Table 1: Two types of parameters and the α values of Grubbs' test for 1,000 replications

Type of parameter	Alpha value	Number outliers	Average of all the G -values for Grubbs' test
Mean	0.01	441	5.3454930514195
Median	0.01	445	5.37140145603596
Mean	0.05	534	4.87943251409262
Median	0.05	535	4.90527097577631
Mean	0.1	591	4.54648039744501
Median	0.1	607	4.52467760964745

Table 2: Sample of ten replications of Grubbs' test out of 1,000 replications

Mean $\alpha = 0.1$	Median $\alpha = 0.1$	Mean $\alpha = 0.05$	Median $\alpha = 0.05$	Mean $\alpha = 0.01$	Median $\alpha = 0.01$
4.99791548	5.00746574	4.99791548	5.00746574	4.99791548	5.0074657
4.72022356	4.67885211	4.72022356	4.6788521	4.72022356	4.6788521
3.13544118	3.19163785	7.89846662	8.00485689	7.89846661	8.00485689
4.29169032	4.31836035	4.29169032	4.31836035	4.29169032	4.31836035
6.39971758	6.56332671	6.39971758	6.56332671	6.39971758	6.56332679
8.72933862	8.79539702	8.72933862	8.79539702	8.70933865	8.79539702
4.93489331	5.00121965	4.93489332	5.00121966	4.98689331	5.00121965
5.42234464	5.43638784	5.42234465	5.43638784	5.42254465	5.43638784
3.29664302	3.28492568	3.29664302	3.28492568	4.37851618	4.4836639

estimation, thus indicating its dependence on the data set. Figure 3 shows the behavior of the result of 1,000 replications, wherein each part of this figure represents the original and the new Grubbs' test for three alpha values. The first part of Figure 3 only has one value, which is quite far from the other values, and the range of all Grubbs' test is close to each other for both versions of Grubbs' test.

Figure 3: The box-plot for original and new Grubbs' test for three α values 0.1, 0.05, 0.01, respectively for 1,000 Monte Carlo sample.

Figure 4: Spread observation example of a group of data as example of generalized ESD data.

Table 3: Sample of ten maximum values of r_{th} for 1,000 replications of generalized ESD test

Mean	Median	Mean	Median	Mean	Median
5.300430002	5.203295069	5.71472549	5.570904725	5.35591713	5.310421837
4.070808367	4.117045592	4.150222854	4.205011857	4.39602947	4.279707365
6.169652668	6.172189401	3.735935686	3.640060263	3.923644425	4.028206728
4.759616896	4.678920536	3.935229529	3.871017171	3.397269384	3.334165472
3.829540283	3.985780779	4.081865981	4.119122537	3.888871123	3.846694817
3.573970163	3.543340603	3.111995699	2.896459924	4.470188945	4.382753211
4.101755091	4.115631057	2.645034796	2.639705743	2.868091054	2.675174872
2.495916551	2.542589952	3.857517887	3.801118877	2.582812931	2.593586935
2.262514987	2.436528915	2.101608211	2.176991513	2.876891594	2.847042216
2.580364591	2.552446223	2.256054084	2.265538408	2.082631391	2.182874679
2.779804583	2.767503326	2.482727728	2.470165789	2.155632263	2.077781431
2.070126694	2.015338046	2.725057463	2.71870878	2.438857826	2.439918636
2.149246808	1.991473785	2.015162636	1.973719327	2.054947758	2.043116464
2.458953457	2.444317153	5.498782864	5.444205319	5.271140234	5.235878777
4.70562353	4.69233363	5.200478464	5.057225327	4.779359114	4.757353968
5.462103129	5.478784101	4.627314023	4.663037715	4.673352835	4.565877198
4.650019999	4.623431136	3.771984071	3.833171563	4.048898887	3.969862562
3.788732048	3.716590666	4.490024378	4.470267213	3.811299799	3.78273964
3.594092799	3.62416543	3.882681257	3.842249739	2.192316979	2.217823645
3.955830893	3.983784999	3.803369769	3.777382906	3.085307907	3.078092184
2.203772269	2.176379838	3.147560491	3.051442616	3.963833845	4.006183014
2.880953678	2.900266821	2.170171417	2.217626525	2.806251077	2.724680092
3.619055938	3.518439247	2.70586475	2.654562996	2.193549521	2.204403805
2.916851336	2.7757971	2.458136407	2.368844298	2.454629576	2.479240315
2.137211278	2.19613043	2.786413065	2.887815709	2.421693815	2.351646994
2.486797174	2.504792535	2.180372308	2.250388542	2.620009583	2.500475795
2.123304242	2.173614836	2.522668822	2.482058356	5.502486098	5.440427526
6.303581351	6.215067768	5.817524369	5.8740713	5.331936524	5.18935738
4.558058988	4.674861556	4.022272734	3.849290853	3.802895094	3.644670821
5.570856767	5.560783171	5.271914513	5.230373984	4.522718631	4.383705974
6.600997557	6.552098733	4.558446943	4.548234141	4.890267956	4.780835832
4.504954764	4.520065448	4.292304867	4.255242213	3.629808288	3.55591935
3.667243481	3.600331585	2.651361851	2.647898658	2.475512044	2.498564178
2.03241704	2.21162707	2.189843656	2.35469121	2.677959848	2.812773134
3.051158696	3.016947219	2.689065776	2.709583109	2.521269187	2.46535682
2.923271844	2.937883779	2.575528	2.537898903	2.552266775	2.525055908
2.223281959	2.242155817	2.445203364	2.42060523	2.5598113	2.510908888
2.047841001	2.005440106	2.062315498	2.255860165	2.01163477	2.202250804
2.241679653	2.213810188	2.468437007	2.46500754	2.511954764	2.47297268
2.127573744	2.089987775	2.310108944	2.291089551	2.5177393	2.488370115

Table 4: Average of sample of all G_i values of test after removing the maximum values satisfies the alternative hypothesis that there are at most r outliers: the median version

	$R = 1$	$R = 2$	$R = 3$	$R = 4$	$R = 5$	$R = 6$	$R = 7$	$R = 8$	$R = 9$	$R = 10$
$n = 90$	0.77967	0.79529	0.81124	0.8075	0.73484	0.78023	0.79627	0.80559	0.76435	0.80517
$n = 91$	0.77395	0.79697	0.81132	0.79130	0.73262	0.77845	0.78228	0.79894	0.7616	0.80052
$n = 92$	0.7673	0.79705	0.80752	0.78165	0.72952	0.77526	0.773	0.79534	0.75770	0.79879
$n = 93$	0.75989	0.79037	0.80414	0.77124	0.72543	0.77370	0.76795	0.76086	0.74969	0.7793
$n = 94$	0.75361	0.77856	0.79324	0.73330	0.72156	0.77267	0.753089	0.72266	0.73878	0.76595
$n = 95$	0.74139	0.76805	0.72740	0.6973	0.7170	0.7709	0.71190	0.69897	0.7258	0.73744
$n = 96$	0.73275	0.72698	0.69824	0.67704	0.69032	0.70193	0.69254	0.68506	0.69328	0.68397
$n = 97$	0.68573	0.69549	0.64749	0.6559	0.68060	0.65674	0.67067	0.65729	0.67543	0.54072
$n = 98$	0.63437	0.66209	0.62416	0.54004	0.6615	0.61387	0.58663	0.61258	0.62984	0.49989
$n = 99$	0.61014	0.59628	0.55033	0.50638	0.63309	0.55083	0.54245	0.57806	0.57951	0.46280

Table 5: Average of sample of all G_i values of test after removing the maximum values satisfies the alternative hypothesis that there are at most r outliers: the mean version

	$R = 1$	$R = 2$	$R = 3$	$R = 4$	$R = 5$	$R = 6$	$R = 7$	$R = 8$	$R = 9$	$R = 10$
$n = 90$	0.78108	0.82073	0.81796	0.80781	0.73548	0.78290	0.80106	0.80650	0.76850	0.79133
$n = 91$	0.77403	0.82063	0.81541	0.79138	0.73261	0.78030	0.78815	0.80158	0.76269	0.7852
$n = 92$	0.76762	0.81219	0.80985	0.78192	0.72980	0.77996	0.78341	0.79826	0.75819	0.78013
$n = 93$	0.76022	0.80191	0.80619	0.77193	0.72611	0.77665	0.77304	0.76417	0.75151	0.77019
$n = 94$	0.75395	0.79511	0.79446	0.73464	0.72156	0.77421	0.76050	0.72779	0.73915	0.7557
$n = 95$	0.74188	0.78524	0.73214	0.70046	0.71713	0.77114	0.72310	0.70423	0.72593	0.73347
$n = 96$	0.73474	0.73621	0.69919	0.67809	0.69210	0.70234	0.69811	0.68588	0.69543	0.70245
$n = 97$	0.68581	0.70112	0.65197	0.65853	0.68516	0.65755	0.67187	0.65990	0.67988	0.67242
$n = 98$	0.63678	0.66413	0.63067	0.54004	0.67089	0.61432	0.58664	0.61277	0.63766	0.62154
$n = 99$	0.61021	0.60396	0.56076	0.50914	0.63527	0.55353	0.54488	0.57860	0.59164	0.57644

5 Generalized ESD test result

Figure 4 is an example of group of data that we use through the simulation study for generalized ESD test. At the generalized ESD procedure we run k Grubbs' test separately. Testing whether $G_i > G_{i-\text{Crit}}$ as we described at Grubbs' test section. But at generalized ESD we adjusted the true size, the value of n replaced by $n - i + 1$. We conclude that there are r outliers, namely $x_1, x_2, \dots, x_r$, where r is the largest value of $i \leq k$ in case that $G_i > G_{i-\text{Crit}}$. If we have no outliers, then $r = 0$; if $G_i > G_{i-\text{Crit}}$ and $h < j$, then both x_h and x_j are outliers even if $G_h \leq G_{h-\text{Crit}}$. The result of Generalized ESD test is shown in Tables 3–5. Where is the Table 3 give a sample pf ten maximum values of r th of 1000 replications. Also, Tables 4 and 5 show the average of sample of all G_i values of test after removing the maximum values that satisfies the hypothesis criteria. The result at Table 5 is maximum than the values in Table 4 which mean give better performance even with small difference value.

6 Conclusion

In this paper, we have proposed an outlier detection method for univariate data. The procedure is based on the Grubbs' and generalized ESD tests with its median instead of the data mean used, whereas the new method that is considered as improver of the classical Grubbs' and generalized ESD used the mean of data. The mean of data is very sensitive to the presence of outlier observation in data set and the median is more robust and more resistant to outlier. Simulation has illustrated that a new version of both tests, performance is robust compared to the classical Grubbs' and generalized tests.

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