Research Article

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A Cauchy-type generalization of Flett's theorem

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Abstract: We prove a Cauchy-type generalization of Flett's theorem and note its geometric interpretations. Several other mean value theorems extending further the result, which involve both real and complex functions, are also proved.

Keywords: Flett's theorem, Cauchy's theorem, Lagrange's theorem, mean value theorems, complex Rolle's theorem, Pawlikowska's extension of Flett's theorem

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1 Introduction

In 1958, Flett proved an interesting mean value theorem, which now bears his name:

Proposition 1. (Flett's theorem, see [1]) *If $f(x)$ is differentiable in $a \leq x \leq b$ and $f'(b) = f'(a)$, then there exists a point ξ , $a < \xi < b$, such that*

$$f'(\xi) = \frac{f(\xi) - f(a)}{\xi - a}. \quad (1)$$

It is actually convenient to assume a slightly more restrictive condition on the function $f(x)$, namely, that $f(x)$ is defined and differentiable on an open interval $I \supset [a, b]$. (Throughout this paper, we shall state our results with such a condition in mind.) This highlights the nice geometric interpretation of Flett's theorem: if a function f has the property that the slopes of the tangents at two points $(a, f(a))$ and $(b, f(b))$ are equal, then there is some point ξ between a and b such that the tangent to the curve $y = f(x)$ at $(\xi, f(\xi))$ passes through $(a, f(a))$. An examination of the original proof reveals that there is nothing special about the left endpoint $(a, f(a))$ and that a symmetric result holds at the right endpoint; thus, Flett's theorem is in reality a statement about the existence of *two* points: under the aforementioned assumptions, there exist $\xi, \eta \in (a, b)$ for which

$$f'(\xi) = \frac{f(\xi) - f(a)}{\xi - a}, \quad f'(\eta) = \frac{f(b) - f(\eta)}{b - \eta}.$$

In the present study, we shall state all results in the usual form – in terms of the endpoint $(a, f(a))$, bearing in mind that all of them could be restated about the other endpoint.

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Sahoo and Riedel developed further Flett's result by removing the boundary condition on the derivatives, proving the following:

Proposition 2. (Generalized Flett's mean value theorem, see [2] or [3]) *Suppose that $f: [a, b] \rightarrow \mathbb{R}$ is differentiable. Then $\exists \xi \in (a, b)$, such that*

$$f'(\xi) = \frac{f(\xi) - f(a)}{\xi - a} + \frac{1}{2} \frac{f'(b) - f'(a)}{b - a} (\xi - a). \quad (2)$$

The natural question arises of how to provide a Cauchy-type generalization of Flett's theorem, similarly to the generalization of the classical mean value theorem to two functions. In that direction, the following result was proved in [4]:

Proposition 3. (Wachnicki [4, Theorem 3]) *Let $f, g: [a, b] \rightarrow \mathbb{R}$ be two differentiable functions. Suppose that $g'(x) \neq 0$ for $x \in [a, b]$, and $\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)}$. Then $\exists \xi \in (a, b)$, such that*

$$\frac{f(\xi) - f(a)}{g(\xi) - g(a)} = \frac{f'(\xi)}{g'(\xi)}. \quad (3)$$

Similarly, Theorem 4 in [4] generalizes (2), again under the assumption that $g'(x) \neq 0$, $x \in [a, b]$.

The condition $g'(x) \neq 0$ is clearly very restrictive, as it forces $g(x)$ to be monotone. It was pointed out by Ivan [5] that under this condition, the two results given by Wachnicki are simply Propositions 1 and 2, applied to the function $F = f \circ g^{-1}$. We mention in passing that Ivan's argument applies without a change to the Cauchy generalization of Lagrange's theorem when that generalization is written in quotient form, as typically presented in introductory calculus textbooks. The aim of this study is to give a generalization of Flett's theorem that avoids these shortcomings as well as to prove some further results of a similar nature.

The following notation shall be used throughout the paper: suppose f and g are two differentiable functions, defined on some open convex set $\mathcal{D} \subseteq \mathbb{R}$ or $\mathcal{D} \subseteq \mathbb{C}$, and let $a \in \mathcal{D}$. Define:

$$\mathbf{f}(\mathbf{a}) = \begin{cases} f'(a), & \text{if } |f'(a)| + |g'(a)| > 0, \\ 1, & \text{if } |f'(a)| + |g'(a)| = 0, \end{cases} \quad \mathbf{g}(\mathbf{a}) = \begin{cases} g'(a), & \text{if } |f'(a)| + |g'(a)| > 0, \\ 1, & \text{if } |f'(a)| + |g'(a)| = 0. \end{cases}$$

2 A Cauchy-type extension for Flett's theorem

In this section, we assume that $I \subseteq \mathbb{R}$ is an open interval, and $[a, b] \subset I$.

Theorem 1. *Let $f, g: I \supset [a, b] \rightarrow \mathbb{R}$ be differentiable, and suppose that*

$$\mathbf{f}(\mathbf{a})g'(b) = f'(b)\mathbf{g}(\mathbf{a}). \quad (4)$$

Then $\exists \xi \in (a, b)$, such that

$$\mathbf{g}(\mathbf{a}) \left[f'(\xi) - \frac{f(\xi) - f(a)}{\xi - a} \right] = \mathbf{f}(\mathbf{a}) \left[g'(\xi) - \frac{g(\xi) - g(a)}{\xi - a} \right]. \quad (5)$$

Proof. Consider the function $\Phi(x) = \mathbf{g}(\mathbf{a})f(x) - \mathbf{f}(\mathbf{a})g(x)$. Suppose first that $f'(a)$ and $g'(a)$ are not both zero. Then (4) becomes $f'(a)g'(b) = f'(b)g'(a)$, and $\Phi(x) = g'(a)f(x) - f'(a)g(x)$. A direct calculation shows that $\Phi'(a) = 0 = \Phi'(b)$, and so by Flett's theorem, there exists $\xi \in (a, b)$ such that $\Phi'(\xi) = \frac{\Phi(\xi) - \Phi(a)}{\xi - a}$, or

$$g'(a)f'(\xi) - f'(a)g'(\xi) = \frac{g'(a)f(\xi) - f'(a)g(\xi) - g'(a)f(a) + f'(a)g(a)}{\xi - a},$$

which is easily rearranged to give

$$g'(a) \left[f'(\xi) - \frac{f(\xi) - f(a)}{\xi - a} \right] = f'(a) \left[g'(\xi) - \frac{g(\xi) - g(a)}{\xi - a} \right]$$

as desired.

In the case when $f'(a) = 0 = g'(a)$, (4) becomes $g'(b) = f'(b)$, and $\Phi(x) = f(x) - g(x)$. Again we have $\Phi'(a) = 0$, $\Phi'(b) = 0$, and Flett's theorem applies: there exists $\xi \in (a, b)$, such that $\Phi'(\xi) = \frac{\Phi(\xi) - \Phi(a)}{\xi - a}$, which now is

$$f'(\xi) - g'(\xi) = \frac{f(\xi) - g(\xi) - f(a) + g(a)}{\xi - a}$$

or

$$f'(\xi) - \frac{f(\xi) - f(a)}{\xi - a} = g'(\xi) - \frac{g(\xi) - g(a)}{\xi - a}.$$

□

The following example illustrates Theorem 1.

Example 1. Take $f(x) = \sin x$ and $g(x) = \cos x$ on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$. Standard calculations show that (4) is met, and (5) becomes $\sin \xi + \cos \xi - \sqrt{2} = (\cos \xi - \sin \xi) \left(\xi - \frac{\pi}{4} \right)$, or

$$\left(1 + \xi - \frac{\pi}{4} \right) \sin \xi + \left(1 - \xi + \frac{\pi}{4} \right) \cos \xi = \sqrt{2}.$$

Consider the function $Q(x) = \left(1 + x - \frac{\pi}{4} \right) \sin x + \left(1 - x + \frac{\pi}{4} \right) \cos x - \sqrt{2}$.

Since $Q\left(\frac{\pi}{2}\right) = 1 + \frac{\pi}{4} - \sqrt{2} > 0$ and $Q(\pi) = \frac{3\pi}{4} - 1 - \sqrt{2} < 0$, there is a zero ξ between $\frac{\pi}{2}$ and π , which is the point guaranteed by Theorem 1. Its numerical value, correct to five decimals, is $\xi = 3.11652$.

Theorem 1 has an interesting geometric interpretation, which generalizes the geometric interpretation of Flett's theorem. Recall that for two nonperpendicular intersecting lines L_1 and L_2 with respective slopes m_1 and m_2 , the angle θ ($0 \leq \theta < \pi$) to which L_1 must be rotated in the positive direction to coincide with L_2 is given by

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

We also adopt the following convention:

$$(1 + m_1 m_2) \tan \theta = m_2 - m_1 \Leftrightarrow L_1 \text{ and } L_2 \text{ are perpendicular.}$$

Given any differentiable function f , let $\theta_f = \theta_f(a, \xi)$ be the angle to which the line through the points $(a, f(a))$ and $(\xi, f(\xi))$ must be rotated counterclockwise to coincide with the tangent line at $(\xi, f(\xi))$. Theorem 1 states that if f and g obey the boundary condition (4), there exists some ξ between a and b , such that

$$g'(a) \left[1 + \frac{f(\xi) - f(a)}{\xi - a} f'(\xi) \right] \tan[\theta_f(a, \xi)] = f'(a) \left[1 + \frac{g(\xi) - g(a)}{\xi - a} g'(\xi) \right] \tan[\theta_g(a, \xi)].$$

For the two functions in Example 1, one obtains $\tan \theta_f \approx -0.547$ and $\tan \theta_g \approx 0.694$.

Theorem 1 also allows for a geometric interpretation in terms of a curve defined parametrically, in the same spirit as the familiar such interpretation for Cauchy's mean-value theorem. Consider a smooth parametric curve $t \mapsto \langle g(t), f(t) \rangle$, $t \in [a, b]$, supposing that (4) holds with $g'(a) \neq 0$, $g'(b) \neq 0$, and the functions $f'(t) - \frac{f(t) - f(a)}{t - a}$ and $g'(t) - \frac{g(t) - g(a)}{t - a}$ are not simultaneously zero in (a, b) . Then (5) guarantees the existence of some $\xi \in (a, b)$, such that

$$\frac{f'(a)}{g'(a)} = \frac{f'(\xi) - \frac{f(\xi) - f(a)}{\xi - a}}{g'(\xi) - \frac{g(\xi) - g(a)}{\xi - a}} = \frac{F'(\xi)}{G'(\xi)},$$

where $F(t) = \frac{f(t) - f(a)}{t - a}$ and $G(t) = \frac{g(t) - g(a)}{t - a}$, $t \in (a, b)$.

We thus obtain the following geometric consequence of Theorem 1: provided a smooth parametric curve with the aforementioned natural restrictions satisfies $\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)}$ (the slopes of the tangents at the two endpoints are equal), then there is a point on an associated parametric graph $t \mapsto \langle G(t), F(t) \rangle$, $t \in (a, b)$, at which point the tangent line has slope equal to the slope at $(g(a), f(a))$.

As an illustration, consider Figures 1 and 2. Figure 1 shows the Lissajous curve $\langle g(t), f(t) \rangle = \langle \cos t, \sin 3t \rangle$ restricted to the interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$, with the tangents drawn at the endpoints. Figure 2 shows the graph of the associated curve:

$$\langle G(t), F(t) \rangle = \left\langle \frac{\cos t - \cos(\pi/4)}{t - \pi/4}, \frac{\sin 3t - \sin(3\pi/4)}{t - \pi/4} \right\rangle,$$

with the point $(G(\xi), F(\xi))$ where the slope is $\frac{F'(\xi)}{G'(\xi)} = \frac{f'(\pi/4)}{g'(\pi/4)}$.

Figure 1: Plot of $\langle \cos t, \sin 3t \rangle$, $t \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$, with tangents $y = 3x - \sqrt{2}$ at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $y = 3x + \sqrt{2}$ at $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Figure 2: Plot of $\langle G(t), F(t) \rangle = \left\langle \frac{\cos t - \cos(\pi/4)}{t - \pi/4}, \frac{\sin 3t - \sin(3\pi/4)}{t - \pi/4} \right\rangle$, $t \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$, with the point $(G(\xi), F(\xi)) = (-0.80114, -2.73618)$ and its tangent $y = 3x - 0.33275$ (correct to five decimals).

Finally, in this section, we generalize Theorem 1 by removing the boundary condition $\mathbf{f}'(\mathbf{a})g'(b) - f'(b)g'(\mathbf{a}) = 0$:

Theorem 2. Let $f, g : I \supset [a, b] \rightarrow \mathbb{R}$ be differentiable. Then $\exists \xi \in (a, b)$, such that

$$\mathbf{g}'(\mathbf{a}) \left[f'(\xi) - \frac{f(\xi) - f(a)}{\xi - a} \right] - \mathbf{f}'(\mathbf{a}) \left[g'(\xi) - \frac{g(\xi) - g(a)}{\xi - a} \right] = \frac{1}{2} \frac{f'(b)g'(\mathbf{a}) - \mathbf{f}'(\mathbf{a})g'(b)}{b - a} (\xi - a). \quad (6)$$

Proof. As in the proof of Theorem 1, consider the function $\Phi(x) = \mathbf{g}'(\mathbf{a})f(x) - \mathbf{f}'(\mathbf{a})g(x)$, and apply the generalized Flett's mean value theorem (Proposition 2): $\exists \xi \in (a, b)$, such that

$$\frac{\Phi(\xi) - \Phi(a)}{\xi - a} - \Phi'(\xi) = -\frac{1}{2} \frac{\Phi(b) - \Phi(a)}{b - a} (\xi - a).$$

This gives

$$\begin{aligned} & \frac{\mathbf{g}'(\mathbf{a})f(\xi) - \mathbf{f}'(\mathbf{a})g(\xi) - \mathbf{g}'(\mathbf{a})f(a) + \mathbf{f}'(\mathbf{a})g(a)}{\xi - a} - \mathbf{g}'(\mathbf{a})f'(\xi) + \mathbf{f}'(\mathbf{a})g'(\xi) \\ &= -\frac{1}{2} \frac{\mathbf{g}'(\mathbf{a})f'(b) - \mathbf{f}'(\mathbf{a})g'(b) - \mathbf{g}'(\mathbf{a})f'(a) + \mathbf{f}'(\mathbf{a})g'(a)}{b - a} (\xi - a) \\ &= -\frac{1}{2} \frac{\mathbf{g}'(\mathbf{a})f'(b) - \mathbf{f}'(\mathbf{a})g'(b)}{b - a} (\xi - a), \end{aligned}$$

and (6) follows. $\square$

3 A survey of several existing Cauchy-type extensions of Flett's theorem

In this section, we briefly depart from the main task in order to survey several interesting theorems, due to Trahan [6], and Hutník and Molnárová [7], which may be compared with Theorem 1. The work [7] especially contains a valuable collection of results on Flett's theorem and its various extensions (in the case of real functions), as well as a comprehensive bibliography. As in [7], we define ${}_a^b \mathfrak{F}(f, g) = \frac{f(b) - f(a)}{g(b) - g(a)}$ and ${}_a^b \mathfrak{F}(f) = \frac{f(b) - f(a)}{b - a}$.

The following two results are due to Trahan:

Proposition 4. (See [6, Theorem 2]; cf. [7, Theorem 3.3]) If $f, g : [a, b] \rightarrow \mathbb{R}$ are differentiable, $g'(a) \neq 0$, $g(x) \neq g(a)$ for $x \in (a, b]$, and

$$\left[\frac{f'(a)}{g'(a)} - {}_a^b \mathfrak{F}(f, g) \right] [[g(b) - g(a)]f'(b) - [f(b) - f(a)]g'(b)] \geq 0,$$

then $\exists \xi \in (a, b]$, such that

$$[g(\xi) - g(a)]f'(\xi) = [f(\xi) - f(a)]g'(\xi).$$

Proposition 5. (See [6, Corollary 3]) If $f, g : [a, b] \rightarrow \mathbb{R}$ are differentiable, $g'(a) \neq 0$, $g(x) \neq g(a)$ for $x \in (a, b]$, $g'(b)[g(b) - g(a)] > 0$, and

$$\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)},$$

then $\exists \xi \in (a, b)$, such that

$$[g(\xi) - g(a)]f'(\xi) = [f(\xi) - f(a)]g'(\xi).$$

It is clear that the inequality assumed in Proposition 4 may be interpreted geometrically, as a (rather involved) combination of various slopes. In the case of $g(x) = x$, that inequality takes the symmetric form:

$$[f'(a) - {}^b_a\mathfrak{F}(f)][f'(b) - {}^b_a\mathfrak{F}(f)] \geq 0,$$

which is an interesting generalization of Flett's theorem. Proposition 5 assumes a boundary condition on the derivatives similar to the one given in Theorem 1, but the conclusion is quite different from that of Theorem 1.

The next results, given by Hutník and Molnárová, also extend or supplement Flett's theorem. They may be stated as follows:

Proposition 6. (See [7, Lemma 3.10 and Theorem 3.11]) *Let f, g be differentiable on $[a, b]$, $g(b) \neq g(a)$, and*

$$[f'(a) - {}^b_a\mathfrak{F}(f, g)g'(a)][f'(b) - {}^b_a\mathfrak{F}(f, g)g'(b)] \geq 0.$$

Then (i) $\exists c \in (a, b)$, such that

$$f(c) - f(a) = {}^b_a\mathfrak{F}(f, g)[g(c) - g(a)],$$

and (ii) $\exists \xi \in (c, b)$, such that

$$f'(\xi) - {}^\xi_a\mathfrak{F}(f) = {}^b_a\mathfrak{F}(f, g)[g'(\xi) - {}^\xi_a\mathfrak{F}(g)].$$

Proposition 7. (See [7, Lemma 3.13 and Theorem 3.14]) *Let f, g be differentiable on $[a, b]$ and twice differentiable at the point a . If $g(a) \neq g(b)$ and there holds the inequality*

$$[f'(a) - {}^b_a\mathfrak{F}(f, g)g'(a)][f''(a) - {}^b_a\mathfrak{F}(f, g)g''(a)] > 0,$$

then (i) $\exists c \in (a, b)$, such that

$$f'(a) - {}^c_a\mathfrak{F}(f) = {}^b_a\mathfrak{F}(f, g)[g'(a) - {}^c_a\mathfrak{F}(g)],$$

and (ii) $\exists \xi \in (a, c)$, such that

$$f'(\xi) - {}^\xi_a\mathfrak{F}(f) = {}^b_a\mathfrak{F}(f, g)[g'(\xi) - {}^\xi_a\mathfrak{F}(g)].$$

Hutník and Molnárová discuss the geometric interpretations of the aforementioned results in the case when $f(a) = g(a)$ and $f(b) = g(b)$: letting $T(f, x_0)(x) = f(x_0) + f'(x_0)(x - x_0)$, Proposition 6 asserts the existence of a point $c \in (a, b)$ such that $f(c) = g(c)$, and the existence of a point $\xi \in (c, b) \subset (a, b)$ such that $T(f, \xi)(a) = T(g, \xi)(a)$, i.e., such that the tangents at $(\xi, f(\xi))$ and $(\xi, g(\xi))$ pass through a common point on the line $x = a$. Similarly, Proposition 7 asserts the existence of a point $c \in (a, b)$ such that $f(c) - g(c) = T(f, a)(c) - T(g, a)(c)$, and the existence of a point $\xi \in (a, c) \subset (a, b)$ such that $T(f, \xi)(a) = T(g, \xi)(a)$. For a further consideration regarding these and various other results, we refer the interested reader to [7] and the references contained therein.

4 A Cauchy-type extension for Flett's theorem in $\mathbb{C}$

Similarly to the theorems of Rolle and Flett, Theorem 1 does not hold true for complex functions. The example given in [3, p. 305] already shows this, but we prefer to give an independent example involving two functions.

Example 2. Let $f(s) = e^s$, $g(s) = s^3 - 3\pi i s^2 - 3s$, on $[0, 2\pi i]$. Condition (4) is met, and we need to show that the equation

$$(-3)\left(\frac{e^s - 1}{s} - e^s\right) = \frac{s^3 - 3\pi i s^2 - 3s}{s} - (3s^2 - 6\pi i s - 3) \quad (7)$$

has no solution in $(0, 2\pi)$. The last equation simplifies to

$$se^s - e^s + 1 = -\frac{2}{3}s^3 + \pi is^2,$$

and since $s = iy$, it is reduced to

$$1 - \cos y - y \sin y + i(y \cos y - \sin y) = i\left(\frac{2}{3}y^3 - \pi y^2\right).$$

Thus, we must consider the system

$$1 - \cos y - y \sin y = 0, \quad y \cos y - \sin y = \frac{2}{3}y^3 - \pi y^2. \quad (8)$$

The first of these gives

$$y = \frac{1 - \cos y}{\sin y} = \tan\left(\frac{y}{2}\right).$$

A standard analysis of the function $y \mapsto y - \tan\left(\frac{y}{2}\right)$, defined on $(0, 2\pi)$, shows that it has a single zero $\tau \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$. (Its approximate numerical value is $\tau \approx 2.33112$.)

On the other hand, substituting $y = \frac{1 - \cos y}{\sin y}$ on the left-hand side of the second equation in (8) gives

$$\frac{1 - \cos y}{\sin y} \cos y - \sin y = \frac{(1 - \cos y) \cos y - \sin^2 y}{\sin y} = \frac{\cos y - 1}{\sin y} = -y,$$

so the equation becomes $\frac{2}{3}y^3 - \pi y^2 + y = 0$. Its solutions are as follows:

$$y = \frac{3\pi \pm \sqrt{9\pi^2 - 24}}{4},$$

none of which is in $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$. Thus, the system (8) has no solution in $(0, 2\pi)$.

Theorem 3. (A complex Cauchy-type extension of Flett's theorem) *Let $s = x + iy$ be a complex variable and let $f(s)$ and $g(s)$ be two holomorphic functions defined on the open convex set $\mathfrak{D} \subseteq \mathbb{C}$. Suppose $A = a_1 + ia_2 \in \mathfrak{D}$, $B = b_1 + ib_2 \in \mathfrak{D}$ and denote by (A, B) the open segment connecting A and B . Suppose that*

$$\mathbf{f}'(\mathbf{A})g'(B) = f'(B)\mathbf{g}'(\mathbf{A}). \quad (9)$$

Then $\exists z_1, z_2 \in (A, B)$ such that

$$\Re\left\{\mathbf{g}'(\mathbf{A})\left[f'(z_1) - \frac{f(z_1) - f(A)}{z_1 - A}\right]\right\} = \Re\left\{\mathbf{f}'(\mathbf{A})\left[g'(z_1) - \frac{g(z_1) - g(A)}{z_1 - A}\right]\right\}, \quad (10)$$

$$\Im\left\{\mathbf{g}'(\mathbf{A})\left[f'(z_2) - \frac{f(z_2) - f(A)}{z_2 - A}\right]\right\} = \Im\left\{\mathbf{f}'(\mathbf{A})\left[g'(z_2) - \frac{g(z_2) - g(A)}{z_2 - A}\right]\right\}. \quad (11)$$

Proof. Let $\Phi(s) = \mathbf{g}'(\mathbf{A})f(s) - \mathbf{f}'(\mathbf{A})g(s)$. As mentioned earlier, the condition (9) means that $\Phi(s) = g'(A)f(s) - f'(A)g(s)$ if $f'(A)$ and $g'(A)$ are not both zero, and $\Phi(s) = f(s) - g(s)$ if $f'(A) = 0 = g'(A)$. As in the proof in Theorem 1, we have $\Phi'(A) = \Phi'(B)$. By [8, Theorem 3], there exist $z_1, z_2 \in (A, B)$ such that

$$\Re[\Phi'(z_1)] = \Re\left[\frac{\Phi(z_1) - \Phi(A)}{z_1 - A}\right], \quad \Im[\Phi'(z_2)] = \Im\left[\frac{\Phi(z_2) - \Phi(A)}{z_2 - A}\right],$$

or

$$\Re\{\mathbf{g}'(\mathbf{A})f'(z_1) - \mathbf{f}'(\mathbf{A})g'(z_1)\} = \Re\left\{\frac{\mathbf{g}'(\mathbf{A})f(z_1) - \mathbf{f}'(\mathbf{A})g(z_1) - \mathbf{g}'(\mathbf{A})f(A) + \mathbf{f}'(\mathbf{A})g(A)}{z_1 - A}\right\},$$

$$\Im\{\mathbf{g}'(\mathbf{A})f'(z_2) - \mathbf{f}'(\mathbf{A})g'(z_2)\} = \Im\left\{\frac{\mathbf{g}'(\mathbf{A})f(z_2) - \mathbf{f}'(\mathbf{A})g(z_2) - \mathbf{g}'(\mathbf{A})f(A) + \mathbf{f}'(\mathbf{A})g(A)}{z_2 - A}\right\}.$$

Since $\Re\{\alpha \pm \beta\} = \Re\{\alpha\} \pm \Re\{\beta\}$ and $\Im\{\alpha \pm \beta\} = \Im\{\alpha\} \pm \Im\{\beta\}$ for any $\alpha, \beta \in \mathbb{C}$, the last two relations can be rearranged, obtaining (10) and (11). $\square$

Example 2 revisited (complex case). Returning to our example and setting $s = iy$ in (7), we get

$$(-3)\left(\frac{e^{iy} - 1}{iy} - e^{iy}\right) = -2(iy)^2 + 3\pi i(iy)$$

or

$$3\left[\cos y - \frac{\sin y}{y} + i\left(\frac{\cos y - 1}{\sin y} + \sin y\right)\right] = 2y^2 - 3\pi y.$$

Equating the real and imaginary parts, respectively, gives

$$\cos y - \frac{\sin y}{y} = \frac{2}{3}y^2 - \pi y \quad (12)$$

and

$$\frac{\cos y - 1}{\sin y} + \sin y = 0. \quad (13)$$

For (12), we consider the function $R(y) = \cos y - \frac{\sin y}{y} - \frac{2}{3}y^2 + \pi y$ and note that $R\left(\frac{3\pi}{2}\right) > 0$, while $R(2\pi) < 0$. Thus, R has a zero in $\left(\frac{3\pi}{2}, 2\pi\right)$. Its numerical value is approximately 4.806302. The solution to (13) as mentioned earlier is approximately 2.33112. Thus, the two complex solutions assured by Theorem 3 are $z_1 \approx 4.806302i$ and $z_2 \approx 2.33112i$.

Theorem 4. (A complex Cauchy-type extension of the generalized Flett's theorem) *Let $s = x + iy$ be a complex variable and let $f(s)$ and $g(s)$ be two holomorphic functions defined on the open convex set $\mathcal{D} \subseteq \mathbb{C}$. Suppose $A = a_1 + ia_2 \in \mathcal{D}$, $B = b_1 + ib_2 \in \mathcal{D}$ and denote by (A, B) the open segment connecting A and B . Then $\exists z_1, z_2 \in (A, B)$, such that*

$$\Re\left\{\mathbf{g}'(\mathbf{A})\left[f'(z_1) - \frac{f(z_1) - f(A)}{z_1 - A}\right]\right\} - \Re\left\{\mathbf{f}'(\mathbf{A})\left[g'(z_1) - \frac{g(z_1) - g(A)}{z_1 - A}\right]\right\} \\ = \frac{1}{2} \frac{\Re\{f'(B)\mathbf{g}'(\mathbf{A}) - \mathbf{f}'(\mathbf{A})g'(B)\}}{B - A}(z_1 - A), \quad (14)$$

$$\Im\left\{\mathbf{g}'(\mathbf{A})\left[f'(z_2) - \frac{f(z_2) - f(A)}{z_2 - A}\right]\right\} - \Im\left\{\mathbf{f}'(\mathbf{A})\left[g'(z_2) - \frac{g(z_2) - g(A)}{z_2 - A}\right]\right\} \\ = \frac{1}{2} \frac{\Im\{f'(B)\mathbf{g}'(\mathbf{A}) - \mathbf{f}'(\mathbf{A})g'(B)\}}{B - A}(z_2 - A). \quad (15)$$

Proof. We have shown in [8] that if a holomorphic function $\Phi(s)$ satisfies the assumptions in Theorem 4, there exist $z_1, z_2 \in (A, B)$ such that

$$\Re\{\Phi'(z_1)\} = \Re\left\{\frac{\Phi(z_1) - \Phi(A)}{z_1 - A} + \frac{1}{2} \frac{\Phi'(B) - \Phi'(A)}{B - A}(z_1 - A)\right\}, \quad (16)$$

$$\Im\{\Phi'(z_2)\} = \Im\left\{\frac{\Phi(z_2) - f(A)}{z_2 - A} + \frac{1}{2} \frac{\Phi'(B) - \Phi'(A)}{B - A} (z_2 - A)\right\}. \quad (17)$$

The method of proof goes back to the beautiful paper by Evard and Jafari [9] who discovered the true form of a complex Rolle's theorem. The idea is to consider the real functions

$$\begin{aligned}\phi(t) &= (b_1 - a_1)\Re[\Phi(A + t(B - A))] + (b_2 - a_2)\Im[\Phi(A + t(B - A))], \\ \psi(t) &= (b_1 - a_1)\Im[\Phi(A + t(B - A))] - (b_2 - a_2)\Re[\Phi(A + t(B - A))], \quad t \in [0, 1],\end{aligned}$$

and to differentiate them, making use of the Cauchy-Riemann equations (see [3,8,9]). Applying the generalized Flett's theorem to ϕ and ψ guarantees the existence of two real numbers $t_1, t_2 \in (0, 1)$ such that

$$\begin{aligned}\phi'(t_1) &= \frac{\phi(t_1) - \phi(0)}{t_1} + \frac{1}{2}[\phi'(1) - \phi'(0)]t_1, \\ \psi'(t_2) &= \frac{\psi(t_2) - \psi(0)}{t_2} + \frac{1}{2}[\psi'(1) - \psi'(0)]t_2.\end{aligned}$$

Then (14) and (15) follow upon setting $z_1 = A + t_1(B - A)$ and $z_2 = A + t_2(B - A)$. Note that (16) and (17) can be rewritten as follows:

$$\begin{aligned}\Re\left\{\Phi'(z_1) - \frac{\Phi(z_1) - \Phi(A)}{z_1 - A}\right\} &= \frac{1}{2} \frac{\Re\{\Phi'(B) - \Phi'(A)\}}{B - A} (z_1 - A), \\ \Im\left\{\Phi'(z_2) - \frac{\Phi(z_2) - f(A)}{z_2 - A}\right\} &= \frac{1}{2} \frac{\Im\{\Phi'(B) - \Phi'(A)\}}{B - A} (z_2 - A).\end{aligned}$$

Now we set $\Phi(s) = \mathbf{g}'(\mathbf{A})f(s) - \mathbf{f}'(\mathbf{A})g(s)$ and apply the aforementioned, from which (14) and (15) follow after some straight-forward manipulations. $\square$

In conclusion, we mention that Theorem 2 can easily be generalized to higher derivatives, by applying a result of Pawlikowska [10, Theorem 2.3] to the function $\Phi(x)$ defined in the proof of Theorem 2. This gives the following:

Theorem 5. Let $f, g : I \subset [a, b] \rightarrow \mathbb{R}$ be n -times differentiable. Then $\exists \xi \in (a, b)$, such that

$$\begin{aligned}& \mathbf{g}'(\mathbf{a})[f(\xi) - f(a)] - \mathbf{f}'(\mathbf{a})[g(\xi) - g(a)] \\ &= \sum_{k=1}^n (-1)^{k-1} \frac{1}{k!} [\mathbf{g}'(\mathbf{a})f^{(k)}(\xi) - \mathbf{f}'(\mathbf{a})g^{(k)}(\xi)](\xi - a)^k \\ &+ (-1)^n \frac{1}{(n+1)!} \frac{\mathbf{g}'(\mathbf{a})[f^{(n)}(b) - f^{(n)}(a)] - \mathbf{f}'(\mathbf{a})[g^{(n)}(b) - g^{(n)}(a)]}{b - a} (\xi - a)^{n+1}.\end{aligned}$$

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