Research Article

Madjda Miloudi*, Samira Saadi, and Mohamed Haiour

L^∞ -error estimates of a finite element method for Hamilton-Jacobi-Bellman equations with nonlinear source terms with mixed boundary condition

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Abstract: In this paper, we introduce a new method to analyze the convergence of the standard finite element method for Hamilton-Jacobi-Bellman equation with noncoercive operators with nonlinear source terms with the mixed boundary conditions. The method consists of combining Bensoussan-Lions algorithm with the characterization of the solution, in both the continuous and discrete contexts, as fixed point of contraction. Optimal error estimates are then derived, first between the continuous algorithm and its finite element counterpart and then between the continuous solution and the approximate solution.

Keywords: algorithm, contraction, finite element, fixed point Hamilton-Jacobi-Bellman equation, L^∞ -error estimate

MSC 2020: 65F30, 47H09, 65L60, 47H10, 60H15, 65M12

1 Introduction

We consider the following Hamilton-Jacobi-Bellman (HJB) equations with nonlinear source terms and mixed boundary conditions: find $u \in W^{2,\infty}(\Omega)$, such that:

$$\begin{cases} \max_{1 \leq i \leq M} (A^i u - f^i(u)) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = \varphi & \text{in } \Gamma_0, \\ u = 0 & \text{on } \Gamma/\Gamma_0, \end{cases} \quad (1)$$

where Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 1$ with smooth boundary Γ , $\Gamma_0 = \{x \in \Gamma \text{ such that } \forall \xi > 0, x + \xi \notin \overline{\Omega}\}$, and $A^1, \dots, A^M$ denote uniformly second-order elliptic operators defined by

$$A^i = - \sum_{1 \leq j, k \leq N} \frac{\partial}{\partial x_j} a_{jk}^i(x) \frac{\partial}{\partial x_k} + \sum_{1 \leq k \leq N} b_k^i(x) \frac{\partial}{\partial x_k} + a_0^i(x).$$

* **Corresponding author: Madjda Miloudi**, Department of Mathematics, University Badji Mokhtar Annaba, Box. 12, Annaba 23000, Algeria, e-mail: miloudimath2@yahoo.fr

Samira Saadi, Mohamed Haiour: Department of Mathematics, University Badji Mokhtar Annaba, Box. 12, Annaba 23000, Algeria

We also define the associated bilinear forms

$$a^i(u, v) = \int_{\Omega} \left(\sum_{1 \leq j, k \leq N} a_{jk}^i(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_k} + \sum_{1 \leq k \leq N} b_k^i(x) \frac{\partial u}{\partial x_k} v + a_0^i(x) uv \right) dx, \quad (2)$$

such that $a_{jk}^i(x), b_k^i(x), a_0^i(x) \in C^2(\bar{\Omega})$, $a_{jk}^i(x) = a_{kj}^i(x)$; $a_0^i(x) \geq \beta > 0$, $x \in \bar{\Omega}$, $\sum_{1 \leq j, k \leq N} a_{jk}^i(x) \xi_j \xi_k \geq \alpha |\xi|^2$, $\forall \xi \in \mathbb{R}^N$, $x \in \bar{\Omega}$, $\alpha > 0$, $\frac{\partial u}{\partial \eta} = \nabla u \cdot \vec{\eta}$, where $\vec{\eta}$ is the normal vector.

$(\cdot, \cdot)_{\Omega}$ is the inner product in $L^2(\Omega)$, $(\cdot, \cdot)_{\Gamma_0}$ stands for the inner product in $L^2(\Gamma_0)$. $K(u) = \left\{ u \in H_0^1(\Omega), \frac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0, u = 0 \text{ on } \Gamma/\Gamma_0 \right\}$.

We specify the following notations: $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|_2$, $\|\cdot\|_{H^1(\Omega)} = \|\cdot\|_1$, and $\|\cdot\|_{L^\infty(\Omega)} = \|\cdot\|_\infty$. $f^i(\cdot) \in L^\infty(\Omega)$ for $i = 1, \dots, M$ are M nonlinear Lipschitz continuous functions with Lipschitz constant c and satisfying condition:

1. $f^i > 0$ and also it is increasing;
2. $\frac{c}{\beta} < 1$.

In this paper, we are concerned with the numerical approximation in the L^∞ norm for the problem (1), and we instead combine, in both the continuous and discrete contexts, the Bensoussan-Lions algorithm with the characterization of the solution as a fixed point of a contraction. We first establish an error estimate between the continuous algorithm and its finite element version and then between the exact solution and the finite element approximate. We exploit this idea to derive an optimal convergence order for the HJB equation.

This method consists, mainly, of combining, in both the continuous and discrete contexts, the concept of fixed point and a geometrical convergence of an iterative scheme approximating the solution. For a computational purpose, this method provides an interesting information as it permits to control the error between the continuous iterative scheme and its finite element counterpart.

While in previous studies, Cortey Dumont and Boulbrachene used the concept of subsolutions to find the estimation of error.

The HJB equation has been analytically studied in [1–4]. For the numerical approximations, Cortey Dumont [5] investigated a finite element approximation and used a subsolution method. Boulbrachene and Haiour [6] investigated a finite element Bensoussan-Lions algorithm version and obtained a quasi-optimal error estimate in the L^∞ -norm. Boulbrachene and Cortey Dumont [7] investigated a finite element method using the concept of subsolutions and discrete regularity and obtained an optimal error estimate in the L^∞ -norm.

Boulaaras and Haiour investigated a finite element of the HJB equation elliptic and parabolic [8,9]. They also studied Schwarz methods of parabolic HJB equation with nonlinear source terms with mixed boundary conditions [10].

This paper is organized as follows: We view the continuous problem in Section 1 and the discrete problem in Section 2. We address the continuous algorithm in Section 3 and the discrete algorithm in Section 4, and we establish, in both the continuous and discrete cases, the geometrical convergence of these algorithms. Finally, in Section 5, we present the finite element error analysis.

It is shown in [3] that (1) can be approximated by the following weakly coupled system of quasi-variational inequalities (QVIs) with mixed boundary conditions

$$\begin{cases} a^i(\xi^i, v - \xi^i) \geq (f^i(u), v - \xi^i)_{\Omega} + (\varphi, v - \xi^i)_{\Gamma_0}, \quad \forall v \in H_0^1(\Omega), \\ \xi^i \leq k + \xi^{i+1}, \quad v \leq k + \xi^{i+1}, \quad i = 1, \dots, M, \\ \xi^{M+1} = \xi^1, \quad \frac{\partial \xi^i}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \xi^i = 0 \text{ on } \Gamma/\Gamma_0, \end{cases} \quad (3)$$

where k is a positive constant. This is precisely stated in the following theorem.

Theorem 1. [3] *The system (3) has a unique solution, which belongs to $W^{2,p}(\Omega)^M$, $2 \leq p < \infty$. Moreover, as $k \rightarrow 0$, each component of this system converges uniformly in $C(\overline{\Omega})$ to the solution u of HJB equation (1).*

Lemma 2. [11] *There exists a constant c independent of k , thus*

$$\|\xi^i - u\|_{\infty} \leq ck, \quad i = 1, \dots, M.$$

Let the mapping

$$\begin{aligned} T : L^{\infty}_+(\Omega) &\rightarrow L^{\infty}(\Omega), \\ \omega &\rightarrow T\omega = \xi, \end{aligned}$$

where ξ is the unique solution of the following HJB equation:

$$\begin{cases} \max_{1 \leq i \leq M} (A^i \xi - f^i(\omega)) = 0 & \text{in } \Omega, \\ \frac{\partial \xi}{\partial \eta} = \varphi & \text{in } \Gamma_0, \\ \xi = 0 & \text{on } \Gamma/\Gamma_0. \end{cases} \quad (4)$$

From [3], (4) can be approximated by the following system of QVIs

$$\begin{cases} a^i(\xi^i, v - \xi^i) \geq (f^i(\omega), v - \xi^i)_{\Omega} + (\varphi, v - \xi^i)_{\Gamma_0}, \quad \forall v \in H_0^1(\Omega), \\ \xi^i \leq k + \xi^{i+1}, \quad v \leq k + \xi^{i+1}, \quad i = 1, \dots, M, \\ \xi^{M+1} = \xi^1, \quad \frac{\partial \xi}{\partial \eta} = \varphi & \text{in } \Gamma_0, \end{cases} \quad (5)$$

where k is a positive constant and we have $\lim_{k \rightarrow 0} \|\xi^i - \xi\|_{C(\Omega)} = 0$, $\forall i = 1, \dots, M$.

Lemma 3. [11] *There exists a constant c independent of k , thus*

$$\|\xi^i - \xi\|_{\infty} \leq ck, \quad i = 1, \dots, M.$$

Let $(M\xi, \varphi)$, $(M\tilde{\xi}, \tilde{\varphi})$, $\xi = (\xi^i)_{1 \leq i \leq M}$, $\tilde{\xi} = (\tilde{\xi}^i)_{1 \leq i \leq M}$ pair of data, and $\xi = \sigma(M\xi, \varphi)$, $\tilde{\xi} = \sigma(M\tilde{\xi}, \tilde{\varphi})$ be the corresponding solutions of the following system of quasi-variational inequalities:

$$\begin{aligned} a^i(\xi^i, v - \xi^i) &\geq (f^i(\xi^i), v - \xi^i)_{\Omega} + (\varphi, v - \xi^i)_{\Gamma_0}, \quad 1 \leq i \leq M, \\ a^i(\tilde{\xi}^i, v - \tilde{\xi}^i) &\geq (f^i(\tilde{\xi}^i), v - \tilde{\xi}^i)_{\Omega} + (\tilde{\varphi}, v - \tilde{\xi}^i)_{\Gamma_0}, \quad 1 \leq i \leq M. \end{aligned}$$

Lemma 4. [10] *If $\varphi \geq \tilde{\varphi}$, then $\sigma(M\xi, \varphi) \geq \sigma(M\tilde{\xi}, \tilde{\varphi})$.*

Lemma 5. *Let $\omega, \tilde{\omega}$ be in $L^{\infty}(\Omega)$ and $(\xi^1, \dots, \xi^M)$, $(\tilde{\xi}^1, \dots, \tilde{\xi}^M)$ be the corresponding solutions to system (5) with right-hand sides $f^i(\omega)$ and $f^i(\tilde{\omega})$, respectively. Then we have*

$$\max_{1 \leq i \leq M} \|\xi^i - \tilde{\xi}^i\|_{\infty} \leq \rho \|\omega - \tilde{\omega}\|_{\infty}, \quad \rho = \frac{c}{\beta} < 1.$$

Proof. Let $\phi^i = \frac{1}{\beta} \|f^i(\omega) - f^i(\tilde{\omega})\|_{\infty}$, $i = 1, \dots, M$.

Then:

$$\begin{aligned} f^i(\omega) &\leq f^i(\tilde{\omega}) + \|f^i(\omega) - f^i(\tilde{\omega})\|_{\infty} \\ &\leq f^i(\tilde{\omega}) + \frac{a_0^i(x)}{\beta} \|f^i(\omega) - f^i(\tilde{\omega})\|_{\infty} \\ &\leq f^i(\tilde{\omega}) + (a_0^i(x))\phi^i, \quad i = 1, \dots, M. \end{aligned}$$

Thus, making use of monotonicity result with respect to right-hand side for system of QVIs related to HJB equation with boundary mixed conditions (see [5]), we get:

$$\xi^i \leq \tilde{\xi}^i + \phi^i,$$

we also get

$$\|\xi^i - \tilde{\xi}^i\|_\infty \leq \phi^i, \quad i = 1, \dots, M,$$

which completes the proof. $\square$

Theorem 6. *The mapping T is a contraction with rate equal to $\rho = \frac{c}{\beta} < 1$, so the solution of HJB equation (1) is its unique fixed point.*

Proof. Let $\xi = T\omega$, $\tilde{\xi} = T\tilde{\omega}$ be solutions of HJB equation (4) with right-hand sides $f^i(\omega)$ and $f^i(\tilde{\omega})$, respectively. Then making use of both Theorem 1, Lemmas 1 and 2, we have

$$\begin{aligned} \|T\omega - T\tilde{\omega}\|_\infty &\leq \|\xi - \tilde{\xi}\|_\infty \\ &\leq \|\xi - \xi^i\|_\infty + \|\xi^i - \tilde{\xi}^i\|_\infty + \|\tilde{\xi}^i - \tilde{\xi}\|_\infty, \\ &\leq ck + \|\xi^i - \tilde{\xi}^i\|_\infty + ck, \quad i = 1, \dots, M. \end{aligned}$$

Hence, passing to the limit, as $k \rightarrow 0$, we get

$$\|T\omega - T\tilde{\omega}\|_\infty \leq \max_{1 \leq i \leq M} \|\xi^i - \tilde{\xi}^i\|_\infty \leq \rho \|\omega - \tilde{\omega}\|_\infty, \quad \rho = \frac{c}{\beta} < 1.$$

Thus, T is a contraction. $\square$

2 The discrete problem

Let Ω be decomposed into triangles, τ_h denote the set of all those elements, and $h > 0$ be the mesh size. We assume that the family τ_h is regular and quasi-uniform. Let

$$V_h = \left\{ v \in C(\bar{\Omega}) \cap H_0^1(\Omega), v|_K \in P_1, \frac{\partial v}{\partial \eta} = \varphi \text{ in } \Gamma_0, v = 0 \text{ on } \Gamma/\Gamma_0 \right\},$$

be the finite element space, where K is a triangle of τ_h and P_1 is the space of polynomials with degree ≤ 1 . Let φ_l , $l = 1, \dots, m(h)$ be the basis functions of the space V_h , and A^i the matrices with generic coefficients

$$(A^i)_{ls} = a^i(\varphi_l, \varphi_s), \quad l, s = 1, \dots, m(h), \quad 1 \leq i \leq M.$$

Let us also define the discrete right-hand sides

$$f^i(u) = (f^i(u), \varphi_l), \quad l = 1, \dots, m(h), \quad 1 \leq i \leq M,$$

and the usual restriction operator r_h :

$$\forall v \in C(\bar{\Omega}) \cap H_0^1(\Omega), \quad r_h v = \sum_{l=1}^{m(h)} v_l \varphi_l.$$

The discrete HJB equation consists of solving the following problem: Find $u_h \in V_h$ solution to:

$$\begin{cases} \max_{1 \leq i \leq M} (A^i u_h - f^i(u_h)) = 0 & \text{on } \Omega, \\ \frac{\partial u_h}{\partial \eta} = \varphi & \text{in } \Gamma_0, \\ u_h = 0 & \text{on } \Gamma/\Gamma_0. \end{cases} \quad (6)$$

Lemma 7. [5] *The matrices A^i , $i = 1, \dots, M$, are M -matrices.*

Theorem 8. [5] *Under condition of Lemma 3, the HJB equation (6) has a unique solution.*

It was shown in [5] that (6) can be approximated by the following discrete weakly coupled system of QVIs

$$\begin{cases} a^i(\xi_h^i, v - \xi_h^i) \geq (f^i(u_h), v - \xi_h^i) + (\varphi, v - \xi_h^i)_{\Gamma_0}, \quad \forall v \in V_h, \\ \xi_h^i \leq k + \xi_h^{i+1}, \quad v \leq k + \xi_h^{i+1}, \quad i = 1, \dots, M, \\ \xi_h^{M+1} = \xi_h^1, \quad \frac{\partial \xi_h^i}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \xi_h^i = 0 \text{ on } \Gamma/\Gamma_0. \end{cases} \quad (7)$$

Theorem 9. [5] *Under condition of Lemma 3, then, the system (7) has a unique solution. Moreover, as $k \rightarrow 0$; each component of the solution of this system converges uniformly in $C(\Omega)$ to the solution u_h of (6).*

We introduce the mapping

$$\begin{aligned} T_h : L^\infty(\Omega) &\rightarrow V_h, \\ \omega &\rightarrow T\omega = \xi_h, \end{aligned}$$

where ξ_h is the unique solution of the following discrete coercive HJB equation

$$\max_{1 \leq i \leq M} (A^i \xi_h - f^i(\omega)) = 0, \quad (8)$$

the discrete coercive HJB equation (8) can be approximated by the following system of QVIs

$$\begin{cases} a^i(\xi_h^i, v - \xi_h^i) \geq (f^i(\omega), v - \xi_h^i) + (\varphi, v - \xi_h^i)_{\Gamma_0}, \quad \forall v \in V_h, \\ \xi_h^i \leq k + \xi_h^{i+1}, \quad v \leq k + \xi_h^{i+1}, \quad i = 1, \dots, M, \\ \xi_h^{M+1} = \xi_h^1, \quad \frac{\partial \xi_h^i}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \xi_h^i = 0 \text{ on } \Gamma/\Gamma_0 \end{cases}$$

and we have [11]:

$$\|\xi_h^i - \xi_h\|_\infty \leq ck, \quad i = 1, \dots, M.$$

Lemma 10. *Under condition of Lemma 3, then, we have*

$$\max_{1 \leq i \leq M} \|\xi^i - \tilde{\xi}^i\|_\infty \leq \rho \|\omega - \tilde{\omega}\|_\infty, \quad \rho = \frac{c}{\beta} < 1, \quad \forall \omega, \tilde{\omega} \in L_+^\infty(\Omega).$$

Proof. Exactly the same as that of Lemma 2. □

Theorem 11. *Under condition of Lemma 3, the mapping T_h is a contraction, so the solution of discrete HJB equation (6) is its unique fixed point.*

Proof. Exactly the same as that of Theorem 2. □

3 A continuous iterative scheme

Starting from $u^0 \in H_0^1(\Omega)$ is the unique solution of the variational equation:

$$a^1(u^0, v) = (f^1(u^0), v), \quad \forall v \in H_0^1(\Omega).$$

We define the sequence $(u^n)_{n \geq 1}$ by:

$$u^n = Tu^{n-1}, \quad \forall n \geq 1,$$

such that each iterate u^n solves the coercive HJB equation:

$$\begin{cases} \max_{1 \leq i \leq M} (A^i(u^n) - f^i(u^{n-1})) = 0 & \text{in } \Omega, \\ \frac{\partial u^n}{\partial \eta} = \varphi & \text{in } \Gamma_0, \\ u^n = 0 & \text{on } \Gamma/\Gamma_0. \end{cases} \quad (9)$$

Theorem 12. *Under condition of the mapping T is a contraction and the fixed point theorem, the sequence $(u^n)_{n \geq 0}$ converges to the unique fixed point u and we have the error bound:*

$$\|u^n - u\|_{\infty} \leq \frac{\rho^n}{1 - \rho} \|u^0 - u^1\|_{\infty}.$$

Let $(\xi^{i,n})_{1 \leq i \leq M}$ be the unique solution of the system of QVIs, which approximated the coercive HJB equation (9):

$$\begin{cases} a^i(\xi^{i,n}, v - \xi^{i,n}) \geq (f^i(u^{n-1}), v - \xi^{i,n}) + (\varphi, v - \xi^{i,n})_{\Gamma_0}, \quad \forall v \in H_0^1(\Omega), \\ \xi^{i,n} \leq k + \xi^{i+1,n}, \quad v \leq k + \xi^{i+1,n}, \quad i = 1, \dots, M, \\ \xi^{M+1,n} = \xi^{1,n}, \quad \frac{\partial \xi^{i,n}}{\partial \eta} = \varphi & \text{in } \Gamma_0, \\ \xi^{i,n} = 0 & \text{on } \Gamma/\Gamma_0, \end{cases}$$

we have [11]:

$$\|\xi^{i,n} - u^n\|_{\infty} \leq ck, \quad i = 1, \dots, M. \quad (10)$$

4 A discrete iterative scheme

Starting from $u_h^0 \in V_h$ is the unique solution of the discrete variational equation:

$$a^1(u_h^0, v) = (f^1(u_h^0), v), \quad \forall v \in V_h.$$

We define the sequence $(u_h^n)_{n \geq 1}$ by:

$$u_h^n = T_h u_h^{n-1}, \quad \forall n \geq 1,$$

such that each iterate u_h^n solves the discrete coercive HJB equation:

$$\begin{cases} \max_{1 \leq i \leq M} (A^i u_h^n - f^i(u_h^{n-1})) = 0, & \text{in } \Omega, \\ \frac{\partial u_h^n}{\partial \eta} = \varphi & \text{in } \Gamma_0, \\ u_h^n = 0 & \text{on } \Gamma/\Gamma_0. \end{cases} \quad (11)$$

Theorem 13. Under condition of the mapping T_h is a contraction and the fixed point theorem, the sequence $(u_h^n)_{n \geq 0}$ converges to the unique fixed point u_h and we have the error bound:

$$\|u_h^n - u_h\|_{\infty} \leq \frac{\rho^n}{1 - \rho} \|u_h^0 - u_h^1\|_{\infty}.$$

Let $(\xi_h^{i,n})_{1 \leq i \leq M}$ be the unique solution of the system of QVIs, which approximated the discrete coercive HJB equation (11):

$$\begin{cases} a^i(\xi_h^{i,n}, v - \xi_h^{i,n}) \geq (f^i(u_h^{n-1}), v - \xi_h^{i,n}) + (\varphi, v - \xi_h^{i,n})_{\Gamma_0}, \quad \forall v \in V_h, \\ \xi_h^{i,n} \leq k + \xi_h^{i+1,n}, \quad v \leq k + \xi_h^{i+1,n}, \quad i = 1, \dots, M, \\ \xi_h^{M+1,n} = \xi_h^{1,n}, \quad \frac{\partial \xi_h^{i,n}}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \xi_h^{i,n} = 0 \text{ on } \Gamma/\Gamma_0, \end{cases}$$

we have [11]:

$$\|\xi_h^{i,n} - u_h^n\|_{\infty} \leq ck, \quad i = 1, \dots, M. \quad (12)$$

5 L^∞ -error estimate

We define the sequence $(\bar{u}_h^n)_{n \geq 0}$ such that:

$$\bar{u}_h^0 = u_h^0, \quad \bar{u}_h^n = T_h u_h^{n-1}, \quad n \geq 1,$$

where $\bar{u}_h^n$ is the unique solution of the discrete HJB equation:

$$\begin{cases} \max_{1 \leq i \leq M} (A^i \bar{u}_h^n - f^i(u_h^{n-1})) = 0, \text{ in } \Omega, \\ \frac{\partial \bar{u}_h^n}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \bar{u}_h^n = 0 \text{ on } \Gamma/\Gamma_0. \end{cases} \quad (13)$$

Let $(\bar{\xi}_h^{i,n})_{1 \leq i \leq M}$ be the unique solution of the system of QVIs, which approximated the discrete coercive HJB equation (13):

$$\begin{cases} a^i(\bar{\xi}_h^{i,n}, v - \bar{\xi}_h^{i,n}) \geq (f^i(u_h^{n-1}), v - \bar{\xi}_h^{i,n}) + (\varphi, v - \bar{\xi}_h^{i,n})_{\Gamma_0}, \quad \forall v \in V_h, \\ \bar{\xi}_h^{i,n} \leq k + \bar{\xi}_h^{i+1,n}, \quad v \leq k + \bar{\xi}_h^{i+1,n}, \quad i = 1, \dots, M, \\ \bar{\xi}_h^{M+1,n} = \bar{\xi}_h^{1,n}, \quad \frac{\partial \bar{\xi}_h^{i,n}}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \bar{\xi}_h^{i,n} = 0 \text{ on } \Gamma/\Gamma_0, \end{cases}$$

we have [11]:

$$\|\bar{\xi}_h^{i,n} - \bar{u}_h^n\|_{\infty} \leq ck, \quad i = 1, \dots, M. \quad (14)$$

Lemma 14. [12] There exists a constant c independent of h such that

$$\|u^0 - u_h^0\|_{\infty} \leq ch^2 |\log h|.$$

Lemma 15. [11], [13] *There exists a constant c independent of both h and n such that*

$$\max_{1 \leq i \leq M} \|\xi^{i,n} - \bar{\xi}_h^{i,n}\|_{\infty} \leq ch^2 |\log h|^3.$$

Theorem 16. *We have*

$$\|u^n - u_h^n\|_{\infty} \leq \frac{1 - \rho^{n+1}}{1 - \rho} ch^2 |\log h|^3, \quad (15)$$

where c is a constant independent of both n and h .

Proof. Combining Lemmas 5, 6, and (12), (14) yields:

$$\begin{aligned} \|u^1 - u_h^1\|_{\infty} &\leq \|u^1 - \bar{u}_h^1\|_{\infty} + \|\bar{u}_h^1 - u_h^1\|_{\infty} \\ &\leq \|u^1 - \bar{u}_h^1\|_{\infty} + \|T_h u^0 - T_h u_h^0\|_{\infty} \\ &\leq \|u^1 - \xi^{i,1}\|_{\infty} + \|\xi^{i,1} - \bar{\xi}_h^{i,1}\|_{\infty} + \|\bar{\xi}_h^{i,1} - \bar{u}_h^1\|_{\infty} + \rho \|u^0 - u_h^0\|_{\infty}, \quad i = 1, \dots, M \\ &\leq ck + \max_{1 \leq i \leq M} \|\xi^{i,1} - \bar{\xi}_h^{i,1}\|_{\infty} + ck + \rho ch^2 |\log h| \\ &\leq ck + ch^2 |\log h|^3 + \rho ch^2 |\log h| \\ &\leq ck + (1 + \rho) ch^2 |\log h|^3 \\ &\leq ck + \frac{1 - \rho^2}{1 - \rho} ch^2 |\log h|^3. \end{aligned}$$

Passing to the limit as $k \rightarrow 0$, we get

$$\|u^1 - u_h^1\|_{\infty} \leq \frac{1 - \rho^2}{1 - \rho} ch^2 |\log h|^3.$$

Now, we assume:

$$\|u^{n-1} - u_h^{n-1}\|_{\infty} \leq \frac{1 - \rho^n}{1 - \rho} ch^2 |\log h|^3.$$

Then, combining Lemmas 5, 6, and (12), (14), we get:

$$\begin{aligned} \|u^n - u_h^n\|_{\infty} &\leq \|u^n - \bar{u}_h^n\|_{\infty} + \|\bar{u}_h^n - u_h^n\|_{\infty}, \\ &\leq \|u^n - \bar{u}_h^n\|_{\infty} + \|T_h u^{n-1} - T_h u_h^{n-1}\|_{\infty}, \\ &\leq \|u^n - \xi^{i,n}\|_{\infty} + \|\xi^{i,n} - \bar{\xi}_h^{i,n}\|_{\infty} + \|\bar{\xi}_h^{i,n} - \bar{u}_h^n\|_{\infty} + \rho \|u^{n-1} - u_h^{n-1}\|_{\infty}, \quad i = 1, \dots, M, \\ &\leq ck + \max_{1 \leq i \leq M} \|\xi^{i,n} - \bar{\xi}_h^{i,n}\|_{\infty} + ck + \rho \frac{1 - \rho^n}{1 - \rho} ch^2 |\log h|^3, \\ &\leq ck + ch^2 |\log h|^3 + \rho \frac{1 - \rho^n}{1 - \rho} ch^2 |\log h|^3, \\ &\leq ck + \left(1 + \rho \frac{1 - \rho^n}{1 - \rho}\right) ch^2 |\log h|^3, \\ &\leq ck + \frac{1 - \rho^{n+1}}{1 - \rho} ch^2 |\log h|^3. \end{aligned}$$

Passing to the limit as $k \rightarrow 0$, we get

$$\|u^n - u_h^n\|_{\infty} \leq \frac{1 - \rho^{n+1}}{1 - \rho} ch^2 |\log h|^3.$$

□

Theorem 17. *We have*

$$\|u - u_h\|_{\infty} \leq ch^2 |\log h|^3,$$

where c is a constant independent of h .

Proof. Indeed, combining Theorems 6, 7, and 8, we have:

$$\begin{aligned} \|u - u_h\|_{\infty} &\leq \|u - u^n\|_{\infty} + \|u^n - u_h^n\|_{\infty} + \|u_h^n - u_h\|_{\infty} \\ &\leq \frac{\rho^n}{1-\rho} \|u^0 - u^1\|_{\infty} + \frac{1-\rho^{n+1}}{1-\rho} ch^2 |\log h|^3 + \frac{\rho^n}{1-\rho} \|u_h^0 - u_h^1\|_{\infty}, \quad \rho < 1. \end{aligned}$$

So, passing to the limit, as $n \rightarrow \infty, \rho^n \rightarrow 0$, we get:

$$\|u - u_h\|_{\infty} \leq \frac{c}{1-\rho} h^2 |\log h|^3.$$

□

6 Conclusion

This paper addresses the finite element of the elliptic HJB equations. The optimal error estimate is derived, combining geometric convergence of an iterative scheme and its finite element error estimate, obtained by means of the concept of Banach fixed point principle.

In light of the findings of this work, we wonder whether these can be exploited to:

1. Extend the study to the noncoercive problem.
2. Extend the study to parabolic HJB equations.

This will be the focus of our attention in future works.

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