Research Article

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Solving system of linear equations via bicomplex valued metric space

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Abstract: In this paper, we prove some common fixed point theorems on bicomplex metric space. Our results generalize and expand some of the literature's well-known results. We also explore some of the applications of our key results.

Keywords: bicomplex valued metric space, common fixed point linear equation

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1 Introduction

Serge [1] made a pioneering attempt in the development of special algebras. He conceptualized commutative generalization of complex numbers as bicomplex numbers, bicomplex numbers, etc. as elements of an infinite set of algebras. Subsequently during the 1930s, other researchers also contributed in this area [2–4]. Priestley [5] proved Cauchy's integral formula as follows.

Theorem 1.1. *Let s be holomorphic inside and on a positively oriented contour γ . Then, if a is inside γ ,*

$$S(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{S(\mathbb{N})}{\mathbb{N} - a} d\mathbb{N}.$$

But unfortunately the next 50 years failed to witness any advancement in this field. Afterward, Price [6] developed the bicomplex algebra and function theory. Recently, renewed interest in this subject finds some significant applications in different fields of mathematical sciences as well as other branches of science and technology. Also one can see the attempts in [7]. An impressive body of work has been developed by a number of researchers. Among them an important work on elementary functions of bicomplex numbers has been carried out by Luna-Elizaarrarás et al. [8]. Choi et al. [9] proved some common fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces. Jebril [10] proved

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some common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces.

In 2017, Choi et al. [11] proved some fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces. In 2020, Agarwal et al. [12] proved a regularity criterion in weak spaces to Boussinesq equations. In 2021, Agarwal et al. [13] proved nonlinear neutral delay differential equations of fourth-order, oscillation of solutions. In 2020, Datta et al. [14] proved fixed point theorems in bicomplex valued metric spaces. In 2021, Beg et al. [15] proved the following fixed point theorems on bicomplex valued metric spaces.

Theorem 1.2. *Let (X, ϱ) be a complete bicomplex valued metric space with degenerated $1 + \varrho(\aleph, \varpi)$ and $\|1 + \varrho(\aleph, \varpi)\| \neq 0$ for all $\aleph, \varpi \in X$ and let $S, \mathcal{T} : X \rightarrow X$ be mappings satisfying the condition*

$$\varrho(S\aleph, \mathcal{T}\varpi) \leq_{i_2} \lambda \varrho(\aleph, \varpi) + \frac{\mu \varrho(\aleph, S\aleph) \varrho(\varpi, \mathcal{T}\varpi)}{1 + \varrho(\aleph, \varpi)}$$

for all $\aleph, \varpi \in X$, where λ, μ are nonnegative real numbers with $\lambda + \sqrt{2}\mu < 1$. Then $S, \mathcal{T}$ have a unique common fixed point.

In this paper, inspired by Theorem 1.2, we prove some common fixed point theorems on bicomplex metric space with an application.

2 Preliminaries

Throughout this paper, we denote the set of real, complex, and bicomplex numbers, respectively, as $\mathbb{C}_0$, $\mathbb{C}_1$, and $\mathbb{C}_2$. Segre [1] defined the bicomplex number as:

$$\sigma = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = (a_1 + i_1 a_2) + i_2 (a_3 + i_1 a_4),$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$, and independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$, we denote the set of bicomplex numbers $\mathbb{C}_2$ is defined as:

$$\mathbb{C}_2 = \{\sigma : \sigma = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2, a_1, a_2, a_3, a_4 \in \mathbb{C}_0\},$$

i.e.,

$$\mathbb{C}_2 = \{\sigma : \sigma = \varphi_1 + i_2 \varphi_2, \varphi_1, \varphi_2 \in \mathbb{C}_1\},$$

where $\varphi_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $\varphi_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$. If $\sigma = \varphi_1 + i_2 \varphi_2$ and $\vartheta = \varpi_1 + i_2 \varpi_2$ be any two bicomplex numbers, then the sum is $\sigma \pm \vartheta = (\varphi_1 + i_2 \varphi_2) \pm (\varpi_1 + i_2 \varpi_2) = \varphi_1 \pm \varpi_1 + i_2 (\varphi_2 \pm \varpi_2)$ and the product is $\sigma \cdot \vartheta = (\varphi_1 + i_2 \varphi_2)(\varpi_1 + i_2 \varpi_2) = (\varphi_1 \varpi_1 - \varphi_2 \varpi_2) + i_2 (\varphi_1 \varpi_2 + \varphi_2 \varpi_1)$.

There are four idempotent elements in $\mathbb{C}_2$, they are $0, 1, e_1 = \frac{1+i_1 i_2}{2}, e_2 = \frac{1-i_1 i_2}{2}$ out of which e_1 and e_2 are nontrivial such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Every bicomplex number $\varphi_1 + i_2 \varphi_2$ can uniquely be expressed as the combination of e_1 and e_2 , namely

$$\sigma = \varphi_1 + i_2 \varphi_2 = (\varphi_1 - i_1 \varphi_2) e_1 + (\varphi_1 + i_1 \varphi_2) e_2.$$

This representation of σ is known as the idempotent representation of bicomplex number and the complex coefficients $\sigma_1 = (\varphi_1 - i_1 \varphi_2)$ and $\sigma_2 = (\varphi_1 + i_1 \varphi_2)$ are known as idempotent components of the bicomplex number σ .

An element $\sigma = \varphi_1 + i_2 \varphi_2 \in \mathbb{C}_2$ is said to be invertible if there exists another element ϑ in $\mathbb{C}_2$ such that $\sigma \vartheta = 1$ and ϑ is said to be inverse (multiplicative) of σ . Consequently, σ is said to be the inverse (multiplicative) of ϑ . An element which has an inverse in $\mathbb{C}_2$ is said to be the nonsingular element of $\mathbb{C}_2$ and an element which does not have an inverse in $\mathbb{C}_2$ is said to be the singular element of $\mathbb{C}_2$.

An element $\sigma = \varphi_1 + i_2\varphi_2 \in \mathbb{C}_2$ is nonsingular if and only if $|\varphi_1^2 + \varphi_2^2| \neq 0$ and singular if and only if $|\varphi_1^2 + \varphi_2^2| = 0$. The inverse of σ is defined as

$$\sigma^{-1} = \vartheta = \frac{\varphi - i_2\varphi_2}{\varphi_1^2 + \varphi_2^2}.$$

Zero is the only element in $\mathbb{C}_0$ which does not have multiplicative inverse and in $\mathbb{C}_1$, $0 = 0 + i0$ is the only element which does not have multiplicative inverse. We denote the set of singular elements of $\mathbb{C}_0$ and $\mathbb{C}_1$ by $\mathfrak{S}_0$ and $\mathfrak{S}_1$, respectively. But there are more than one element in $\mathbb{C}_2$ which do not have multiplicative inverse, we denote this set by $\mathfrak{S}_2$ and clearly $\mathfrak{S}_0 = \mathfrak{S}_1 \subset \mathfrak{S}_2$.

A bicomplex number $\sigma = \alpha_1 + \alpha_2 i_1 + \alpha_3 i_2 + \alpha_4 i_1 i_2 \in \mathbb{C}_2$ is said to be degenerated if the matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$$

is degenerated. In that case σ^{-1} exists and it is also degenerated.

The norm $\|\cdot\|$ of $\mathbb{C}_2$ is a positive real valued function and $\|\cdot\| : \mathbb{C}_2 \rightarrow \mathbb{C}_0^+$ is defined by

$$\begin{aligned} \|\sigma\| &= \|\varphi_1 + i_2\varphi_2\| = \{|\varphi_1|^2 + |\varphi_2|^2\}^{\frac{1}{2}} = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2)^{\frac{1}{2}}, \\ \|\varphi_1 + i_2\varphi_2\| &= \left[\frac{|(\varphi_1 - i_1\varphi_2)|^2 + |(\varphi_1 + i_1\varphi_2)|^2}{2} \right]^{\frac{1}{2}}, \end{aligned}$$

where $\sigma = \alpha_1 + \alpha_2 i_1 + \alpha_3 i_2 + \alpha_4 i_1 i_2 = \varphi_1 + i_2\varphi_2 \in \mathbb{C}_2$.

The linear space $\mathbb{C}_2$ with respect to defined norm is a norm linear space, also $\mathbb{C}_2$ is complete, therefore $\mathbb{C}_2$ is the Banach space. If $\sigma, \vartheta \in \mathbb{C}_2$, then $\|\sigma\vartheta\| \leq \sqrt{2}\|\sigma\|\|\vartheta\|$ holds instead of $\|\sigma\vartheta\| \leq \|\sigma\|\|\vartheta\|$, therefore, $\mathbb{C}_2$ is not the Banach algebra. The partial order relation $\preceq_{i_2}$ on $\mathbb{C}_2$ is defined as: Let $\mathbb{C}_2$ be the set of bicomplex numbers and $\sigma = \varphi_1 + i_2\varphi_2, \vartheta = \varpi_1 + i_2\varpi_2 \in \mathbb{C}_2$ then $\sigma \preceq_{i_2} \vartheta$ if and only if $\varphi_1 \preceq \varpi_1$ and $\varphi_2 \preceq \varpi_2$, i.e., $\sigma \preceq_{i_2} \vartheta$ if one of the following conditions is satisfied:

- (A) $\varphi_1 = \varpi_1, \varphi_2 = \varpi_2$;
- (B) $\varphi_1 < \varpi_1, \varphi_2 = \varpi_2$;
- (C) $\varphi_1 = \varpi_1, \varphi_2 < \varpi_2$; and
- (D) $\varphi_1 < \varpi_1, \varphi_2 < \varpi_2$.

In particular, we can write $\sigma \preceq_{i_2} \vartheta$ if $\sigma \preceq_{i_2} \vartheta$ and $\sigma \neq \vartheta$, i.e., one of (B), (C), and (D) is satisfied and we will write $\sigma <_{i_2} \vartheta$ if only (D) is satisfied.

For any two bicomplex numbers $\sigma, \vartheta \in \mathbb{C}_2$ we can verify the followings:

- (S1) $\sigma \preceq_{i_2} \vartheta \Rightarrow \|\sigma\| \leq \|\vartheta\|$;
- (S2) $\|\sigma + \vartheta\| \leq \|\sigma\| + \|\vartheta\|$;
- (S3) $\|a\sigma\| = a\|\sigma\|$, where a is a nonnegative real number;
- (S4) $\|\sigma\vartheta\| \leq \sqrt{2}\|\sigma\|\|\vartheta\|$ and the equality holds only when at least one of σ and ϑ is degenerated;
- (S5) $\|\sigma^{-1}\| = \|\sigma\|^{-1}$, if σ is a degenerated bicomplex number with $0 < \sigma$;
- (S6) $\left\| \frac{\sigma}{\vartheta} \right\| = \frac{\|\sigma\|}{\|\vartheta\|}$, if ϑ is a degenerated bicomplex number.

Now, let us recall some basic concepts and notations, which will be used in the sequel.

Definition 2.1. [15] Let X be a nonvoid set, whereas $\mathbb{C}_2$ be the set of bicomplex numbers. Suppose that the mapping $\varrho : X \times X \rightarrow \mathbb{C}_2$ satisfies the following conditions:

- (A1) $0 \preceq_{i_2} \varrho(\aleph, \varpi)$, for all $\aleph, \varpi \in X$ and $\varrho(\aleph, \varpi) = 0$ if and only if $\aleph = \varpi$;
- (A2) $\varrho(\aleph, \varpi) = \varrho(\varpi, \aleph)$, for all $\aleph, \varpi \in X$;
- (A3) $\varrho(\aleph, \varpi) \preceq_{i_2} \varrho(\aleph, \eta) + \varrho(\eta, \varpi)$, for all $\aleph, \varpi, \eta \in X$.

Then ϱ is called the bicomplex valued metric on X , and (X, ϱ_b) is called the bicomplex valued metric space.

Example 2.1. Let $X = [0, 1]$ and $\varrho : X \times X \rightarrow \mathbb{C}_2$ be defined by $\varrho(\aleph, \varpi) = |\aleph - \varpi|e^{i\frac{\pi}{6}}$. Then (X, ϱ) is a bicomplex valued metric space.

Definition 2.2. Let (X, ϱ) be a bicomplex valued metric space and $\mathcal{B} \subseteq X$,

(B1) $\alpha \in \mathcal{B}$ is called an interior point of a set $\mathcal{B}$ whenever there is $0 \prec_{i_2} q \in \mathbb{C}_2$ such that

$$\mathcal{N}(\alpha, q) \subseteq \mathcal{B},$$

where $\mathcal{N}(\alpha, q) = \{\varpi \in X : \varrho(\alpha, \varpi) \preceq_{i_2} q\}$.

(B2) A point $\aleph \in X$ is called a limit point of $\mathcal{B}$ whenever for every $0 \prec_{i_2} q \in \mathbb{C}_2$,

$$\mathcal{N}(\aleph, q) \cap (\mathcal{B} - \aleph) \neq \emptyset.$$

(B3) A subset $\mathcal{A} \subseteq X$ is called open whenever each element of $\mathcal{A}$ is an interior point of $\mathcal{A}$. A subset $\mathcal{B} \subseteq X$ is called closed whenever each limit point of $\mathcal{B}$ belongs to $\mathcal{B}$. The family

$$\mathcal{F} = \{\mathcal{N}(\aleph, q) : \aleph \in X, 0 \prec_{i_2} q\}$$

is a sub-basis for a topology on X . We denote this bicomplex topology by τ_{trc} . Indeed, the topology τ_{trc} is Hausdorff.

Definition 2.3. [15] Let (X, ϱ) be a bicomplex valued metric space. A sequence $\{\aleph_{\mathfrak{k}}\}$ in X is said to be a convergent and converges to $\aleph \in X$ if for every $0 \prec_{i_2} \varepsilon \in \mathbb{C}_2$ there exists $\mathfrak{k}_0 \in \mathbb{N}$ such that $\varrho(\aleph_{\mathfrak{k}}, \aleph) \prec_{i_2} \varepsilon$, for all $\mathfrak{k} \geq \mathfrak{k}_0$ and it is denoted by $\lim_{\mathfrak{k} \rightarrow \infty} \aleph_{\mathfrak{k}} = \aleph$.

Lemma 2.2. [15] Let (X, ϱ) be a bicomplex valued metric space. A sequence $\{\aleph_{\mathfrak{k}}\} \in X$ converges to $\aleph \in X$ iff $\lim_{\mathfrak{k} \rightarrow \infty} \|\varrho(\aleph_{\mathfrak{k}}, \aleph)\| = 0$.

Definition 2.4. [15] Let (X, ϱ) be a bicomplex valued metric space. A sequence $\{\aleph_{\mathfrak{k}}\}$ in X is said to be a Cauchy sequence in (X, ϱ) if for any $0 \prec_{i_2} \varepsilon \in \mathbb{C}_2$, there exists $\mathfrak{h} \in \mathbb{N}$ such that $\varrho(\aleph_{\mathfrak{k}}, \aleph_{\mathfrak{k}+\mathfrak{m}}) \prec_{i_2} \varepsilon$ for all $\mathfrak{k}, \mathfrak{m} \in \mathbb{N}$ and $\mathfrak{k}, \mathfrak{m} \geq \mathfrak{h}$.

Definition 2.5. [15] Let (X, ϱ) be a bicomplex valued metric space. Let $\{\aleph_{\mathfrak{k}}\}$ be any sequence in X . Then, if every Cauchy sequence in X is convergent in X , then (X, ϱ) is said to be a complete bicomplex valued metric space.

Lemma 2.3. [15] Let (X, ϱ) be a bicomplex valued metric space and $\{\aleph_{\mathfrak{k}}\}$ be a sequence in X . Then $\{\aleph_{\mathfrak{k}}\}$ is a Cauchy sequence in X iff $\lim_{\mathfrak{k} \rightarrow \infty} \|\varrho(\aleph_{\mathfrak{k}}, \aleph_{\mathfrak{k}+\mathfrak{m}})\| = 0$.

Definition 2.6. Let S and $\mathcal{T}$ be self-mappings of nonvoid set X . A point $\aleph \in X$ is called a common fixed point of S and $\mathcal{T}$ if $\aleph = S\aleph = \mathcal{T}\aleph$.

3 Main result

In this section, we prove common fixed point theorem in a bicomplex valued metric space using rational-type contraction condition.

Theorem 3.1. If S and $\mathcal{T}$ are self-mappings defined on a complete bicomplex valued metric space (X, ϱ) satisfying the condition

$$\varrho(S\aleph, \mathcal{T}\varpi) \preceq_{i_2} \lambda \varrho(\aleph, \varpi) + \frac{\mu \varrho(\aleph, S\aleph) \varrho(\varpi, \mathcal{T}\varpi) + \gamma \varrho(\varpi, S\aleph) \varrho(\aleph, \mathcal{T}\varpi)}{1 + \varrho(\aleph, \varpi)}$$

for all $\aleph, \varpi \in \mathcal{X}$ where λ, μ, γ are nonnegative reals with $\lambda + \sqrt{2}\mu + \sqrt{2}\gamma < 1$, then $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

Proof. Let $\aleph_0$ be an arbitrary point in $\mathcal{X}$ and define $\aleph_{2\ell+1} = \mathcal{S}\aleph_{2\ell}$, $\aleph_{2\ell+2} = \mathcal{T}\aleph_{2\ell+1}$, $\ell = 0, 1, 2, \dots$. Then

$$\begin{aligned} \varrho(\aleph_{2\ell+1}, \aleph_{2\ell+2}) &= \varrho(\mathcal{S}\aleph_{2\ell}, \mathcal{T}\aleph_{2\ell+1}) \\ &\leq_{i_2} \lambda \varrho(\aleph_{2\ell}, \aleph_{2\ell+1}) + \frac{\mu \varrho(\aleph_{2\ell}, \mathcal{S}\aleph_{2\ell}) \varrho(\aleph_{2\ell+1}, \mathcal{T}\aleph_{2\ell+1}) + \gamma \varrho(\aleph_{2\ell}, \mathcal{T}\aleph_{2\ell+1}) \varrho(\aleph_{2\ell+1}, \mathcal{S}\aleph_{2\ell})}{1 + \varrho(\aleph_{2\ell}, \aleph_{2\ell+1})}. \end{aligned}$$

Since $\aleph_{2\ell+1} = \mathcal{S}\aleph_{2\ell}$ implies $\varrho(\aleph_{2\ell+1}, \mathcal{S}\aleph_{2\ell}) = 0$,

$$\varrho(\aleph_{2\ell+1}, \aleph_{2\ell+2}) \leq_{i_2} \lambda \varrho(\aleph_{2\ell}, \aleph_{2\ell+1}) + \frac{\mu \varrho(\aleph_{2\ell}, \aleph_{2\ell+1}) \varrho(\aleph_{2\ell+1}, \aleph_{2\ell+2})}{1 + \varrho(\aleph_{2\ell}, \aleph_{2\ell+1})},$$

which implies that

$$\|\varrho(\aleph_{2\ell+1}, \aleph_{2\ell+2})\| \leq \lambda \|\varrho(\aleph_{2\ell}, \aleph_{2\ell+1})\| + \frac{\sqrt{2}\mu \|\varrho(\aleph_{2\ell}, \aleph_{2\ell+1})\| \|\varrho(\aleph_{2\ell+1}, \aleph_{2\ell+2})\|}{\|1 + \varrho(\aleph_{2\ell}, \aleph_{2\ell+1})\|}.$$

Since $\|1 + \varrho(\aleph_{2\ell}, \aleph_{2\ell+1})\| > \|\varrho(\aleph_{2\ell}, \aleph_{2\ell+1})\|$,

$$\|\varrho(\aleph_{2\ell+1}, \aleph_{2\ell+2})\| \leq \lambda \|\varrho(\aleph_{2\ell}, \aleph_{2\ell+1})\| + \sqrt{2}\mu \|\varrho(\aleph_{2\ell+1}, \aleph_{2\ell+2})\|,$$

so that

$$\|\varrho(\aleph_{2\ell+1}, \aleph_{2\ell+2})\| \leq \frac{\lambda}{1 - \sqrt{2}\mu} \|\varrho(\aleph_{2\ell}, \aleph_{2\ell+1})\|.$$

Also,

$$\begin{aligned} \varrho(\aleph_{2\ell+2}, \aleph_{2\ell+3}) &= \varrho(\mathcal{T}\aleph_{2\ell+1}, \mathcal{S}\aleph_{2\ell+2}) \\ &= \varrho(\mathcal{S}\aleph_{2\ell+2}, \mathcal{T}\aleph_{2\ell+1}) \\ &\leq_{i_2} \lambda \varrho(\aleph_{2\ell+2}, \aleph_{2\ell+1}) + \frac{\mu \varrho(\aleph_{2\ell+2}, \mathcal{S}\aleph_{2\ell+2}) \varrho(\aleph_{2\ell+1}, \mathcal{T}\aleph_{2\ell+1}) + \gamma \varrho(\aleph_{2\ell+1}, \mathcal{S}\aleph_{2\ell+2}) \varrho(\aleph_{2\ell+2}, \mathcal{T}\aleph_{2\ell+1})}{1 + \varrho(\aleph_{2\ell+2}, \aleph_{2\ell+1})}. \end{aligned}$$

Since $\aleph_{2\ell+2} = \mathcal{T}\aleph_{2\ell+1}$ implies $\varrho(\aleph_{2\ell+2}, \mathcal{T}\aleph_{2\ell+1}) = 0$,

$$\varrho(\aleph_{2\ell+2}, \aleph_{2\ell+3}) \leq_{i_2} \lambda \varrho(\aleph_{2\ell+2}, \aleph_{2\ell+1}) + \frac{\mu \varrho(\aleph_{2\ell+2}, \mathcal{S}\aleph_{2\ell+2}) \varrho(\aleph_{2\ell+1}, \mathcal{T}\aleph_{2\ell+1})}{1 + \varrho(\aleph_{2\ell+2}, \aleph_{2\ell+1})},$$

which implies that

$$\|\varrho(\aleph_{2\ell+2}, \aleph_{2\ell+3})\| \leq \lambda \|\varrho(\aleph_{2\ell+2}, \aleph_{2\ell+1})\| + \frac{\sqrt{2}\mu \|\varrho(\aleph_{2\ell+2}, \aleph_{2\ell+3})\| \|\varrho(\aleph_{2\ell+1}, \aleph_{2\ell+2})\|}{\|1 + \varrho(\aleph_{2\ell+2}, \aleph_{2\ell+1})\|}.$$

As $\|1 + \varrho(\aleph_{2\ell+2}, \aleph_{2\ell+1})\| > \|\varrho(\aleph_{2\ell+2}, \aleph_{2\ell+1})\|$, therefore

$$\|\varrho(\aleph_{2\ell+2}, \aleph_{2\ell+3})\| \leq \frac{\lambda}{1 - \sqrt{2}\mu} \|\varrho(\aleph_{2\ell+1}, \aleph_{2\ell+2})\|.$$

Putting $\mathfrak{h} = \frac{\lambda}{1 - \sqrt{2}\mu}$, we have (for all ℓ)

$$\|\varrho(\aleph_\ell, \aleph_{\ell+1})\| \leq \mathfrak{h} \|\varrho(\aleph_{\ell-1}, \aleph_\ell)\| \leq \mathfrak{h}^2 \|\varrho(\aleph_{\ell-2}, \aleph_{\ell-1})\| \leq \dots \leq \mathfrak{h}^\ell \|\varrho(\aleph_0, \aleph_1)\|.$$

Therefore, for any $m > \ell$, we have

$$\begin{aligned} \|\varrho(\aleph_\ell, \aleph_m)\| &\leq \|\varrho(\aleph_\ell, \aleph_{\ell+1})\| + \|\varrho(\aleph_{\ell+1}, \aleph_{\ell+2})\| + \|\varrho(\aleph_{\ell+2}, \aleph_{\ell+3})\| + \dots + \|\varrho(\aleph_{m-1}, \aleph_m)\| \\ &\leq [\mathfrak{h}^\ell + \mathfrak{h}^{\ell+1} + \mathfrak{h}^{\ell+2} + \dots + \mathfrak{h}^{m-1}] \|\varrho(\aleph_0, \aleph_1)\| \\ &\leq \left[\frac{\mathfrak{h}^\ell}{1 - \mathfrak{h}} \right] \|\varrho(\aleph_0, \aleph_1)\|, \end{aligned}$$

which implies that

$$\|q(N_k, N_m)\| \leq \left[\frac{h^k}{1-h} \right] \|q(N_0, N_1)\| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

In view of Lemma 2.3, the sequence $\{N_k\}$ is Cauchy. Since X is complete, there exists some $l \in X$ such that $N_k \rightarrow l$ as $k \rightarrow \infty$. On the contrary, let $l \neq Sl$ so that $0 < \eta = q(l, Sl)$ and henceforth we can have

$$\begin{aligned} \eta &= q(l, Sl) \leq_{i_2} q(l, \mathcal{T}N_{2k+1}) + q(\mathcal{T}N_{2k+1}, Sl) \\ &\leq_{i_2} q(l, N_{2k+2}) + \lambda q(l, N_{2k+1}) + \frac{\mu q(l, Sl)q(N_{2k+1}, \mathcal{T}N_{2k+1}) + \gamma q(N_{2k+1}, Sl)q(l, \mathcal{T}N_{2k+1})}{1 + q(l, N_{2k+1})} \\ &\leq_{i_2} q(l, N_{2k+2}) + \lambda q(l, N_{2k+1}) + \frac{\mu\eta \cdot q(N_{2k+1}, \mathcal{T}N_{2k+1}) + \gamma q(N_{2k+1}, Sl)q(l, \mathcal{T}N_{2k+1})}{1 + q(l, N_{2k+1})}. \end{aligned}$$

Also, for every k , we have

$$\|q(l, Sl)\| \leq \|q(l, N_{2k+2})\| + \lambda \|q(l, N_{2k+1})\| + \frac{\mu\eta \|q(N_{2k+1}, N_{2k+2})\| + \sqrt{2}\gamma \|q(N_{2k+1}, Sl)\| \|q(l, N_{2k+2})\|}{\|1 + q(l, N_{2k+1})\|}.$$

As $k \rightarrow \infty$, we get

$$\|q(l, Sl)\| = 0,$$

which is a contradiction so that $l = Sl$. Similarly, one can also show that $l = \mathcal{T}l$.

To prove the uniqueness of common fixed point, let l^* (in X) be another common fixed point of S and $\mathcal{T}$, i.e., $l^* = Sl^* = \mathcal{T}l^*$. Then

$$q(l, l^*) = q(Sl, \mathcal{T}l^*) \leq_{i_2} \lambda q(l, l^*) + \frac{\mu q(l, Sl)q(l^*, \mathcal{T}l^*) + \gamma q(l^*, Sl)q(l, \mathcal{T}l^*)}{1 + q(l, l^*)} = \lambda q(l, l^*) + \frac{\gamma q(l^*, l)q(l, l^*)}{1 + q(l, l^*)},$$

which implies that

$$\|q(l, l^*)\| \leq \lambda \|q(l, l^*)\| + \frac{\sqrt{2}\gamma \|q(l^*, l)\| \|q(l, l^*)\|}{\|1 + q(l, l^*)\|}.$$

Since $\|1 + q(l, l^*)\| > \|q(l, l^*)\|$,

$$\|q(l, l^*)\| \leq (\lambda + \sqrt{2}\gamma) \|q(l, l^*)\|,$$

which is a contradiction so that $l = l^*$ (as $\lambda + \sqrt{2}\gamma < 1$). This completes the proof of the theorem. $\square$

Example 3.2. Consider $X = [0, 1]$ and $q(N, \omega) = |N - \omega|e^{i_2\theta}$, $0 \leq \theta \leq \frac{\pi}{2}$ with the order $N \leq_{i_2} \omega$ iff $\omega \leq N$. Then $\leq_{i_2}$ is a partial order in X . Define the functions $S, \mathcal{T} : X \rightarrow X$ by

$$SN = \begin{cases} \frac{N}{4}, & \text{if } N \in \left[0, \frac{1}{2}\right), \\ \frac{N}{5}, & \text{if } N \in \left[\frac{1}{2}, 1\right] \end{cases}$$

and

$$\mathcal{T}N = \begin{cases} \frac{N}{6}, & \text{if } N \in \left[0, \frac{1}{3}\right), \\ \frac{N}{7}, & \text{if } N \in \left[\frac{1}{3}, 1\right]. \end{cases}$$

Clearly, (X, q) is a complete bicomplex valued metric space. Now, we consider four cases:

Case I:

Let $N \in \left[0, \frac{1}{2}\right)$ and $\omega \in \left[0, \frac{1}{3}\right)$, then $SN = \frac{N}{4}$ and $\mathcal{T}\omega = \frac{\omega}{6}$.

Now,

$$\begin{aligned}\varrho(S\aleph, \mathcal{T}\varpi) &= \left| \frac{\aleph}{4} - \frac{\varpi}{6} \right| e^{i_2\theta} = \left| \frac{6\aleph - 4\varpi}{24} \right| e^{i_2\theta}, \\ \varrho(\aleph, S\aleph) &= \left| \aleph - \frac{\aleph}{4} \right| e^{i_2\theta} = \left| \frac{3\aleph}{4} \right| e^{i_2\theta}, \\ \varrho(\varpi, \mathcal{T}\varpi) &= \left| \varpi - \frac{\varpi}{6} \right| e^{i_2\theta} = \left| \frac{5\varpi}{6} \right| e^{i_2\theta}, \\ \varrho(\aleph, \mathcal{T}\varpi) &= \left| \aleph - \frac{\varpi}{6} \right| e^{i_2\theta} = \left| \frac{6\aleph - \varpi}{6} \right| e^{i_2\theta}, \\ \varrho(S\aleph, \varpi) &= \left| \frac{\aleph}{4} - \varpi \right| e^{i_2\theta} = \left| \frac{\aleph - 4\varpi}{4} \right| e^{i_2\theta},\end{aligned}$$

and

$$\varrho(\aleph, \varpi) = |\aleph - \varpi| e^{i_2\theta}.$$

Case II:

Let $\aleph \in \left[0, \frac{1}{2}\right)$ and $\varpi \in \left[\frac{1}{3}, 1\right]$, then $S\aleph = \frac{\aleph}{4}$ and $\mathcal{T}\varpi = \frac{\varpi}{7}$.

Now,

$$\begin{aligned}\varrho(S\aleph, \mathcal{T}\varpi) &= \left| \frac{\aleph}{4} - \frac{\varpi}{7} \right| e^{i_2\theta} = \left| \frac{7\aleph - 4\varpi}{28} \right| e^{i_2\theta}, \\ \varrho(\aleph, S\aleph) &= \left| \aleph - \frac{\aleph}{4} \right| e^{i_2\theta} = \left| \frac{3\aleph}{4} \right| e^{i_2\theta}, \\ \varrho(\varpi, \mathcal{T}\varpi) &= \left| \varpi - \frac{\varpi}{7} \right| e^{i_2\theta} = \left| \frac{6\varpi}{7} \right| e^{i_2\theta}, \\ \varrho(\aleph, \mathcal{T}\varpi) &= \left| \aleph - \frac{\varpi}{7} \right| e^{i_2\theta} = \left| \frac{7\aleph - \varpi}{7} \right| e^{i_2\theta}, \\ \varrho(S\aleph, \varpi) &= \left| \frac{\aleph}{4} - \varpi \right| e^{i_2\theta} = \left| \frac{\aleph - 4\varpi}{4} \right| e^{i_2\theta},\end{aligned}$$

and

$$\varrho(\aleph, \varpi) = |\aleph - \varpi| e^{i_2\theta}.$$

Case III:

Let $\aleph \in \left[\frac{1}{2}, 1\right]$ and $\varpi \in \left[0, \frac{1}{3}\right)$, then $S\aleph = \frac{\aleph}{5}$ and $\mathcal{T}\varpi = \frac{\varpi}{6}$.

Now,

$$\begin{aligned}\varrho(S\aleph, \mathcal{T}\varpi) &= \left| \frac{\aleph}{5} - \frac{\varpi}{6} \right| e^{i_2\theta} = \left| \frac{6\aleph - 5\varpi}{30} \right| e^{i_2\theta}, \\ \varrho(\aleph, S\aleph) &= \left| \aleph - \frac{\aleph}{5} \right| e^{i_2\theta} = \left| \frac{4\aleph}{5} \right| e^{i_2\theta}, \\ \varrho(\varpi, \mathcal{T}\varpi) &= \left| \varpi - \frac{\varpi}{6} \right| e^{i_2\theta} = \left| \frac{5\varpi}{6} \right| e^{i_2\theta}, \\ \varrho(\aleph, \mathcal{T}\varpi) &= \left| \aleph - \frac{\varpi}{6} \right| e^{i_2\theta} = \left| \frac{6\aleph - \varpi}{6} \right| e^{i_2\theta}, \\ \varrho(S\aleph, \varpi) &= \left| \frac{\aleph}{5} - \varpi \right| e^{i_2\theta} = \left| \frac{\aleph - 5\varpi}{5} \right| e^{i_2\theta},\end{aligned}$$

and

$$\varrho(\aleph, \varpi) = |\aleph - \varpi| e^{i_2\theta}.$$

Case IV:

Let $\aleph \in \left[\frac{1}{2}, 1\right]$ and $\varpi \in \left[\frac{1}{3}, 1\right]$, then $S\aleph = \frac{\aleph}{5}$ and $\mathcal{T}\varpi = \frac{\varpi}{7}$.

Now,

$$\begin{aligned}\varrho(S\aleph, \mathcal{T}\varpi) &= \left| \frac{\aleph}{5} - \frac{\varpi}{7} \right| e^{i_2\theta} = \left| \frac{7\aleph - 5\varpi}{35} \right| e^{i_2\theta}, \\ \varrho(\aleph, S\aleph) &= \left| \aleph - \frac{\aleph}{5} \right| e^{i_2\theta} = \left| \frac{4\aleph}{5} \right| e^{i_2\theta}, \\ \varrho(\varpi, \mathcal{T}\varpi) &= \left| \varpi - \frac{\varpi}{7} \right| e^{i_2\theta} = \left| \frac{6\varpi}{7} \right| e^{i_2\theta}, \\ \varrho(\aleph, \mathcal{T}\varpi) &= \left| \aleph - \frac{\varpi}{7} \right| e^{i_2\theta} = \left| \frac{7\aleph - \varpi}{7} \right| e^{i_2\theta}, \\ \varrho(S\aleph, \varpi) &= \left| \frac{\aleph}{5} - \varpi \right| e^{i_2\theta} = \left| \frac{\aleph - 5\varpi}{5} \right| e^{i_2\theta},\end{aligned}$$

and

$$\varrho(\aleph, \varpi) = |\aleph - \varpi| e^{i_2\theta}.$$

It is easy to verify that, all the above cases, the following conditions hold.

$$\varrho(S\aleph, \mathcal{T}\varpi) \leq_{i_2} \lambda \varrho(\aleph, \varpi) + \frac{\mu \varrho(\aleph, S\aleph) \varrho(\varpi, \mathcal{T}\varpi) + \gamma \varrho(\varpi, S\aleph) \varrho(\aleph, \mathcal{T}\varpi)}{1 + \varrho(\aleph, \varpi)}.$$

Hence, the conditions of Theorem 3.1 are satisfied. Therefore, 0 is the unique common fixed point of S and $\mathcal{T}$.

By setting $S = \mathcal{T}$ in Theorem 3.1, one deduces the following :

Corollary 3.3. If $\mathcal{T} : X \rightarrow X$ is a self-mapping defined on a complete bicomplex valued metric space (X, ϱ) satisfying the condition

$$\varrho(\mathcal{T}\aleph, \mathcal{T}\varpi) \leq_{i_2} \lambda \varrho(\aleph, \varpi) + \frac{\mu \varrho(\aleph, \mathcal{T}\aleph) \varrho(\varpi, \mathcal{T}\varpi) + \gamma \varrho(\varpi, \mathcal{T}\aleph) \varrho(\aleph, \mathcal{T}\varpi)}{1 + \varrho(\aleph, \varpi)},$$

for all $\aleph, \varpi \in X$, where λ, μ, γ are nonnegative reals with $\lambda + \sqrt{2}\mu + \sqrt{2}\gamma < 1$, then $\mathcal{T}$ has a unique fixed point.

Theorem 3.4. Let (X, ϱ) be a complete bicomplex valued metric space where in mapping $S, \mathcal{T} : X \rightarrow X$ satisfy the inequality

$$\varrho(S\aleph, \mathcal{T}\varpi) \leq_{i_2} \begin{cases} \lambda \varrho(\aleph, \varpi) + \mu \frac{\varrho(\aleph, S\aleph) \varrho(\varpi, \mathcal{T}\varpi) + \varrho(\varpi, S\aleph) \varrho(\aleph, \mathcal{T}\varpi)}{\varrho(S\aleph, \aleph) + \varrho(\mathcal{T}\varpi, \varpi)} \\ + \gamma \frac{\varrho(\aleph, S\aleph) \varrho(\aleph, \mathcal{T}\varpi) + \varrho(\varpi, S\aleph) \varrho(\varpi, \mathcal{T}\varpi)}{\varrho(S\aleph, \varpi) + \varrho(\mathcal{T}\varpi, \aleph)}, & \text{if } \mathcal{D} \neq 0, \mathcal{D}_1 \neq 0, \\ 0, & \text{if } \mathcal{D} = 0 \text{ or } \mathcal{D}_1 = 0, \end{cases} \quad (1)$$

for all $\aleph, \varpi \in X$, where $\mathcal{D} = \varrho(S\aleph, \aleph) + \varrho(\mathcal{T}\varpi, \varpi)$ and $\mathcal{D}_1 = \varrho(S\aleph, \varpi) + \varrho(\mathcal{T}\varpi, \aleph)$ and λ, μ, γ are nonnegative reals with $\lambda + \sqrt{2}\mu + \gamma < 1$. Then $S, \mathcal{T}$ have unique common fixed point.

Proof. Let $\aleph_0$ be an arbitrary point in X . Define $\aleph_{2\mathfrak{k}+1} = S\aleph_{2\mathfrak{k}}$ and $\aleph_{2\mathfrak{k}+2} = \mathcal{T}\aleph_{2\mathfrak{k}+1}$, $\mathfrak{k} = 0, 1, 2, \dots$. Now, we distinguish two cases. First, if (for $\mathfrak{k} = 0, 1, 2, \dots$) $\varrho(S\aleph_{2\mathfrak{k}}, \aleph_{2\mathfrak{k}}) + \varrho(\mathcal{T}\aleph_{2\mathfrak{k}+1}, \aleph_{2\mathfrak{k}+1}) \neq 0$ and $\varrho(S\aleph_{2\mathfrak{k}}, \aleph_{2\mathfrak{k}+1}) + \varrho(\mathcal{T}\aleph_{2\mathfrak{k}+1}, \aleph_{2\mathfrak{k}}) \neq 0$, then

$$\begin{aligned} \varrho(N_{2\ell+1}, N_{2\ell+2}) &= \varrho(SN_{2\ell}, \mathcal{T}N_{2\ell+1}) \\ &\leq_{i_2} \lambda \varrho(N_{2\ell}, N_{2\ell+1}) + \mu \frac{\varrho(N_{2\ell}, SN_{2\ell})\varrho(N_{2\ell+1}, \mathcal{T}N_{2\ell+1}) + \varrho(N_{2\ell+1}, SN_{2\ell})\varrho(N_{2\ell}, \mathcal{T}N_{2\ell+1})}{\varrho(SN_{2\ell}, N_{2\ell}) + \varrho(\mathcal{T}N_{2\ell+1}, N_{2\ell+1})} \\ &\quad + \gamma \frac{\varrho(N_{2\ell}, SN_{2\ell})\varrho(N_{2\ell}, \mathcal{T}N_{2\ell+1}) + \varrho(N_{2\ell+1}, SN_{2\ell})\varrho(N_{2\ell+1}, \mathcal{T}N_{2\ell+1})}{\varrho(SN_{2\ell}, N_{2\ell+1}) + \varrho(\mathcal{T}N_{2\ell+1}, N_{2\ell})}. \end{aligned}$$

Since $N_{2\ell+1} = SN_{2\ell}$ and $N_{2\ell+2} = \mathcal{T}N_{2\ell+1}$,

$$\begin{aligned} \varrho(N_{2\ell+1}, N_{2\ell+2}) &\leq_{i_2} \lambda \varrho(N_{2\ell}, N_{2\ell+1}) + \mu \frac{\varrho(N_{2\ell}, N_{2\ell+1})\varrho(N_{2\ell+1}, N_{2\ell+2}) + \varrho(N_{2\ell+1}, N_{2\ell+1})\varrho(N_{2\ell}, N_{2\ell+2})}{\varrho(N_{2\ell+1}, N_{2\ell}) + \varrho(N_{2\ell+2}, N_{2\ell+1})} \\ &\quad + \gamma \frac{\varrho(N_{2\ell}, N_{2\ell+1})\varrho(N_{2\ell}, N_{2\ell+2}) + \varrho(N_{2\ell+1}, N_{2\ell+1})\varrho(N_{2\ell+1}, N_{2\ell+2})}{\varrho(N_{2\ell+1}, N_{2\ell+1}) + \varrho(N_{2\ell+2}, N_{2\ell})}, \end{aligned}$$

or

$$\varrho(N_{2\ell+1}, N_{2\ell+2}) \leq_{i_2} \lambda \varrho(N_{2\ell}, N_{2\ell+1}) + \mu \frac{\varrho(N_{2\ell}, N_{2\ell+1})\varrho(N_{2\ell+1}, N_{2\ell+2})}{\varrho(N_{2\ell+1}, N_{2\ell}) + \varrho(N_{2\ell+2}, N_{2\ell+1})} + \gamma \frac{\varrho(N_{2\ell}, N_{2\ell+1})\varrho(N_{2\ell}, N_{2\ell+2})}{\varrho(N_{2\ell+2}, N_{2\ell})},$$

which implies that

$$\|\varrho(N_{2\ell+1}, N_{2\ell+2})\| \leq \lambda \|\varrho(N_{2\ell}, N_{2\ell+1})\| + \sqrt{2}\mu \frac{\|\varrho(N_{2\ell}, N_{2\ell+1})\| \|\varrho(N_{2\ell+1}, N_{2\ell+2})\|}{\|\varrho(N_{2\ell+1}, N_{2\ell})\| + \|\varrho(N_{2\ell+2}, N_{2\ell+1})\|} + \gamma \|\varrho(N_{2\ell}, N_{2\ell+1})\|.$$

Since $\|\varrho(N_{2\ell+1}, N_{2\ell})\| + \|\varrho(N_{2\ell+2}, N_{2\ell+1})\| > \|\varrho(N_{2\ell+1}, N_{2\ell})\|$,

$$\|\varrho(N_{2\ell+1}, N_{2\ell+2})\| \leq \lambda \|\varrho(N_{2\ell}, N_{2\ell+1})\| + \sqrt{2}\mu \|\varrho(N_{2\ell+1}, N_{2\ell+2})\| + \gamma \|\varrho(N_{2\ell}, N_{2\ell+1})\|,$$

so that

$$\|\varrho(N_{2\ell+1}, N_{2\ell+2})\| \leq \frac{\lambda + \gamma}{1 - \sqrt{2}\mu} \|\varrho(N_{2\ell}, N_{2\ell+1})\|.$$

Also

$$\begin{aligned} \varrho(N_{2\ell+2}, N_{2\ell+3}) &= \varrho(SN_{2\ell+2}, \mathcal{T}N_{2\ell+1}) \leq_{i_2} \lambda \varrho(N_{2\ell+2}, N_{2\ell+1}) \\ &\quad + \mu \frac{\varrho(N_{2\ell+2}, SN_{2\ell+2})\varrho(N_{2\ell+1}, \mathcal{T}N_{2\ell+1}) + \varrho(N_{2\ell+1}, SN_{2\ell+2})\varrho(N_{2\ell+2}, \mathcal{T}N_{2\ell+1})}{\varrho(SN_{2\ell+2}, N_{2\ell+2}) + \varrho(\mathcal{T}N_{2\ell+1}, N_{2\ell+1})} \\ &\quad + \gamma \frac{\varrho(N_{2\ell+2}, SN_{2\ell+2})\varrho(N_{2\ell+2}, \mathcal{T}N_{2\ell+1}) + \varrho(N_{2\ell+1}, SN_{2\ell+2})\varrho(N_{2\ell+1}, \mathcal{T}N_{2\ell+1})}{\varrho(SN_{2\ell+2}, N_{2\ell+1}) + \varrho(\mathcal{T}N_{2\ell+1}, N_{2\ell+2})}. \end{aligned}$$

Since $N_{2\ell+3} = SN_{2\ell+2}$ and $N_{2\ell+2} = \mathcal{T}N_{2\ell+1}$, we get

$$\begin{aligned} \varrho(N_{2\ell+2}, N_{2\ell+3}) &\leq_{i_2} \lambda \varrho(N_{2\ell+2}, N_{2\ell+1}) + \mu \frac{\varrho(N_{2\ell+2}, N_{2\ell+3})\varrho(N_{2\ell+1}, N_{2\ell+2}) + \varrho(N_{2\ell+1}, N_{2\ell+3})\varrho(N_{2\ell+2}, N_{2\ell+2})}{\varrho(N_{2\ell+3}, N_{2\ell+2}) + \varrho(N_{2\ell+2}, N_{2\ell+1})} \\ &\quad + \gamma \frac{\varrho(N_{2\ell+2}, N_{2\ell+3})\varrho(N_{2\ell+2}, N_{2\ell+2}) + \varrho(N_{2\ell+1}, N_{2\ell+3})\varrho(N_{2\ell+1}, N_{2\ell+2})}{\varrho(N_{2\ell+3}, N_{2\ell+1}) + \varrho(N_{2\ell+2}, N_{2\ell+2})}, \end{aligned}$$

or

$$\varrho(N_{2\ell+2}, N_{2\ell+3}) \leq_{i_2} \lambda \varrho(N_{2\ell+2}, N_{2\ell+1}) + \mu \frac{\varrho(N_{2\ell+2}, N_{2\ell+3})\varrho(N_{2\ell+1}, N_{2\ell+2})}{\varrho(N_{2\ell+3}, N_{2\ell+2}) + \varrho(N_{2\ell+2}, N_{2\ell+1})} + \gamma \frac{\varrho(N_{2\ell+1}, N_{2\ell+3})\varrho(N_{2\ell+1}, N_{2\ell+2})}{\varrho(N_{2\ell+1}, N_{2\ell+3})},$$

which implies that

$$\|\varrho(N_{2\ell+2}, N_{2\ell+3})\| \leq \lambda \|\varrho(N_{2\ell+2}, N_{2\ell+1})\| + \sqrt{2}\mu \frac{\|\varrho(N_{2\ell+2}, N_{2\ell+3})\| \|\varrho(N_{2\ell+1}, N_{2\ell+2})\|}{\|\varrho(N_{2\ell+3}, N_{2\ell+2})\| + \|\varrho(N_{2\ell+2}, N_{2\ell+1})\|} + \gamma \|\varrho(N_{2\ell+1}, N_{2\ell+2})\|.$$

Since $\|\varrho(N_{2\ell+3}, N_{2\ell+2})\| + \|\varrho(N_{2\ell+2}, N_{2\ell+1})\| > \|\varrho(N_{2\ell+1}, N_{2\ell+2})\|$,

$$\|q(N_{2k+2}, N_{2k+3})\| \leq \lambda \|q(N_{2k+2}, N_{2k+1})\| + \sqrt{2}\mu \frac{\|q(N_{2k+2}, N_{2k+3})\| \|q(N_{2k+1}, N_{2k+2})\|}{\|q(N_{2k+2}, N_{2k+1})\|} + \gamma \|q(N_{2k+1}, N_{2k+2})\|$$

or

$$\|q(N_{2k+2}, N_{2k+3})\| \leq \frac{\lambda + \gamma}{1 - \sqrt{2}\mu} \|q(N_{2k+1}, N_{2k+2})\|.$$

Now, with $\eta = \frac{\lambda + \gamma}{1 - \sqrt{2}\mu}$, we have (for all k)

$$\|q(N_k, N_{k+1})\| \leq \eta \|q(N_{k-1}, N_k)\| \leq \dots \leq \eta^k \|q(N_0, N_1)\|.$$

So, for any $m > k$, we have

$$\begin{aligned} \|q(N_k, N_m)\| &\leq \|q(N_k, N_{k+1})\| + \|q(N_{k+1}, N_{k+2})\| + \dots + \|q(N_{m-1}, N_m)\| \\ &\leq [\eta^k + \eta^{k+1} + \dots + \eta^{m-1}] \|q(N_0, N_1)\| \\ &\leq \left[\frac{\eta^k}{1 - \eta} \right] \|q(N_0, N_1)\|, \end{aligned}$$

and henceforth

$$\|q(N_k, N_m)\| \leq \left[\frac{\eta^k}{1 - \eta} \right] \|q(N_0, N_1)\| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

On using Lemma 2.3, we concluded that $\{N_k\}$ is a Cauchy sequence. Since X is a complete, there exists $l \in X$ such that $N_k \rightarrow l$ as $k \rightarrow \infty$. Now, we assert that $l = Sl$, otherwise $0 <_{i_2} \eta = q(l, Sl)$ and we have

$$\begin{aligned} \eta &= q(l, Sl) \leq_{i_2} q(l, \mathcal{T}N_{2k+1}) + q(\mathcal{T}N_{2k+1}, Sl) \\ &\leq_{i_2} q(l, N_{2k+2}) + \lambda q(l, N_{2k+1}) + \mu \frac{q(l, Sl)q(N_{2k+1}, \mathcal{T}N_{2k+1}) + q(N_{2k+1}, Sl)q(l, \mathcal{T}N_{2k+1})}{q(Sl, l) + q(\mathcal{T}N_{2k+1}, N_{2k+1})} \\ &\quad + \gamma \frac{q(l, Sl)q(l, \mathcal{T}N_{2k+1}) + q(N_{2k+1}, Sl)q(N_{2k+1}, \mathcal{T}N_{2k+1})}{q(Sl, N_{2k+1}) + q(N_{2k+2}, l)}, \end{aligned}$$

which implies that

$$\begin{aligned} \|\eta\| &= \|q(l, Sl)\| \\ &\leq \|q(l, N_{2k+2})\| + \lambda \|q(l, N_{2k+1})\| + \mu \frac{\eta \|q(N_{2k+1}, N_{2k+2})\| + \sqrt{2} \|q(N_{2k+1}, Sl)\| \|q(l, N_{2k+2})\|}{\|q(Sl, l) + q(N_{2k+2}, N_{2k+1})\|} \\ &\quad + \gamma \frac{\eta \|q(l, N_{2k+2})\| + \sqrt{2} \|q(N_{2k+1}, Sl)\| \|q(N_{2k+1}, N_{2k+2})\|}{\|q(Sl, N_{2k+1}) + q(N_{2k+2}, l)\|}, \end{aligned}$$

a contradiction, so that $\|\eta\| = \|q(l, Sl)\| = 0$, i.e., $l = Sl$. It follows, similarly, that $l = \mathcal{T}l$.

We now prove that S and $\mathcal{T}$ have a unique common fixed point. For this, assume that l^* in X is an another common fixed point of S and $\mathcal{T}$. Then we have

$$Sl^* = \mathcal{T}l^* = l^*.$$

Since $\mathcal{D} = q(Sl, l) + q(\mathcal{T}l^*, l^*) = 0$, by definition of contraction condition

$$q(l, l^*) = q(Sl, \mathcal{T}l^*) = 0,$$

so that $l = l^*$, which proves the uniqueness of common fixed point.

Second, we consider the case: $(q(SN_{2k}, N_{2k}) + q(\mathcal{T}N_{2k+1}, N_{2k+1}))(q(SN_{2k}, N_{2k+1}) + q(\mathcal{T}N_{2k+1}, N_{2k})) = 0$ (for any k) implies $q(SN_{2k}, \mathcal{T}N_{2k+1}) = 0$. Now, if $q(SN_{2k}, N_{2k}) + q(\mathcal{T}N_{2k+1}, N_{2k+1}) = 0$, then $N_{2k} = SN_{2k} = N_{2k+1} = \mathcal{T}N_{2k} = N_{2k+2}$. Thus, we have $N_{2k+1} = SN_{2k} = N_{2k}$, so there exist k_1 and m_1 such that $k_1 = Sm_1 = m_1$. Using foregoing arguments, one can also show that there exist k_2 and m_2 such that $k_2 = \mathcal{T}m_2 = m_2$. As $q(Sm_1, m_1) + q(\mathcal{T}m_2, m_2) = 0$, (due to definition) implies $q(Sm_1, \mathcal{T}m_2) = 0$, so that $k_1 = Sm_1 = \mathcal{T}m_2 = k_2$ which in turn yields that $k_1 = Sm_1 = S k_1$. Similarly, one can also have $k_2 = \mathcal{T}k_2$. As $k_1 = k_2$, implies $S k_1 = \mathcal{T}k_1 = k_1$, therefore $k_1 = k_2$ is a common fixed point of S and $\mathcal{T}$.

We now prove that S and $\mathcal{T}$ have unique common fixed point. For this, assume that $\mathfrak{k}_1^*$ in X is an another common fixed point of S and $\mathcal{T}$. Then we have

$$S\mathfrak{k}_1^* = \mathcal{T}\mathfrak{k}_1^* = \mathfrak{k}_1^*.$$

$$\text{Since } \mathcal{D} = \varrho(S\mathfrak{k}_1, \mathfrak{k}_1) + \varrho(\mathcal{T}\mathfrak{k}_1^*, \mathfrak{k}_1^*) = 0,$$

$$\varrho(\mathfrak{k}_1, \mathfrak{k}_1^*) = \varrho(S\mathfrak{k}_1, \mathcal{T}\mathfrak{k}_1^*) = 0.$$

This implies that $\mathfrak{k}_1^* = \mathfrak{k}_1$.

If $\varrho(S\mathfrak{N}_{2\mathfrak{k}}, \mathfrak{N}_{2\mathfrak{k}+1}) + \varrho(\mathcal{T}\mathfrak{N}_{2\mathfrak{k}+1}, \mathfrak{N}_{2\mathfrak{k}}) = 0$, implies that $\varrho(S\mathfrak{N}_{2\mathfrak{k}}, \mathcal{T}\mathfrak{N}_{2\mathfrak{k}+1}) = 0$, then also proof can be completed in the preceding lines. This completes the proof of the theorem. $\square$

By setting $S = \mathcal{T}$, we get the following.

Corollary 3.5. Let (X, ϱ) be a complete bicomplex valued metric space and let the mapping $\mathcal{T} : X \rightarrow X$ satisfy:

$$\varrho(\mathcal{T}\mathfrak{N}, \mathcal{T}\mathfrak{w}) \preceq_{i_2} \begin{cases} \lambda \varrho(\mathfrak{N}, \mathfrak{w}) + \mu \frac{\varrho(\mathfrak{N}, \mathcal{T}\mathfrak{N})\varrho(\mathfrak{w}, \mathcal{T}\mathfrak{w}) + \varrho(\mathfrak{w}, \mathcal{T}\mathfrak{N})\varrho(\mathfrak{N}, \mathcal{T}\mathfrak{w})}{\varrho(\mathcal{T}\mathfrak{N}, \mathfrak{N}) + \varrho(\mathcal{T}\mathfrak{w}, \mathfrak{w})} \\ + \gamma \frac{\varrho(\mathfrak{N}, \mathcal{T}\mathfrak{N})\varrho(\mathfrak{N}, \mathcal{T}\mathfrak{w}) + \varrho(\mathfrak{w}, \mathcal{T}\mathfrak{N})\varrho(\mathfrak{w}, \mathcal{T}\mathfrak{w})}{\varrho(\mathcal{T}\mathfrak{N}, \mathfrak{w}) + \varrho(\mathcal{T}\mathfrak{w}, \mathfrak{N})}, & \text{if } \mathcal{D} \neq 0, \mathcal{D}_1 \neq 0, \\ 0, & \text{if } \mathcal{D} = 0 \text{ or } \mathcal{D}_1 = 0 \end{cases} \quad (2)$$

for all $\mathfrak{N}, \mathfrak{w} \in X$, where $\mathcal{D} = \varrho(\mathcal{T}\mathfrak{N}, \mathfrak{N}) + \varrho(\mathcal{T}\mathfrak{w}, \mathfrak{w})$ and $\mathcal{D}_1 = \varrho(\mathcal{T}\mathfrak{N}, \mathfrak{w}) + \varrho(\mathcal{T}\mathfrak{w}, \mathfrak{N})$ and λ, μ, γ are nonnegative reals with $\lambda + \sqrt{2}\mu + \gamma < 1$. Then $\mathcal{T}$ has unique fixed point.

Example 3.6. Consider $X = \{0, \frac{1}{3}, 3\}$ with the order $\mathfrak{N} \preceq_{i_2} \mathfrak{w}$ iff $\mathfrak{w} \leq \mathfrak{N}$. Then $\preceq_{i_2}$ is a partial order in X . Define a mapping $\varrho : X \times X \rightarrow \mathbb{C}_2$ by $\varrho(\mathfrak{N}, \mathfrak{w}) = (1 + i_2)|\mathfrak{N} - \mathfrak{w}|$, for all $\mathfrak{N}, \mathfrak{w} \in X$, where $|\cdot|$ is the usual real modulus. Then (X, ϱ) is a complete bicomplex valued metric space. Now, we consider a self-mapping $\mathcal{T} : X \rightarrow X$ defined by

$$\mathcal{T}(0) = 0, \quad \mathcal{T}\left(\frac{1}{3}\right) = 0 \text{ and } \mathcal{T}(3) = \frac{1}{3}.$$

Let $\lambda = \frac{1}{6}$, $\mu = \frac{1}{5}$ and $\gamma = \frac{1}{9}$, then $\lambda + \sqrt{2}\mu + \gamma = \frac{1}{6} + \frac{\sqrt{2}}{5} + \frac{1}{9} < 1$. Then, clearly

$$0 = \varrho(\mathcal{T}\mathfrak{N}, \mathcal{T}\mathfrak{w}) \preceq_{i_2} \lambda \varrho(\mathfrak{N}, \mathfrak{w}) + \frac{\mu \varrho(\mathfrak{N}, \mathcal{T}\mathfrak{N})\varrho(\mathfrak{w}, \mathcal{T}\mathfrak{w}) + \gamma \varrho(\mathfrak{w}, \mathcal{T}\mathfrak{N})\varrho(\mathfrak{N}, \mathcal{T}\mathfrak{w})}{1 + \varrho(\mathfrak{N}, \mathfrak{w})}.$$

Hence, the conditions of Corollary 3.3 are satisfied. Therefore, 0 is the unique fixed point of $\mathcal{T}$.

Example 3.7. Let $X = \mathcal{B}(0, q)$, $q > 1$, for all $\mathfrak{N}, \mathfrak{w} \in X$ with the order $\mathfrak{N} \preceq_{i_2} \mathfrak{w}$ iff $\mathfrak{w} \leq \mathfrak{N}$. Then $\preceq_{i_2}$ is a partial order in X . Define $\varrho : X \times X \rightarrow \mathbb{C}_2$ by :

$$\varrho(\mathfrak{N}(t), \mathfrak{w}(t)) = \frac{i_2}{2\pi} \left| \int_{\mathcal{E}} \frac{\mathfrak{N}(t)}{t} - \int_{\mathcal{E}} \frac{\mathfrak{w}(t)}{t} \right|,$$

a complete bicomplex valued metric space, where $\mathcal{E}$ is a closed path in X containing a zero. We prove that ϱ is a bicomplex metric space. For this,

$$\begin{aligned} \varrho(\mathfrak{N}(t), \mathfrak{w}(t)) &= \frac{i_2}{2\pi} \left| \int_{\mathcal{E}} \frac{\mathfrak{N}(t)}{t} - \int_{\mathcal{E}} \frac{\mathfrak{w}(t)}{t} \right| \\ &= \frac{i_2}{2\pi} \left| \int_{\mathcal{E}} \frac{\mathfrak{N}(t)}{t} - \int_{\mathcal{E}} \frac{\eta(t)}{t} + \int_{\mathcal{E}} \frac{\eta(t)}{t} - \int_{\mathcal{E}} \frac{\mathfrak{w}(t)}{t} \right| \end{aligned}$$

$$\begin{aligned} & \leq_{i_2} \frac{i_2}{2\pi} \left| \int_{\varepsilon} \frac{\aleph(t)}{t} - \int_{\varepsilon} \frac{\eta(t)}{t} \right| + \frac{i_3}{2\pi} \left| \int_{\varepsilon} \frac{\eta(t)}{t} - \int_{\varepsilon} \frac{\varpi(t)}{t} \right| \\ & = \varrho(\aleph(t), \eta(t)) + \varrho(\eta(t), \varpi(t)). \end{aligned}$$

Now we define the mapping $S, \mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ by:

$$S\aleph(t) = t, \quad \mathcal{T}\varpi(t) = e^t - 1.$$

Using the Cauchy integral formula when the mappings S and $\mathcal{T}$ are analytic, we get:

$$\begin{aligned} \varrho(S\aleph(t), \mathcal{T}\varpi(t)) &= \frac{i_2}{2\pi} \left| \int_{\varepsilon} \frac{t}{t} - \int_{\varepsilon} \frac{e^t - 1}{t} \right| = 0, \\ \varrho(\aleph(t), \varpi(t)) &= \frac{i_2}{2\pi} \left| \int_{\varepsilon} \frac{\aleph(t)}{t} - \int_{\varepsilon} \frac{\varpi(t)}{t} \right|, \\ \varrho(\aleph(t), S\aleph(t)) &= \frac{i_2}{2\pi} \left| \int_{\varepsilon} \frac{\aleph(t)}{t} \right|, \\ \varrho(\varpi(t), \mathcal{T}\varpi(t)) &= \frac{i_2}{2\pi} \left| \int_{\varepsilon} \frac{\varpi(t)}{t} - \int_{\varepsilon} \frac{e^t - 1}{t} \right| = \frac{i_2}{2\pi} \left| \int_{\varepsilon} \frac{\varpi(t)}{t} \right|, \\ \varrho(\aleph(t), S\varpi(t)) &= \frac{i_2}{2\pi} \left| \int_{\varepsilon} \frac{\aleph(t)}{t} - \int_{\varepsilon} \frac{t}{t} \right| = \frac{i_2}{2\pi} \left| \int_{\varepsilon} \frac{\aleph(t)}{t} \right|, \\ \varrho(\aleph(t), \mathcal{T}\varpi(t)) &= \frac{i_2}{2\pi} \left| \int_{\varepsilon} \frac{\aleph(t)}{t} - \int_{\varepsilon} \frac{e^t - 1}{t} \right| = \frac{i_2}{2\pi} \left| \int_{\varepsilon} \frac{\aleph(t)}{t} \right|. \end{aligned}$$

Clearly,

$$\varrho(S\aleph, \mathcal{T}\varpi) \leq_{i_2} \lambda \varrho(\aleph, \varpi) + \frac{\mu \varrho(\aleph, S\aleph) \varrho(\varpi, \mathcal{T}\varpi) + \gamma \varrho(\varpi, S\aleph) \varrho(\aleph, \mathcal{T}\varpi)}{1 + \varrho(\aleph, \varpi)},$$

for all $\aleph, \varpi \in \mathcal{X}$, where λ, μ, γ are nonnegative reals with $\lambda + \sqrt{2}\mu + \sqrt{2}\gamma < 1$. Therefore, all the conditions of Theorem 3.1 are satisfied, then the mappings S and $\mathcal{T}$ have a unique common fixed point in $\mathcal{X}$.

4 Application

In this section, we give an application using Corollary 3.3.

Theorem 4.1. Let $\mathcal{X} = \mathbb{C}^{\mathfrak{k}}$ be a bicomplex valued metric space with the metric

$$\varrho(\aleph, \varpi) = \sum_{i=1}^{\mathfrak{k}} (|\aleph_i - \varpi_i| + i_3 |\aleph_i - \varpi_i|),$$

where $\aleph, \varpi \in \mathcal{X}$. If

$$\sum_{j=1}^{\mathfrak{k}} |\lambda_{ij}| \leq_{i_2} \lambda < 1, \text{ for all } i = 1, 2, \dots, \mathfrak{k},$$

then the linear system

$$\begin{cases} b_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1\ell}x_\ell \\ b_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2\ell}x_\ell \\ \vdots \\ b_\ell = a_{\ell 1}x_1 + a_{\ell 2}x_2 + \cdots + a_{\ell \ell}x_\ell \end{cases}$$

of ℓ linear equations in ℓ unknowns has a unique solution.

Proof. Define $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ by

$$\mathcal{T}(\mathcal{N}) = \mathcal{A}\mathcal{N} + \mathbf{b},$$

where $\mathcal{N} = (x_1, x_2, x_3, \dots, x_\ell) \in \mathbb{C}^\ell$, $\mathbf{b} = (b_1, b_2, \dots, b_\ell) \in \mathbb{C}^\ell$ and

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1\ell} \\ a_{21} & a_{22} & \cdots & a_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\ell 1} & a_{\ell 2} & \cdots & a_{\ell \ell} \end{pmatrix}.$$

Now,

$$\begin{aligned} \varrho(\mathcal{T}(\mathcal{N}), \mathcal{T}(\mathcal{W})) &= \sum_{j=1}^{\ell} (|\lambda_{ij}(x_j - w_j)| + i_3 |\lambda_{ij}(x_j - w_j)|) \\ &\leq_{i_2} \sum_{j=1}^{\ell} |\lambda_{ij}| \left(\sum_{j=1}^{\ell} (|x_j - w_j| + i_3 |x_j - w_j|) \right) \\ &\leq_{i_2} \lambda \sum_{j=1}^{\ell} (|x_j - w_j| + i_3 |x_j - w_j|) \\ &= \lambda \varrho(\mathcal{N}, \mathcal{W}) \\ &= \lambda \varrho(\mathcal{N}, \mathcal{W}) + \frac{\mu \varrho(\mathcal{N}, \mathcal{T}\mathcal{N}) \varrho(\mathcal{W}, \mathcal{T}\mathcal{W}) + \gamma \varrho(\mathcal{W}, \mathcal{T}\mathcal{N}) \varrho(\mathcal{N}, \mathcal{T}\mathcal{W})}{1 + \varrho(\mathcal{N}, \mathcal{W})}. \end{aligned}$$

Hence, all the conditions of Corollary 3.3 are satisfied with $\lambda = \frac{1}{6}$, $\mu = 0$, $\gamma = 0$, $\lambda + \sqrt{2}\mu + \sqrt{2}\gamma < 1$, and so the linear system of equation has a unique solution. $\square$

5 Conclusion and future work

In this paper, we proved some common fixed point theorems on bicomplex valued metric space. An illustrative example and application on bicomplex valued metric space is given. Furthermore, one can prove common fixed theorems on bicomplex b-metric space, bicomplex metric-like space, bicomplex b-metric-like space, bicomplex bipolar metric space, bicomplex partial metric space, bicomplex partial b-metric space under some rational-type contraction mappings.

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