Research Article

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Duality for convolution on subclasses of analytic functions and weighted integral operators

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Abstract: In this article, we investigate a class of analytic functions defined on the unit open disc \( \mathcal{U} = \{ z : |z| < 1 \} \), such that for every \( f \in \mathcal{P}(\alpha, \beta, \gamma) \), \( 0 \leq \beta \leq 1 \), and \( |z| < 1 \), the inequality

\[
\Re \left\{ f'(z) + \frac{1 - Y zf''(z)}{aY} \right\} > 0
\]

holds. We find conditions on the numbers \( \alpha, \beta, \) and \( \gamma \) such that \( \mathcal{P}(\alpha, \beta, \gamma) \subseteq \mathcal{SP}(\lambda) \), for \( \lambda \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), where \( \mathcal{SP}(\lambda) \) denotes the set of all \( \lambda \)-spirallike functions. We also make use of Ruscheweyh’s duality theory to derive conditions on the numbers \( \alpha, \beta, \gamma \) and the real-valued function \( \varphi \) so that the integral operator \( V_\varphi(f) \) maps the set \( \mathcal{P}(\alpha, \beta, \gamma) \) into the set \( \mathcal{SP}(\lambda) \), provided \( \varphi \) is non-negative normalized function \( \int_0^1 \varphi(t) dt = 1 \) and

\[
V_\varphi(f)(z) = \int_0^1 \varphi(t) \frac{f(tz)}{t} dt.
\]

Keywords: convolution, Hadamard product, duality principle, \( \lambda \)-spirallike functions, dual set, Gaussian hypergeometric function

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1 Introduction

Let \( \mathcal{A} \) be the set of analytic functions defined on the open unit disc \( \mathcal{U} = \{ z \in \mathbb{C} : |z| < 1 \} \) possessing the property that \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) and \( \lambda \) be a real number. Then, by \( \mathcal{SP}(\lambda) \) we denote the subclass of \( \mathcal{A} \) of all \( \lambda \)-spirallike functions for which \( \lambda \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \) (see, e.g., [1–6]). Every analytic function \( f \) in \( \mathcal{A} \) is a convex \( \lambda \)-spirallike function on \( \mathcal{U} \) if and only if \( zf''(z) \) is a \( \lambda \)-spirallike function on \( \mathcal{U} \). For \( a > 0 \), \( 0 \leq \beta \leq 1 \), and \( 0 < \gamma \leq 1 \), we denote by \( \mathcal{P}(\alpha, \beta, \gamma) \) the set of all functions \( f \) in \( \mathcal{A} \) provided

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For \( f \) and \( g \) in \( \mathcal{A} \) (or \( \mathcal{A}_0 \), \( \mathcal{A}_0 = \{ g : g(z) = \frac{f(z)}{z} ; f \in \mathcal{A} \} \)), \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \), the convolution product (Hadamard product) of \( f \) and \( g \), denoted by \( f \ast g \), is a function in \( \mathcal{A} \) (or \( \mathcal{A}_0 \)) defined by

\[
(f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad z \in \mathcal{U}.
\]

For a set \( \mathcal{V} \subseteq \mathcal{A}_0 \), the dual set \( \mathcal{V}^* \) is defined as \( \mathcal{V}^* = \{ g \in \mathcal{A}_0 : (f \ast g)(z) \neq 0, \forall f \in \mathcal{V}, z \in \mathcal{U} \} \). The second dual or the dual hull of \( \mathcal{V} \) is defined by \( \mathcal{V}^{**} = \langle \mathcal{V}^* \rangle \). Indeed, we have, \( \mathcal{V} \subseteq \mathcal{V}^{**} \) (see, e.g., [7] and [8]). Let \( (x)_k \) denote the Pochhammer symbol [6]

\[
(x)_0 = 1 \quad \text{and} \quad (x)_k = x(x+1)\ldots(x+k-1), \quad \text{for } k \in \mathbb{N}.
\]

Then, the Gaussian hypergeometric function is defined by ([1–27])

\[
_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k.
\]

The classical theory of integral transforms and their applications have been studied for a long time, and they are applied in many fields of mathematics [28–32]. Several integral transforms are extended to various spaces of distributions [29], tempered distributions [33], distributions of compact support [34], ultradistributions [35], and many others. In a classical sense, Fourier and Ruscheweyh introduced the integral transform \( V_\phi : \mathcal{A} \rightarrow \mathcal{A} \),

\[
V_\phi(f)(z) = \int_0^1 \varphi(t) \frac{f(tz)}{t} \, dt,
\]

where \( \varphi \) is a real-valued integrable function satisfying the normalizing condition [4]

\[
\int_0^1 \varphi(t) \, dt = 1.
\]

In the literature, the integral operator \( V_\phi(f) \) has been discussed by many authors on various choices of \( \varphi \) (see, e.g., [9,11,12,36,37]). In what follows, we introduce the function \( g = g_{a,x}^\lambda \) as a solution to the differential equation

\[
g(t) + \frac{1 - y}{ay} g'(t) = \frac{2 - e^{-it} - te^{-it}}{e^{it}(1 + t)} - \frac{2(1 - e^{-it}) \log(1 + t)}{e^{-it} t},
\]

which can be expressed in terms of an integral equation as

\[
g(t) = \frac{ay}{1 - y} \int_0^1 \frac{2 - e^{-it} - te^{-it}}{e^{it}(1 + t)} - \frac{2(1 - e^{-it}) \log(1 + t)}{e^{-it} t} \, ds.
\]

(1)

However, we find conditions on \( a, \beta, \) and \( y \) so that \( P_d(\beta, y) \subseteq \text{SP}(\lambda) \), for given \( |\lambda| < \frac{\pi}{2} \). We refer to the monographs [38] and [10] for more details on a variety of sufficient conditions on the \( \lambda \)-spirallike functions. For detailed analysis of various integral operators we refer readers to [14–19] and references cited therein. In Section 2, we present several lemmas which simplify our results. Section 3 is devoted for our main results and applications. One more result establishes the inclusion \( P_d(\beta, y) \subseteq \text{SP}(\lambda) \) for \( a > 0, 0 \leq \beta < 1, 0 < y < 1, \lambda \) being real number but \( |\lambda| < \frac{\pi}{2} \). In another conclusion, we impose conditions on the set \( P_d(\beta, y) \) to be univalent. Several remarks, corollaries, and theorems are also derived in some detail.
2 Preliminary lemmas

The following are preliminary lemmas which are very useful in our next analysis.

Lemma 2.1. (Duality principle, see [8]). Let \( V \subseteq A_0 \) be a compact subset having the property
\[
 f \in V \Rightarrow \forall |x| \leq 1 : f_x \in V, f_x(z) = f(xz).
\]
(2)
Then, for all continuous linear functionals \( \varphi \) on \( A \) we have \( \varphi(V) = \varphi(V^*) \) and \( c\varphi(V) \subseteq \varphi(V^*) \), where \( c\varphi \) stands for the closed convex hull of the set.

Lemma 2.2. [7] Let \( f \in A \). Then, the function \( f \) belongs to \( SP(\lambda) \) if and only if
\[
\frac{1}{z}(f(z) \ast h(z)) \neq 0, \quad z \in U,
\]
where
\[
h(z) = \frac{z + e^{-\lambda z}}{1 - e^{-\lambda z}}, \quad |x| = 1.
\]

Lemma 2.3. [13] Let \( 0 \leq \gamma < 1 \) and \( \beta \in \mathbb{R} \) such that \( \beta \neq 1 \). Let
\[
V_{\beta,\gamma} = \left\{ y(1 - \beta)\frac{1 + xz}{1 - xz} + (1 - y)(1 - \beta)\frac{1 + yz}{1 - yz} + \beta, \quad |x| = |y| = 1, \quad z \in U \right\}.
\]
Then, we have
\[
V_{\beta,\gamma}^* = \left\{ f \in A_0 : \exists \alpha \in \mathbb{R}, \quad \text{Re} \left( g(z) - \frac{1 - 2\beta}{2(1 - \beta)} \right) > 0, \quad g(z) = f_x(z), \quad |x| \leq 1 \right\}
\]
and
\[
V_{\beta,\gamma}^{**} = \left\{ f \in A_0; \quad \text{Re} \left( \frac{g(z) - \beta}{1 - \beta} \right) > 0, \quad g(z) = f_x(z), \quad |x| \leq 1 \right\}.
\]

Lemma 2.4. Let \( 0 \leq \gamma < 1 \) and \( \beta \in \mathbb{R}, \beta \neq 1 \), and \( V_{\beta,\gamma} \) be given by (3). Then, we have
\[
\frac{\Gamma_1(g)}{\Gamma_2(g)} = \frac{\Gamma_1(f)}{\Gamma_2(f)},
\]
for every \( g \in V_{\beta,\gamma}^* \) and some \( f \in V_{\beta,\gamma} \), where \( \Gamma_1, \Gamma_2 \) are continuous linear functionals on \( V_{\beta,\gamma} \) with \( \Gamma_2(V_{\beta,\gamma}) \neq 0 \).

It is clear from the context that the set \( V_{\beta,\gamma} \) in (3) does not satisfy the property (2), i.e., if \( f \in V_{\beta,\gamma} \) then \( f(xz) \in V_{\beta,\gamma} \) for all \( |x| \leq 1 \), which is the requirement of the Duality Principle. Therefore, the Duality Principle can be stated with a slightly weaker condition, but more complicated, when \( V_{\beta,\gamma} \) satisfies [7]. In the present article, we apply the duality principle on the set \( V_{\beta,\gamma} \) and hence we will not state it in its most general form.

3 Main results

In this section, we discuss various results involving spiral-like functions, hypergeometric functions, and certain class of integral transforms. We immense our section by establishing the following theorem.
Theorem 3.1. Let $\alpha > 0$, $0 \leq \beta < 1$, $0 < y < 1$, $|x| = 1$, and $\lambda$ be a real number such that $|\lambda| < \frac{\pi}{2}$. Then $P_0(\beta, y) \subseteq \mathcal{SP}(\lambda)$ if and only if

$$F(x, z) = ay \sum_{k=1}^{\infty} \frac{k(x + 1) + 1 + e^{-2\lambda k}}{(k + 1)(ay + (1 - y)k)}z^k,$$

where

$$\text{Re}[F(x, z)] > \frac{1 + \cos 2\lambda}{2(1 - \beta)}.$$  \hspace{1cm} (5)

Proof. Let $f$ be a function in the class $P_0(\beta, y)$ and $g_{a, y}(z) = f'(z) + \frac{1 - y}{ay}z f''(z)$. Then, we have $g_{a, y} \in V_{\beta, y}^*$. If $f(z) = \sum_{k=1}^{\infty} a_k z^k$, $a_1 = 1$, then

$$g_{a, y}(z) = 1 + \sum_{k=1}^{\infty} \left( k + 1 \right) a_k \left( 1 + \frac{\lambda}{ay} \right) z^k,$$

$$= 1 + \sum_{k=1}^{\infty} \left( k + 1 \right) a_k \left( 1 + \frac{\lambda}{ay} \right) z^k,$$

$$= \sum_{k=0}^{\infty} \left( k + 1 \right) a_k \left( 1 + \frac{\lambda}{ay} \right) z^k,$$

$$= \sum_{k=1}^{\infty} k a_k \left( 1 + \frac{\lambda}{ay} \right) (k - 1) z^{k-1}.$$

Hence, we have

$$f(z) = \sum_{k=1}^{\infty} a_k z^{k-1} = g_{a, y}(z) \cdot \sum_{k=1}^{\infty} \frac{ay}{k(ay + (1 - y)(k - 1))} z^{k-1}.$$

Therefore, we obtain a one-to-one correspondence between $P_0(\beta, y)$ and $V_{\beta, y}^*$. Thus, by aid of Theorem 2.2, $P_0(\beta, y) \subseteq \mathcal{SP}(\lambda)$ if and only if

$$g_{a, y}(z) = \sum_{k=1}^{\infty} \frac{ay z^{k-1}}{k(ay + (1 - y)(k - 1))} \cdot \frac{1 + x - e^{2\lambda k}}{1 - z^2} \neq 0, \quad \forall g \in V, |x| = 1, \forall z \in \mathcal{U}. \hspace{1cm} (6)$$

For $z \in \mathcal{U}$, let us consider the continuous linear functional $\lambda_z : \mathcal{A}_0 \longrightarrow \mathbb{C}$ such that

$$\lambda_z(h) = h(z) \cdot \sum_{k=1}^{\infty} \frac{ay z^{k-1}}{k(ay + (1 - y)(k - 1))} \cdot \frac{1 + x - e^{2\lambda k}}{1 - z^2} \neq 0.$$

By the Duality Principle, we obtain $\lambda(z) = \lambda_z(V_{\beta, y}^*)$. Therefore, (6) holds if and only if

$$\left( 1 + 2(1 - \beta) \sum_{k=1}^{\infty} z^k \right) \cdot \left( 1 + \sum_{k=1}^{\infty} \frac{ay z^{k-1}}{(k + 1)(ay + (1 - y)k)} z^k \right) \cdot \left( 1 + \sum_{k=1}^{\infty} k \cdot 1 + \frac{x - e^{2\lambda k}}{1 - e^{2\lambda k}} \right) z^k \neq 0.$$

Hence, we have

$$1 + 2(1 - \beta) \sum_{k=1}^{\infty} \frac{k(x + 1) + 1 + e^{-2\lambda k}}{1 + e^{-2\lambda k}} z^k \neq 0. \hspace{1cm} (7)$$

Using the properties of the convolution, we reformulate (7) as

$$ay \sum_{k=1}^{\infty} \frac{k(x + 1) + 1 + e^{-2\lambda k}}{k(ay + (1 - y)k)} z^k \neq -1 + \frac{e^{-2\lambda}}{2(1 - \beta)}. \hspace{1cm} (8)$$

For all $z \in \mathcal{U}, |x| = 1$, the equality on the right side of (8) takes its value on the line $\text{Re} w = \frac{1 + \cos 2\lambda}{2(1 - \beta)}$. So (8) is equivalent to (4). □
Corollary 3.2. The function \( F(x, z) \) can be expressed in terms of Gaussian hypergeometric function as

\[
F(x, z) = -(1 + e^{2i\lambda}) + \frac{ay(e^{-2i\lambda} - x)}{ay + y - 1} \frac{1}{z} \log \frac{1}{1 - z} + \sum_{k=1}^{\infty} \frac{(ay(1 + e^{2i\lambda}) + ay(x + 1))}{ay + y - 1} \binom{x}{k} z^k.
\]

Proof. By taking into account definitions and following simple computations we obtain

\[
F(x, z) = -\frac{ay(e^{-2i\lambda} - x)}{ay + y - 1} \sum_{k=1}^{\infty} \frac{1}{(k + 1) \frac{ay}{1 - y} + k} z^k + \sum_{k=1}^{\infty} \frac{(ay(1 + e^{2i\lambda}) + ay(x + 1))}{ay + y - 1} \frac{1}{z} \log \frac{1}{1 - z} + \sum_{k=1}^{\infty} \frac{ay}{ay + y - 1} z^k.
\]

That is,

\[
F(x, z) = -\frac{ay(e^{2i\lambda} + 1) - (1 - y)(1 + e^{-2i\lambda})}{ay + y + 1} + \frac{ay(e^{2i\lambda} - x)}{ay} + y - 1
\]

i.e.

\[
= -(1 + e^{-2i\lambda}) + \frac{ay(e^{-2i\lambda} - x)}{ay + y - 1} \log \frac{1}{1 - z}
\]

\[
+ \frac{ay}{ay + y - 1} \binom{x}{k} z^k.
\]

The following remark expresses a new form of Inequality (5).

Remark 3.3. The following inequality holds.

\[
\frac{1 + \cos 2\lambda}{2ay(1 - \beta)} + \text{Re} \left( \sum_{k=1}^{\infty} \frac{1}{ay + (1 - y)k} z^k \right) \geq \left| \sum_{k=1}^{\infty} \frac{k}{(k + 1)(ay + (1 - y)k)} z^k \right| + \left| \sum_{k=1}^{\infty} \frac{1}{(k + 1)(ay + (1 - y)k)} z^k \right|.
\]

Proof. Following the previous analysis, we write

\[
\frac{1 + \cos 2\lambda}{2ay(1 - \beta)} + \text{Re} \left( \sum_{k=1}^{\infty} \frac{k}{(k + 1)(ay + (1 - y)k)} z^k \right) + \text{Re} \left( \sum_{k=1}^{\infty} \frac{1}{(k + 1)(ay + (1 - y)k)} z^k \right) \geq \text{Re} \left( -x \sum_{k=1}^{\infty} \frac{k}{(k + 1)(ay + (1 - y)k)} z^k \right) + \text{Re} \left( -e^{-2i\lambda} \sum_{k=1}^{\infty} \frac{1}{(k + 1)(ay + (1 - y)k)} z^k \right).
\]

This indeed satisfies the above inequality when

\[
\frac{1 + \cos 2\lambda}{2ay(1 - \beta)} + \text{Re} \left( \sum_{k=1}^{\infty} \frac{1}{ay + (1 - y)k} z^k \right) \geq \left| \sum_{k=1}^{\infty} \frac{k}{(k + 1)(ay + (1 - y)k)} z^k \right| + \left| \sum_{k=1}^{\infty} \frac{1}{(k + 1)(ay + (1 - y)k)} z^k \right|.
\]

Therefore, we obtain

\[
\frac{1 + \cos 2\lambda}{2ay(1 - \beta)} + \text{Re} \left( \sum_{k=1}^{\infty} \frac{1}{ay + (1 - y)k} z^k \right) \geq \left| \sum_{k=1}^{\infty} \frac{1}{ay + (1 - y)k} z^k \right|.
\]
Theorem 3.4. Let \( f \in P_{\alpha}(\beta, y) , \alpha > 0 , 0 \leq \beta \leq 1 , \) and \( 0 < y \leq 1. \) Define

\[
K_{\alpha, y} = \int_0^1 \frac{dt}{1 - t + \frac{1}{\alpha y}}, \tag{9}
\]

Then, the function \( f \) belongs to \( P_{\alpha}(0, 1) \) and, hence, it is univalent for

\[
\beta \geq \frac{1 - 2K_{\alpha, y}}{2(1 - K_{\alpha, y})}.
\]

Proof. Let \( \alpha > 0 \) and \( y > 0. \) Define

\[
\phi(z) = 1 + \sum_{n=1}^{\infty} \left( 1 + \frac{1 - y}{\alpha y} \right) z^n
\]

and

\[
\psi(z) = 1 + \sum_{n=1}^{\infty} \frac{ay}{(1 - y)n} z^n = 1 + \sum_{n=1}^{\infty} \frac{z^n}{1 + \frac{1 - y}{\alpha y}} = \int_0^1 \frac{1}{1 - t + \frac{1}{\alpha y}z} dt.
\]

By using change of variables, we rewrite \( \psi(z) \) in the form

\[
\psi(z) = \begin{cases} 
\frac{1}{1 - z}, & y = 1, \\
\frac{ay}{1 - y} \int_0^1 \frac{s^{\frac{1}{\alpha y} - 1}}{1 - sz} ds, & 0 < y < 1.
\end{cases} \tag{10}
\]

In view of these representations, we can write

\[
f'(z) + \frac{1 - y}{\alpha y}zf''(z) = f'(z) \ast \phi(z) \quad \text{and} \quad \left( f'(z) + \frac{1 - y}{\alpha y}zf''(z) \right) \ast \psi(z) = f'(z). \tag{11}
\]

Now, we let \( f \in P_{\alpha}(\beta, y). \) Then, in view of the Duality Principle (see Lemma 2.1), we may restrict our attention to the function \( f \in P_{\alpha}(\beta, y) \) for which

\[
f'(z) + \frac{1 - y}{\alpha y}zf''(z) = y(1 - \beta)\frac{1 + xz}{1 - xz} + (1 - \beta)(1 - y)\frac{1 + yz}{1 - yz} + \beta.
\]

Thus, in view of (11), the preceding observation reveals

\[
f'(z) = \left( y(1 - \beta)\frac{1 + xz}{1 - xz} + (1 - \beta)(1 - y)\frac{1 + yz}{1 - yz} + \beta \right) \ast \psi(z). \tag{12}
\]

Hence, equation (12) is equivalent to

\[
f'(z) = \left( y\frac{1 + xz}{1 - xz} + (1 - y)\frac{1 + yz}{1 - yz} \right) \ast \left( (1 - \beta)\psi(z) + \beta \right),
\]

\[
= \left( y\frac{1 + xz}{1 - xz} + (1 - y)\frac{1 + yz}{1 - yz} \right) \ast \left( \int_0^1 \left( (1 - \beta)\frac{1}{1 - t + \frac{1}{\alpha y}z} + \beta \right) dt \right) \tag{13}
\]

\[
= \left( y\frac{1 + xz}{1 - xz} + (1 - y)\frac{1 + yz}{1 - yz} \right) \ast G(z),
\]

where

\[
G(z) = \int_0^1 \left( (1 - \beta)\frac{1}{1 - t + \frac{1}{\alpha y}z} + \beta \right) dt.
\]
Therefore, we have
\[
\text{Re}G(z) \geq \int_0^1 \left( 1 - \beta \right) \frac{1}{1 - t^2 \pi^2} + \beta \, dt = (1 - \beta)K_{a,y} + \beta,
\]
where \(K_{a,y}\) satisfies (9). Note that if \(\beta \geq (1 - 2K_{a,y})2(1 - K_{a,y})\), then \(\text{Re}G(z) \geq 1/2\). Also, it is well-known that functions with real parts greater than 1/2 preserve the closed convex hull under convolution [10, p.23]. Therefore, from (13) we write
\[
f^*(z) = y \left( \frac{2}{1 - xz} - 1 \right) \ast G(z) + (1 - y) \left( \frac{2}{1 - yz} - 1 \right) \ast G(z) = 2yG(xz) - y + 2(1 - y)G(yz) - (1 - y)
\]
\[
= 2yG(xz) + 2(1 - y)G(yz) - 1.
\]
But, since \(\text{Ref}^*(z) > 0\), we have \(f \in P_0(0, 1)\).

**Theorem 3.5.** Let \(f \in P_0(\beta, y)\), \(\alpha > 0\), \(0 \leq \beta \leq 1\), \(0 < y < 1\). Let the following integral equation
\[
\frac{\beta}{1 - \beta} = -\int_0^1 \varphi(t)g(t) \, dt
\]
hold. Then, for every real \(\lambda, |\lambda| < \frac{\pi}{2}\), we have \(V_\lambda(f) \in \text{SP}(\lambda)\) if and only if
\[
\text{Re} \int_0^1 \Pi_\lambda(t) \left( \frac{h(tz)}{tz} - \frac{-t + e^{-i\lambda}(1 + t)}{e^{-i\lambda}(1 + t)^2} \right) \, dt \geq 0,
\]
where
\[
\Pi_\lambda(t) = \int_\mathbb{T} A_\lambda(s) s^{-iz} \, ds, \quad A_\lambda(t) = \int_\mathbb{T} \varphi(s) \, ds(y > 0)
\]
and
\[
h(z) = z + \frac{z - e^{i\lambda}}{(1 - z)^2}, \quad |\lambda| = 1, \quad z \in \mathbb{U}.
\]

**Proof.** Let \(\alpha > 0\), \(y > 0\), and \(F(z) = V_\varphi(f)\). Then \(F \in \text{SP}(\lambda)\) if and only if
\[
0 \neq \frac{F(z)}{z} \ast h(z) = \int_0^1 \varphi(t) \frac{f(tz)}{t} \, dt \ast \frac{h(z)}{z} = \int_0^1 \varphi(t) \frac{f(tz)}{1 - tz} \, dt \ast \frac{f(z)}{z} \ast \frac{h(z)}{z}.
\]
Hence, by applying equation (12), equation (15) is equivalent to
\[
(1 - \beta) \int_0^1 \varphi(t) \frac{1}{2} \int_0^z \left( \frac{h(tw)}{tw} - g(t) \right) \, dw \, dt \ast \psi(z) \ast \left( \frac{1 + xz}{1 - xz} + (1 - y) \frac{1 + yz}{1 - yz} \right) \neq 0,
\]
where \(g(t)\) is defined by (1). Equation (16) indeed holds if and only if
\[
\text{Re} \left( (1 - \beta) \int_0^1 \varphi(t) \frac{1}{2} \int_0^z \left( \frac{h(tw)}{tw} - g(t) \right) \, dw \, dt \ast \psi(z) \right) > \frac{1}{2}.
\]
Now, by using the fact \((1 - \beta) \int_0^1 \varphi(t)(1 - g(t)) \, dt = 1\) we derive
\[
\Re \left\{ \int_0^1 \phi(t) \left( \int_0^t \left( \frac{h(tw)}{tw} - \frac{1 + g(t)}{2} \right) dw \right) dt \ast \psi(z) \right\} \geq 0.
\]

Using the definition of \( \psi \) given by (10), the above expression becomes
\[
\Re \left\{ \int_0^1 \phi(t) \left( \int_0^t \frac{1}{z} \int_0^{tw} \frac{ay}{1 - y} \left( \frac{h(tw)}{tw} - \frac{1 + g(t)}{2} \right) ds dw dt \right) \right\} \geq 0.
\]

By the definition of \( g \) presented in (1), it is easy to see that
\[
\frac{1 + g(t)}{2} = \frac{ay}{1 - y} \int_0^{tw} \frac{1}{1 - \frac{2}{t} - e^{-it} - e^{-it}e^{-it}} - \frac{2(e^{-it} - 1) \log 1 + st}{e^{-it}} ds.
\]

By substituting in the left-hand side of the previous inequality and using the change of variables \( v = st \) in the resulting equation, it follows that
\[
\Re \left\{ \int_0^1 \phi(t) \left( \int_0^{tw} \frac{1}{z} \int_0^{tw} \frac{h(tw)}{tw} - t \left( \frac{1}{2} + \frac{2 - e^{-it} - e^{-it}e^{-it}}{e^{-it}(1 + v)} - \frac{2(e^{-it} - 1) \log 1 + v}{e^{-it}} \right) dw \right) dv dt \right\} \geq 0.
\]

By integrating by parts, inequality (17) yields
\[
\Re \left\{ \int_0^1 \Lambda_\psi(t) \left( \frac{1}{z} \int_0^{tw} \frac{h(tw)}{tw} - t \left( \frac{1}{2} + \frac{2 - e^{-it} - e^{-it}e^{-it}}{e^{-it}(1 + t)} - \frac{2(e^{-it} - 1) \log 1 + t}{e^{-it}} \right) \right) - \frac{h(tz)}{t} \left( \frac{1}{e^{-it}(1 + t)^2} \right) dt \right\} \geq 0,
\]

where \( \Lambda_\psi(t) = \int_{S_{\psi}} a(s) d\sigma \). Once again, integrating by part suggests the following compact form:
\[
\Re \left\{ \int_0^1 \Pi_\psi(t) \left( \frac{h(tz)}{tz} - \frac{t + e^{-it}(1 + t)^2}{e^{-it}(1 + t)^2} \right) dt \right\} \geq 0,
\]

where \( \Pi_\psi(t) = \int_0^1 \Lambda_\psi(s) ds dt \). Thus, \( F \in \text{SP}(\lambda) \). To extend our result from the particular choice of \( f \) given in (12) to all of \( P_\beta(\beta, \gamma) \) we refer to Lemma 2.4. Note that \( V_\psi \) defines a linear functional and the condition on the \( \lambda \)-spirallikeness \( \Re \left( \frac{e^{itf(z)}}{F(z)} > 0 \right) > 0 \) is expressed as a quotient of two linear functionals. Therefore, Lemma 2.4 can be applied. Finally, to prove the sharpness, let \( f \in P_\beta(\beta, \gamma) \) be in the form
\[
f'(z) + \frac{1 - y}{ay} zf''(z) = y(1 - \beta) \frac{1 + xz}{1 - xz} + (1 - y)(1 - \beta) \frac{1 + yz}{1 - yz} + \beta.
\]

Let \( x = y = 1 \). Using a series expansion, we obtain
\[
f(z) = z + 2(1 - \beta) \sum_{n=2}^{\infty} \frac{ay}{n(ay + (n - 1)(1 - y))} z^n.
\]

Therefore, we can write
\[
F(z) = V_\phi(f)(z) = \int_0^1 \phi(t) \frac{f(tz)}{t} dt = z + 2(1 - \beta) \sum_{n=2}^{\infty} \frac{ay \mu_n}{n(ay + (n - 1)(1 - y))} z^n,
\]

where \( \mu_n = \int_0^1 \Lambda(t) t^{n-1} dt \). Furthermore, it is easy to write \( g(t) \) in (1) in a series expansion as
\[
g(t) = 1 + \frac{-2}{e^{it}} \sum_{n=1}^{\infty} \left( -1 \right)^n \frac{n + 2 - e^{-it}}{n + 1} \frac{ay}{ay + n(1 - y)} t^n.
\]
Now by equations (14) and (18), we have

\[
\frac{\beta}{1 - \beta} = -\int_0^1 \varphi(t)g(t)\,dt
\]

\[
= -\int_0^1 \varphi(t) \left(1 + \frac{-2}{e^{-it}} \sum_{n=1}^{\infty} (-1)^n \frac{n^2 - e^{-it}}{n + 1} \frac{ay}{ay + n(1 - y)} t^n \right)\,dt
\]

\[
= -1 - \frac{2}{e^{-it}} \sum_{n=1}^{\infty} (-1)^n \frac{n + 2 - e^{-it}}{n + 1} \frac{ay \mu_{n+1}}{ay + n(1 - y)}.
\]

Hence, we obtain

\[
\frac{1}{1 - \beta} = -\frac{2}{e^{-it}} \sum_{n=1}^{\infty} (-1)^n \frac{n + 1 - e^{-it} ay \mu_n}{n ay + n(1 - y)}.
\]

Computing \(F'(1)\), indeed gives

\[
F'(1) = 1 + 2(1 - \beta) \sum_{n=1}^{\infty} \frac{(-1)^{n-1} ay \mu_n}{n(n - 1)(1 - y) + ay}
\]

\[
= (1 - e^{-it}) + 2(1 - \beta)(1 - e^{-it}) \sum_{n=2}^{\infty} \frac{(-1)^{n-1} ay \mu_n}{n(n - 1)(1 - y) + ay}
\]

\[
= (1 - e^{-it}) \left(1 + 2(1 - \beta) \sum_{n=2}^{\infty} \frac{(-1)^{n-1} ay \mu_n}{n(n - 1)(1 - y) + ay}\right)
\]

\[
= -(1 - e^{-it}) F(-1).
\]

Hence, we obtain

\[
\text{Re} \left\{ e^{it} \frac{F'(1)}{F(1)} \right\} < 0.
\]

This implies that our result is sharp for the \(\lambda\)-spirallike function. \(\square\)

### 4 Concluding remarks

In this article, a class of analytic functions was discussed on a unit open disc \(U = \{z : |z| < 1\}\). Certain conditions on the numbers \(\alpha, \beta, \) and \(y\) were imposed so that \(P_{\alpha}(\beta, y)\) defines a subset of the set \(\text{SP}(\lambda)\) of \(\lambda\)-spirallike functions for all \(\lambda \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\). Ruscheweyh’s duality theory was employed in predicting conditions on the numbers \(\alpha, \beta, y\) and the real-valued functions \(\varphi\) so that the integral transform \(V_{\varphi}(f)\) maps \(P_{\alpha}(\beta, y)\) into \(\text{SP}(\lambda)\) for nonnegative and normalized functions \(\varphi\).

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