Research Article

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The study of solutions for several systems of PDDEs with two complex variables

https://doi.org/10.1515/dema-2022-0241
received October 16, 2022; accepted May 6, 2023

Abstract: The purpose of this article is to describe the properties of the pair of solutions of several systems of Fermat-type partial differential difference equations. Our theorems exhibit the forms of finite order transcendental entire solutions for these systems, which are some extensions and improvement of the previous theorems given by Xu, Cao, Liu, etc. Furthermore, we give a series of examples to show that the existence conditions and the forms of transcendental entire solutions with finite order of such systems are precise.

Keywords: partial differential difference equation, entire solution, Nevanlinna theory

MSC 2020: 30D 35, 35M 30, 39A 45

1 Introduction

Let us first recall the classical results that the entire solutions of the functional equation

\[ f^2 + g^2 = 1, \]

(1.1)

are \( f = \cos a(z), g = \sin a(z) \) was proved by Gross [1], where \( a(z) \) is an entire function. This simple-looking nonlinear functional equation (1.1) can be called as the Fermat-type functional equation, analogous with the equation \( x^2 + y^2 = z^2 \) in Fermat’s last theorem in number theory. As a matter of fact, we can find that the study of the Fermat-type functional equations can be tracked back to more than 60 years ago or even earlier [2,3].

In the past 30 years, there were lots of research focusing on the solutions of functional equation (1.1), readers can refer to [4–17]. For example, Khavinson [11] in 1995 proved that any entire solutions of the partial differential equations

\[ \left( \frac{\partial f}{\partial z_1} \right)^2 + \left( \frac{\partial f}{\partial z_2} \right)^2 = 1, \]

(1.2)

in \( \mathbb{C}^2 \) are necessarily linear. It should be noted that equation (1.2) is called as eiconal equation. Later, Saleeby [18,19] further proved that the entire solution of equation (1.2) is of the form \( f(z_1, z_2) = \varphi_1 + \varphi_2 + \eta \). After theirs works, Li and co-authors [20–22] further discussed a series of deformation forms of Fermat-type partial differential equations and gave a number of important and interesting results about the existence and the forms of solutions for these partial differential equations.

Theorem A. [20, Corollary 2.3] Let \( P(z_1, z_2) \) and \( Q(z_1, z_2) \) be arbitrary polynomials in \( \mathbb{C}^2 \). Then, \( f \) is an entire solution of the equation

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\[ \left( \frac{p \partial f(z_1, z_2)}{\partial z_1} \right)^2 + \left( \frac{q \partial f(z_1, z_2)}{\partial z_2} \right)^2 = 1, \]  

(1.3)

if and only if \( f = c_0z_1 + c_0z_2 + c_1 \) is a linear function, where \( c_i \)’s are constants, and exactly one of the following holds:

(i) \( c_0 = 0 \) and \( Q \) is a constant satisfying that \((c_0Q)^2 = 1\);
(ii) \( c_2 = 0 \) and \( P \) is a constant satisfying that \((c_0P)^2 = 1\);
(iii) \( c_0 \neq 0 \) and \( P \) and \( Q \) are both constants satisfying that \((c_0P)^2 + (c_0Q)^2 = 1\).

In the past 40 years, the Nevanlinna theory and the difference Nevanlinna theory of Meromorphic function with several complex variables have been developed rapidly [23–29]. Especially, Korhonen [30, Theorem 3.1] in 2012 established a logarithmic difference lemma for meromorphic functions in several variables of hyper order <2/3. Later, Cao and Korhonen [23] proved that the logarithmic difference lemma for meromorphic functions holds under the condition “hyper order <1.” By making use of the Nevanlinna theory and difference Nevanlinna theory of several complex variables [23,30], Xu and Cao [31,32] discussed the transcendental solutions of several Fermat-type partial differential difference equations. An equation is called partial differential difference equation, if the equation includes partial derivatives, shifts or differences of \( f \), which can be called PDDE for short.

**Theorem B.** [31, Theorem 1.2] Let \( c = (c_1, c_2) \in \mathbb{C}^2 \). Then, any transcendental entire solution with finite order of the PDEE

\[ \left( \frac{\partial f(z_1, z_2)}{\partial z_1} \right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1, \]  

(1.4)

has the form of \( f(z_1, z_2) = \sin(Az_1 + B) \), where \( A \) is a constant on \( \mathbb{C} \) satisfying \( A e^{i\alpha_1} = 1 \), and \( B \) is a constant on \( \mathbb{C} \); as a special case, whenever \( c_1 = 0 \), we have \( f(z_1, z_2) = \sin(\alpha_1 + B) \).

In 2020, the author of this article and his colleagues [33] studied the finite order transcendental entire solutions when equation (1.4) turns to the system of Fermat-type PDDEs and obtained Theorem C.

**Theorem C.** [33, Theorem 1.3] Let \( c = (c_1, c_2) \in \mathbb{C}^2 \). Then, any pair of transcendental entire solutions with finite order for the system of Fermat-type PDDEs

\[ \left( \frac{\partial f_1(z_1, z_2)}{\partial z_1} \right)^2 + f_1(z_1 + c_1, z_2 + c_2)^2 = 1, \]  

\[ \left( \frac{\partial f_2(z_1, z_2)}{\partial z_1} \right)^2 + f_2(z_1 + c_1, z_2 + c_2)^2 = 1, \]  

(1.5)

have the following forms:

\[ (f_1(z), f_2(z)) = \left( \frac{e^{iL(z)+B_1} + e^{-iL(z)+B_1}}{2}, \ \frac{A_{21}e^{iL(z)+B_1} + A_{22}e^{-iL(z)+B_1}}{2} \right), \]

where \( L(z) = a_1z_1 + a_2z_2, B_1 \) is a constant in \( \mathbb{C} \), and \( a_1, c, A_{21}, A_{22} \) satisfy one of the following cases:

(i) \( A_{21} = -i, A_{22} = i, \) and \( a_1 = i, L(c) = \left\{ 2k + \frac{1}{2} \right\} \pi i, \) or \( a_1 = -i, L(c) = \left\{ 2k - \frac{1}{2} \right\} \pi i; \)

(ii) \( A_{21} = i, A_{22} = -i, \) and \( a_1 = i, L(c) = \left\{ 2k - \frac{1}{2} \right\} \pi i, \) or \( a_1 = -i, L(c) = \left\{ 2k + \frac{1}{2} \right\} \pi i; \)

(iii) \( A_{21} = 1, A_{22} = 1, \) and \( a_1 = i, L(c) = 2k\pi i, \) or \( a_1 = -i, L(c) = (2k + 1)\pi i; \)

(iv) \( A_{21} = -1, A_{22} = -1, \) and \( a_1 = i, L(c) = (2k + 1)\pi i, \) or \( a_1 = -i, L(c) = 2k\pi i. \)
To the best of our knowledge, there are few results about the study of systems of this Fermat-type PDDE with several complex variables. Moreover, it appears that the study of such fields has not been addressed in the literature before. Based on these, we are mainly concerned with the solutions of complex Fermat-type PDDEs, and describe the existence and form of the pair of the finite order transcendental solutions of the systems of PDDEs with constant coefficients

\begin{equation}
\begin{aligned}
\left[ \mu f_1(z) + \lambda \frac{\partial f_1}{\partial z_1} \right]^2 + [a f_2(z + c) - \beta f_1(z)]^2 &= 1, \\
\left[ \mu f_2(z) + \lambda \frac{\partial f_2}{\partial z_1} \right]^2 + [a f_1(z + c) - \beta f_2(z)]^2 &= 1,
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\left[ \mu f_1(z) + \lambda_i \frac{\partial f_1}{\partial z_1} + \lambda_2 \frac{\partial f_1}{\partial z_2} \right]^2 + [a f_2(z + c) - \beta f_1(z)]^2 &= 1, \\
\left[ \mu f_2(z) + \lambda_i \frac{\partial f_1}{\partial z_1} + \lambda_2 \frac{\partial f_1}{\partial z_2} \right]^2 + [a f_1(z + c) - \beta f_2(z)]^2 &= 1,
\end{aligned}
\end{equation}

where \(a, \beta, \mu, \lambda, \lambda_i, \alpha, \gamma\) are constants in \(C\). Obviously, equation (1.4) and system (1.5) are the special cases of systems (1.6) and (1.7). The article is organized as follows. We will introduce our main results about the existence and the forms of entire solutions for (1.6) and (1.7) in Section 2, which generalize the previous theorems given by Xu et al. [31–33]. Meantime, we give a series of examples to explain that our results about the forms of solutions of such systems are precise. The proofs of Theorems 1.6 and 1.7 are given in Sections 4 and 5, respectively.

\section{Results and examples}

The first main theorem is about the existence and the forms of the solutions for system (1.6).

\textbf{Theorem 2.1.} Let \(c = (c_1, c_2) \in C^2\), \(c_2 \neq 0\), and \(a, \beta, \mu, \lambda\) be nonzero constants in \(C\). Let \((f_1(z_1, z_2), f_2(z_1, z_2))\) be a pair of transcendental entire solution with finite order of system (1.6). Then, \((f_1(z_1, z_2), f_2(z_1, z_2))\) must satisfy one of the following cases:

(i)

\begin{align}
&f_1(z_1, z_2) = \frac{\eta_1}{\mu} - 1 e^{\mu z_1 + \frac{\eta_1}{\mu} z_2 + D_1}, \quad f_2(z_1, z_2) = \frac{\eta_1}{\mu} - 1 e^{-\mu z_1 - \frac{\eta_1}{\mu} z_2 + D_2},
\end{align}

where \(\eta_1, \eta_2, a, D_1, D_2 \in C\) satisfy

\begin{equation}
e^{D_1-D_2} = 1, \quad y = \frac{\mu \eta_1 + \log \frac{\eta_1}{\mu} + k\pi}{\eta_1}, \quad k \in Z, (2.1)
\end{equation}

and one of the following cases:

(i) \(\delta_1 = -\eta_1 = \pm 1, a = \mp \beta\) or \(\delta_1 = \eta_1 = \pm 1, a = \mp \beta\);

(ii) \(\delta_1 = 0\) and \(\eta_1^2 = \frac{\mu^2}{\mu^2 + (a - \beta)^2}\), or \(\delta_2 = -\eta_1\) and \(\eta_2^2 = \frac{\mu^2}{\mu^2 + (a + \beta)^2}\);

(iii) \(\mu^2 = \alpha^2 - \beta^2, \delta_1^2 + \delta_2^2 = 1, \eta_1^2 + \eta_2^2 = 1,\) and \(a = \pm (\eta_1^2 - 1)(\beta^2 - \alpha^2)\);

(ii) if \(a_1 \neq \pm \frac{\mu}{\eta_1}\), then
\[ f_1(z_1, z_2) = \frac{1}{2(\lambda a_1 + \mu)} e^{a_1 z_1 + a_2 z_2 + b_1} - \frac{1}{2(\lambda a_1 - \mu)} e^{-a_1 z_1 - a_2 z_2 - b_1} - \theta(z_2) e^\frac{\mu}{2} z_1 + y z_2 + D_1, \]
\[ f_2(z_1, z_2) = \frac{1}{2(\lambda a_1 + \mu)} e^{a_1 z_1 + a_2 z_2 + b_2} - \frac{1}{2(\lambda a_1 - \mu)} e^{-a_1 z_1 - a_2 z_2 - b_2} - \theta(z_1 + c_2) e^\frac{\mu}{2} z_1 + y z_2 + D_2, \]

where \( \theta(z_2) \) is a finite order period entire function with period \( 2c_2 \), and \( a_1, a_2, b_1, b_2, y, D_1, D_2 \in \mathbb{C} \) satisfy (2.1),
\[ e^{2(b_1 - b_2)} = 1, \] (2.2)

and
\[ a_1 = -\beta i + \sqrt{\mu^2 - \alpha^2} \lambda, \quad e^{2(a_1 q_1 + a_2 q_2)} = \frac{(\lambda a_1 + \mu + \beta i)^2}{-\alpha^2}; \]

if \( a_1 = \frac{\mu}{\lambda} \), then
\[ f_1(z_1, z_2) = \frac{1}{4\mu} e^{a_1 z_1 + a_2 z_2 + b_1} + \frac{z_1}{2\lambda} e^{-a_1 z_1 - a_2 z_2 - b_1} - \theta(z_2) e^{\frac{\mu}{2} z_1 + y z_2 + D_1}, \]
\[ f_2(z_1, z_2) = \frac{1}{4\mu} e^{a_1 z_1 + a_2 z_2 + b_2} + \frac{z_1}{2\lambda} e^{-a_1 z_1 - a_2 z_2 - b_2} - \theta(z_1 + c_2) e^{\frac{\mu}{2} z_1 + y z_2 + D_1}, \]

where \( a_1, a_2, b_1, b_2, y \in \mathbb{C} \) satisfy (2.1), (2.2) and
\[ 2\mu\beta i = \beta^2 - \alpha^2, \quad \beta q_1 = \lambda i, \quad e^{2(a_1 q_1 + a_2 q_2)} = 1 - \frac{2\mu}{\beta} i; \] (2.3)

if \( a_1 = -\frac{\mu}{\lambda} \), then
\[ f_1(z_1, z_2) = \frac{z_1}{2\lambda} e^{a_1 z_1 + a_2 z_2 + b_1} + \frac{1}{4\mu} e^{-a_1 z_1 - a_2 z_2 - b_1} - \theta(z_2) e^{\frac{\mu}{2} z_1 + y z_2 + D_1}, \]
\[ f_2(z_1, z_2) = \frac{z_1}{2\lambda} e^{a_1 z_1 + a_2 z_2 + b_2} + \frac{1}{4\mu} e^{-a_1 z_1 - a_2 z_2 - b_2} - \theta(z_1 + c_2) e^{\frac{\mu}{2} z_1 + y z_2 + D_1}, \]

where \( a_1, a_2, b_1, b_2, y \in \mathbb{C} \) satisfy (2.1), (2.2) and
\[ -2\mu\beta i = \beta^2 - \alpha^2, \quad \beta q_1 = -\lambda i, \quad e^{-2(a_1 q_1 + a_2 q_2)} = 1 + \frac{2\mu}{\beta} i. \] (2.4)

The following examples show the existence of transcendental entire solutions of system (1.6).

**Example 2.1.** Let
\[ f_1(z_1, z_2) = \frac{1}{2} - \frac{1}{2} e^{-2\lambda z_1 + 3z_2}, \quad f_2(z_1, z_2) = -\frac{1}{2} - \frac{1}{2} e^{-2\lambda z_1 + 3z_2}. \]

Thus, \((f_1, f_2)\) is a pair of finite order transcendental entire solution of system (1.6) with \( \lambda = 1, \mu = 2, \alpha = 1, \beta = -1, \) and \((q_1, q_2) = (\pi i, \pi i)\).

**Example 2.2.** Let
\[ f_1(z_1, z_2) = 1 - e^{-z_1 + z_2}, \quad f_2(z_1, z_2) = 1 - e^{-z_1 + z_2}. \]

Thus, \((f_1, f_2)\) is a pair of finite order transcendental entire solution of system (1.6) with \( \lambda = \mu = \alpha = \beta = 1, \) and \((q_1, q_2) = (\pi i, \pi i)\).

**Example 2.3.** Let
\[ f_1(z_1, z_2) = \frac{1}{\sqrt{3}} - e^{\frac{1}{2} z_1 + z_2}, \quad f_2(z_1, z_2) = -\frac{1}{\sqrt{3}} - e^{\frac{1}{2} z_1 + z_2}. \]
Thus, \((f_1, f_2)\) is a pair of finite order transcendental entire solution of system (1.6) with \(\lambda = 2, \mu = 1, a = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \beta = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\), and \((\alpha_1, \alpha_2) = (\pi, \pi i)\).

**Example 2.4.** Let
\[
f_1(z_1, z_2) = \frac{\sqrt{2}}{2} + \frac{i}{\sqrt{2}} e^{-\sqrt{2}z_1 + (\sqrt{2} - 0.5)z_2}, \quad f_2(z_1, z_2) = 1 - \frac{\sqrt{2}}{2} i + \frac{i}{\sqrt{2}} e^{-\sqrt{2}z_1 + (\sqrt{2} - 0.5)z_2}.
\]
Thus, \((f_1, f_2)\) is a pair of finite order transcendental entire solution of system (1.6) with \(\lambda = \mu = \sqrt{2}i, a = i, \beta = 1, \) and \((\alpha_1, \alpha_2) = (\pi, \pi i)\).

**Example 2.5.** Let
\[
f_1(z_1, z_2) = \frac{1}{2\sqrt{2}(\sqrt{2} + 1)} e^{2z_1 + a_2z_2} - \frac{1}{2\sqrt{2}(\sqrt{2} - 1)} e^{-2z_1 - a_2z_2} - \sin(\pi z_2) e^{-\sqrt{2}z_1 + (\sqrt{2} + 0.5)z_2},
\]
\[
f_2(z_1, z_2) = -\frac{1}{2\sqrt{2}(\sqrt{2} + 1)} e^{2z_1 + a_2z_2} + \frac{1}{2\sqrt{2}(\sqrt{2} - 1)} e^{-2z_1 - a_2z_2} + \sin(\pi z_2) e^{-\sqrt{2}z_1 + (\sqrt{2} + 0.5)z_2},
\]
where \(a_2 = \log(\sqrt{2} + 1) - \frac{\pi}{2}i\). Thus, \((f_1, f_2)\) is a pair of finite order transcendental entire solution of system (1.6) with \(\lambda = \mu = \sqrt{2}, a = 1, \beta = i, \) and \((\alpha_1, \alpha_2) = (\pi, 1)\).

**Example 2.6.** Let \(y = -\frac{1}{4} \ln 2 + \frac{\pi}{8} i + \frac{1}{2} i, a_2 = \frac{1}{4} \ln 2 - \frac{\pi}{8} i - \frac{1}{2} i\) and
\[
f_1(z_1, z_2) = \frac{1}{4} e^{\lambda_1 + a_2z_2} - i z_2 e^{-\lambda_1 - a_2z_2} - \cos(\pi z_2) e^{-\lambda_1 - a_2z_2},
\]
\[
f_2(z_1, z_2) = \frac{1}{4} e^{\lambda_1 + a_2z_2} - i z_2 e^{-\lambda_1 - a_2z_2} + \cos(\pi z_2) e^{-\lambda_1 - a_2z_2}.
\]
Thus, \((f_1, f_2)\) is a transcendental entire solution of (1.6) with \(\lambda = -2i, \mu = 1, a = 2^i e^{-\frac{\pi}{2}i}, \beta = 2, (\alpha_1, \alpha_2) = (1, 1), \) and \(\rho(f) = 1\).

For \(a = 1\) and \(\beta = 0\) in system (1.6), we have

**Corollary 2.1.** Let \(c = (\alpha_1, \alpha_2) \in C^2, \alpha_2 \neq 0, \) and \(\mu, \lambda \) be nonzero constants. If \((f_1, f_2)\) are a pair of finite order transcendental entire solution of the following system:
\[
\begin{align*}
\left[ \mu f_1(z) + \lambda \frac{\partial f_1}{\partial z_1} \right]^2 + f_2(z + c)^2 &= 1, \\
\left[ \mu f_2(z) + \lambda \frac{\partial f_2}{\partial z_1} \right]^2 + f_1(z + c)^2 &= 1,
\end{align*}
\]
then, \((f_1, f_2)\) must be of the form
\[
f_1(z_1, z_2) = \frac{1}{2(\lambda a_1 + \mu)} e^{a_2 z_1 + a_2 z_2 + b_1} + \frac{1}{2(\mu - \lambda a_1)} e^{-a_2 z_1 - a_2 z_2 - b_1},
\]
\[
f_2(z_1, z_2) = \frac{1}{2(\lambda a_1 + \mu)} e^{a_2 z_1 + a_2 z_2 + b_1} + \frac{1}{2(\mu - \lambda a_1)} e^{-a_2 z_1 - a_2 z_2 - b_1},
\]
where \(a_1, a_2, b_1, b_2 \in C\) satisfy \(a_1^2 = \frac{\mu^2 - 1}{\mu}\) and
\[
e^{2(\lambda a_1 + a_2 z_2)} = (\mu + \lambda a_1)^2 = \frac{1}{(\mu - \lambda a_1)^2}, \quad e^{2(\lambda b_1 + b_2)} = -1,
\]
\[ e^{2(a_1 \alpha + a_2 \gamma)} = -(\mu + \lambda a_1)^2 = -\frac{1}{(\mu - \lambda a_1)^2}, \quad e^{2(b_1 + b_2)} = 1. \]  

(2.7)

**Theorem 2.2.** Let \( c = (\alpha_1, \alpha_2) \in \mathbb{C}^2 \), \( \alpha, \beta, \mu, \lambda_1, \lambda_2 \) be nonzero constants in \( \mathbb{C} \), \( s_1 = \lambda_2 z_1 - \lambda_2 z_2 \), and \( s_0 = \lambda_1 z_2 - \lambda_2 \alpha_2 \neq 0 \). Let \((f_1, f_2)\) be a pair of the finite order transcendental entire solutions of system (1.7). Then, \((f_1, f_2)\) must satisfy one of the following cases:

(i) 
\[ f_1(z_1, z_2) = \frac{\eta_1}{\mu} - \frac{1}{\mu} e^{-\frac{\mu}{\alpha} y \lambda_2 z_1 - \lambda_2 z_2} D_1, \]
\[ f_2(z_1, z_2) = \frac{\delta_1}{\mu} - \frac{1}{\mu} e^{-\frac{\mu}{\alpha} y \lambda_2 z_1 - \lambda_2 z_2} D_2, \]

where \( \eta_1, \delta_1, y, D_1, D_2 \in \mathbb{C} \) satisfy

\[ e^{D_1-D_2} = 1, \quad y = \frac{\mu a_1 + \log \frac{\beta}{\alpha} + 2k\pi i}{\lambda_1 z_2 - \lambda_2 \alpha_2}, \quad k \in \mathbb{Z}, \]  

(2.8)

and one of the following cases:

(i) \( \delta_1 = -\eta_1 = \pm 1 \) and \( \alpha = -\beta \) or \( \delta_1 = \eta_1 = \pm 1, \alpha = \beta; \)

(ii) \( \delta_1 = \eta_1 \) and \( \eta_1^2 = \frac{\mu^2}{\mu^2 + (\alpha - \beta)^2} \) or \( \delta_1 = -\eta_1 \) and \( \eta_1^2 = \frac{\mu^2}{\mu^2 + (\alpha + \beta)^2}; \)

(iii) \( \mu^2 = \alpha^2 - \beta^2, \delta_1^2 + \delta_2^2 = 1, \eta_1^2 + \eta_2^2 = 1, \) and \( \alpha \delta_2 = -\beta \eta_2 = \sqrt{(\eta_2^2 - 1)(\beta^2 - \alpha^2)} \);

(ii) If \( \mu^2 \neq (\lambda_1 a_1 + \lambda_2 a_2)^2 \), then

\[ f_1(z_1, z_2) = \frac{1}{2(\lambda_1 a_1 + \lambda_2 a_2 - \mu)} e^{\frac{\mu^2}{\alpha} y \lambda_1 a_1 - \lambda_2 a_2} - \frac{1}{2(\lambda_1 a_1 + \lambda_2 a_2 - \mu)} e^{-\frac{\mu^2}{\alpha} y \lambda_2 a_2 - \lambda_1 a_1} - \vartheta(s_1) e^{-\frac{\mu^2}{\alpha} y s_1 + D_1}, \]
\[ f_2(z_1, z_2) = \frac{1}{2(\lambda_1 a_1 + \lambda_2 a_2 + \mu)} e^{\frac{\mu^2}{\alpha} y \lambda_1 a_1 - \lambda_2 a_2} - \frac{1}{2(\lambda_1 a_1 + \lambda_2 a_2 + \mu)} e^{-\frac{\mu^2}{\alpha} y \lambda_2 a_2 - \lambda_1 a_1} - \vartheta(s_1 + s_0) e^{\frac{\mu^2}{\alpha} y s_1 + D_2}, \]

where \( \vartheta(s_1) \) is a finite order period entire functions with period \( 2s_0 \), and \( a_1, a_2, b_1, b_2, y, D_1, D_2 \) satisfy (2.2), (2.8), and

\[(\lambda_1 a_1 + \lambda_2 a_2 + \beta i)^2 = \mu^2 - \alpha^2, \quad e^{2(a_1 \alpha + a_2 \gamma)} = \frac{\lambda_1 a_1 + \lambda_2 a_2 + \beta i + \mu}{\lambda_1 a_1 + \lambda_2 a_2 + \beta i - \mu}; \]  

(2.9)

if \( \mu = \lambda_1 a_1 + \lambda_2 a_2 \), then

\[ f_1(z_1, z_2) = \frac{1}{4\mu} e^{\frac{\mu^2}{\alpha} y \lambda_1 a_1 + \lambda_2 a_2 + \beta i} + \frac{z_1}{2\lambda_1} e^{-\frac{\mu^2}{\alpha} y \lambda_2 a_2 - \lambda_1 a_1} - \vartheta(s_1) e^{\frac{\mu^2}{\alpha} y s_1 + D_1}, \]
\[ f_2(z_1, z_2) = \frac{1}{4\mu} e^{\frac{\mu^2}{\alpha} y \lambda_1 a_1 + \lambda_2 a_2 + \beta i} + \frac{z_1}{2\lambda_1} e^{-\frac{\mu^2}{\alpha} y \lambda_2 a_2 - \lambda_1 a_1} - \vartheta(s_1 + s_0) e^{\frac{\mu^2}{\alpha} y s_1 + D_2}, \]

where \( a_1, a_2, b_1, b_2, y \in \mathbb{C} \) satisfy (2.2), (2.8), and

\[ 2\mu \beta i = \beta^2 - \alpha^2, \quad \beta i = \lambda_1 i, \quad e^{2(a_1 \alpha + a_2 \gamma)} = 1 - \frac{2\mu}{\beta i}; \]  

(2.10)

if \( \mu = -\lambda_1 a_1 + \lambda_2 a_2 \), then

\[ f_1(z_1, z_2) = \frac{z_1}{2\lambda_1} e^{\frac{\mu^2}{\alpha} y \lambda_1 a_1 + \lambda_2 a_2 + \beta i} + \frac{1}{4\mu} e^{-\frac{\mu^2}{\alpha} y \lambda_2 a_2 - \lambda_1 a_1} - \vartheta(s_1) e^{\frac{\mu^2}{\alpha} y s_1 + D_1}, \]
\[ f_2(z_1, z_2) = \frac{z_1}{2\lambda_1} e^{\frac{\mu^2}{\alpha} y \lambda_1 a_1 + \lambda_2 a_2 + \beta i} + \frac{1}{4\mu} e^{-\frac{\mu^2}{\alpha} y \lambda_2 a_2 - \lambda_1 a_1} - \vartheta(s_1 + s_0) e^{\frac{\mu^2}{\alpha} y s_1 + D_2}, \]

where \( a_1, a_2, b_1, b_2, y \in \mathbb{C} \) satisfy (2.2), (2.8), and
\[-2\mu i = \beta^2 - \alpha^2, \quad \beta \varrho = -\lambda i, \quad e^{2i(\alpha \varrho + \beta \varrho \epsilon)} = 1 + \frac{2\mu}{\beta}i. \quad (2.11)\]

The following examples show the existence of transcendental entire solutions of system (1.7).

**Example 2.7.** Let
\[f_1(z_1, z_2) = \frac{1}{2} - \frac{1}{2} e^{-z_1 + z_2}, \quad f_2(z_1, z_2) = \frac{1}{2} - \frac{1}{2} e^{-z_1 + z_2}.\]
Thus, \((f_1, f_2)\) is a pair of finite order transcendental entire solution of system (1.7) with \(\lambda_1 = 1, \lambda_2 = -1, \mu = 2, \alpha = 1, \beta = 1, \) and \((\varrho_1, \varrho_2) = (\kappa, \varpi), \kappa \in \mathbb{C} \setminus \{0\}\).

**Example 2.8.** Let
\[f_1(z_1, z_2) = \frac{\sqrt{2}}{4} - \frac{1}{2} e^{-z_1 + z_2}, \quad f_2(z_1, z_2) = -\frac{\sqrt{2}}{4} - \frac{1}{2} e^{-z_1 + z_2}.\]
Thus, \((f_1, f_2)\) is a pair of finite order transcendental entire solution of system (1.7) with \(\lambda_1 = 1, \lambda_2 = -1, \mu = 2, \alpha = 1, \beta = 1, \) and \((\varrho_1, \varrho_2) = (\kappa, \varpi), \kappa \in \mathbb{C} \setminus \{0\}\).

**Example 2.9.** Let
\[f_1(z_1, z_2) = \frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}} e^{\left(\frac{\pi}{2} + \frac{1}{2}\right)z_1 + \left(\frac{\pi}{2} + \frac{1}{2}\right)z_2}, \quad f_2(z_1, z_2) = \frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{2}} i - \frac{1}{\sqrt{2}} e^{\left(\frac{\pi}{2} + \frac{1}{2}\right)z_1 + \left(\frac{\pi}{2} + \frac{1}{2}\right)z_2}.\]
Thus, \((f_1, f_2)\) is a pair of finite order transcendental entire solution of system (1.7) with \(\lambda_1 = 1, \lambda_2 = -1, \mu = \sqrt{2}, \alpha = 1, \beta = 1, \) and \((\varrho_1, \varrho_2) = (\pi i, \pi i)\).

**Example 2.10.** Let
\[f_1(z_1, z_2) = \frac{1}{2\sqrt{3}} e^{\left(\frac{\pi}{2} i + \frac{1}{2}\right)z_1 + \left(\frac{\pi}{2} i + \frac{1}{2}\right)z_2} + \frac{z_1}{2\sqrt{3}} e^{-\frac{\pi}{2} i + \frac{1}{2}} - \cos \left(\frac{\pi i}{\kappa}(z_1 - z_2)\right) e^{-\frac{\pi}{2} i + \frac{1}{2}} e^{(\pi i + A)(z_1 - z_2)}, \]
\[f_2(z_1, z_2) = \frac{1}{2\sqrt{3}} e^{\left(\frac{\pi}{2} i + \frac{1}{2}\right)z_1 + \left(\frac{\pi}{2} i + \frac{1}{2}\right)z_2} + \frac{z_1}{2\sqrt{3}} e^{-\frac{\pi}{2} i + \frac{1}{2}} - \cos \left(\frac{\pi i}{\kappa}(z_1 - z_2)\right) e^{-\frac{\pi}{2} i - \frac{1}{2}} e^{(\pi i + A)(z_1 - z_2)}, \]
where
\[A = \frac{\sqrt{3}}{2} + \frac{1}{2} i + \frac{\pi}{3\sqrt{3}} i, \quad \kappa = \sqrt{3} \left[\frac{9}{10} - \frac{\sqrt{3} + 4\pi}{10} i\right].\]
Thus, \((f_1, f_2)\) is a pair of finite order transcendental entire solution of system (1.7) with \(\lambda_1 = \sqrt{3}, \lambda_2 = \sqrt{3}, \mu = \frac{\sqrt{3}}{2}, \alpha = 1, \beta = \frac{1}{2} + \frac{\sqrt{3}}{2} i, \) and
\[(\varrho_1, \varrho_2) = \left\{\frac{3}{2} + \frac{\sqrt{3}}{2} i, \frac{3}{5} + \left(\frac{4\sqrt{3} + 2\pi}{5}\right) i\right\}.\]

For \(\alpha = 1\) and \(\beta = 0\) in system (1.7), we have

**Corollary 2.2.** Let \(c = (\varrho_1, \varrho_2) \in \mathbb{C}^2, \mu, \lambda_1, \lambda_2\) be nonzero constants, and \(\lambda_1 \varrho - \lambda_2 \varrho \neq 0.\) If \((f_1, f_2)\) are a pair of finite order transcendental entire solution of the following system:
\[
\mu f_2(z) + \lambda_1 \frac{\partial f_1}{\partial z_1} + \lambda_2 \frac{\partial f_1}{\partial z_2} + f_2(z + c)^2 = 1,
\]

(2.12)

then, \((f_1, f_2)\) must be of the form

\[
f_1(z_1, z_2) = \frac{1}{2(\lambda_1 a_1 + \lambda_2 a_2 + \mu)} \exp^{a_1 z_1 + a_2 z_2 + b_1} + \frac{1}{2(\mu - \lambda_1 a_1 - \lambda_2 a_2)} \exp^{-a_1 z_1 - a_2 z_2 - b_1},
\]

\[
f_2(z_1, z_2) = \frac{1}{2(\lambda_1 a_1 + \lambda_2 a_2 + \mu)} \exp^{a_1 z_1 + a_2 z_2 + b_1} + \frac{1}{2(\mu - \lambda_1 a_1 - \lambda_2 a_2)} \exp^{-a_1 z_1 - a_2 z_2 - b_1},
\]

where \(a_1, a_2, b_1, b_2 \in \mathbb{C}\) satisfy \((\lambda_1 a_1 + \lambda_2 a_2)^2 = \mu^2 - 1\) and

\[
e^{2(\mu a_1 + a_2 c_2)} = (\mu + \lambda_1 a_1 + \lambda_2 a_2)^2 = \frac{1}{(\mu - \lambda_1 a_1 - \lambda_2 a_2)^2},
\]

\[
e^{2(b_1 - b_2)} = -1,
\]

(2.13)

or

\[
e^{2(\mu a_1 + a_2 c_1)} = -(\mu + \lambda_1 a_1 + \lambda_2 a_2)^2 = -\frac{1}{(\mu - \lambda_1 a_1 - \lambda_2 a_2)^2},
\]

\[
e^{2(b_1 - b_2)} = 1.
\]

(2.14)

### 3 Some lemmas

The following lemmas play the key role in proving our results.

**Lemma 3.1.** [27,28] For an entire function \(F\) on \(\mathbb{C}^n, F(0) \neq 0\) and put \(\rho(n_F) = \rho < \infty\). Then, there exist a canonical function \(f_F\) and a function \(g_F \in \mathbb{C}^n\) such that \(F(z) = f_F(z)e^{g_F(z)}\). For the special case \(n = 1\), \(f_F\) is the canonical product of Weierstrass.

**Remark 3.1.** Here denote \(\rho(n_F)\) to be the order of the counting function of zeros of \(F\).

**Lemma 3.2.** [3] If \(g\) and \(h\) are entire functions on the complex plane \(\mathbb{C}\) and \(g(h)\) is an entire function of finite order, then there are only two possible cases: either

(a) the internal function \(h\) is a polynomial and the external function \(g\) is of finite order; or else

(b) the internal function \(h\) is not a polynomial but a function of finite order, and the external function \(g\) is of zero order.

**Lemma 3.3.** [34, Lemma 3.1] Let \(f_j(\neq 0), j = 1, 2, 3\), be meromorphic functions on \(\mathbb{C}^m\) such that \(f_1\) is not constant, and \(f_1 + f_2 + f_3 = 1\), and such that

\[
\sum_{j=1}^{3} \left[ N_j(r, \frac{1}{f_j}) + 2N(r, f_j) \right] < \lambda T(r, f_1) + O(\log^+ T(r, f_1)),
\]

for all \(r\) outside possibly a set with finite logarithmic measure, where \(\lambda < 1\) is a positive number. Then, either \(f_2 = 1\) or \(f_3 = 1\).

**Remark 3.2.** Here \(N_j(r, \frac{1}{f_j})\) is the counting function of the zeros of \(f_j |z| \leq r\), where the simple zero is counted once, and the multiple zero is counted twice.

**Lemma 3.4.** Let \(c = (c_1, c_2) \in \mathbb{C}^2\) and \(a, \beta\) be two nonzero constants. Let

...
where $\xi$ is a constant, $\theta_j(z_j), j = 1, 2$ are finite order entire functions, and $\psi_j(z_j), j = 1, 2$ are polynomials in $z_j$. If $(g_1, g_2)$ is a pair of solutions of system

\[
\begin{align*}
ag_1(z + c) - \beta g_1(z) &= 0, \\
ag_2(z + c) - \beta g_2(z) &= 0,
\end{align*}
\]

then, $(g_1, g_2)$ can be expressed as the form of

\[
g_1(z) = \theta_1(z) e^{\xi_1 + \psi_1(z_1)}, \quad g_2(z) = \theta_2(z) e^{\xi_2 + \psi_2(z_2)},
\]

where $\theta_j(z_j)$ are finite order entire period function with the period $c_j$ and $\gamma$ is a constant such that

\[
e^{\gamma - \gamma} = \frac{\beta}{\alpha} e^{-\gamma}.
\]

**Proof.** Substituting $g_1, g_2$ into system (3.1), we have

\[
\begin{align*}
e^{\psi_1(z_1 + c_1) - \psi(z_1)} &= \frac{\theta_1(z_1)}{\theta_1(z_1 + c_1)} \frac{\beta}{\alpha} e^{-\gamma}, \\
e^{\psi_1(z_1 + c_1) - \psi_1(z_1)} &= \frac{\theta_1(z_1)}{\theta_1(z_1 + c_1)} \frac{\beta}{\alpha} e^{-\gamma},
\end{align*}
\]

which implies

\[
e^{\psi(z_1 + 2c_1) - \psi(z_1)} = \frac{\theta_1(z_1)}{\theta_1(z_1 + 2c_1)} \frac{\beta}{\alpha} e^{2\gamma}, \quad j = 1, 2. \tag{3.3}
\]

Noting that $\psi_j(z_j)$ is a polynomial, we will consider two cases as follows.

**Case 1.** Suppose that $e^{\psi(z_1 + 2c_1) - \psi(z)}$ is a constant. In view of (3.3), it follows that $\frac{\theta_j(z_j)}{\theta_j(z_j + 2c_j)}$ is a nonzero constant for $j = 1, 2$. Set

\[
\frac{\theta_j(z_j)}{\theta_j(z_j + 2c_j)} = d_j, \quad j = 1, 2. \tag{3.4}
\]

In view of (3.3), we can deduce that

\[
\psi_j(z_j) = h z_j + D_j, \quad j = 1, 2, \tag{3.5}
\]

where $h, D_j, (j = 1, 2)$ are constants and

\[
h = \gamma + \frac{\log d}{2c_2}, \quad \gamma = \frac{-c_1 + \log \frac{\beta}{\alpha}}{c_2}. \tag{3.6}
\]

If $d = 1$, then $\theta_j(z_j)(j = 1, 2)$ are finite order entire period functions with period $2c_2$ and $h = \gamma = \frac{-c_1 + \log \frac{\beta}{\alpha}}{c_2}$.

Substituting (3.5) and (3.6) in (3.2), we have

\[
\theta_j(z_j) e^{D_j} = \theta_1(z_1 + c_1) e^{D_1}.
\]

Thus, in view of (3.5) and (3.6), it follows that

\[
g_1(z) = \theta_1(z_1) e^{\xi_1 + \psi_1(z_1)},
\]

and

\[
g_2(z) = \theta_2(z_2) e^{\xi_2 + \psi_2(z_2)} = \theta_1(z_1 + c_1) e^{\xi_1 + \psi_1(z_1 + D_1)}.
\]

If $d \neq 1$, it follows from (3.4) that

\[
g_1(z) = \theta_1(z_1) e^{\xi_1 + \psi_1(z_1)},
\]

and

\[
g_2(z) = \theta_2(z_2) e^{\xi_2 + \psi_2(z_2)} = \theta_1(z_1 + c_1) e^{\xi_1 + \psi_1(z_1 + D_1)}.
\]
\[ \theta_j(z) = e^{\frac{\log z}{C_j} \phi_j}, \quad j = 1, 2, \]  

(3.7)

where \( \phi_j, j = 1, 2 \) are constants. Substituting (3.5)–(3.7) in (3.2), we have \( e^{D_1-D_1} = e^{\phi_1} \), which implies

\[ e^{D_1+\phi_1-(D_2+\phi_2)} = 1. \]

(3.8)

For convenience, let \( B_1 = D_1 + \phi_1 \) and \( B_2 = D_2 + \phi_2 \). In view of (3.5)–(3.8), it follows that

\[ g_1(z) = \theta_1(z)e^{\phi_{11}+\phi_{12}+D_1} = e^{\phi_{11}+\phi_{12}+B_1}, \]

and

\[ g_2(z) = \theta_2(z)e^{\phi_{21}+\phi_{22}+D_2} = e^{\phi_{21}+\phi_{22}+B_2}, \]

where \( B_1, B_2 \) are constants satisfying (3.8).

**Case 2.** Suppose that \( e^{(\phi_{11}+\phi_{12})Z}+\phi_{12}(j = 1, 2) \) are not constants. Noting that \( \phi_j(z), j = 1, 2, \) are nonconstant polynomials, we can deduce from (3.3) that \( \phi_j(z), j = 1, 2 \) are finite order transcendental entire functions. Thus, there exist two functions \( q_1(z), q_2(z) \) such that

\[ \theta_j(z) = e^{q_j(z)}, \quad j = 1, 2. \]

(3.9)

This leads to

\[ g_1(z) = e^{\phi_{11}+\mu_1(z)}, \quad g_2(z) = e^{\phi_{11}+\mu_2(z)}, \]

(3.10)

where \( \mu_j(z) = \phi_j(z) + q_j(z) \). Substituting \( g_1, g_2 \) in (3.1), we have

\[
\begin{align*}
& e^{\phi_{11}(z_2)+\phi_{12}(z_2)-\mu_1(z_2)} = \frac{\beta}{a} e^{-c_1 \xi}, \\
& e^{\phi_{11}(z_2)+\phi_{12}(z_2)-\mu_2(z_2)} = \frac{\beta}{a} e^{-c_2 \xi},
\end{align*}
\]

(3.11)

which implies that

\[ \mu_1(z_1) = y z_2 + D_j, \quad j = 1, 2, \]

(3.12)

where \( y, D_j, (j = 1, 2) \) are constants and

\[ y = -c_1 \xi + \log \frac{\beta}{a}, \quad e^{D_1-D_2} = 1. \]

(3.13)

In view of (3.10) and (3.12), we have

\[ g_1(z) = e^{\phi_{11}+\phi_{12}+D_1}, \quad g_2(z) = e^{\phi_{11}+\phi_{12}+D_2}, \]

(3.14)

where \( y, D_1, D_2 \) satisfy (3.13).

Therefore, this completes the proof of Lemma 3.4. □

**4 The proof of Theorem 2.1**

**Proof.** Let \((f_1, f_2)\) be a pair of transcendental entire solutions of finite order for system (1.6). Thus, we will consider the following two cases.

(i) If \( \phi f_1(z) + \lambda \frac{\partial f_1}{\partial z_1} \) is a constant. Denote

\[ \mu f_1(z) + \lambda \frac{\partial f_1}{\partial z_1} = \eta_1. \]

(4.1)
In view of (1.6), it follows that
\[ a f_z(z + c) - \beta f_z(z) = \eta_2, \]
where \( \eta_2 \) is a constant satisfying
\[ \eta_1^2 + \eta_2^2 = 1. \]

From (4.1) and (4.2), we have
\[ \mu f_z + \lambda \frac{\partial f_z}{\partial z_1} = \beta \left[ a f_z(z - c) + \lambda \frac{\partial f_z(z - c)}{\partial z_2} \right] + \mu \frac{\eta_2}{\alpha} = \eta_1 \beta + \mu \frac{\eta_2}{\alpha}. \]
This shows that \( \mu f_z + \lambda \frac{\partial f_z}{\partial z_1} \) is a constant. Let
\[ \mu f_z + \lambda \frac{\partial f_z}{\partial z_1} = \delta_1, \]
then \( a f_z(z + c) - \beta f_z(z) = \delta_2, \)
then it follows
\[ \delta_1^2 + \delta_2^2 = 1, \quad \delta_1 = \frac{\beta}{\alpha} \eta_1 + \frac{\mu}{\alpha} \eta_2. \]

Solving equations (4.1) and (4.5), we have
\[ f_1(z_1, z_2) = \frac{\eta_1}{\mu} - \frac{1}{\mu} e^{\bar{\gamma} z_1 + \phi(z_2)}, \quad f_2(z_1, z_2) = \frac{\delta_1}{\mu} - \frac{1}{\mu} e^{\mu z_1 + \phi(z_2)}, \]
where \( \phi(z_2), \phi(z_2) \) are entire functions in \( z_2 \). Substituting (4.8) in (4.2) and (4.6), we have
\[ \begin{align*}
& a \left[ \frac{\delta_1}{\mu} - \frac{1}{\mu} e^{\bar{\gamma} z_1 + \phi(z_2)} \right] - \beta \left[ \frac{\eta_1}{\mu} - \frac{1}{\mu} e^{\mu z_1 + \phi(z_2)} \right] = \eta_2, \\
& a \left[ \frac{\eta_1}{\mu} - \frac{1}{\mu} e^{\mu z_1 + \phi(z_2)} \right] - \beta \left[ \frac{\delta_1}{\mu} - \frac{1}{\mu} e^{\bar{\gamma} z_1 + \phi(z_2)} \right] = \delta_2,
\end{align*} \]
which implies
\[ \begin{align*}
& a \delta_1 - \beta \eta_1 = \mu \eta_2, \\
& e^{\bar{\gamma} z_1 + \phi(z_2)} = \frac{\beta}{\alpha} e^{\phi(z_2)}, \\
& a \eta_1 - \beta \delta_1 = \mu \delta_2, \\
& e^{\phi(z_1 + z_2)} = \frac{\beta}{\alpha} e^{\phi(z_2)}. \quad (4.9)
\end{align*} \]

Thus, it yields that
\[ \phi_1(z_2) = y z_2 + D_1, \quad \phi_2(z_2) = y z_2 + D_2, \]
where \( y, D_1, D_2 \) are constants satisfying
\[ e^{2(D_1 - D_2)} = 1, \quad y = \frac{\mu \delta_1 - \log \frac{\beta}{\alpha} + k \pi i}{\bar{\gamma}}, \quad k \in \mathbb{Z}. \]

Moreover, it follows from (7.7) and (4.9) that
\[ \eta_1 = \frac{\beta}{\alpha} \delta_1 + \frac{\mu}{\alpha} \delta_2, \quad (\delta_2^2 - \eta_2^2) [\mu^2 - (a^2 - \beta^2)] \frac{a^2 - \beta^2}{\mu^2} = 0. \]
(a) If \( \delta_2 = \eta_2 \), it follows from (4.7) and (4.9) that
\[
\delta_1 = \pm \eta_1, \quad (\eta_1 - \delta_1)\left(1 + \frac{\beta}{a}\right) = 0.
\] (4.13)

If \( \delta_1 = \eta_1 \) and \( a = \beta \), then \( \delta_2 = \eta_2 = 0 \) and \( \delta_1 = \pm 1 \).

If \( \delta_1 = \eta_1 \) and \( a \neq \beta \), then it follows from (4.12) that \( \eta_2 = \frac{a - \beta}{\eta_1} \).
Substituting this into (4.3), we have
\[
\delta_1^2 = \eta_1^2 = \frac{\mu^2}{(a - \beta)^2}.
\] If \( \delta_1 = -\eta_1 \), then it follows from (4.13) that \( a = -\beta \). Thus, we can deduce from (4.7) or (4.12) that \( \delta_2 = \eta_2 = 0 \), which implies that \( \delta_1 = -\eta_1 = 1 \) or \( \delta_1 = -\eta_1 = -1 \).

(b) If \( \delta_2 = -\eta_2 \), it follows from (4.2), (4.7), and (4.9) that
\[
\delta_1 = \pm \eta_1, \quad (\eta_1 + \delta_1)\left(1 - \frac{\beta}{a}\right) = 0.
\] (4.14)

If \( \delta_1 = -\eta_1 \) and \( a = -\beta \), then \( \delta_2 = \eta_2 = 0 \). Then, it yields that \( \delta_1 = -\eta_1 = 1 \) or \( \delta_1 = -\eta_1 = -1 \).

If \( \delta_1 = -\eta_1 \) and \( a \neq -\beta \), then it follows from (4.12) that \( \eta_2 = \frac{-a + \beta}{\eta_1} \).
Substituting this into (4.3), we have
\[
\delta_1^2 = \eta_1^2 = \frac{\mu^2}{(a + \beta)^2}.
\]
If \( \delta_1 = \eta_1 \), then it follows from (4.14) that \( a = \beta \). Thus, we can deduce from (4.7) or (4.12) that \( \delta_2 = \eta_2 = 0 \), which implies that \( \delta_1 = \eta_1 = \pm 1 \).

(c) If \( \mu^2 = (a^2 - \beta^2) \), it follows from \( \mu \neq 0 \) that \( a \neq \pm \beta \). Then we have
\[
\alpha \delta_2 = -\beta \eta_2 \pm \sqrt{(\eta_2^2 - 1)(\beta^2 - a^2)}.
\] (4.15)

Therefore, from (4.8), (4.10), (4.11) and (a), (b), (c), we obtain the conclusion (i) of Theorem 2.1.

(ii) If \( \mu f_1(z) + \lambda \frac{\partial f_1}{\partial z_1} \) is a nonconstant, then it yields that \( af_1(z + c) - \beta f_1(z) \), \( \mu f_2 + \lambda \frac{\partial f_2}{\partial z_1} \), and \( af_1(z + c) - \beta f_2(z) \) are all nonconstant. Otherwise, if one of these terms is a constant, we can deduce that \( \mu f_1(z) + \lambda \frac{\partial f_1}{\partial z_1} \) is a constant. This is a contradiction. Thus, we can rewrite (1.6) as the form
\[
\begin{bmatrix}
\mu f_1(z) + \lambda \frac{\partial f_1}{\partial z_1} + i(af_2(z + c) - \beta f_1(z)) \\
\mu f_1(z) + \lambda \frac{\partial f_1}{\partial z_1} - i(af_2(z + c) - \beta f_1(z))
\end{bmatrix} = \begin{bmatrix}
\mu f_2(z) + \lambda \frac{\partial f_2}{\partial z_1} + i(af_1(z + c) - \beta f_2(z)) \\
\mu f_2(z) + \lambda \frac{\partial f_2}{\partial z_1} - i(af_1(z + c) - \beta f_2(z))
\end{bmatrix} = 1.
\]

Since \( f_1, f_2 \) are entire functions, it follows that \( \mu f_1(z) + \lambda \frac{\partial f_1}{\partial z_1} + i(af_2(z + c) - \beta f_1(z)) \), \( \mu f_1(z) + \lambda \frac{\partial f_1}{\partial z_1} - i(af_2(z + c) - \beta f_1(z)) \), \( \mu f_2(z) + \lambda \frac{\partial f_2}{\partial z_1} + i(af_1(z + c) - \beta f_2(z)) \) and \( \mu f_2(z) + \lambda \frac{\partial f_2}{\partial z_1} - i(af_1(z + c) - \beta f_2(z)) \) do not exist zeros and poles. By Lemmas 3.1 and 3.2, there exist two nonconstant polynomials \( p(z), q(z) \) in \( \mathbb{C}^2 \) such that
\[
\begin{align*}
\mu f_1(z) + \lambda \frac{\partial f_1}{\partial z_1} + i(af_2(z + c) - \beta f_1(z)) &= e^{p(z)}, \\
\mu f_1(z) + \lambda \frac{\partial f_1}{\partial z_1} - i(af_2(z + c) - \beta f_1(z)) &= e^{-p(z)}, \\
\mu f_2(z) + \lambda \frac{\partial f_2}{\partial z_1} + i(af_1(z + c) - \beta f_2(z)) &= e^{q(z)}, \\
\mu f_2(z) + \lambda \frac{\partial f_2}{\partial z_1} - i(af_1(z + c) - \beta f_2(z)) &= e^{-q(z)}.
\end{align*}
\]
The above equations lead to
\[ \mu f_1(z) + \frac{\partial f_1}{\partial z_1} = \frac{1}{2}(e^p + e^{-p}), \quad (4.16) \]
\[ a f_2(z + c) - \beta f_2(z) = \frac{1}{2i}(e^p - e^{-p}), \quad (4.17) \]
\[ \mu f_2(z) + \frac{\partial f_2}{\partial z_2} = \frac{1}{2}(e^q + e^{-q}), \quad (4.18) \]
\[ a f_1(z + c) - \beta f_1(z) = \frac{1}{2i}(e^q - e^{-q}). \quad (4.19) \]

In view of (4.17) and (4.18), we can deduce that
\[ a f_2(z + c) - \beta f_2(z) = \frac{a}{2i}\left(e^{(z+c)+p(z)} - e^{(z+c)-p(z)}\right). \quad (4.20) \]

In view of (4.16) and (4.17), we have
\[ a f_1(z + c) - \beta f_1(z) = \frac{b}{2i}\left(e^{(z+c)+p(z)} - e^{(z+c)-p(z)}\right). \quad (4.21) \]

By combining with (4.20) and (4.21), we have
\[ \frac{\lambda a}{\partial z_1} + \frac{\beta i + \mu}{\partial z_1} e^{(z+c)+p(z)} + \frac{\lambda b}{\partial z_1} - \frac{\beta i - \mu}{\partial z_1} e^{(z+c)-p(z)} - e^{2q(z+c)} \equiv 1. \quad (4.22) \]

Similar to the above argument, we can deduce from (4.16), (4.18), and (4.19) that
\[ \frac{\lambda a}{\partial z_2} + \frac{\beta i + \mu}{\partial z_2} e^{(z+c)+q(z)} + \frac{\lambda b}{\partial z_2} - \frac{\beta i - \mu}{\partial z_2} e^{(z+c)-q(z)} - e^{2q(z+c)} \equiv 1. \quad (4.23) \]

By Lemma 3.3, we can deduce from (4.22) and (4.23) that
\[ \frac{\lambda a}{\partial z_1} + \frac{\beta i - \mu}{\partial z_1} e^{(z+c)-p(z)} \equiv 1 \quad \text{or} \quad \frac{\lambda b}{\partial z_1} + \frac{\beta i + \mu}{\partial z_1} e^{(z+c)+p(z)} \equiv 1, \]

and
\[ \frac{\lambda a}{\partial z_2} + \frac{\beta i + \mu}{\partial z_2} e^{(z+c)-q(z)} \equiv 1 \quad \text{or} \quad \frac{\lambda b}{\partial z_2} + \frac{\beta i - \mu}{\partial z_2} e^{(z+c)+q(z)} \equiv 1. \]

Now we will consider four cases as follows.

**Case 1.**

\[
\begin{align*}
\frac{\lambda a}{\partial z_1} + \frac{\beta i - \mu}{\partial z_1} e^{(z+c)-p(z)} & \equiv 1, \\
\frac{\lambda b}{\partial z_2} + \frac{\beta i + \mu}{\partial z_2} e^{(z+c)-q(z)} & \equiv 1.
\end{align*}
\quad (4.24)
\]

In view of (4.24), it follows that \( q(z + c) - p(z) = d_1 \) and \( p(z + c) - q(z) = d_2 \), where \( d_1, d_2 \) are constants in \( C \).

Thus, it yields that \( q(z + 2c) - q(z) = d_1 + d_2 \) and \( p(z + 2c) - p(z) = d_1 + d_2 \). Since \( p, q \) are polynomials in \( C^2 \), it follows that \( p(z) = L(z) + H(c_2z_1 - c_2z_2) + b_1 \) and \( q(z) = L(z) + H(c_2z_1 - c_2z_2) + b_2 \), where \( L(z) \) is a linear form of \( L(z) = a_2z_1 + a_2z_2 \), \( H(s) \) is a polynomial in \( s = c_2z_1 - c_2z_2, a_1, a_2, b_1, b_2 \) are constants. Substituting \( p(z), q(z) \) into (4.24), we have
\[
\frac{\lambda a_1 + \lambda c H' + \beta i - \mu}{ai} e^{L(c)+b_2-b_1} \equiv 1, \quad \frac{\lambda a_1 + \lambda c H' + \beta i - \mu}{ai} e^{L(c)+b_2-b_1} \equiv 1. \tag{4.25}
\]

By combining with \( \lambda \neq 0 \) and \( c \neq 0 \), it follows from (4.25) that \( \deg H \leq 1 \). Thus, we still write \( p(z) \), \( q(z) \) as the forms of \( p(z) = L(z) + b_1 \) and \( q(z) = L(z) + b_2 \). In view of (4.22)–(4.24), we have

\[
\begin{align*}
\frac{\lambda a_1 + \beta i + \mu}{ai} e^{L(c)+b_2-b_1} & \equiv 1, \\
\frac{\lambda a_1 + \beta i + \mu}{ai} e^{L(c)+b_2-b_1} & \equiv 1, \\
\frac{\lambda a_1 + \beta i - \mu}{ai} e^{L(c)+b_2-b_1} & \equiv 1, \\
\frac{\lambda a_1 + \beta i - \mu}{ai} e^{L(c)+b_2-b_1} & \equiv 1.
\end{align*}
\tag{4.26}
\]

Thus, we can deduce from (4.26) that

\[
(\lambda a_1 + \beta i)^2 = \mu^2 - a^2, \quad e^{2(b_2-b_1)} = 1, \quad e^{2L(c)} = \frac{-(\lambda a_1 + \beta i + \mu)^2}{(\lambda a_1 + \beta i - \mu)^2} = -a^2.
\tag{4.27}
\]

If \( a_1 \neq \pm \frac{\mu}{\lambda} \), solving equations (4.16) and (4.18), we have

\[
\begin{align*}
f_1(z_1, z_2) = \frac{e^{a_1 z_1 + a_2 z_2 + b_1}}{2(\lambda a_1 + \mu)} - \frac{e^{-a_1 z_1 - a_2 z_2 - b_1}}{2(\lambda a_1 - \mu)} - \partial(z_2)e^{-\frac{\mu}{\lambda}z_2 + \phi_1(z_2)}, \\
f_2(z_1, z_2) = \frac{e^{a_1 z_1 + a_2 z_2 + b_1}}{2(\lambda a_1 + \mu)} - \frac{e^{-a_1 z_1 - a_2 z_2 - b_1}}{2(\lambda a_1 - \mu)} - \partial(z_2)e^{-\frac{\mu}{\lambda}z_2 + \phi_2(z_2)},
\end{align*}
\tag{4.28}
\]

where \( \partial(z_2), \phi_1(z_2) \) are finite order entire functions and \( \phi_1(z_2), \phi_2(z_2) \) are polynomials in \( z_2 \). Substituting (4.28) and (4.29) in (4.17) and (4.19), and combining with (4.26) and (4.27), by Lemma 3.4, we have

\[
\begin{align*}
f_1(z_1, z_2) = \frac{e^{a_1 z_1 + a_2 z_2 + b_1}}{2(\lambda a_1 + \mu)} - \frac{e^{-a_1 z_1 - a_2 z_2 - b_1}}{2(\lambda a_1 - \mu)} - \partial(z_2)e^{-\frac{\mu}{\lambda}z_1 y_1 + z_1 D_1}, \\
f_2(z_1, z_2) = \frac{e^{a_1 z_1 + a_2 z_2 + b_2}}{2(\lambda a_1 + \mu)} - \frac{e^{-a_1 z_1 - a_2 z_2 - b_2}}{2(\lambda a_1 - \mu)} - \partial(z_2 + c_2)e^{-\frac{\mu}{\lambda}z_1 y_1 + z_2 D_2},
\end{align*}
\tag{4.30}
\]

where \( \partial(z_2) \) is a finite order period entire function with period \( 2c_2 \), and \( y, D_1, D_2 \) satisfy (4.11) and (4.27).

If \( a_1 = \pm \frac{\mu}{\lambda} \), solving equations (4.16) and (4.18), similar to the argument as in case \( a_1 \neq \pm \frac{\mu}{\lambda} \), we have

\[
\begin{align*}
f_1(z_1, z_2) = \frac{1}{4\lambda} e^{a_1 z_1 + a_2 z_2 + b_1} + \frac{Z_1}{2\lambda} e^{-a_1 z_1 - a_2 z_2 - b_1} - \partial(z_2)e^{-\frac{\mu}{\lambda}z_1 y_1 + z_1 D_1}, \\
f_2(z_1, z_2) = \frac{1}{4\lambda} e^{a_1 z_1 + a_2 z_2 + b_2} + \frac{Z_2}{2\lambda} e^{-a_1 z_1 - a_2 z_2 - b_2} - \partial(z_2 + c_2)e^{-\frac{\mu}{\lambda}z_1 y_1 + z_2 D_2},
\end{align*}
\tag{4.32}
\]

where \( \partial(z_2) \) is a finite order period entire function with period \( 2c_2 \), and \( y, D_1, D_2 \) satisfy (4.11) and (4.27).

Substituting (4.32) and (4.33) in (4.17) and (4.19), and combining with (4.26) and (4.27), we have \( \beta c_1 = \lambda i \).

If \( a_1 = -\frac{\mu}{\lambda} \), solving equations (4.16) and (4.18), we have

\[
\begin{align*}
f_1(z_1, z_2) = \frac{1}{4\lambda} e^{-a_1 z_1 - a_2 z_2 - b_1} + \frac{Z_1}{2\lambda} e^{a_1 z_1 + a_2 z_2 + b_1} - \partial(z_2)e^{-\frac{\mu}{\lambda}z_1 y_1 + z_1 D_1}, \\
f_2(z_1, z_2) = \frac{1}{4\lambda} e^{-a_1 z_1 - a_2 z_2 - b_2} + \frac{Z_2}{2\lambda} e^{a_1 z_1 + a_2 z_2 + b_2} - \partial(z_2 + c_2)e^{-\frac{\mu}{\lambda}z_1 y_1 + z_2 D_2},
\end{align*}
\tag{4.34}
\]

where \( \partial(z_2) \) is a finite order period entire function with period \( 2c_2 \), and \( y, D_1, D_2 \) satisfy (4.11) and (4.27). Similar to the above argument, we have \( \beta c_1 = -\lambda i \).
Case 2.

\[
\begin{align*}
\begin{cases}
\lambda \frac{\partial p}{\partial z} + \beta i - \mu e^{q(z+c)-p(z)} \equiv 1, \\
\lambda \frac{\partial q}{\partial z} + \beta i + \mu e^{p(z+c)+q(z)} \equiv 1.
\end{cases}
\end{align*}
\]

Thus, it follows that \( q(z+c) - p(z) = d_1 \) and \( p(z+c) + q(z) = d_2 \), where \( d_1, d_2 \) are constants. Hence, we have \( q(z+2c) + q(z) = d_1 + d_2 \), which is a contradiction with the assumption of \( q(z) \) being nonconstant polynomial in \( \mathbb{C}^2 \).

Case 3.

\[
\begin{align*}
\begin{cases}
\lambda \frac{\partial p}{\partial z} + \beta i + \mu e^{q(z+c)+p(z)} \equiv 1, \\
\lambda \frac{\partial q}{\partial z} + \beta i - \mu e^{p(z+c)-q(z)} \equiv 1.
\end{cases}
\end{align*}
\]

Thus, it follows that \( q(z+c) + p(z) = d_1 \) and \( p(z+c) - q(z) = d_2 \), where \( d_1, d_2 \) are constants. Hence, we have \( p(z+2c) + p(z) = d_1 + d_2 \), which is a contradiction with the assumption of \( p(z) \) being nonconstant polynomial in \( \mathbb{C}^2 \).

Case 4.

\[
\begin{align*}
\begin{cases}
\lambda \frac{\partial p}{\partial z} + \beta i + \mu e^{q(z+c)+p(z)} \equiv 1, \\
\lambda \frac{\partial q}{\partial z} + \beta i + \mu e^{p(z+c)+q(z)} \equiv 1.
\end{cases}
\end{align*}
\]

In view of (4.36), it follows that \( q(z+c) + p(z) = d_1 \) and \( p(z+c) + q(z) = d_2 \), where \( d_1, d_2 \) are constants in \( \mathbb{C} \). Thus, it yields that \( q(z+2c) - q(z) = d_1 - d_2 \) and \( p(z+2c) - p(z) = d_1 - d_2 \). Since \( p, q \) are polynomials in \( \mathbb{C}^2 \), it follows that \( p(z) = L(z) + H(c_2 z_1 - c_2 z_2 + b_1) \) and \( q(z) = -L(z) - H(c_2 z_1 - c_2 z_2 + b_2) \), where \( L(z) \) is a linear form of \( L(z) = a_1 z_1 + a_2 z_2 \), \( H(s) \) is a polynomial in \( s = c_2 z_1 - c_2 z_2, a_1, a_2, b_1, b_2 \) are constants. Similar to the argument as in Case 1, we can obtain that \( p(z) = L(z) + b_1 = a_1 z_1 + a_2 z_2 + b_1 \) and \( q(z) = -L(z) + b_2 = -a_1 z_1 - a_2 z_2 + b_2 \). In view of (4.22), (4.23), and (4.36), it follows

\[
\begin{align*}
\begin{cases}
\lambda a_1 + \beta i + \mu e^{-L(z) + b_1} \equiv 1, \\
-\lambda a_1 + \beta i + \mu e^{L(z) - b_2} \equiv 1, \\
\lambda a_1 + \beta i - \mu e^{-L(z) - b_1} \equiv 1, \\
-\lambda a_1 + \beta i - \mu e^{L(z) + b_2} \equiv 1.
\end{cases}
\end{align*}
\]

Thus, it leads to

\[
\begin{align*}
\frac{-\lambda a_1 + \beta i + \mu}{a i} - \frac{-\lambda a_1 + \beta i - \mu}{a i} &= \frac{\lambda a_1 + \beta i + \mu}{a i} - \frac{\lambda a_1 + \beta i - \mu}{a i}.
\end{align*}
\]

By combining with \( a \neq 0, \beta \neq 0, \) and \( \lambda \neq 0 \), we have \( a_i = 0 \). Then, \( p(z) = a_2 z_2 + b_1 \) and \( q(z) = -a_2 z_2 + b_2 \). In view of (4.37), it follows

\[
\mu^2 = a^2 - \beta^2, \quad e^{2a \sqrt{a_i}} \equiv 1, \quad e^{2(b_1 + b_2)} = \frac{-a^2}{(\beta i + \mu)^2} = \frac{(\beta i - \mu)^2}{-a^2}.
\]

Solving equations (4.16) and (4.18), we have
\[ f_1(z_1, z_2) = \frac{1}{2\mu}(e^{a_1z_2+b_1} + e^{-a_1z_2-b_1}) - \partial_1(z_2)e^{\mu z_1}\phi_1(z_2), \]
\[ f_2(z_1, z_2) = \frac{1}{2\mu}(e^{a_2z_2+b_2} + e^{-a_2z_2-b_2}) - \partial_2(z_2)e^{\mu z_1}\phi_2(z_2), \]
where \( \partial_1(z_2), \partial_2(z_2) \) are finite order entire functions and \( \phi_1(z_2), \phi_2(z_2) \) are polynomials in \( z_2 \). Similar to the above argument in Case 1, it follows from (4.39) and (4.40) that
\[ f_1(z_1, z_2) = \frac{1}{2\mu}(e^{a_1z_2+b_1} + e^{-a_1z_2-b_1}) - \partial_1(z_2)e^{\mu z_1}\phi_1(z_2), \]
\[ f_2(z_1, z_2) = \frac{1}{2\mu}(e^{a_2z_2+b_2} + e^{-a_2z_2-b_2}) - \partial_2(z_2)e^{\mu z_1}\phi_2(z_2), \]
where \( \partial_1(z_2) \) is a finite order period entire function with period 2\( D_2 \), and \( \gamma_1, \gamma_2 \) satisfy (4.11) and (4.27). In fact, we can see that the forms of solutions are included in case that \( \alpha \neq \pm \frac{\beta}{\mu} \) in Case 1.

Therefore, this completes the proof of Theorem 2.1. \( \square \)

5 The proof of Theorem 2.2

**Proof.** Let \((f_1, f_2)\) be a pair of transcendental entire solutions of finite order for system (1.7). Thus, we will consider the following two cases.

(i) If \( \mu f_1(z) + \lambda_1 \frac{\partial f_1}{\partial z_1} + \lambda_2 \frac{\partial f_1}{\partial z_2} \) is a constant. Denote
\[ \mu f_1(z) + \lambda_1 \frac{\partial f_1}{\partial z_1} + \lambda_2 \frac{\partial f_1}{\partial z_2} = \eta_1, \] (5.1)
In view of (1.7), it follows that
\[ \alpha f_2(z + c) - \beta f_1(z) = \eta_2, \] (5.2)
where \( \eta_2 \) is a constant satisfying (4.3).

From (5.1) and (5.2), we have
\[ \mu f_2 + \lambda_1 \frac{\partial f_2}{\partial z_1} + \lambda_2 \frac{\partial f_2}{\partial z_2} = \frac{\beta}{a} \left[ \mu f_1(z - c) + \lambda_1 \frac{\partial f_1(z - c)}{\partial z_1} + \lambda_2 \frac{\partial f_1(z - c)}{\partial z_2} \right] + \frac{\eta_2}{a} \] (5.3)
\[ = \eta_1 + \frac{\beta}{a}. \]
This shows that \( \mu f_2 + \lambda_1 \frac{\partial f_2}{\partial z_1} + \lambda_2 \frac{\partial f_2}{\partial z_2} \) is a constant. Let
\[ \mu f_2 + \lambda_1 \frac{\partial f_2}{\partial z_1} + \lambda_2 \frac{\partial f_2}{\partial z_2} = \delta_1, \] (5.4)
then \( \alpha f_1(z + c) - \beta f_2(z) \) is a constant. Denote
\[ \alpha f_1(z + c) - \beta f_2(z) = \delta_2, \] (5.5)
then, we have (4.7). The characteristic equations of (5.1) are
\[ \frac{dz_1}{dt} = \lambda_1, \quad \frac{dz_2}{dt} = \lambda_2, \quad \frac{df_1}{dt} = \eta_1 - \mu f_1. \]
Using the initial conditions: \( z_1 = 0, z_2 = s_1, \) and \( f_1 = f_1(0, s_1) = \psi(s_1) \) with a parameter \( s \), we obtain the following parametric representation for the solutions of the characteristic equations: \( z_1 = \lambda_1 t, z_2 = \lambda_2 t + s_1, \) and
\[ f_1(z_1, z_2) = \frac{\eta_1}{\mu} - \frac{1}{\mu} e^{\frac{\mu}{2} z_1 + \varphi_1(s_1)}, \quad (5.6) \]

where \( \varphi_1(s_1) \) is an entire function in \( s_1 = \lambda_1 z_1 - \lambda_2 z_2 \). Similarly, solving equation (5.4), we have

\[ f_2(z_1, z_2) = \frac{\delta_1}{\mu} - \frac{1}{\mu} e^{\frac{\mu}{2} z_1 + \varphi_2(s_1)}, \quad (5.7) \]

where \( \varphi_2(s_1) \) is an entire function in \( s_1 = \lambda_1 z_1 - \lambda_2 z_2 \). Substituting (5.6) and (5.7) in (5.2) and (5.5), we have

\[
\begin{align*}
\alpha \delta_1 - \beta \eta_1 &= \mu \eta_2, \\
\frac{e^{\varphi_2(s_1 + s_2)}}{\varphi_1(s_1)} &= \frac{\beta}{\alpha} e^{\xi_1}, \\
\alpha \eta_1 - \beta \delta_1 &= \mu \delta_2, \\
\frac{e^{\varphi_1(s_1 + s_2)}}{\varphi_2(s_1)} &= \frac{\beta}{\alpha} e^{\xi_1}.
\end{align*}
\]

which implies

\[
\left\{ \begin{array}{l}
\alpha \delta_1 - \beta \eta_1 = \mu \eta_2, \\
\frac{e^{\varphi_2(s_1 + s_2)}}{\varphi_1(s_1)} = \frac{\beta}{\alpha} e^{\xi_1}, \\
\alpha \eta_1 - \beta \delta_1 = \mu \delta_2, \\
\frac{e^{\varphi_1(s_1 + s_2)}}{\varphi_2(s_1)} = \frac{\beta}{\alpha} e^{\xi_1}.
\end{array} \right. \quad (5.8)
\]

Thus, it yields that

\[ \varphi_1(s_1) = \gamma s_1 + D_1, \quad \varphi_2(s_1) = \gamma s_1 + D_2, \quad (5.9) \]

where \( D_1, D_2 \) are constants and satisfying

\[ e^{2(D_1 - D_2)} = 1, \quad \gamma = \frac{\mu}{\lambda_1 c_1 - \log \frac{\alpha}{\beta} + k \pi i}{\lambda_2 q_1 - \lambda_4 q_2}, \quad k \in \mathbb{Z}. \quad (5.10) \]

Moreover, from (4.7) and (5.8), we have (4.12). Thus, from (5.6), (5.7), and (5.9), by using the same argument as in the proof of Theorem 2.1 (i), we can obtain the conclusions of Theorem 2.2 (i).

(ii) If \( \mu f_1(z) + \frac{\lambda_1 \frac{\partial f_1}{\partial z_1}}{\partial z_1} + \frac{\lambda_2 \frac{\partial f_1}{\partial z_2}}{\partial z_2} \) is a nonconstant, then it yields that \( a f_2(z + c) - \beta f_1(z) \), \( \mu f_2 + \frac{\lambda_1 \frac{\partial f_2}{\partial z_1}}{\partial z_1} + \frac{\lambda_2 \frac{\partial f_2}{\partial z_2}}{\partial z_2} \) and \( \mu f_1(z + c) - \beta f_2(z) \) are all nonconstant. Otherwise, if one of these terms is a constant, we can deduce that \( \mu f_1(z) + \frac{\lambda_1 \frac{\partial f_1}{\partial z_1}}{\partial z_1} + \frac{\lambda_2 \frac{\partial f_1}{\partial z_2}}{\partial z_2} \) is a constant. This is a contradiction. Thus, similar to the argument as in the proof of Theorem 2.1 (ii), there exists two nonconstant polynomials \( p(z), q(z) \) in \( \mathbb{C}^2 \) such that

\[ \frac{\lambda_1 \frac{\partial p}{\partial z_1} + \lambda_2 \frac{\partial p}{\partial z_2} + \beta i + \mu}{ai} e^{\varphi_1(s_1 + s_2)} = \frac{\lambda_1 \frac{\partial p}{\partial z_1} + \lambda_2 \frac{\partial p}{\partial z_2} + \beta i - \mu}{ai} e^{\varphi_2(s_1 + s_2)} \equiv 1. \quad (5.11) \]

\[ \frac{\lambda_1 \frac{\partial q}{\partial z_1} + \lambda_2 \frac{\partial q}{\partial z_2} + \beta i + \mu}{ai} e^{\varphi_1(s_1 + s_2)} = \frac{\lambda_1 \frac{\partial q}{\partial z_1} + \lambda_2 \frac{\partial q}{\partial z_2} + \beta i - \mu}{ai} e^{\varphi_2(s_1 + s_2)} \equiv 1. \quad (5.12) \]

In view of (5.11) and (5.14), we have

\[ \frac{\lambda_1 \frac{\partial p}{\partial z_1} + \lambda_2 \frac{\partial p}{\partial z_2} + \beta i + \mu}{ai} e^{\varphi_1(s_1 + s_2)} + \frac{\lambda_1 \frac{\partial q}{\partial z_1} + \lambda_2 \frac{\partial q}{\partial z_2} + \beta i - \mu}{ai} e^{\varphi_2(s_1 + s_2)} = e^{2\varphi_1(s_1 + s_2)} \equiv 1. \quad (5.15) \]
\[ \frac{\lambda_1 \frac{\partial q}{\partial z_1} + \lambda_2 \frac{\partial q}{\partial z_2} + \beta i + \mu}{ai} e^{p(z+c)q(z)} + \frac{\lambda_1 \frac{\partial q}{\partial z_1} + \lambda_2 \frac{\partial q}{\partial z_2} + \beta i - \mu}{ai} e^{p(z+c)-q(z)} - e^{2p(z+c)} \equiv 1. \] (5.16)

By Lemma 3.3, we can deduce from (5.15) and (5.16) that
\[ \frac{\lambda_1 \frac{\partial p}{\partial z_1} + \lambda_2 \frac{\partial p}{\partial z_2} + \beta i - \mu}{ai} e^{q(z+c)-p(z)} \equiv 1 \quad \text{or} \quad \frac{\lambda_1 \frac{\partial p}{\partial z_1} + \lambda_2 \frac{\partial p}{\partial z_2} + \beta i + \mu}{ai} e^{q(z+c)+p(z)} \equiv 1, \]
and
\[ \frac{\lambda_1 \frac{\partial q}{\partial z_1} + \lambda_2 \frac{\partial q}{\partial z_2} + \beta i - \mu}{ai} e^{q(z+c)-q(z)} \equiv 1 \quad \text{or} \quad \frac{\lambda_1 \frac{\partial q}{\partial z_1} + \lambda_2 \frac{\partial q}{\partial z_2} + \beta i + \mu}{ai} e^{q(z+c)+q(z)} \equiv 1. \]

Now we will consider four cases as follows.

**Case 1.**
\[ \begin{aligned}
\frac{\lambda_1 \frac{\partial p}{\partial z_1} + \lambda_2 \frac{\partial p}{\partial z_2} + \beta i - \mu}{ai} e^{q(z+c)-p(z)} \equiv 1, \\
\frac{\lambda_1 \frac{\partial q}{\partial z_1} + \lambda_2 \frac{\partial q}{\partial z_2} + \beta i - \mu}{ai} e^{q(z+c)-q(z)} \equiv 1. 
\end{aligned} \] (5.17)

In view of (5.17), it follows that \( q(z+c) - p(z) = d_1 \) and \( p(z+c) - q(z) = d_2 \), where \( d_1, d_2 \) are constants in \( \mathbb{C} \). Thus, it yields that \( q(z+2c) - q(z) = d_1 + d_2 \) and \( p(z+2c) - p(z) = d_1 + d_2 \). Since \( p, q \) are polynomials in \( \mathbb{C}^2 \), it follows that \( p(z) = L(z) + H(c_1z + c_2z) + b_1 \) and \( q(z) = L(z) + H(c_3z + c_4z) + b_2 \), where \( L(z) \) is a linear form of \( L(z) = a_1z + a_2z \), \( H(z) \) is a polynomial in \( s \) and \( a_1, a_2, b_1, b_2 \) are constants. Substituting \( p(z), q(z) \) in (5.17), we have
\[ \frac{\lambda_1 a_1 + \lambda_2 a_2 + (\lambda_1 c_2 - \lambda_2 c_1)H' + \beta i - \mu}{ai} e^{L(c)+b_1-b_1} \equiv 1, \] (5.18)
\[ \frac{\lambda_1 a_1 + \lambda_2 a_2 + (\lambda_1 c_2 - \lambda_2 c_1)H' + \beta i - \mu}{ai} e^{L(c)+b_1-b_1} \equiv 1. \] (5.19)

By combining with \( \lambda_1 c_2 - \lambda_2 c_1 \neq 0 \), it follows from (5.18) and (5.19) that \( \deg H \leq 1 \). Thus, we still write \( p(z), q(z) \) as the forms of \( p(z) = L(z) + b_1 \) and \( q(z) = L(z) + b_2 \). In view of (5.15)–(5.17), we have
\[ \begin{aligned}
\frac{\lambda_1 a_1 + \lambda_2 a_2 + \beta i + \mu}{ai} e^{-L(c)+b_1-b_1} \equiv 1, \\
\frac{\lambda_1 a_1 + \lambda_2 a_2 + \beta i + \mu}{ai} e^{-L(c)+b_1-b_1} \equiv 1, \\
\frac{\lambda_1 a_1 + \lambda_2 a_2 + \beta i - \mu}{ai} e^{L(c)+b_1-b_1} \equiv 1, \\
\frac{\lambda_1 a_1 + \lambda_2 a_2 + \beta i - \mu}{ai} e^{L(c)+b_1-b_1} \equiv 1. 
\end{aligned} \] (5.20)

Thus, we can deduce from (5.20) that
\[ (\lambda_1 a_1 + \lambda_2 a_2 + \beta i)^2 = \mu^2 - a^2, \quad e^{2b_1-b_1} = 1, \] (5.21)
and
\[ e^{2L(c)} = \frac{-a^2}{(\lambda_1 a_1 + \lambda_2 a_2 + \beta i)^2} = \frac{(\lambda_1 a_1 + \lambda_2 a_2 + \beta i + \mu)^2}{-a^2}. \] (5.22)

If \( \lambda_1 a_1 + \lambda_2 a_2 \neq \pm \mu \), solving equations (5.11) and (5.13), we have
\[ f_1(z, z) = \frac{e^{\lambda_1 a_1 + \lambda_2 a_2 + b_1}}{2(\lambda_1 a_1 + \lambda_2 a_2 + \mu)} - \frac{e^{-\lambda_1 a_1 - \lambda_2 a_2 - b_1}}{2(\lambda_1 a_1 + \lambda_2 a_2 - \mu)} - b(1) e^{z\lambda_1 + \phi(s)}, \] (5.23)
where \( \vartheta(s) \) is a finite order period entire function with period \( s_2 \), and \( \alpha, \beta \) satisfy (5.10), (5.21), and (5.22). If \( \lambda_1 \alpha_1 + \lambda_2 \alpha_2 = -\mu \), solving equations (5.11) and (5.13), we have

\[
f_1(z_1, z_2) = \frac{1}{4\mu} e^{-a_1 z_1^* - a_2 z_2^* + b_1} - \frac{z_1}{2\lambda_1} e^{-a_1 z_1^* - a_2 z_2^* - b_1} - \vartheta(s_1 + s_0) e^{-\frac{\mu}{\pi} \bar{s}_1 y_1 + D_1},
\]

\[
f_2(z_1, z_2) = \frac{1}{4\mu} e^{-a_1 z_1^* - a_2 z_2^* + b_1} + \frac{z_1}{2\lambda_1} e^{-a_1 z_1^* - a_2 z_2^* - b_1} - \vartheta(s_1 + s_0) e^{-\frac{\mu}{\pi} \bar{s}_1 y_1 + D_1},
\]

Thus, it follows that \( q(z + c) - p(z) = d_1 \) and \( q(z + c) + q(z) = d_2 \) where \( d_1, d_2 \) are constants. Hence, we have \( q(z + 2c) + q(z) = d_1 + d_2 \), which is a contradiction with the assumption of \( q(z) \) being nonconstant polynomial in \( \mathbb{C}^2 \).

**Case 3.**

\[
\begin{vmatrix}
\lambda_1 \frac{\partial p}{\partial z_1} + \lambda_2 \frac{\partial p}{\partial z_2} + \beta i + \mu \\
- \frac{\partial q(z + c) + q(z)}{ai} \\
\end{vmatrix} + \begin{vmatrix}
\lambda_1 \frac{\partial q}{\partial z_1} + \lambda_2 \frac{\partial q}{\partial z_2} + \beta i + \mu \\
- \frac{\partial q(z + c) - q(z)}{ai} \\
\end{vmatrix} \equiv 0.
\]

Thus, it follows that \( q(z + c) + p(z) = d_1 \) and \( q(z + c) - p(z) = d_2 \) where \( d_1, d_2 \) are constants. Hence, we have \( q(z + 2c) + q(z) = d_1 + d_2 \), which is a contradiction with the assumption of \( p(z) \) being nonconstant polynomial in \( \mathbb{C}^2 \).
Case 4.

\[
\begin{align*}
\lambda_1 \frac{\partial p}{\partial z_1} + \lambda_2 \frac{\partial p}{\partial z_2} + \beta \frac{\partial q}{\partial z_1} + \mu &= 1, \\
\lambda_1 \frac{\partial q}{\partial z_1} + \lambda_2 \frac{\partial q}{\partial z_2} + \beta \frac{\partial p}{\partial z_1} + \mu &= 1.
\end{align*}
\]  

(5.31)

In view of (5.31), it follows that \( q(z + c) + p(z) = d_1 \) and \( p(z + c) + q(z) = d_2 \), where \( d_1, d_2 \) are constants in \( \mathbb{C} \).

Thus, it yields that \( q(z + 2c) - q(z) = d_1 - d_2 \) and \( p(z + 2c) - p(z) = d_1 - d_2 \). Since \( p, q \) are polynomials in \( \mathbb{C}^2 \), it follows that \( p(z) = L(z) + H(c z_1 - a z_2) + b_1 \) and \( q(z) = -L(z) - H(c z_4 - a z_2) + b_2 \), where \( L(z) \) is a linear form of \( L(z) = a_1 z_1 + a_2 z_2 \), \( H(s) \) is a polynomial in \( s \) and \( a_1, a_2, b_1, b_2 \) are constants. Similar to the argument as in Case 1, we can obtain that \( p(z) = L(z) + b_1 = a_1 z_1 + a_2 z_2 + b_1 \) and \( q(z) = -L(z) + b_2 = a_1 z_4 - a_2 z_2 + b_1 \).

By using the same argument as in Case 4 of Theorem 2.1, we can obtain that the forms of solutions are included in case that \( \lambda_1 a_1 + \lambda_2 a_2 \neq \pm \mu \) in Case 1 of Theorem 2.2.

Therefore, this completes the proof of Theorem 2.2. □

**Acknowledgments:** The authors are very thankful to referees for their valuable comments which improved the presentation of the paper.

**Funding information:** This work was supported by the National Natural Science Foundation of China (12161074), the Talent Introduction Research Foundation of Suqian University (106-CK00042/028), and the Suqian Sci & Tech Program (Grant No. K202009).

**Author contributions:** Conceptualization, H. Y. Xu; writing-original draft preparation, H. Y. Xu, Y. H. Xu and X. L. Liu; writing—review and editing, H. Y. Xu, Y. H. Xu and X. L. Liu; and funding acquisition, H. Y. Xu and Y. H. Xu.

**Conflict of interest:** The authors declare that none of the authors have any competing interests in the manuscript.

**Ethical approval:** The conducted research is not related to either human or animal use.

**Data availability statement:** All data generated or analyzed during this study are included in this published article.

**References**


E. G. Saleebey, On entire and meromorphic solutions of $\lambda u^k + \sum_{i=1}^{m} u_i^n = 1$, Complex. Variables Theory Appl. 49 (2004), 101–107.


B. Q. Li, Entire solutions of $\left(u_{i,j}\right)^m + \left(u_{i,j}\right)^n = e^f$, Nagoya Math. J. 178 (2005), 151–162.


Y. M. Chiang and S. J. Feng, On the Nevanlinna characteristic of $f(z + \eta)$ and difference equations in the complex plane, Ramanujan J. 16 (2008), 105–129.


W. Stoll, Holomorphic Functions of Finite Order in Several Complex Variables, American Mathematical Society, Providence, 1974.


