Research Article

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Generalized Stević-Sharma operators from the minimal Möbius invariant space into Bloch-type spaces

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Abstract: The aim of this study is to investigate the boundedness, essential norm, and compactness of generalized Stević-Sharma operator from the minimal Möbius invariant space into Bloch-type space.

Keywords: generalized Stević-Sharma operator, minimal Möbius invariant space, Bloch-type space, essential norm

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1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{N}$ the set of positive integers. Denote by $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the family of all analytic self-maps of $\mathbb{D}$.

The set of all conformal automorphisms of $\mathbb{D}$ forms a group, called the Möbius group, and is denoted by $\text{Aut}(\mathbb{D})$. It is well known from complex analysis that every element of $\text{Aut}(\mathbb{D})$ has the form $e^{i\theta} \sigma_w(z)$, where $\theta$ is a real number and

$$\sigma_w(z) = \frac{w - z}{1 - wz}, \quad w \in \mathbb{D},$$

is a special automorphism of $\mathbb{D}$ exchanging the points $w$ and 0. Let $X$ be a linear space of analytic functions on $\mathbb{D}$. Then, $X$ is said to be Möbius invariant if for all $f \in X$ and $v \in \text{Aut}(\mathbb{D})$, $f \circ v \in X$ and satisfies that $||f \circ v||_X = ||f||_X$ (see [1]). A typical example of Möbius invariant space is the analytic Besov space $B_p$: Recall that for $1 < p < \infty$, a function $f \in H(\mathbb{D})$ belongs to $B_p$ if

$$\int_\mathbb{D} |f(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty,$$

where $dA$ is the normalized Lebesgue area measure on $\mathbb{D}$. Note that when $p = 2$, $B_2$ is known as the Dirichlet space, which is the only Möbius invariant Hilbert space (see [2]).

The analytic Besov space $B_1$ consists of all $f \in H(\mathbb{D})$, which have a representation as:

$$f(z) = \sum_{n=1}^{\infty} a_n \sigma_n(z),$$

for some sequences $\{a_n\}_{n \in \mathbb{N}} \in l^1$ and $\{\lambda_n\}_{n \in \mathbb{N}}$ in $\mathbb{D}$. The norm in $B_1$ is defined by:

$$||f||_{B_1} = \inf \left\{ \sum_{n=1}^{\infty} |a_n| : f(z) = \sum_{n=1}^{\infty} a_n \sigma_n(z) \right\}.$$
By [1], we know that the space $B_1$ is the minimal Möbius invariant space, as it is contained in any Möbius invariant space. Furthermore, $B_1$ is identical with the set of $f \in H(D)$ for which $f'' \in L^1(D, dA)$, and there exist constants $C_1$ and $C_2$ such that

$$C_1||f||_{B_1} \leq |f(0)| + |f'(0)| + \int_D |f''(z)|dA(z) \leq C_2||f||_{B_1}.$$

For more studies of $B_1$ space, see also [3–8].

Suppose that $\mu$ is a weight, namely, a strictly positive continuous function on $D$. We also assume that $\mu$ is radial: $\mu(z) = \mu(|z|)$ for any $z \in D$. An $f \in H(D)$ is said to belong to the Bloch-type space $B_\mu$, if

$$\sup_{z \in D} \mu(z)|f'(z)| < \infty.$$

$B_\mu$ is a Banach space under the norm $||f||_{B_\mu} = |f(0)| + \sup_{z \in D} \mu(z)|f'(z)|$. When $\mu(z) = 1 - |z|^2$, the induced space $B_{\mu}$ reduces to the classical Bloch space, which is the maximal Möbius invariant space [9]. For some results on the Bloch-type spaces and operators on them, see, for instance, [4,10–14].

Suppose that $\varphi \in S(D)$ and $u \in H(D)$, the composition and multiplication operators on $H(D)$ are defined, respectively, by:

$$C_\varphi f(z) = f(\varphi(z)) \quad \text{and} \quad M_u f(z) = u(z)f(z),$$

where $f \in H(D)$ and $z \in D$. The product of these two operators is known as the weighted composition operator $W_{u,\varphi} = u(z)f(\varphi(z))$. It is important to provide function theoretic characterizations when $\varphi$ and $u$ induce a bounded or compact weighted composition operator on various function spaces. See [7,15] for more research about the (weighted) composition operators acting on several spaces of analytic functions.

The differentiation operator $D$, which is defined by $Df(z) = f'(z)$ for $f \in H(D)$, plays an important role in operator theory and dynamical system.

The first papers on product-type operators including the differentiation operator dealt with the operators $DC_\varphi$ and $C_\varphi D$ (see, for example, [11,16–19]). In [20,21], Stević and co-workers introduced the so-called Stević-Sharma operator as follows:

$$T_{u,\varphi} f(z) = u(z)f(\varphi(z)) + \nu(z)f'(\varphi(z)), \quad f \in H(D),$$

where $u, \nu \in H(D)$ and $\varphi \in S(D)$. By taking some specific choices of the involving symbols, we can easily obtain the general product-type operators:

$$M_u C_\varphi = T_{u,0,\varphi}, \quad C_\varphi M_u = T_{u,\varphi,0}, \quad M_u D = T_{u,u,0}, \quad D M_u = T_{u,u,0}, \quad C_\varphi D = T_{0,1,\varphi},$$

$$DC_\varphi = T_{0,0,\varphi}, \quad M_u C_\varphi D = T_{u,0,0}, \quad M_u D C_\varphi = T_{u,u,0}, \quad C_\varphi M_u D = T_{0,u,0}, \quad C_\varphi D M_u = T_{0,u,0},$$

$$D M_u C_\varphi = T_{0,u,0}, \quad C_\varphi D M_u = T_{u,u,u}, \quad D C_\varphi M_u = T_{0,u,u}.$$

Recently, there has been an increasing interest in studying the Stević-Sharma operator between various spaces of analytic function. For instance, the boundedness, compactness, and essential norm of $T_{u,v,\varphi}$ on the weighted Bergman space were characterized by Stević et al. in [20,21]. Wang et al. in [22] considered the difference of two Stević-Sharma operators and investigated its boundedness, compactness, and order boundedness between Banach spaces of analytic functions. Zhu et al. in [14] provided some necessary and sufficient conditions for $T_{u,v,\varphi}$ to be bounded or compact when considered as an operator from the analytic Besov space $B_\mu$ into Bloch space. Abbasi et al. in [23] generalized the Stević-Sharma operator as follows:

$$T_{u,v,\varphi}^m f(z) = u(z)f(\varphi(z)) + \nu(z)f^{(m)}(\varphi(z)), \quad m \in \mathbb{N},$$

and studied its boundedness, compactness, and essential norm from Hardy space into the $n$th weighted-type space, which was introduced by Stević in [24] (see also [25]). Note that when $m = 1$, we obtain the Stević-Sharma operator $T_{u,v,\varphi}$. Some more related results can be found (see, e.g., [4,5,8,10–14,26–32] and references therein).

Motivated by the aforementioned studies, here we investigate the boundedness and essential norm of the generalized Stević-Sharma operator $T_{u,v,\varphi}^m$ from the minimal Möbius invariant space $B_1$ into the Bloch-type space $B_\mu$. As a corollary, we give the characterizations of its compactness.
Recall that the essential norm of a bounded linear operator \( T : X \to Y \) is the distance from \( T \) to the compact operators \( K : X \to Y \), that is,
\[
\|T\|_{e, X \to Y} = \inf\{\|T - K\|_{X \to Y} : K \text{ is compact}\},
\]
where \( X \) and \( Y \) are the Banach spaces. Note that \( \|T\|_{e, X \to Y} = 0 \) if and only if \( T : X \to Y \) is compact.

Throughout this article, for nonnegative quantities \( X \) and \( Y \), we use the abbreviation \( \lesssim X \) and \( \gtrsim Y \) if there exists a positive constant \( C \) independent of \( X \) and \( Y \) such that \( X \leq CY \). Moreover, we write \( X \approx Y \) if \( \lesssim X \approx Y \).

### 2 Auxiliary results

In this section, we state several auxiliary results that are needed in the proofs of our main results. The following lemma can be found, for example, in [8] (see also [33]).

**Lemma 1.** Let \( k \in \mathbb{N} \), then
\[
|f|_w \lesssim ||f||_{B_1} \quad \text{and} \quad (1 - |z|^2)^k |f^{(k)}(z)| \lesssim ||f||_{B_1}
\]
for each \( f \in B_1 \).

For any \( w \in \mathbb{D} \) and \( j \in \mathbb{N} \), set
\[
f_{j,w}(z) = \frac{(1 - |w|^2)^j}{(1 - |w|^2)^j}, \quad z \in \mathbb{D}.
\]
It is easily seen that \( f_{j,w} \in B_1 \) and \( \sup_{w \in \mathbb{D}} ||f_{j,w}||_{B_1} \leq 1 \) for each \( j \in \mathbb{N} \). Moreover, \( f_{j,w} \) converges to 0 uniformly on compact subsets of \( \mathbb{D} \) as \( |w| \to 1 \).

**Lemma 2.** Let \( m \in \mathbb{N} \) and \( m > 1 \). For any \( w \in \mathbb{D} \setminus \{0\} \) and \( i, k \in \{0, 1, m, m + 1\} \), there exists a function \( g_{i,w} \in B_1 \) such that
\[
g_{i,w}^{(k)}(w) = \frac{\mathbb{W}^k \delta_{ik}}{(1 - |w|^2)^k},
\]
where \( \delta_{ik} \) is the Kronecker delta.

**Proof.** For any \( w \in \mathbb{D} \setminus \{0\} \) and constants \( c_1, c_2, c_3, \) and \( c_4 \), let
\[
g_w(z) = \sum_{j=1}^{4} c_j f_{j,w}(z),
\]
where \( f_{j,w} \) is defined in (1). For each \( i \in \{0, 1, m, m + 1\} \), the system of linear equations
\[
\begin{align*}
g_w^{(0)}(w) &= c_1 + c_2 + c_3 + c_4 = \delta_{i0}, \\
g_w^{(1)}(w) &= (c_1 + 2c_2 + 3c_3 + 4c_4) \frac{\mathbb{W}}{1 - |w|^2} = \frac{\mathbb{W} \delta_{i1}}{1 - |w|^2}, \\
g_w^{(m)}(w) &= \left( m! c_1 + (m + 1)! c_2 + \frac{(m + 2)!}{2} c_3 + \frac{(m + 3)!}{6} c_4 \right) \frac{\mathbb{W}^m}{(1 - |w|^2)^m} = \frac{\mathbb{W}^m \delta_{im}}{(1 - |w|^2)^m}, \\
g_w^{(m+1)}(w) &= \left( (m + 1)! c_1 + (m + 2)! c_2 + \frac{(m + 3)!}{2} c_3 + \frac{(m + 4)!}{6} c_4 \right) \frac{\mathbb{W}^{m+1}}{(1 - |w|^2)^{m+1}} = \frac{\mathbb{W}^{m+1} \delta_{i(m+1)}}{(1 - |w|^2)^{m+1}}.
\end{align*}
\]
has a unique solution \( c_1^i, c_2^i, c_3^i, \) and \( c_4^i \), which is independent of \( w \), since the determinant of the system
For such \( c^j_j \in \{1, 2, 3, 4\} \), the function

\[
g_{i,w}(z) = \sum_{j=1}^{4} c^j_{f_{j,w}}(z)
\]

satisfies the desired result.

By a similar argument, we can obtain the following lemma.

**Lemma 3.** For any \( w \in \mathbb{D} \setminus \{0\} \) and \( i, k \in \{0, 1, 2\} \), there exists a function \( h_{i,w} \in B_1 \) such that

\[
h_{i,w}^{(k)}(z) = \frac{\overline{w}^k \delta_{ik}}{(1 - |w|^2)^k},
\]

where \( \delta_{ik} \) is the Kronecker delta.

In order to estimate the essential norm of \( T^m_{u,v,\varphi} : B_1 \to B_\mu \), we need the following two lemmas. The first one characterizes the compactness in terms of sequential convergence, whose proof is similar to that of [15, Proposition 3.11], so we omit the details.

**Lemma 4.** Let \( m \in \mathbb{N} \), \( u, v \in H(\mathbb{D}) \), and \( \varphi \in S(\mathbb{D}) \). Then, the operator \( T^m_{u,v,\varphi} : B_1 \to B_\mu \) is compact if and only if for each bounded sequence \( \{f_n\}_{n \in \mathbb{N}} \) in \( B_1 \) converges to zero uniformly on compact subsets of \( \mathbb{D} \) as \( n \to \infty \), we have

\[
|T^m_{u,v,\varphi}f_n|_{\mathcal{B}_\mu} \to 0 \text{ as } n \to \infty.
\]

**Lemma 5.** [8] Every bounded sequence in \( B_1 \) has a subsequence that converges uniformly in \( \mathbb{D} \) to a function in \( B_1 \).

### 3 Main results

In this section, we formulate our main results. For simplicity of the expressions, we write

\[
A_j(z) = |u(z)| \varphi(z)|, \quad A_{m}(z) = |v(z)|, \quad A_{m+1}(z) = |v(z)\varphi(z)|.
\]

We first give several characterizations of the generalized Stević-Sharma operator \( T^m_{u,v,\varphi} : B_1 \to B_\mu \) to be bounded.

**Theorem 1.** Let \( u, v \in H(\mathbb{D}) \), \( \varphi \in S(\mathbb{D}) \), \( m \in \mathbb{N} \), \( m > 1 \), and \( \mu \) be a radial weight. Then, the following statements are equivalent.

(i) The operator \( T^m_{u,v,\varphi} : B_1 \to B_\mu \) is bounded.

(ii) \( u \in B_\mu \),

\[
\sum_{j=1}^{4} \sup_{w \in \mathbb{D}} ||T^m_{u,v,\varphi}f_{j,w}||_{\mathcal{B}_\mu} < \infty,
\]

and

\[
\frac{1}{12}m!(m+1)!m^2(m-1)(m+1) \neq 0.
\]
\[ \sum_{i \in \{1, m, m+1\}} \sup_{z \in D} \mu(z)A_i(z) < \infty, \]

where \( f_{j,w} \) are defined in (1).

(iii) \( u \in B_{\mu} \), and

\[ \sum_{i \in \{1, m, m+1\}} \sup_{z \in D} \frac{\mu(z)A_i(z)}{(1 - |\varphi(z)|^2)^i} < \infty. \]

**Proof.** (i) \( \Rightarrow \) (ii). Suppose that \( T_{u,v,\varphi}^m : B_1 \to B_{\mu} \) is bounded. Taking \( f_j(z) = 1 \in B_1 \) we obtain, \( T_{u,v,\varphi}^m f_j = u \in B_{\mu} \), that is,

\[ \sup_{z \in D} \mu(z)|u'(z)| < \infty. \quad (2) \]

For each \( w \in D \) and \( j \in \{1, 2, 3, 4\} \), \( |f_{j,w}|_{B_1} \leq 1 \) and hence by the boundedness of \( T_{u,v,\varphi}^m \) we have \( ||T_{u,v,\varphi}^m f_{j,w}||_{B_{\mu}} < \infty \). Therefore,

\[ \sum_{j=1}^4 \sup_{w \in D} ||T_{u,v,\varphi}^m f_{j,w}||_{B_{\mu}} < \infty. \]

Taking \( f_1(z) = z \in B_1 \) and using the boundedness of \( T_{u,v,\varphi}^m : B_1 \to B_{\mu} \), we obtain

\[ \infty > ||T_{u,v,\varphi}^m f_1||_{B_{\mu}} \geq \sup_{z \in D} \mu(z)(T_{u,v,\varphi}^m f_1)'(z) \]
\[ = \sup_{z \in D} \mu(z)|u'(z)|\varphi(z) + u(z)\varphi'(z)| \]
\[ \geq \sup_{z \in D} \mu(z)|u'(z)|\varphi(z) - \sup_{z \in D} \mu(z)|u'(z)|\varphi(z)|, \]

which along with (2) and the fact that \( |\varphi(z)| < 1 \), it follows that

\[ \sup_{z \in D} \mu(z)|u'(z)| |u(z)| < \infty. \quad (3) \]

Applying the operator \( T_{u,v,\varphi}^m \) for \( f_m(z) = z^m \in B_1 \) yields

\[ \infty > ||T_{u,v,\varphi}^m f_m||_{B_{\mu}} \geq \sup_{z \in D} \mu(z)(T_{u,v,\varphi}^m f_m)'(z) = \sup_{z \in D} \mu(z)|u'(z)\varphi(z)^m + m\varphi(z)|\varphi(z)^{m-1} + m!\varphi(z)|. \]

Using (2), (3), the fact that \( |\varphi(z)| < 1 \), and the triangle inequality, we obtain

\[ \sup_{z \in D} \mu(z)|\varphi(z)^m| < \infty. \quad (4) \]

By choosing \( f_{m+1}(z) = z^{m+1} \in B_1 \), we conclude that

\[ \infty > ||T_{u,v,\varphi}^m f_{m+1}||_{B_{\mu}} \geq \sup_{z \in D} \mu(z)(T_{u,v,\varphi}^m f_{m+1})'(z) \]
\[ = \sup_{z \in D} \mu(z)|u'(z)\varphi(z)^{m+1} + (m + 1)u(z)\varphi(z)|\varphi(z)^m + (m + 1)!\varphi'(z)|\varphi(z) + (m + 1)!\varphi(z)|. \]

By using (2), (3), and (4), in the same manner, we obtain

\[ \sup_{z \in D} \mu(z)|\varphi(z)^m| < \infty. \quad (5) \]

Combining (3), (4), and (5), we deduce that

\[ \sum_{i \in \{1, m, m+1\}} \sup_{z \in D} \mu(z)A_i(z) < \infty. \]

(ii) \( \Rightarrow \) (iii). Assume that (ii) holds. By Lemma 2, for each \( i \in \{1, m+1\} \) and \( \varphi(w) \neq 0 \), there exist constants \( c_1, c_2, c_3, \) and \( c_4 \) such that
\[
g_{j,\phi(w)}(z) = \sum_{j=1}^{4} c_{f_{j,\phi(w)}}(z) \in B_1, 
\]

and
\[
g^{(k)}_{j,\phi(w)}(w) = \frac{\phi(w)^k}{\delta_{k}}. 
\]

where \( f_{j,w} \) are defined in (1) and \( k \in \{0, 1, m, m + 1\} \). Then,
\[
\sum_{j=1}^{\infty} \mu_{w}(w) T_{u,\phi(w)} f_{j,\phi(w)}(w) \leq \sup_{w \in \mathbb{D}} \mu_{w}(w) (w) \cdot |f_{j,\phi(w)}(w)|. 
\]

From (7) and (ii), for each \( i \in \{1, m, m + 1\} \), we have
\[
\sup_{|\phi(w)| \leq \frac{1}{2}} \mu_{w}(w) \cdot A_{i}(w) < \infty.
\]

and
\[
\sup_{|\phi(w)| \leq \frac{1}{2}} \mu_{w}(w) \cdot A_{i}(w) \leq \sup_{w \in \mathbb{D}} \mu_{w}(w) A_{i}(w) < \infty.
\]

Therefore,
\[
\sum_{i=1}^{m} \sup_{w \in \mathbb{D}} \mu_{z}(w) A_{i}(z) < \infty.
\]

(iii) \( \Rightarrow \) (i). Suppose that (iii) holds. For any \( f \in B_1 \), by Lemma 1, we have
\[
\mu(z) |(T_{u,\phi(w)} f)(z)| \leq \mu(z) u(z) |f(\phi(z))| + \sum_{i=1}^{m} \mu(z) A_{i}(z) |f^{(i)}(\phi(z))| \\
\leq \left(\|u\|_{S_p} + \sum_{i=1}^{m} \frac{\mu(z) A_{i}(z)}{(1 - |\phi(z)|^2)^i}\right) \|f\|_{B_1}.
\]

Moreover,
\[
|(T_{u,\phi(w)} f)(0)| = |u(0) f(\phi(0)) + v(0) f^{(m)}(\phi(0))| \leq \left(\|u(0)\| + \frac{|v(0)|}{(1 - |\phi(0)|^2)^m}\right) \|f\|_{B_1}.
\]

Thus, \( T_{u,\phi(w)} : B_1 \to B_{\mu} \) is bounded. The proof is completed. \( \square \)

By using Lemma 3 instead of Lemma 2, the following result may be proved in much the same way as Theorem 1.

**Theorem 2.** Let \( u, v \in H(\mathbb{D}) \), \( \phi \in S(\mathbb{D}) \), and \( \mu \) be a radial weight. Then, the following statements are equivalent.

(i) The operator \( T_{u,\phi} : B_1 \to B_{\mu} \) is bounded.

(ii) \( u \in B_{\mu} \),
\[
\sum_{j=1}^{m} \sup_{w \in \mathbb{D}} \|T_{u,\phi} f_{j,\phi}\|_{S_p} < \infty,
\]

and
\[\sup_{z \in D} |\varphi'(z)| + \sup_{z \in D} |\varphi'(z)| < \infty.\]

(iii) \(u \in B_\mu,\) and
\[\sup_{z \in D} \frac{\mu(z) |u(z)\varphi'(z) + v'(z)|}{1 - |\varphi(z)|^2} + \sup_{z \in D} \frac{\mu(z) |v(z)\varphi'(z)|}{1 - |\varphi(z)|^2} < \infty.
\]

Now, we estimate the essential norm of \(T_{u,v,\varphi}^m\) acting from the minimal Möbius invariant space to the Bloch-type space. Then, we obtain some equivalence conditions for compactness of \(T_{u,v,\varphi}^m\).

**Theorem 3.** Let \(u, v \in H(D), \varphi \in S(D),\) \(m \in \mathbb{N}, m > 1,\) and \(\mu\) be a radial weight such that \(T_{u,v,\varphi}^m : B_1 \to B_\mu\) is bounded. Then,
\[
|T_{u,v,\varphi}^m|_{e,B_1 \to B_\mu} = \sum_{j=1}^{4} \limsup_{|w| \to 1} \|T_{u,v,\varphi}^m f_{j,w} \|_{B_\mu} = \sum_{i=|\{1,m,m+1\}|}^{4} \limsup_{|w| \to 1} \frac{\mu(z) A_i(z)}{1 - |\varphi(z)|^2},
\]
where \(f_{j,w}\) are defined in (1).

**Proof.** We first show that
\[
|T_{u,v,\varphi}^m|_{e,B_1 \to B_\mu} \geq \sum_{j=1}^{4} \limsup_{|w| \to 1} \|T_{u,v,\varphi}^m f_{j,w} \|_{B_\mu}.
\]
It is obvious that for each \(j \in \{1, 2, 3, 4\}\) and \(w \in D,\) \(\|f_{j,w}\|_{B_\mu} \leq 1.\) Moreover, \(f_{j,w}\) converge to zero uniformly on compact subsets of \(D.\) For any compact operator \(K\) from \(B_1\) into \(B_\mu,\) by using some standard arguments (see, e.g., [34,35]), we obtain
\[
\lim_{|w| \to 1} \|Kf_{j,w}\|_{B_\mu} = 0.
\]
It follows that
\[
|T_{u,v,\varphi}^m - K|_{e,B_1 \to B_\mu} \geq \limsup_{|w| \to 1} \|T_{u,v,\varphi}^m - Kf_{j,w}\|_{B_\mu} 
\geq \limsup_{|w| \to 1} \|T_{u,v,\varphi}^m f_{j,w}\|_{B_\mu} - \limsup_{|w| \to 1} \|Kf_{j,w}\|_{B_\mu}.
\]
Therefore,
\[
|T_{u,v,\varphi}^m|_{e,B_1 \to B_\mu} = \inf_{K} \|T_{u,v,\varphi}^m - K\|_{e,B_1 \to B_\mu} \geq \sum_{j=1}^{4} \limsup_{|w| \to 1} \|T_{u,v,\varphi}^m f_{j,w}\|_{B_\mu}.
\]

Next, we prove that
\[
|T_{u,v,\varphi}^m|_{e,B_1 \to B_\mu} \geq \sum_{i=|\{1,m,m+1\}|}^{4} \limsup_{|w| \to 1} \frac{\mu(z) A_i(z)}{1 - |\varphi(z)|^2}.
\]
Let \(\{z_j\}\) be a sequence in \(D\) such that \(|\varphi(z_j)| \to 1\) as \(j \to \infty.\) Since \(T_{u,v,\varphi}^m : B_1 \to B_\mu\) is bounded, for any compact operator \(K : B_1 \to B_\mu\) and \(i \in \{1, m, m + 1\},\) applying Lemma 4 and (7), we obtain
\[
|T_{u,v,\varphi}^m - K|_{e,B_1 \to B_\mu} \geq \limsup_{j \to \infty} \|T_{u,v,\varphi} g_{i,\varphi(z_j)}\|_{B_\mu} - \limsup_{j \to \infty} \|Kg_{i,\varphi(z_j)}\|_{B_\mu} 
\geq \limsup_{j \to \infty} \frac{\mu(z_j) A_i(z_j) |\varphi(z_j)|^i}{1 - |\varphi(z_j)|^2},
\]
where \(g_{i,\varphi(z_j)}\) are defined in (6). Therefore,
\[
|T_{u,v,\varphi}^m|_{e,B_1 \to B_\mu} \geq \limsup_{j \to \infty} \frac{\mu(z_j) A_i(z_j) |\varphi(z_j)|^i}{1 - |\varphi(z_j)|^2} = \limsup_{|\varphi(z)| \to 1} \frac{\mu(z) A_i(z)}{|\varphi(z)|^2},
\]
from which we have
\[ ||T^m_{u,v,\varphi}||_{B_1 \rightarrow B_1} \leq \sum_{i \in [1, m, m+1]} \limsup_{|\varphi(z)| \to 1} \frac{\mu(z)A_i(z)}{1 - |\varphi(z)|^2}. \]  
(9)

Combining (8) and (9) yields
\[ ||T^m_{u,v,\varphi}||_{B_1 \rightarrow B_1} \leq \min \left\{ 4 \sum_{j=1}^{\infty} \limsup_{|w| \to 1} ||T^m_{u,v,\varphi}f_j||_{B_1}, \sum_{i \in [1, m, m+1]} \limsup_{|\varphi(z)| \to 1} \frac{\mu(z)A_i(z)}{1 - |\varphi(z)|^2} \right\}. \]

It is sufficient to show that
\[ ||T^m_{u,v,\varphi}||_{B_1 \rightarrow B_1} \leq \min \left\{ 4 \sum_{j=1}^{\infty} \limsup_{|w| \to 1} ||T^m_{u,v,\varphi}f_j||_{B_1}, \sum_{i \in [1, m, m+1]} \limsup_{|\varphi(z)| \to 1} \frac{\mu(z)A_i(z)}{1 - |\varphi(z)|^2} \right\}. \]

Define \( K_r f(z) = f_r(z) = f(rz) \), where \( 0 \leq r < 1 \). Then, \( K_r : B_1 \rightarrow B_1 \) is a compact operator with \( ||K_r|| \leq 1 \) and \( f_r \to f \) uniformly on compact subsets of \( D \) as \( r \to 1 \) clearly. Let \( \{r_j\} \subseteq (0, 1) \) be a sequence such that \( r_j \to 1 \) as \( j \to \infty \). Then, for each \( j \in \mathbb{N} \), \( T^m_{u,v,\varphi}K_r : B_1 \rightarrow B_1 \) is compact, and so
\[ ||T^m_{u,v,\varphi}||_{B_1 \rightarrow B_1} \leq \limsup_{j \to \infty} ||T^m_{u,v,\varphi} - T^m_{u,v,\varphi}K_r||_{B_1 \rightarrow B_1}. \]

Therefore, we only need to show that
\[
\limsup_{j \to \infty} ||T^m_{u,v,\varphi} - T^m_{u,v,\varphi}K_r||_{B_1 \rightarrow B_1} \leq \min \left\{ 4 \sum_{j=1}^{\infty} \limsup_{|w| \to 1} ||T^m_{u,v,\varphi}f_j||_{B_1}, \sum_{i \in [1, m, m+1]} \limsup_{|\varphi(z)| \to 1} \frac{\mu(z)A_i(z)}{1 - |\varphi(z)|^2} \right\}. \]
(10)

For every \( f \in B_1 \) such that \( ||f||_{B_1} \leq 1 \), we have
\[ ||(T^m_{u,v,\varphi} - T^m_{u,v,\varphi}K_r)f||_{B_1} \leq ||T^m_{u,v,\varphi}f(0) - T^m_{u,v,\varphi}f_r(0)|| + \sup_{z \in D} \mu(z)||T^m_{u,v,\varphi}f - T^m_{u,v,\varphi}f_r||(z) \]
\[ \leq |(f - f_r)(\varphi(0))u(0)| + |(f - f_r)^{m}(\varphi(0))v(0)| + \sup_{z \in D} \mu(z)||(f - f_r)(\varphi(z))u(z)|| \]
\[ + \sup_{|\varphi(z)| \leq N} \mu(z) \sum_{i \in [1, m, m+1]} |(f - f_r)^{(i)}(\varphi(z))|A_i(z) \]
\[ + \sup_{|\varphi(z)| > N} \mu(z) \sum_{i \in [1, m, m+1]} |(f - f_r)^{(i)}(\varphi(z))|A_i(z), \]
(11)
where \( N \in \mathbb{N} \) such that \( r_j \geq \frac{2}{3} \) for all \( j \geq N \). Furthermore, we have \( (f - f_r)^{(i)} \to 0 \) uniformly on compact subsets of \( D \) as \( j \to \infty \) for any nonnegative integer \( t \). Now, Theorem 1 implies
\[ \limsup_{j \to \infty} E_0 = \limsup_{j \to \infty} E_2 = 0. \]
(12)

From Lemma 5,
\[ \lim_{j \to \infty} E_1 \leq ||u||_{B_1} \limsup_{j \to \infty} |(f - f_r)(z)| = 0. \]
(13)

Finally, we estimate \( E_3 \),
\[ E_3 \leq \sum_{i \in [1, m, m+1]} \sup_{|\varphi(z)| \leq \gamma_N} \mu(z)|(f - f_r)^{(i)}(\varphi(z))|A_i(z) + \sum_{i \in [1, m, m+1]} \sup_{|\varphi(z)| > \gamma_N} \mu(z)|(f - f_r)^{(i)}(\varphi(z))|A_i(z). \]
(14)

For each \( i \in [1, m, m+1] \), using Lemma 1, (6), and (7), we obtain
Taking the limits as $N \to \infty$ in (15) and (16), we obtain

$$\limsup_{j \to \infty} F_i \leq \sum_{j=1}^{4} \limsup_{|\varphi(z)| \to N} m \|T_{u,v,\varphi} f_j \| s_{\mu}$$

and

$$\limsup_{j \to \infty} F_i \leq \limsup_{|\varphi(z)| \to -1} \frac{\mu(z)A_i(z)}{1 - |\varphi(z)|^2}.\quad (17)$$

Similarly, we have

$$\limsup_{j \to \infty} G_i \leq \sum_{j=1}^{4} \limsup_{|\varphi(z)| \to N} m \|T_{u,v,\varphi} f_j \| s_{\mu} \quad \text{and} \quad \limsup_{j \to \infty} G_i \leq \limsup_{|\varphi(z)| \to -1} \frac{\mu(z)A_i(z)}{1 - |\varphi(z)|^2}.\quad (18)$$

Therefore, by (11)–(14) and (17)–(19), we obtain

$$\limsup_{j \to \infty} \|T_{u,v,\varphi} \Delta_m \| s_{\mu} = \limsup_{j \to \infty} \|T_{u,v,\varphi} f_j \| s_{\mu} \leq \sum_{j=1}^{4} \limsup_{|\varphi(z)| \to N} m \|T_{u,v,\varphi} f_j \| s_{\mu},$$

and

$$\limsup_{j \to \infty} \|T_{u,v,\varphi} \Delta_m \| s_{\mu} = \limsup_{j \to \infty} \|T_{u,v,\varphi} f_j \| s_{\mu} \leq \sum_{|z| \in [1, m+1]} \limsup_{|\varphi(z)| \to N} \frac{\mu(z)A_i(z)}{1 - |\varphi(z)|^2}.\quad (19)$$

From the last two inequalities, we obtain (10) and the proof is completed. \(\square\)

**Corollary 1.** Let $u, v \in H(D), \varphi \in S(D), m \in \mathbb{N}, m > 1$, and $\mu$ be a radial weight. Suppose that $T_{u,v,\varphi}^m : B_1 \to B_\mu$ is bounded, then the following statements are equivalent.

(i) The operator $T_{u,v,\varphi}^m : B_1 \to B_\mu$ is compact.

(ii) \[ \sum_{j=1}^{4} \limsup_{|\varphi(z)| \to N} m \|T_{u,v,\varphi} f_j \| s_{\mu} = 0. \]

(iii) \[ \sum_{|z| \in [1, m+1]} \limsup_{|\varphi(z)| \to N} \frac{\mu(z)A_i(z)}{1 - |\varphi(z)|^2} = 0. \]

By the same method as in the proof of Theorem 3, we can obtain the following results for the case $m = 1$, namely, the Stević-Sharma operator.
Theorem 4. Let $u, v \in H(D), \varphi \in S(D)$, and $\mu$ be a radial weight such that $T_{u,v,\varphi} : B_1 \to B_{\mu}$ is bounded. Then,

$$
\|T_{u,v,\varphi}\|_{B_1 \to B_{\mu}} = \sum_{j=1}^{3} \limsup_{|w| \to 1} \|T_{u,v,\varphi} f_j, w\|_{S_\mu}
$$

$$
= \limsup_{|\varphi(z)| \to 1} \frac{\mu(z) |u(z)\varphi'(z) + v'(z)|}{1 - |\varphi(z)|^2} + \limsup_{|\varphi(z)| \to 1} \frac{\mu(z) |v(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^2}.
$$

Corollary 2. Let $u, v \in H(D), \varphi \in S(D)$, and $\mu$ be a radial weight. Suppose that $T_{u,v,\varphi} : B_1 \to B_{\mu}$ is bounded, then the following statements are equivalent.

(i) The operator $T_{u,v,\varphi} : B_1 \to B_{\mu}$ is compact.

(ii) \[
\sum_{j=1}^{3} \limsup_{|w| \to 1} \|T_{u,v,\varphi} f_j, w\|_{S_\mu} = 0.
\]

(iii) \[
\limsup_{|\varphi(z)| \to 1} \frac{\mu(z) |u(z)\varphi'(z) + v'(z)|}{1 - |\varphi(z)|^2} + \limsup_{|\varphi(z)| \to 1} \frac{\mu(z) |v(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^2} = 0.
\]

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