Research Article

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Spectral collocation method for convection-diffusion equation

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Abstract: Spectral collocation method, named linear barycentric rational interpolation collocation method (LBRICM), for convection-diffusion (C-D) equation with constant coefficient is considered. We change the discrete linear equations into the matrix equation. Different from the classical methods to solve the C-D equation, we solve the C-D equation with the time variable and space variable obtained at the same time. Furthermore, the convergence rate of the C-D equation by LBRICM is proved. Numerical examples are presented to test our analysis.

Keywords: linear rational interpolation, convection-diffusion equation, Barycentric form, error estimate, matrix equation

MSC 2020: 65D05, 65L60, 31A30

1 Introduction

In this study, we consider the convection-diffusion (C-D) equation with constant coefficient:

\[
Du = \frac{\partial u(x, t)}{\partial t} + q \frac{\partial u(x, t)}{\partial x} + s \frac{\partial^2 u(x, t)}{\partial x^2} = f(x, t),
\]

where \((x, t) \in [a, b] \times (0, T]\) and \(q\) and \(s\) are the constants related to fluid diffusion coefficient and concentration, respectively.

There are lots of methods to invest the C-D equation, such as finite difference method [1]. Spectral methods [2] have been widely used in lots of scientific engineering area.

In [3], nonlinear C-D equation is studied by the finite-difference lattice Boltzmann model. In reference [4], error estimate of a nonlinear C-D equation by finite element method is presented. In [5], the nonlinear C-D equation was studied and posteriori error estimates were also given. In [6], some spectral and pseudospectral approximations of Jacobi and Legendre type are considered for the C-D equation. In [7], a high-order accurate method for solving the one-dimensional heat and advection-diffusion equations is presented. In reference [8], C-D equation was solved by new formulas of the high-order derivatives of the fifth-kind Chebyshev polynomials. In [9], a transient C-D equation is considered. Time-fractional diffusion equation was solved by Petrov-Galerkin Lucas polynomials in [10]. In reference [11], the time-fractional diffusion equation was dealt by the explicit Chebyshev-Galerkin scheme. In reference [12], fractional diffusion-wave equation was dealt by the shifted fifth-kind Chebyshev polynomials. In reference [13], two-dimensional nonlinear reaction-diffusion equation with Riesz space-fractional was solved by the Legendre-Chebyshev spectral collocation method. In reference [14], fractional calculus approach was used for oxygen diffusion from capillary to tissues.

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As for the barycentric interpolation collocation method, there are barycentric Lagrange interpolation collocation method and barycentric rational interpolation collocation method. We can obtain high accuracy by changing the parameter of barycentric rational interpolation polynomial; in the following, we take the barycentric rational interpolation basis function to approximate the unknown function. Similarly, as the finite difference methods, linear barycentric rational method (LBRM) has been studied [15–19] by choosing the parameter in weight function. The LBRM has been used to solve certain problems such as delay Volterra integro-differential equations [20,21], Volterra integral equations [22,23], boundary value problem [24], plane elastic problems [25], incompressible plane elastic problems [26], nonlinear problems [27], and so on [28,29]. Linear barycentric rational collocation method (LBRCM) to solve biharmonic equation [30,31], fractional differential equations [32–34], telegraph equation [35], Volterra integro-differential equation [36], and heat conduction equation [37] are studied.

In this study, we use the linear barycentric rational collocation method to solve the C-D equation. With the help of convergence rate of interpolation barycentric rational function, both the convergence rate of space and time of linear barycentric rational collocation method for the CD equation can be obtained at the same time.

This article is organized as follows: in Section 2, the differentiation matrices and collocation scheme for the C-D equation are presented. In Section 3, the convergence rate of space and time is proved. At last, some numerical examples are listed to illustrate our theorem.

2 Collocation scheme for C-D equation

The area \( \Omega \) is partitioned into \( a = x_1 < x_2 < \ldots < x_m = b, 0 = t_1 < t_2 < \ldots < t_n = T \) with \( h = \frac{b-a}{m} \) and \( \tau = \frac{T}{n} \) being the uniform partition. However, for the quasi-uniform partition, the second kind of Chebyshev point \( x = \cos(x_i), t = \cos(t) \) is chosen.

Let

\[
u(x, t) = \sum_{j=1}^{m} L_j(x)u_j(t), \quad (2)\]

where

\[u(x_i, t) = u_i(t), \quad i = 1, 2, \ldots, m\]

and

\[
L_j(x) = \frac{x - x_j}{\sum_{k=1}^{m} x - x_k}, \quad (3)
\]

where

\[
\lambda_k = \sum_{j \in J_k} (-1)^j \prod_{i=j, j \neq k}^{J_k} \frac{1}{x_k - x_i}, \quad J_k = \{j \in \{0, 1, \ldots, l - d_k\} : k - d_k \leq j \leq k\}
\]

is the basis function [36].

Then, we change equation (1) into the following:

\[
\sum_{j=1}^{m} \delta_j u_j(t) + q \sum_{j=1}^{m} C_{ij}^{(1)}u_j(t) + s \sum_{j=1}^{m} C_{ij}^{(2)}u_j(t) = f_i(t), \quad (4)
\]

where \( L_j(x_i) = \delta_j, L_j^{(2)}(x_i) = C_{ij}^{(2)} \) and \( f(x_i, t) = f_i(t), \quad i = 1, 2, \ldots, m \). Then, we have
\[
\begin{bmatrix}
\dot{u}_1(t) \\
\vdots \\
\dot{u}_m(t)
\end{bmatrix} + \begin{bmatrix}
C^{(1)}_{11} & \cdots & C^{(1)}_{1m} \\
\vdots & \ddots & \vdots \\
C^{(1)}_{m1} & \cdots & C^{(1)}_{mm}
\end{bmatrix} + \begin{bmatrix}
C^{(2)}_{11} & \cdots & C^{(2)}_{1m} \\
\vdots & \ddots & \vdots \\
C^{(2)}_{m1} & \cdots & C^{(2)}_{mm}
\end{bmatrix} \begin{bmatrix}
\dot{u}_1(t) \\
\vdots \\
\dot{u}_m(t)
\end{bmatrix} = \begin{bmatrix}
f_1(t) \\
\vdots \\
f_m(t)
\end{bmatrix},
\]

(5)

where

\[
u(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T, \quad f(t) = [f_1(t), f_2(t), \ldots, f_m(t)]^T.
\]

(6)

In order to obtain the time discrete, we take the barycentric interpolation function to the \( t \) similarly as \( x \):

\[
u_j = u_k(t_j) = u(x_j, t), \quad i, j = 1, 2, \ldots, m, n
\]

(7)

and

\[
u_k(t) = \sum_{k=1}^{n} L_k(t) u_{i_k},
\]

(8)

then we have

\[
\begin{bmatrix}
\sum_{k=1}^{n} L_k(t_j) u_{i_k}
\end{bmatrix} + \begin{bmatrix}
C^{(1)}_{11} & \cdots & C^{(1)}_{1m} \\
\vdots & \ddots & \vdots \\
C^{(1)}_{m1} & \cdots & C^{(1)}_{mm}
\end{bmatrix} \begin{bmatrix}
\sum_{k=1}^{n} L_k(t_j) u_{i_k}
\end{bmatrix} + \begin{bmatrix}
C^{(2)}_{11} & \cdots & C^{(2)}_{1m} \\
\vdots & \ddots & \vdots \\
C^{(2)}_{m1} & \cdots & C^{(2)}_{mm}
\end{bmatrix} \begin{bmatrix}
\sum_{k=1}^{n} L_k(t_j) u_{i_k}
\end{bmatrix} = \begin{bmatrix}
f_1(t_j) \\
\vdots \\
f_m(t_j)
\end{bmatrix},
\]

(9)

where \( j = 1, 2, \ldots, n \).

We transform the linear equations into the matrix form as:

\[
\left[ I_n \otimes D^{(1)} + q \left( C^{(1)} \otimes I_n \right) + s \left( C^{(2)} \otimes I_n \right) \right] U = F
\]

(10)

and

\[
LU = F,
\]

(11)

where \( L = \left( I_n \otimes D^{(1)} + q \left( C^{(1)} \otimes I_n \right) + s \left( C^{(2)} \otimes I_n \right) \right) \) and \( \otimes \) denotes the Kronecker product \([32,38]\).

### 3 Convergence and error analysis

We define the error of \( u(x) \) with \( r(x) \) as:

\[
e(x) = u(x) - r(x) = (x - x_1) \cdots (x - x_{i+1}) u[x, x_{i+1}, \ldots, x_{i+d}, x]
\]

(12)

and

\[
|e(x)| = \frac{\sum_{i=0}^{n-d} \lambda_i(x) u(x) - p_i(x)}{\sum_{i=0}^{n-d} \lambda_i(x)} = \frac{A(x)}{B(x)} = O(h^{d+1}),
\]

(13)

where

\[
A(x) = \sum_{i=0}^{n-d} (-1)^i u[x, \ldots, x_{i+d}, x]
\]

(14)

and

\[
B(x) = \sum_{i=0}^{n-d} \lambda_i(x),
\]

(15)

Spectral collocation method for convection-di

**diffusion equation**
where
\[ \lambda_i(x) = \frac{(-1)^i}{(x-x_i)\cdots(x-x_{i+d})} \]
and
\[ |e_k(x)| \leq Ch^{d_k+1-k}, u \in C^{d+k}[a, b], k = 0, 1, \ldots. \]

We first define barycentric rational interpolants as:
\[
r_{nm}(x, t) = \sum_{i=-d}^{m+d} \lambda_i(x) \sum_{j=-d_j}^{m+d} \lambda_j(t) u_{ij} \frac{w_{ij}}{\sum_{i=-d}^{m+d} \sum_{j=-d_j}^{m+d} w_{ij}(x-x_i)(t-t_j)},
\]
where
\[
w_{ij} = (-1)^{i-d_j-d} \sum_{k_i \in J_i, k_j \in J_j, k_h \in \mathcal{K}_{i,j}, k_{h_i} \in \mathcal{K}_{i,h_i}} \frac{1}{(x-x_{i,h_i})} \prod_{h_i \in \mathcal{K}_{i,h_i}} \frac{1}{|t_h| - |t_h|}
\]
and \( J_i = \{ k_i \in I_m : i - d_i \leq k_i \leq i \}, I_m = \{ 0, \ldots, m - d_i \} \).

The error of \( u(x, t) \) with \( r_{nm}(x, t) \) is defined as:
\[
e(x, t) = u(x, t) - r_{nm}(x, t)
= (x-x_i)\cdots(x-x_{i+d})u[x_i, x_{i+1}, \ldots, x_{i+d}, x]
+ (t-t_j)\cdots(t-t_{j+d_j})u[t_j, t_{j+1}, \ldots, t_{j+d_j}, t].
\]

Now, we present the following theorem.

**Theorem 1.** For the error \( e(x, t) \) in (19) and \( u(x, t) \), we have
\[ |e(x, t)| \leq C(h^{d+1} + t^{d+1}). \]

**Proof.** For \( (x, t) \in \Omega \) and \( e(x, t) \), we have
\[
e(x, t) = u(x, t) - r_{nm}(x, t)
= \sum_{i=0}^{m-d_i} \lambda_i(x) \sum_{j=0}^{m-d_j} \lambda_j(t) (u(x, t) - r_{nm}(x, t))
\]
and
\[
u(x, t) - r_{nm}(x, t) = u(x, t) - u(x_m, t) + u(x_m, t) - r_{nm}(x, t)
= (x-x_i)\cdots(x-x_{i+d})u[x_i, x_{i+1}, \ldots, x_{i+d}, x_m, t] + (t-t_j)\cdots(t-t_{j+d_j})u[t_j, t_{j+1}, \ldots, t_{j+d_j}, x, t].
\]

We reach that
\[
u(x, t) - r_{nm}(x, t) = \sum_{i=0}^{m-d_i} (-1)^i \lambda_i(x) \sum_{j=0}^{m-d_j} \lambda_j(t)
+ \sum_{j=0}^{m-d_j} (-1)^j u[t_j, t_{j+1}, \ldots, t_{j+d_j}, x, t].
\]

By the similarly analysis in Floater and Kai [29], we have
\[
\left| \sum_{i=0}^{m-d_i} \lambda_i(x) \right| \geq \frac{1}{d_i! h^{d+1}}
\]
and
\[
\left| \sum_{j=0}^{m-d_j} \lambda_j(t) \right| \geq \frac{1}{d_j! t^{d+1}}.
\]
Combining (23), (24), and (25) together, the proof of Theorem 1 is completed.

\[ \square \]

**Corollary 1.** For the \( e(x, t) \) defined in (19), we have

\[
\begin{aligned}
\|e(x, t)\| \leq C(h^{d_1} + \tau^{d_2-1}), & \quad u(x, t) \in \Omega, \\
\|e_{xx}(x, t)\| \leq C(h^{d_1} + \tau^{d_2}), & \quad u(x, t) \in \Omega, \\
\|e_{xx}(x, t)\| \leq C(h^{d_1-1} + \tau^{d_2+1}), & \quad u(x, t) \in \Omega.
\end{aligned}
\]  

(26)

Let \( u(x, t) \) be the analysis solution of (1), \( u(x_m, t_n) \) is its approximate value, there holds

\[
\begin{aligned}
\lim_{m,n \to \infty} u(x_m, t_n) &= u(x, t), \\
Du(x_m, t_n) &= f(x, t),
\end{aligned}
\]  

(27) \hspace{1cm} (28)

and

\[
\lim_{m,n \to \infty} Du(x_m, t_n) = f(x, t).
\]  

(29)

Based on the aforementioned lemma, we obtain the following theorem.

**Theorem 2.** Let operator \( D \) be defined as (28) and \( f(x, t) \in \Omega \), and we have

\[ |u(x, t) - u(x_m, t_n)| \leq C(h^{d_1-1} + \tau^{d_2}). \]

**Proof.** As

\[
Du(x, t) - Du(x_m, t_n) = u(x, t) - u(x, t) + q[u(x, t) - u(x, t)] + s[u_{xx}(x, t) - u_{xx}(x, t)] \\
= R_1(x, t) + R_2(x, t) + R_3(x, t),
\]  

(30)

where

\[
R_1(x, t) = u(x, t) - u(x, t), \\
R_2(x, t) = q[u(x, t) - u(x, t)], \\
R_3(x, t) = s[u_{xx}(x, t) - u_{xx}(x, t)].
\]

As for the \( R_i \), we have

\[
R_1(x, t) = u(x, t) - u(x, t) = u(x, t) - u(x, t) \\
= \sum_{i=0}^{m-d_1} (-1)^i u_i[x_{i+1}, \ldots, x_{i+d_l}, x, t] + \sum_{j=0}^{n-d_2} (-1)^{j} u_{xx}[t_{j+1}, \ldots, t_{j+d_3}, x_m, t] \\
= e_i(x, t) + e_{xx}(x_m, t).
\]  

(31)

By the corollary, we obtain

\[ |R_1(x, t)| \leq |e_i(x, t) + e_{xx}(x_m, t)| \leq C(h^{d_1+1} + \tau^{d_2}). \]

(32)

Similarly, for \( R_2(x, t) \) and \( R_3(x, t) \), we have

\[
R_2(x, t) = u(x, t) - u(x, t) = e_i(x, t) + e_{xx}(x_m, t),
\]

(33) \hspace{1cm} (34)

and

\[
R_3(x, t) = u_{xx}(x, t) - u_{xx}(x, t) = e_{xx}(x, t) + e_{xx}(x, t),
\]

(35) \hspace{1cm} (36)

\[
|R_3(x, t)| \leq |e_{xx}(x, t) + e_{xx}(x_m, t)| \leq C(h^{d_1} + \tau^{d_2+1}).
\]

(37) \hspace{1cm} (38)
Combining the identity (30), (32), (34), and (36), the proof of theorem is complete. □

4 Numerical examples

Three examples are presented to illustrate our theorem. All examples were performed on personal computer by Matlab 2013a with a (configuration: Intel(R) Core(TM) i5-8265U CPU @ 1.60GHz 1.80 GHz).

Example 1. Consider the C-D equation:

\[ u_t(x, t) = u_{xx}(x, t) - u_x(x, t) + f_b(x, t), \]

with the condition

\[ u(x, 0) = 0, u_x(1, t) + b_0 u(1, t) = 0, u_x(0, t) - u(0, t) = 0. \]

Its analysis solutions is

\[ u(x, t) = [3e^x - e(1 + x + x^2)]e^{-\beta t} (1 - e^{-t}). \]

In this example, we test the linear barycentric rational with the equidistant nodes. Table 1 shows that the convergence rate is \(O(h^d_1)\) with \(d_2 = 9\) first given for the space area for \(t = 1\). In Table 2, for the space area partition \(d_1 = 9\) first given, the convergence rate of times is \(O(\tau^{d_1})\), which agrees with our theorem analysis.

We choose \(m = 40, n = 40, d_1 = 9, d_2 = 9, \text{ and } \beta = 1\) to test our algorithm. Figure 1 shows the error estimate of equidistant nodes, and Figure 2 shows the error estimate of Chebyshev nodes. From Figures 1 and 2, we know that the barycentric rational interpolation collocation method has higher accuracy under the condition of Chebyshev nodes, and the barycentric rational interpolation collocation method also has higher accuracy under the condition of equidistant nodes.

In Table 3, we test the linear barycentric rational with the Chebyshev nodes, which shows the convergence rate is \(O(h^{d_1+1})\) with \(d_1 = 9\) first given for the space area. In Table 4, for \(d_2 = 9\) first given, the convergence rate of times is \(O(\tau^{d_1+1})\), which agrees with our theorem analysis.

**Table 1:** Convergence rate of equidistant nodes with \(d_2 = 9\) and \(t = 1\)

<table>
<thead>
<tr>
<th>(m, n)</th>
<th>(d_1 = 1)</th>
<th>(h^a)</th>
<th>(d_1 = 2)</th>
<th>(h^a)</th>
<th>(d_1 = 3)</th>
<th>(h^a)</th>
<th>(d_1 = 4)</th>
<th>(h^a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 × 10</td>
<td>3.5548 × 10^{-3}</td>
<td>3.3300 × 10^{-4}</td>
<td>1.1116 × 10^{-4}</td>
<td>1.0400 × 10^{-5}</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 × 20</td>
<td>1.4005 × 10^{-3}</td>
<td>1.3438</td>
<td>2.2907</td>
<td>1.0650 × 10^{-5}</td>
<td>3.3837</td>
<td>5.8204 × 10^{-7}</td>
<td>4.1593</td>
<td></td>
</tr>
<tr>
<td>40 × 40</td>
<td>5.3926 × 10^{-4}</td>
<td>1.3769</td>
<td>2.3453</td>
<td>1.0249 × 10^{-6}</td>
<td>3.3774</td>
<td>4.2937</td>
<td></td>
<td></td>
</tr>
<tr>
<td>80 × 80</td>
<td>2.0200 × 10^{-4}</td>
<td>1.4166</td>
<td>9.5236 × 10^{-8}</td>
<td>3.4127</td>
<td>1.4282 × 10^{-5}</td>
<td>3.3770</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 2:** Convergence rate of equidistant nodes with \(d_1 = 9\) and \(t = 1\)

<table>
<thead>
<tr>
<th>(m, n)</th>
<th>(d_2 = 1)</th>
<th>(h^a)</th>
<th>(d_2 = 2)</th>
<th>(h^a)</th>
<th>(d_2 = 3)</th>
<th>(h^a)</th>
<th>(d_2 = 4)</th>
<th>(h^a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 × 10</td>
<td>1.0353 × 10^{-2}</td>
<td>2.1124 × 10^{-3}</td>
<td>6.3532 × 10^{-4}</td>
<td>1.7118 × 10^{-5}</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 × 20</td>
<td>4.7873 × 10^{-3}</td>
<td>1.1127</td>
<td>2.0312</td>
<td>1.0670 × 10^{-5}</td>
<td>3.1096</td>
<td>1.1376 × 10^{-6}</td>
<td>3.9115</td>
<td></td>
</tr>
<tr>
<td>40 × 40</td>
<td>2.3078 × 10^{-3}</td>
<td>1.0528</td>
<td>2.0176</td>
<td>8.3487 × 10^{-6}</td>
<td>3.0802</td>
<td>7.1789 × 10^{-8}</td>
<td>3.9860</td>
<td></td>
</tr>
<tr>
<td>80 × 80</td>
<td>1.1946 × 10^{-3}</td>
<td>1.0244</td>
<td>2.0092</td>
<td>1.0137 × 10^{-6}</td>
<td>3.0391</td>
<td>4.4931 × 10^{-9}</td>
<td>3.9980</td>
<td></td>
</tr>
</tbody>
</table>
In Table 5, we show that the error estimate of time equals $t = 1, 10, 40, 80$, and $100$ with $m = 40, n = 40, d_1 = 9, d_2 = 9, \text{ and } \beta = 1$, and the error estimate is given as blow for the equidistant nodes and Chebyshev nodes.

**Example 2.** Consider the C-D equation:

$$\frac{u(x, t)}{x} = u_{xx}(x, t) - u(x, t) + f(x, t), \quad 0 < x < 1; \ t > 0$$

$$u(x, 0) = 0, \quad 0 < x < 1$$

$$u(x, 1) + 2u(1, t) = 0, \quad u_0(0, t) - \frac{3}{2}u(0, t) = 0, \quad t > 0.$$

**Figure 1:** Error estimate of equidistant nodes with $m = 40, n = 40, d_1 = 9, d_2 = 9, \text{ and } \beta = 1$.

**Figure 2:** Error estimate of Chebyshev nodes with $m = 40, n = 40, d_1 = 9, d_2 = 9, \text{ and } \beta = 1$. 
Its analysis solutions is

\[ u(x, t) = \left( 1 + \frac{1}{2}x - x^2 \right) e^t e^{-\beta t} \]

and

Table 3: Convergence rate of Chebyshev nodes with \( d_1 = 9 \) and \( t = 1 \)

<table>
<thead>
<tr>
<th>( m, n )</th>
<th>( d_1 = 1 )</th>
<th>( h^a )</th>
<th>( d_1 = 2 )</th>
<th>( h^a )</th>
<th>( d_1 = 3 )</th>
<th>( h^a )</th>
<th>( d_1 = 4 )</th>
<th>( h^a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 \times 10</td>
<td>1.1867 \times 10^{-1}</td>
<td>9.1568 \times 10^{-5}</td>
<td>1.7898 \times 10^{-5}</td>
<td>1.1576 \times 10^{-6}</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 \times 20</td>
<td>2.6617 \times 10^{-1}</td>
<td>2.1565</td>
<td>1.0834 \times 10^{-5}</td>
<td>1.0571 \times 10^{-5}</td>
<td>4.0816</td>
<td>4.9817 \times 10^{-8}</td>
<td>4.5383</td>
<td></td>
</tr>
<tr>
<td>40 \times 40</td>
<td>6.0449 \times 10^{-5}</td>
<td>2.1386</td>
<td>1.2475 \times 10^{-6}</td>
<td>3.1185</td>
<td>5.9525 \times 10^{-8}</td>
<td>4.1505</td>
<td>1.4384 \times 10^{-9}</td>
<td>5.1141</td>
</tr>
<tr>
<td>80 \times 80</td>
<td>1.4033 \times 10^{-5}</td>
<td>2.1069</td>
<td>1.4806 \times 10^{-7}</td>
<td>3.0748</td>
<td>3.5266 \times 10^{-9}</td>
<td>4.0771</td>
<td>8.8800 \times 10^{-10}</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 4: Convergence rate of Chebyshev nodes with \( d_1 = 9 \) and \( t = 1 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( d_1 = 1 )</th>
<th>( h^a )</th>
<th>( d_1 = 2 )</th>
<th>( h^a )</th>
<th>( d_1 = 3 )</th>
<th>( h^a )</th>
<th>( d_1 = 4 )</th>
<th>( h^a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 \times 10</td>
<td>2.4674 \times 10^{-3}</td>
<td>7.7254 \times 10^{-5}</td>
<td>8.2887 \times 10^{-6}</td>
<td>1.0909 \times 10^{-6}</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 \times 20</td>
<td>1.0512 \times 10^{-3}</td>
<td>1.2309</td>
<td>1.8978 \times 10^{-5}</td>
<td>2.0253</td>
<td>6.8668 \times 10^{-7}</td>
<td>3.5934</td>
<td>1.7571 \times 10^{-9}</td>
<td>9.2781</td>
</tr>
<tr>
<td>40 \times 40</td>
<td>3.7121 \times 10^{-4}</td>
<td>1.5017</td>
<td>1.8696 \times 10^{-6}</td>
<td>3.3435</td>
<td>1.8229 \times 10^{-8}</td>
<td>5.2353</td>
<td>3.7351 \times 10^{-11}</td>
<td>5.5559</td>
</tr>
<tr>
<td>80 \times 80</td>
<td>1.2165 \times 10^{-4}</td>
<td>1.6095</td>
<td>6.5999 \times 10^{-6}</td>
<td>—</td>
<td>7.6684 \times 10^{-6}</td>
<td>—</td>
<td>4.4403 \times 10^{-6}</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 5: Convergence rate of long time with \( m = 40, n = 40, d_1 = 9, d_2 = 9 \), and \( \beta = 1 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( t = 1 )</th>
<th>( t = 10 )</th>
<th>( t = 40 )</th>
<th>( t = 80 )</th>
<th>( t = 100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equidistant nodes</td>
<td>4.2413 \times 10^{-11}</td>
<td>3.4452 \times 10^{-6}</td>
<td>3.8497 \times 10^{-3}</td>
<td>6.4714 \times 10^{-3}</td>
<td>5.0873 \times 10^{-3}</td>
</tr>
<tr>
<td>Chebyshev nodes</td>
<td>1.5197 \times 10^{-10}</td>
<td>1.2543 \times 10^{-7}</td>
<td>3.6741 \times 10^{-3}</td>
<td>1.3245 \times 10^{-10}</td>
<td>3.104 \times 10^{-10}</td>
</tr>
</tbody>
</table>

Figure 3: Convergence rate of equidistant nodes with \( m = 40, n = 40, d_1 = 9, d_2 = 9 \), and \( \beta = 0.6 \).
Figure 4: Error estimate of Chebyshev nodes with $m = 40$, $n = 40$, $d_1 = 9$, $d_2 = 9$, and $\beta = 0.6$.

Table 6: Convergence rate of equidistant nodes with $d_1 = 9$ and $t = 1$

<table>
<thead>
<tr>
<th>$m$, $n$</th>
<th>$d_1 = 1$</th>
<th>$h^a$</th>
<th>$d_1 = 2$</th>
<th>$h^a$</th>
<th>$d_1 = 3$</th>
<th>$h^a$</th>
<th>$d_1 = 4$</th>
<th>$h^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 × 10</td>
<td>2.0117 × 10^{-2}</td>
<td>9.9996 × 10^{-4}</td>
<td>2.6462 × 10^{-4}</td>
<td>9.8527 × 10^{-6}</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 × 20</td>
<td>6.7076 × 10^{-3}</td>
<td>1.3910</td>
<td>2.4016 × 10^{-5}</td>
<td>3.4619</td>
<td>5.3162 × 10^{-7}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40 × 40</td>
<td>2.8333 × 10^{-3}</td>
<td>1.4368</td>
<td>2.1654 × 10^{-6}</td>
<td>3.4713</td>
<td>2.5515 × 10^{-8}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>80 × 80</td>
<td>1.0279 × 10^{-3}</td>
<td>1.4628</td>
<td>1.9398 × 10^{-7}</td>
<td>3.4807</td>
<td>2.0626 × 10^{-9}</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7: Convergence rate of equidistant nodes with $d_2 = 9$ and $t = 1$

<table>
<thead>
<tr>
<th>$m$, $n$</th>
<th>$d_1 = 1$</th>
<th>$h^a$</th>
<th>$d_1 = 2$</th>
<th>$h^a$</th>
<th>$d_1 = 3$</th>
<th>$h^a$</th>
<th>$d_1 = 4$</th>
<th>$h^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 × 10</td>
<td>2.6369 × 10^{-2}</td>
<td>3.4104 × 10^{-3}</td>
<td>2.0623</td>
<td>6.6503 × 10^{-5}</td>
<td>3.1775</td>
<td>2.1560 × 10^{-6}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 × 20</td>
<td>6.0571 × 10^{-3}</td>
<td>3.9190 × 10^{-4}</td>
<td>2.0218</td>
<td>3.2950 × 10^{-6}</td>
<td>3.0737</td>
<td>6.6715 × 10^{-8}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40 × 40</td>
<td>1.4660 × 10^{-3}</td>
<td>4.7384 × 10^{-5}</td>
<td>2.0157</td>
<td>3.9582 × 10^{-7}</td>
<td>3.0402</td>
<td>2.0590 × 10^{-9}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>80 × 80</td>
<td>3.5956 × 10^{-4}</td>
<td>1.0149</td>
<td>3.8076 × 10^{-6}</td>
<td>1.4882</td>
<td>1.0407 × 10^{-6}</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 8: Convergence rate of Chebyshev nodes with $d_2 = 9$ and $t = 10$

<table>
<thead>
<tr>
<th>$m$, $n$</th>
<th>$d_1 = 1$</th>
<th>$h^a$</th>
<th>$d_1 = 2$</th>
<th>$h^a$</th>
<th>$d_1 = 3$</th>
<th>$h^a$</th>
<th>$d_1 = 4$</th>
<th>$h^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 × 10</td>
<td>3.5254 × 10^{-2}</td>
<td>1.4203 × 10^{-3}</td>
<td>1.5504 × 10^{-3}</td>
<td>1.5309 × 10^{-3}</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20 × 20</td>
<td>1.2567 × 10^{-2}</td>
<td>1.4882</td>
<td>1.0407 × 10^{-4}</td>
<td>6.3260 × 10^{-7}</td>
<td>7.9371</td>
<td>5.5968 × 10^{-6}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40 × 40</td>
<td>3.9582 × 10^{-3}</td>
<td>1.6667</td>
<td>9.8234 × 10^{-6}</td>
<td>3.4051</td>
<td>1.0552 × 10^{-7}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>80 × 80</td>
<td>1.7563 × 10^{-3}</td>
<td>1.1723</td>
<td>1.0063 × 10^{-6}</td>
<td>3.2871</td>
<td>3.5781 × 10^{-7}</td>
<td>1.2045 × 10^{-6}</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ f(x, t) = \frac{1}{2} e^x e^{-\beta t} (2 + x - 2x^2 - 2\beta t - \beta x t + 2\beta x^2 t + 3t + 4xt). \]
Table 9: Convergence rate of Chebyshev nodes with \( d_1 = 9 \) and \( t = 10 \)

<table>
<thead>
<tr>
<th>( m, n )</th>
<th>( d_2 = 1 )</th>
<th>( h^a )</th>
<th>( d_2 = 2 )</th>
<th>( h^a )</th>
<th>( d_2 = 3 )</th>
<th>( h^a )</th>
<th>( d_2 = 4 )</th>
<th>( h^a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 \times 10</td>
<td>5.2543 \times 10^{-2}</td>
<td>3.7959 \times 10^{-2}</td>
<td>2.4572 \times 10^{-2}</td>
<td>1.2281 \times 10^{-2}</td>
<td>1.4743 \times 10^{-3}</td>
<td>7.4472 \times 10^{-3}</td>
<td>3.4116 \times 10^{-3}</td>
<td>2.8485 \times 10^{-3}</td>
</tr>
<tr>
<td>20 \times 20</td>
<td>1.8911 \times 10^{-2}</td>
<td>4.7752 \times 10^{-3}</td>
<td>2.8884 \times 10^{-4}</td>
<td>3.8009 \times 10^{-4}</td>
<td>1.9856 \times 10^{-3}</td>
<td>1.6058 \times 10^{-3}</td>
<td>5.6804 \times 10^{-5}</td>
<td>4.6564</td>
</tr>
<tr>
<td>40 \times 40</td>
<td>4.7752 \times 10^{-3}</td>
<td>1.9993</td>
<td>2.9940 \times 10^{-4}</td>
<td>3.9966</td>
<td>1.2625 \times 10^{-3}</td>
<td>2.9940 \times 10^{-4}</td>
<td>2.3166 \times 10^{-4}</td>
<td>4.6159</td>
</tr>
<tr>
<td>80 \times 80</td>
<td>1.1944 \times 10^{-3}</td>
<td>1.9993</td>
<td>2.9940 \times 10^{-4}</td>
<td>3.9966</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 10: Convergence rate of long time with \( m = 40, n = 40, d_1 = 9, d_2 = 9, \) and \( \beta = 1 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( t = 1 )</th>
<th>( t = 10 )</th>
<th>( t = 40 )</th>
<th>( t = 80 )</th>
<th>( t = 100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equidistant nodes</td>
<td>1.1725 \times 10^{-9}</td>
<td>8.0311 \times 10^{-9}</td>
<td>1.2664 \times 10^{-4}</td>
<td>3.2401 \times 10^{-3}</td>
<td>7.0226 \times 10^{-3}</td>
</tr>
<tr>
<td>Chebyshev nodes</td>
<td>3.8538 \times 10^{-9}</td>
<td>1.4320 \times 10^{-9}</td>
<td>3.7099 \times 10^{-3}</td>
<td>5.5803 \times 10^{-3}</td>
<td>2.3399 \times 10^{-2}</td>
</tr>
</tbody>
</table>

We still choose \( m = 40, n = 40, d_1 = 9, d_2 = 9, \) and \( \beta = 1 \) to test our theorem. Figure 3 shows the error estimate of equidistant nodes, and Figure 4 shows the error figure of Chebyshev nodes. From Figures 3 and 4, we know that the barycentric rational interpolation collocation method has higher accuracy under the condition of Chebyshev nodes, and the barycentric rational interpolation collocation method also has higher accuracy under the condition of equidistant nodes.

In this example, in order to test our theorem analysis, we have first given \( d_1 = 9 \) and \( t = 1 \) to be the exact solution. Table 6 shows that the convergence rate of equidistant nodes is \( O(t^d) \), which agrees with our theorem analysis. In Table 7, in order to test the convergence rate of space, \( d_2 = 9 \) is first given to be the exact solution, and the convergence rate of times is \( O(h^d) \), which is one order higher than our theorem analysis.

In Table 8, we test the linear barycentric rational with the Chebyshev nodes, and shows that the convergence rate is \( O(h^{d+1}) \) with \( d_2 = 9 \) and \( t = 10 \) first given for the space area. In Table 9, for the space area partition \( d_1 = 9 \) first given, the convergence rate of times is \( O(t^{d+1}) \), which is one order higher than our theorem analysis.

Table 11: Convergence rate of equidistant nodes with \( d_1 = 9 \) and \( t = 1 \)

<table>
<thead>
<tr>
<th>( m, n )</th>
<th>( d_2 = 1 )</th>
<th>( h^a )</th>
<th>( d_2 = 2 )</th>
<th>( h^a )</th>
<th>( d_2 = 3 )</th>
<th>( h^a )</th>
<th>( d_2 = 4 )</th>
<th>( h^a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 \times 10</td>
<td>8.0557 \times 10^{-4}</td>
<td>1.2053 \times 10^{-5}</td>
<td>3.0259 \times 10^{-6}</td>
<td>1.6513 \times 10^{-7}</td>
<td>4.1120 \times 10^{-4}</td>
<td>0.9702</td>
<td>1.8156</td>
<td>4.1782 \times 10^{-7}</td>
</tr>
<tr>
<td>20 \times 20</td>
<td>1.7830 \times 10^{-4}</td>
<td>1.2039</td>
<td>8.0203 \times 10^{-7}</td>
<td>3.1130</td>
<td>9.0948 \times 10^{-10}</td>
<td>3.9805</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40 \times 40</td>
<td>7.0234 \times 10^{-5}</td>
<td>1.3457</td>
<td>1.6862 \times 10^{-7}</td>
<td>2.2499</td>
<td>4.8670 \times 10^{-9}</td>
<td>3.3107</td>
<td>1.4183 \times 10^{-9}</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 12: Convergence rate of equidistant nodes with \( d_1 = 9 \) and \( t = 1 \)

<table>
<thead>
<tr>
<th>( m, n )</th>
<th>( d_1 = 1 )</th>
<th>( h^a )</th>
<th>( d_1 = 2 )</th>
<th>( h^a )</th>
<th>( d_1 = 3 )</th>
<th>( h^a )</th>
<th>( d_1 = 4 )</th>
<th>( h^a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 \times 10</td>
<td>3.9671 \times 10^{-4}</td>
<td>7.6555 \times 10^{-6}</td>
<td>2.4860 \times 10^{-6}</td>
<td>1.9775 \times 10^{-7}</td>
<td>1.2339 \times 10^{-3}</td>
<td>1.6848</td>
<td>2.9400</td>
<td>1.5130 \times 10^{-7}</td>
</tr>
<tr>
<td>20 \times 20</td>
<td>3.7144 \times 10^{-3}</td>
<td>1.7643</td>
<td>1.6644 \times 10^{-8}</td>
<td>3.7662</td>
<td>8.2156 \times 10^{-10}</td>
<td>—</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In Table 10, we show that the error estimate of time equals $t = 1, 10, 40, 80, 100$ with $d_1 = m_{40}$, $d_2 = n_{40}$, $d_9 = d_{91}$, and $d_2 = d_{92}$, and $\beta = 1$, the error estimate is given for the equidistant nodes and Chebyshev nodes. It is shown that linear barycentric rational collocation method is still effective for the long time.

**Example 3.** Consider the nonlinear C-D equation:

$$u_t(x, t) = (u + u^2)u_{xx}(x, t) + (2u + 1)u^3_t(x, t) - 2uu_t(x, t) + 2u(1 - 2u - 6u^2),$$

$$0 < x < 1; \quad t > 0; u(x, 0) = e^{2x}, \quad u(x, 1) = e^{2x+2}.$$ 

Its analysis solutions is

$$u(x, t) = e^{2x+2},$$

and the boundary condition and initial condition are also given by the analysis solution.

In this example, we take direct linearization scheme with $u_0$:

$$u_t(x, t) = (u_0 + u_0^2)u_{xx}(x, t) + (2u_0 + 1)u_0^3(x, t) - 2u_0u_t(x, t) + 2u_0(1 - 2u_0 - 6u_0^2)$$

and we test our algorithm to solve the nonlinear C-D equation. In order to test our theorem analysis, we have first given $d_1 = 9$ and $t = 1$ to be the exact solution. Table 11 shows that the convergence rate of equidistant nodes is $O(\tau^{d_1})$, which agrees with our theorem analysis. In Table 12, in order to test the convergence rate of space, $d_2 = 9$ is first given to be the exact solution and the convergence rate of times is $O(h^{d_2+1})$, which is one order higher than our theorem analysis. From this example, the same convergence rate is the same as the linear C-D equation, while the theorem analysis is beyond our goal and will be given in another study.

Figure 5 shows the error estimate of equidistant nodes. From Figure 5, we know that the barycentric rational interpolation collocation method has higher accuracy under the condition of equidistant nodes.
5 Conclusion

In this study, (1+1)-dimensional C-D equation has been solved by LBRICM, and the discrete C-D equation can be written to matrix equation using the Kronecker product. For the linear C-D equation, the convergence rate of space and time at the same time is proved with the constant coefficient. However, for the nonlinear C-D equation, the convergence rate is the same. As for (2+1)-dimensional linear and nonlinear C-D equation, magneto-micropolar equations, and fractional C-D equation, we will study in further article to overcome the difficulty of the full coefficient matrix.

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Conflict of interest: Authors declare that they have no conflicts of interest.

Data availability statement: No data is associated with this research.

References


