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Risk bounds with additional information on functionals of the risk vector

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Abstract: We consider the problem of determining risk bounds for the Value at Risk for risk vectors X where besides the marginal distributions also information on the distribution or on the expectation of some functionals $T_j(X)$, $1 \leq j \leq m$, is available. In particular this formulation includes the case where information on subgroup sums or maxima or on the correlations or covariances is available. Based on the method of dual bounds we obtain improved risk bounds compared to the marginal case. In general the explicit calculation of the dual bounds poses a challenge. We discuss various forms of relaxation of these bounds which are accessible and in some cases even lead to sharp bounds.

Keywords: risk bounds, Value-at-Risk, dependence uncertainty, Fréchet class

MSC: 91B30 (primary); 60E15 (secondary)

1 Introduction

Besides the marginal information on the risk $X = (X_1, \dots, X_n)$ a useful additional source for reducing dependence uncertainty is to use additional information on some functionals of the risk vector. This information may be available in insurance type hierarchical models as information on the aggregation of some branches of the company evaluated by statistical analysis. E.g. as additional information the distribution of some subgroup sums $\sum_{j \in I_i} X_j \sim Q_i$ might be given where $I_i \subset \{1, \dots, n\}$, $1 \leq i \leq m$, or the worst case distribution of some subgroups $\max_{j \in I_i} X_j \sim Q_i$ might be known. Alternatively, information on (some) correlations $\tau_{ij} = \text{Corr}(X_i, X_j)$ or covariances $\sigma_{ij} = \text{Cov}(X_i, X_j)$ might be available.

In more general terms let $T_i : \mathbb{R}^n \rightarrow \mathbb{R}^1$, $i \in K$, be a class of measurable real functions. Assume that the distribution Q_i of $T_i(X)$ is known for $i \in K$. In this paper we concentrate on the case where $K = \{1, \dots, m\}$ is finite. Several parts of the paper have however a direct extension to the case where K is an infinite set. Under this information our aim is to derive upper resp. lower risk bounds for the Value at Risk on the aggregated risk $\sum_{i=1}^n X_i$. This formulation includes for the case $T_i(X) = \sum_{j \in I_i} X_j$ resp. $T_i(X) = \max_{j \in I_i} X_j$ the situation described above. If we also allow higher dimensional functions $T_i : \mathbb{R}^n \rightarrow \mathbb{R}^{n_i}$, then we can in this way also describe the case where higher dimensional marginals are known, considering $T_i(X) = (X_j)_{j \in I_i}$, where $\mathcal{E} = \{I_i, i \in K\}$ is a marginal system and $Q_i = P^{X_{I_i}}$ are the corresponding higher dimensional marginal distributions.

Deriving risk bounds under additional information on the dependence has been the subject of several recent papers. In particular we mention information on variance bounds in [3], dependence information in [4, 6, 14, 15, 20, 27] and the assumption that the distribution is known on subsets in [5, 13, 19]. Some survey papers on this subject are available by [25, 26].

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In this paper we extend the method of dual bounds as described in [18] and [11] in the context of marginal information to the case where additional information is available. In several of the above mentioned papers in a first step exact dual representations of the VaR or max risk bounds have been derived, which however turn out to be too involved for determining solutions explicitly. Based on these results then in a second step some relaxed versions of these duality results have been introduced which are accessible by analytical or numerical methods. We discuss various forms of relaxation. In some cases even a strong form of relaxation by omitting the marginal information and using only the information from the additional constraints gives good results. We concentrate in this paper on the problem of getting usable bounds and do not elaborate on exact duality results. We obtain a unified approach which yields improved risk bounds depending on the degree and quality of the additional information. In some cases we even get sharp bounds.

2 Functionals of the risk vector

Knowing higher order marginals P_I , $I \in \mathcal{E}$ implies that also the distribution of all functionals $T(X) = T_I(X_I)$ is known. This assumption is in applications often too strong. In this section we assume that for some real functionals $T_1, \dots, T_m$ the distribution of $T_i(X)$, $1 \leq i \leq m$ is known.

We call this the (DF)-assumption for functionals T_i :

$$T_i(X) \sim Q_i, \quad 1 \leq i \leq m. \quad (2.1)$$

We also consider a weaker form of this assumption, the (EF)-assumption

$$ET_i(X) = a_i, \quad 1 \leq i \leq m, \quad (2.2)$$

and the corresponding $(EF^{\leq})$ -assumption

$$ET_i(X) \leq a_i, \quad 1 \leq i \leq m. \quad (2.3)$$

The (DF)- and (EF)-assumptions are quite flexible and widely applicable and allow a unified derivation of upper and lower tail-risk bounds and, therefore, of reduced upper and lower VaR-bounds. We generally assume that the constraints in (2.1) resp. (2.2) resp. (2.3) are compatible with the marginal constraints $X_i \sim F_i$, $1 \leq i \leq n$, i.e., there exists a random vector X in $\mathbb{R}^n$ (resp. a probability measure on $\mathbb{R}^n$) satisfying both types of constraints. Determining the distributional compatibility may pose a challenge. So e.g. it may be hard to determine joint mixability for given marginals corresponding to one additional constraint of the form (EF) with $T_1(x) = 1_{\{\sum_{i=1}^n X_i = c\}}$ and $ET_1(X) = a_1 = 1$ (see e.g. [28]).

Several methods in the literature for VaR-reduction can be subsumed under this kind of functional assumptions. Assuming that $\text{Var}(S_n) \leq \sigma^2$ as in [4] or a related higher moment assumptions $ES_n^k \leq c_k$ is exactly of the form $(EF^{\leq})$ for one functional $T_1(X) = (\sum_{i=1}^n X_i)^2$ resp. $(\sum_{i=1}^n X_i)^k$. Assuming that the distributions G_r of the weighted subgroup sums

$$Y_r = \sum_{j \in I_r} \frac{1}{\eta_j} X_j = T_r(X), \quad 1 \leq r \leq m, \quad (2.4)$$

with $\eta_j := |\{r; j \in I_r\}|$ and $\mathcal{E} = \{I_1, \dots, I_m\}$ are known, is of the form (DF) and is a weakening of the assumption of knowing the multivariate distributions F_I of X_I for $I \in \mathcal{E}$ as in [10], or [17]. Similarly we can consider the case

$$T_r(X) = \max_{j \in I_r} X_j, \quad 1 \leq r \leq m, \quad (2.5)$$

of knowing the distributions of the subgroup maxima. Assuming knowledge of the covariances or correlations corresponds to (EF) in the case where

$$T_{i,j}(X) = X_i X_j, \quad 1 \leq i < j \leq n \quad (2.6)$$

and $a_{i,j} = \sigma_{i,j} + \mu_i \mu_j$ are the corresponding mixed expectations.

We denote by $M^{\text{DF}}, m^{\text{DF}}$ resp. $M^{\text{EF}}, m^{\text{EF}}$ resp. $M^{\text{EF} \leq}, m^{\text{EF} \leq}$ the corresponding maximal resp. minimal tail risk probabilities for the sum and the corresponding VaR bounds by $\text{VaR}_\alpha^{\text{MF}}$, etc. The maximal tail risk probability $M^{\text{DF}}(s)$ is defined as

$$M^{\text{DF}}(s) = \sup \left\{ P \left(\sum_{i=1}^n X_i \geq s \right); X_i \sim F_i, 1 \leq i \leq n, T_i(X) \sim Q_i, 1 \leq i \leq m \right\}, \quad (2.7)$$

where $X = (X_1, \dots, X_n)$. The other quantities are defined similarly. In the following subsections we determine upper and lower dual bounds for the tail risk.

2.1 Dual bounds under the (DF) assumption

Under the (DF)-assumption we introduce dual problems of the form

$$\begin{aligned} U^{\text{DF}}(s) &= \inf \left\{ \sum_{i=1}^n \int f_i dF_i + \sum_{j=1}^m \int g_j dQ_j; (f_i, g_j) \in \bar{A}^{\text{DF}}(s) \right\} \\ \text{and } I^{\text{DF}}(s) &= \sup \left\{ \sum_{i=1}^n \int f_i dF_i + \sum_{j=1}^m \int g_j dQ_j; (f_i, g_j) \in \underline{A}^{\text{DF}}(s) \right\} \end{aligned} \quad (2.8)$$

where

$$\begin{aligned} \bar{A}^{\text{DF}}(s) &= \left\{ (f_i, g_j); f_i \in L^1(F_i), g_j \in L^1(Q_j), \right. \\ &\quad \left. \sum_{i=1}^n f_i(x_i) + \sum_{j=1}^m g_j \circ T_j(x) \geq 1_{[s, \infty)} \left(\sum_{j=1}^n X_j \right) \right\} \end{aligned}$$

and

$$\begin{aligned} \underline{A}^{\text{DF}}(s) &= \left\{ (f_i, g_j); f_i \in L^1(F_i), g_j \in L^1(Q_j), \right. \\ &\quad \left. \sum_{i=1}^n f_i(x_i) + \sum_{j=1}^m g_j \circ T_j(x) \leq 1_{[s, \infty)} \left(\sum_{j=1}^n X_j \right) \right\}, \end{aligned}$$

and $F_i \sim X_i$.

Under several regularity conditions strong duality theorems for these functionals can be proved. For some examples see f.e. [25]. We formulate the simple to verify upper and lower bounds properties of these dual functionals and abstain from discussing the more involved strong duality results.

Proposition 2.1 (Upper and lower bounds under (DF)). *Assume that the risk vector satisfies assumption (DF) in (2.1) for functionals $T_1, \dots, T_m$. Then the following improved tail risk bounds hold:*

$$M^{\text{DF}}(s) \leq U^{\text{DF}}(s) \quad \text{and} \quad m^{\text{DF}}(s) \geq I^{\text{DF}}(s). \quad (2.9)$$

Proof. By assumption (DF) holds for any $(f_i, g_j) \in \bar{A}^{\text{DF}}(s)$,
i.e. $\sum_{i=1}^n f_i(x_i) + \sum_{j=1}^m g_j \circ T_j(x) \geq 1_{[s, \infty)}(\sum_{j=1}^n X_j)$,

$$\begin{aligned} P \left(\sum_{i=1}^n X_i \geq s \right) &\leq E \sum_{i=1}^n f_i(X_i) + \sum_{j=1}^m E g_j \circ T_j(X) \\ &= \sum_{i=1}^n \int f_i dF_i + \sum_{j=1}^m \int g_j dQ_j. \end{aligned}$$

This implies taking sup on the left-hand side and inf on the right-hand side, $M^{\text{DF}}(s) \leq U^{\text{DF}}(s)$. The inequality $m^{\text{DF}} \geq I^{\text{DF}}$ follows similarly. $\square$

Remark 2.2 (VaR bounds). The dual bounds for the tail risk in (2.9) directly imply corresponding upper and lower bounds for $\text{VaR}_\alpha^{\text{DF}}$, the Value at Risk for the sum $\sum_{i=1}^n X_i$ under assumption (DF). This results from the following simple argument.

If $P(X \geq t) \leq h(t)$ for some decreasing, right continuous function h , then

$$\begin{aligned} \text{VaR}_\alpha(X) &= \inf\{t; P(X \leq t) \geq \alpha\} \\ &= \inf\{t; P(X > t) \leq 1 - \alpha\} \\ &= \inf\{t; P(X \geq t) \leq 1 - \alpha\} \\ &\leq \inf\{t; h(t) \leq 1 - \alpha\} := h^{-1}(1 - \alpha). \end{aligned} \quad (2.10)$$

As consequence also all further tail risk bounds in this paper imply directly corresponding VaR bounds.

The dual problems in (2.9) are typically difficult to solve exactly. Therefore, it seems reasonable to relax the dual problems by restricting the marginal functions in the dual formulation to some more manageable classes like e.g. piecewise linear functions. This strategy has been successfully introduced in the pure marginal case in [9] (see also [17, 18] and [11]). The admissibility condition for pairs (f_i, g_j) however still is not easy to characterize in general in the case of additional constraints. In some cases it may be useful to relax the dual problems strongly by omitting the simple marginal information, i.e. considering admissible dual functions of the form $(0, g_j)$. Define

$$\begin{aligned} \tilde{U}^{\text{DF}}(s) &= \inf \left\{ \sum_{j=1}^m \int g_j dQ_j; \sum_{j=1}^m g_j \circ T_j(x) \geq 1_{[s, \infty)} \left(\sum_{j=1}^n x_j \right) \right\} \\ \text{and } \tilde{I}^{\text{DF}}(s) &= \sup \left\{ \sum_{j=1}^m \int g_j dQ_j; \sum_{j=1}^m g_j \circ T_j(x) \leq 1_{[s, \infty)} \left(\sum_{j=1}^n x_j \right) \right\}. \end{aligned} \quad (2.11)$$

Corollary 2.3 (Relaxed upper and lower bounds under (DF)). *Under assumption (DF) for $T_1, \dots, T_m$ holds:*

$$M^{\text{DF}}(s) \leq \tilde{U}^{\text{DF}}(s) \quad \text{and} \quad m^{\text{DF}}(s) \geq \tilde{I}^{\text{DF}}(s). \quad (2.12)$$

Proof. Corollary 2.3 follows from Proposition 2.1 noting that by definition of $\tilde{U}^{\text{DF}}$ and $\tilde{I}^{\text{DF}}$ it holds that

$$U^{\text{DF}}(s) \leq \tilde{U}^{\text{DF}}(s) \quad \text{and} \quad I^{\text{DF}}(s) \geq \tilde{I}^{\text{DF}}(s). \quad \square \quad (2.13)$$

This reduction of the problem by restriction to the additional constraints is in general too strong but in some cases sharpness of the reduced bounds can be seen directly and the reduced dual bounds in (2.9) and (2.12) can be reduced to simple marginal bounds. We consider the subgroup sum case in (2.4) for a marginal system $\mathcal{E} = \{I_1, \dots, I_m\}$ and the weighted sum

$$T_r(x) = \sum_{j \in I_r} \frac{1}{\eta_j} X_j =: Y_r. \quad (2.14)$$

Here η_j counts the number of subsets which have j as an element. In the case of non-overlapping sets $\{I_r\}$ with $\bigcup_{r=1}^m I_r = \{1, \dots, n\}$ holds $\eta_j = 1, \forall j$ and the weighted sum Y_r is identical to the partial sum over subgroup I_r .

Denote $H_r = F_{Y_r}$ the partial sum distribution of Y_r and $\mathcal{H} = \mathcal{F}(H_1, \dots, H_m)$ the corresponding simple marginal systems. Under assumption (DF) the distributions H_r of Y_r are known and we obtain:

Theorem 2.4 (Upper and lower bounds with partial sum information). *Let the risk vector X satisfy assumption (DF) for the partial sum functionals in (2.14). Then:*

$$\begin{aligned} a) \quad & U^{\text{DF}}(s) \leq \tilde{U}^{\text{DF}}(s) \quad \text{and} \quad I^{\text{DF}}(s) \geq \tilde{I}^{\text{DF}}(s) \\ b) \quad & M_{\mathcal{E}}(s) \leq M^{\text{DF}}(s) \leq M_{\mathcal{H}}(s) = \tilde{U}^{\text{DF}}(s) \\ & \text{and} \quad m_{\mathcal{E}}(s) \geq m^{\text{DF}}(s) \geq m_{\mathcal{H}}(s) = \tilde{I}^{\text{DF}}(s) \end{aligned} \quad (2.15)$$

c) If the marginal system is non-overlapping, then

$$\begin{aligned} M_{\mathcal{E}}(s) &= M^{\text{DF}}(s) = M_{\mathcal{H}}(s) = \tilde{U}^{\text{DF}}(s) \\ \text{and } m_{\mathcal{E}}(s) &= m^{\text{DF}}(s) = m_{\mathcal{H}}(s) = \tilde{I}^{\text{DF}}(s). \end{aligned} \quad (2.16)$$

Proof. a), b) The proof of a) and b) follows by combining Proposition 2.1 and Corollary 2.3 with the arguments used in the proof of Theorem 3.5 in [17] for the inequality $M_{\mathcal{E}}(s) \leq M_{\mathcal{H}}(s)$. In particular note that $\sum_{i=1}^n X_i = \sum_{r=1}^m Y_r$ and that by assumption (DF) we have that $F_{Y_r} = H_r$. The equality $M_{\mathcal{H}}(s) = \tilde{U}^{\text{DF}}(s)$ is the classical strong duality for simple marginal systems.

c) From b) we have the inequality

$$M_{\mathcal{E}}(s) \leq M_{\mathcal{H}}(s).$$

Conversely, if $Y = (Y_1, \dots, Y_m)$ is any vector with distribution function $F_Y \in \mathcal{H}$, then by a classical result on stochastic equations there exist $X_{I_r} \sim F_{I_r}$ such that $\sum_{j \in I_r} X_j = Y_r$ a.s., $1 \leq r \leq m$. This implies the converse inequality

$$M_{\mathcal{H}}(s) \leq M_{\mathcal{E}}(s).$$

Since for the simple marginal system the strong duality theorem holds we obtain

$$M_{\mathcal{H}}(s) = \tilde{U}^{\text{DF}}(s)$$

and thus equalities in (2.15) are obtained. The case of the lower bounds is similar. $\square$

In consequence of Theorem 2.4 we only need the weaker assumption (DF) of knowledge of the distribution of the subgroup sums in order to derive the same bounds as in Theorems 3.3 and 3.5 in [17] given there under the stronger assumption of knowledge of the higher order marginals.

2.2 Dual bounds under the (EF) and (EF $^{\leq}$) assumption

Under the (EF)- resp. (EF $^{\leq}$)-assumption for functionals $T_1, \dots, T_m$ we similarly to the (DF)-case introduce dual functionals

$$U^{\text{EF}}(s) = \inf \left\{ \sum_{i=1}^n \int f_i dF_i + \sum_{j=1}^m \lambda_j a_j; (f_i, \lambda_j) \in \bar{A}^{\text{EF}}(s) \right\} \quad (2.17)$$

$$\text{and } I^{\text{EF}}(s) = \sup \left\{ \sum_{i=1}^n \int f_i dF_i + \sum_{j=1}^m \lambda_j a_j; (f_i, \lambda_j) \in \underline{A}^{\text{EF}}(s) \right\} \quad (2.18)$$

and, similarly, in the inequality case $U^{\text{EF}^{\leq}}, I^{\text{EF}^{\leq}}$, where

$$\begin{aligned} \bar{A}^{\text{EF}}(s) &= \left\{ (f_i, \lambda_j); f_i \in L^1(F_i), \lambda_j \in \mathbb{R}, \sum_{i=1}^n f_i(x_i) + \sum_{j=1}^m \lambda_j T_j(x) \geq 1_{[s, \infty)} \left(\sum_{j=1}^n x_j \right) \right\}, \\ \bar{A}^{\text{EF}^{\leq}}(s) &= \left\{ (f_i, \lambda_j); f_i \in L^1(F_i), \lambda_j \in \mathbb{R}_+, \sum_{i=1}^n f_i(x_i) + \sum_{j=1}^m \lambda_j T_j(x) \geq 1_{[s, \infty)} \left(\sum_{j=1}^n x_j \right) \right\} \end{aligned}$$

and $\underline{A}^{\text{EF}}(s)$ and $\underline{A}^{\text{EF}^{\leq}}(s)$ are defined similarly.

Proposition 2.5 (Upper and lower bounds under (EF) resp. (EF $^{\leq}$)). Assume that the risk vector satisfies:

a) Assumption (EF) for the functionals $T_1, \dots, T_m$, then

$$M^{\text{EF}}(s) \leq U^{\text{EF}}(s) \quad \text{and} \quad m^{\text{EF}}(s) \geq I^{\text{EF}}(s). \quad (2.19)$$

b) Assumption $(EF^{\leq})$ for $T_1, \dots, T_m$, then

$$M^{EF^{\leq}}(s) \leq U^{EF^{\leq}}(s) \quad \text{and} \quad m^{EF^{\leq}}(s) \geq I^{EF^{\leq}}(s). \quad (2.20)$$

Proof.

a) This follows as in Proposition 2.1 using that for $(f_i, \lambda_j) \in \bar{A}^{EF}(s)$

$$\sum_{i=1}^n f_i(x_i) + \sum_{j=1}^m \lambda_j T_j(x) \geq 1_{[s, \infty)} \left(\sum_{i=1}^n X_i \right). \quad (2.21)$$

b) Under $(EF^{\leq})$ we have $ET_j(X) \leq a_j$ and, therefore, since $\lambda_j \geq 0$ we have $E\lambda_j T_j(X) \leq \lambda_j a_j$ and thus the inequality (2.21) implies

$$P\left(\sum_{j=1}^n X_j \geq s\right) \leq \sum_{i=1}^n \int f_i dF_i + \sum_{j=1}^m \lambda_j a_j.$$

□

In some cases it may be useful to omit the marginal information and use strongly relaxed duals as in (2.11) and Corollary 2.3. Define

$$\tilde{M}^{EF}(s) = \inf \left\{ \sum_{j=1}^m \lambda_j a_j; \lambda_j \in \mathbb{R}, \sum_{j=1}^m \lambda_j T_j(x) \geq 1_{[s, \infty)} \left(\sum_{i=1}^n X_i \right) \right\} \quad (2.22)$$

and, similarly, $\tilde{I}^{EF}(s)$, $\tilde{U}^{EF}(s)$ and $\tilde{I}^{EF^{\leq}}(s)$.

Corollary 2.6 (Relaxed upper and lower bounds under (EF) resp. $(EF^{\leq})$).

a) Under assumption (EF) for $T_1, \dots, T_m$ holds

$$M^{EF}(s) \leq \tilde{U}^{EF}(s) \quad \text{and} \quad m^{EF}(s) \geq \tilde{I}^{EF}(s). \quad (2.23)$$

b) Under assumption $(EF^{\leq})$ for $T_1, \dots, T_m$ holds

$$M^{EF^{\leq}}(s) \leq \tilde{U}^{EF^{\leq}}(s) \quad \text{and} \quad m^{EF^{\leq}}(s) \geq \tilde{I}^{EF^{\leq}}(s). \quad (2.24)$$

Remark 2.7.

a) The dual method to derive upper and lower bounds for the tail risks under the assumptions (EF), $(EF^{\leq})$ and (DF) has immediate extensions to derive upper and lower bounds for the expectation $E\varphi(X)$ of a function φ of the risk vector. We just have to change the admissible class of dual functions f.e. change $\bar{A}^{DF}(s)$ to

$$\bar{A}^{DF}(\varphi) := \left\{ (f_i, g_j); \sum_{i=1}^n f_i(x_i) + \sum_{j=1}^m g_j(T_j(x)) \geq \varphi(x) \right\}. \quad (2.25)$$

Denoting the maximal expectation of φ under (DF) by $M^{DF}(\varphi)$, we obtain similarly to Proposition 2.1:

$$M^{DF}(\varphi) \leq U^{DF}(\varphi) \quad \text{and} \quad m^{DF}(\varphi) \geq I^{DF}(\varphi). \quad (2.26)$$

Similar bounds are valid under (EF) and $(EF^{\leq})$.

b) The strong relaxation of the upper and lower risk bounds as in Corollary 2.3, Theorem 2.4 and in Corollary 2.6 is achieved by throwing away marginal information and basing the bounds just on the additional generalized moment constraints. This reduction may work well in some cases as in the subgroup sum example in Theorem 2.4, in the subgroup maxima example in Theorem 2.8 or in the example with variance bounds

as shown in [3]. In general however this reduction changes the optimization problem and gives only ad hoc lower resp. upper bounds with no quality guarantee. By this strong reduction the original problem is changed to a pure moment problem in the spirit of Chebyshev's inequality. There is a huge literature on this type of moment problems with many results for the solution available. For some recent work in this direction see e.g. [16] or [12]. For the marginal problem with additional constraints it seems adequate to reduce the marginal functions (f_j) to some manageable classes (as e.g. piecewise linear functions) and to combine this reduction with the additional moment constraint functions (g_j).

2.3 Some examples for risk bounds

In this subsection we consider some examples of applications of the dual bounds to obtain risk bounds under the (DF) and (EF) conditions.

A) Tail risk under subgroup max information

We consider some applications of assumption (DF) with maxima or with subgroup maxima information. Let $\mathcal{E} = \{I_1, \dots, I_m\}$ be a non-overlapping system with $\bigcup_{j=1}^m I_j = \{1, \dots, n\}$ and consider the subgroup maxima

$$T_r(X) := \max_{j \in I_r} X_j. \quad (2.27)$$

Under assumption (DF) for $T_1, \dots, T_m$, i.e. knowing the distribution G_r of the subgroup maxima

$$G_r \sim T_r(X) \quad (2.28)$$

we obtain improved bounds for the distribution function or the survival function (tail risk) of the max in comparison to the case of marginal information only, dealt with in [22] (see also Corollary 2.20 in [24]).

Theorem 2.8 (Tail risk of max under subgroup max information). *Let the risk vector X satisfy condition (DF) for the subgroup maxima $T_1, \dots, T_m$ in (2.27), then*

$$\left(\sum_{r=1}^m G_r(t) - (m-1) \right)_+ \leq F_{\max_{1 \leq i \leq n} X_i}(t) \leq \min_{r \leq m} G_r(t) \quad (2.29)$$

and the bounds in (2.29) are sharp.

Proof. By definition of the subgroup maxima T_r we have the basic equality

$$\max_{1 \leq i \leq n} X_i = \max_{r=1, \dots, m} T_r(X). \quad (2.30)$$

Since $T_r(X) \sim G_r$, we obtain from the Hoeffding–Fréchet bounds that the bounds in (2.29) are valid. Defining $Y_1, \dots, Y_m$ as maximally dependent vector with marginal distributions $G_1, \dots, G_m$ we obtain (see [22])

$$\max_{i \leq m} Y_i \sim \left(\sum_{r=1}^m G_r(t) - (m-1) \right)_+. \quad (2.31)$$

Similarly, considering $Z_1, \dots, Z_m$, the comonotonic vector with marginals G_i , the upper bound is attained

$$\max_{i \leq m} Z_i \sim \min_{r \leq m} G_r(t). \quad (2.32)$$

By a well-known result on stochastic equations it is possible to construct a vector X with marginals F_i such that a.s. $\max_{j \in I_r} X_j = Y_r$ and similarly it is possible to construct a vector X with marginals F_i such that a.s. $\max_{j \in I_r} X_j = Z_r$. This implies that the bounds in (2.29) are sharp. $\square$

Remark 2.9. a) By (2.29) holds for the maximal tail risk $\bar{F}_{\max_{i \leq n} X_i}(t)$:

$$\max_{j \leq m} \bar{G}_r(t) \leq \bar{F}_{\max_{i \leq n} X_i}(t) \leq \min \left\{ \left(m - \sum_{r=1}^m G_r(t) \right)_+, 1 \right\}. \quad (2.33)$$

This is an improvement over the sharp simple marginal bound

$$\max_{i \leq n} \bar{F}_i(t) \leq \bar{F}_{\max_{i \leq n} X_i}(t) \leq \min \left(n - \sum_{i=1}^n F_i(t), 1 \right). \quad (2.34)$$

b) Theorem 2.8 also results from an application of the relaxed dual bounds as in Corollary 2.6 based on the inequality

$$\max_{1 \leq i \leq n} X_i \leq \inf_{v \in \mathbb{R}^m} \sum_{r=1}^m (v_r + (T_r(X) - v_r)_+) \quad (2.35)$$

and noting, that the upper bound is attained for the maximally dependent vector $T_r(X) = Y_r$, $1 \leq r \leq m$. This gives the lower bound in (2.36) while the upper bound is a direct consequence of the stochastic ordering result

$$T_r(X) \geq_{\text{st}} Z_r, \quad 1 \leq r \leq m. \quad (2.36)$$

c) For the upper tail risk of the aggregated sum given the assumption (EF) for the subgroup max functionals $T_r(X) = \max_{j \in I_r} X_j$ by the reduced form of the dual functionals it is indicated to consider inequalities of the form

$$\sum_{i=1}^n x_i \leq \sum_{r=1}^m \alpha_r \max_{j \in I_r} x_j. \quad (2.37)$$

Assuming that $x_i \geq 0$ this inequality implies $\alpha_r \geq n_r = |I_r|$ and as consequence this implies the tail risk bound for the sum

$$M^{\text{EF}}(s) \leq M_{\mathcal{H}}(s), \quad (2.38)$$

where $\mathcal{H} = \{H_1, \dots, H_m\}$, $H_r(t) = G_r\left(\frac{t}{n_r}\right)$. The upper bound in (2.38) can be evaluated by the RA-algorithm but it seems typically to be too rough based on the rough inequality in (2.37).

The tail risks problem in Remark 2.9 c) with maximal subgroup information seems to be better dealable with by the method based on knowledge of a distribution function on a subset as dealt with in [19] and [14]. Note that knowing the distribution G_r of $Y_r = \max_{j \in I_r} X_j$ amounts to knowing the distribution function $F_{X_{I_r}} = F^{(r)}$ on the subset $S = \{(t, \dots, t); t \in \mathbb{R}\}$ since

$$F_{X_{I_r}}((t, \dots, t)) = P(X_j \leq t; j \in I_r) = P(\max_{j \in I_r} X_j \leq t) = G_r(t). \quad (2.39)$$

This implies by the improved Hoeffding–Fréchet bounds in [20] and [14] that

$$\underline{F}_r^S(x) \leq F^{(r)}(x) \leq \bar{F}_r^S(x), \quad x \in \mathbb{R}^{I_r}, \quad (2.40)$$

where

$$\begin{aligned} \bar{F}_r^S(x) &= \min \left(\min_{j \in I_r} F_j(x_j), \inf_{t \in \mathbb{R}} \left\{ G_r(t) + \sum_{j \in I_r} (F_j(x_j) - F_j(t))_+ \right\} \right) \\ \text{and } \underline{F}_r^S(x) &= \max \left(0, \sum_{j \in I_r} F_j(x_j) - (n_r - 1), \sup_t \left\{ G_r(t) + \sum_{j \in I_r} (F_j(t) - F_j(x_j))_+ \right\} \right). \end{aligned} \quad (2.41)$$

The bounds in (2.40) imply that the distribution function $F = F_X$ of the risk vector is bounded by

$$\underline{F}^S(x) := \left(\sum_{r=1}^m \underline{F}_r^S(x_{I_r}) - (n-1) \right)_+ \leq F(x) \leq \min_{r=1, \dots, m} \bar{F}_r^S(x_{I_r}) =: \bar{F}^S(x). \quad (2.42)$$

These estimates allow to apply the method of improved standard bounds. This method was introduced in [29] and further developed in [7, 8, 10, 23], see also [17, Theorem 3.1]. Let for a real function f on $\mathbb{R}^n$, $\bigvee f$ denote the sup convolution defined as

$$\bigvee f(s) = \sup \left\{ f(u); u \in \mathbb{R}^n, \sum_{i=1}^n u_i = s \right\}.$$

Inf and sup convolutions appear typically in the literature on improved standard bounds.

Theorem 2.10 (Tail risk of sum under subgroup max information). *Let the risk vector X satisfy condition (DF) for the subgroup maxima $T_1, \dots, T_m$ in (2.27), then*

$$P\left(\sum_{i=1}^n X_i \leq s\right) \geq \bigvee \underline{F}_r^S(s), \quad (2.43)$$

where $\bigvee$ denotes the sup-convolution.

Remark 2.11. Similarly we get a lower bound for the tail risk of the sum. Denoting the survival function of $F^{(r)}$ by $\hat{F}^r$ and defining

$$\hat{F}_r^S(x) = \max \left(0, \sum_{j \in I_r} \bar{F}_j(x_j) - (n_r - 1), \sup_t \left(G_r(t) - \sum_{j \in I_r} (\bar{F}_j(t) - \bar{F}_j(x_j))_+ \right) \right)$$

we obtain

$$P\left(\sum_{i=1}^n X_i \geq s\right) \geq \bigvee \hat{F}_r^S(s). \quad (2.44)$$

As consequence (2.43) directly implies a lower bound for the Value at Risk of the sum.

B) Stop loss premia for a portfolio with additional sum information

We next consider an example for the application of the (DF) condition to an optimization problem for stop loss premia. Assume that $X = (X_1, X_2)$ is a risk vector with marginals $X_1 \sim F_1$, $X_2 \sim F_2$ and assume that the distribution of $T(X) = X_1 + X_2$ is known to be G_1 , i.e. we make the (DF) assumption for $T_1 = T$. Our aim is to determine under this condition the maximum stop loss premium for the portfolio $2X_1 + X_2$, i.e. to determine $M^{\text{DF}}(\varphi_s)$ for $\varphi_s(x) = (2x_1 + x_2 - s)_+$. With the mean excess function defined for a random variable X by $\pi_X(t) = E(X - t)_+$ the problem can be written in the form

$$\pi_{2X_1+X_2}(t) = \max \quad (2.45)$$

under assumption (DF), as specified above.

Note that for all $a_1, a_2, a_3 \geq 0$ with

$$a_1 + a_3 = 2 \text{ and } a_2 + a_3 = 1 \text{ and for all } u = (u_i) \text{ with } u_1 + u_2 + u_3 = s \quad (2.46)$$

we have

$$(2x_1 + x_2 - s)_+ \leq (a_1 x_1 - u_1)_+ + (a_2 x_2 - u_2)_+ + (a_3(x_1 + x_2) - u_3)_+. \quad (2.47)$$

This inequality implies taking expectations and infima

$$\begin{aligned} M^{\text{DF}}(\varphi_s) &= \sup\{E(2X_1 + X_2 - s)_+; X \text{ satisfies (DF)}\} \\ &\leq \inf \left\{ a_1 \pi_{X_1} \left(\frac{u_1}{a_1} \right) + a_2 \pi_{X_2} \left(\frac{u_2}{a_2} \right) + a_3 \pi_{X_1+X_2} \left(\frac{u_3}{a_3} \right); a, u \text{ satisfying (2.46)} \right\} \end{aligned} \quad (2.48)$$

By assumption (DF) for T the excess functions are known. This dual problem can be solved for distributions with analytical form of the mean excess functions involved.

Similar dual bounds are obtained for variations of the problem. If e.g. $X = (X_1, X_2, X_3)$ is a risk vector with marginals F_i and $T(X) = X_1 + X_2 \sim G_1$ and the aim is to determine bounds for $\pi_{2X_1+X_2+X_3}$ then we modify the constraints in (2.46) to

$$a_1 + a_3 = 2, \quad a_2 + a_3 = 1 \text{ and for all } (u_i) \text{ with } u_1 + u_2 + u_3 + u_4 = s.$$

We then have

$$(2x_1 + x_2 + x_3 - s)_+ \leq (a_1 x_1 - u_1)_+ + (a_2 x_2 - u_2)_+ + (a_3(x_1 + x_2) - u_3)_+ + (x_3 - u_4)_+. \quad (2.49)$$

As consequence we obtain with $\varphi_s(x) = (2x_1 + x_2 + x_3 - s)_+$

$$\begin{aligned} M^{\text{DF}}(\varphi_s) &\leq \inf \left\{ a_1 \pi_{X_1} \left(\frac{u_1}{a_1} \right) + a_2 \pi_{X_2} \left(\frac{u_2}{a_2} \right) + \pi_{X_3}(u_4) + a_3 \pi_{X_1+X_2} \left(\frac{u_3}{a_3} \right); \right. \\ &\quad \left. a, u \text{ satisfying (2.49)} \right\}. \end{aligned} \quad (2.50)$$

This example directly extends to the case of general risk vectors $X = (X_1, \dots, X_n)$.

C) Risk bounds with covariance information

In the following application for the method of dual bounds we consider the case where additional to the marginals also the covariances

$$\sigma_{ij} = \text{Cov}(X_i, X_j) = EX_i X_j - \mu_i \mu_j, \quad \mu_i = EX_i, \quad (2.51)$$

are specified. This corresponds to assumption (EF) for $T_{ij}(X) = X_i X_j$ with $s_{ij} = EX_i X_j = \sigma_{ij} + \mu_i \mu_j$. From Proposition 2.5 we obtain the improved upper bounds formulated here for a function φ of the risk vector (as in Remark 2.7).

Theorem 2.12 (Risk bound with covariance information). *a) Let the risk vector X satisfy additionally the moment information (EF) $EX_i X_j = s_{ij}$, $1 \leq i \leq j \leq n$, then for a risk function φ holds*

$$\begin{aligned} M^{\text{EF}}(\varphi) &\leq U^{\text{EF}}(\varphi) = \inf \left\{ \sum_{i=1}^n \int f_i dF_i + \sum_{i,j=1}^n \alpha_{ij} s_{ij}; f_i \in L^1(F_i), \alpha_{ij} \in \mathbb{R}, \right. \\ &\quad \left. \varphi(x) \leq \sum_{i=1}^n f_i(x_i) + \sum \alpha_{ij} x_i x_j \right\}. \end{aligned} \quad (2.52)$$

b) Under (EF) the inequality (2.52) holds with $\alpha_{ij} \in \mathbb{R}_+$.

Remark 2.13. a) For certain classes of functions φ the exact duality in (2.52) is stated in [25].

b) Considering as in [3] $\varphi = \mathbb{1}_{\{\sum_{i=1}^n x_i \geq s\}}$, the tail risk of the sum functional and assuming that besides the marginals F_i it is known that

$$ES_n^2 \leq s^2 \quad (2.53)$$

or, equivalently, $\text{Var } S_n \leq \alpha^2 = s^2 - \mu^2$, $\mu = ES_n$, then the dual corresponding to (2.52) simplifies to the form

$$U^{\text{EF}}(s) = \inf \left\{ \sum_{i=1}^n \int f_i dF_i + \alpha s^2; \alpha \geq 0, f_i \in L^1(F_i), \right. \\ \left. \mathbb{1}_{\{\sum_{i=1}^n x_i \geq s\}} \leq \sum_{i=1}^n f_i(x_i) + \alpha \left(\sum_{i=1}^n x_i \right)^2 \right\}. \quad (2.54)$$

In [4] good upper bounds for this case are given. In comparison (2.54) gives theoretical sharp upper bounds which can be evaluated however only in a relaxed form.

D) Final remarks

- a) As mentioned in Remark 2.7 a) the dual risk bounds also apply in order to derive upper and lower bounds for the expectation $E\varphi(X)$ of a function φ of the risk vector. This allows also to determine risk bounds for risk measures given by maximal or minimal expectations of functions of the risk vector.

If we take e.g. the Tail Value at Risk

$$\text{TVaR}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \text{VaR}_u(X) du, \quad (2.55)$$

then by the Rockafellar–Uryasev representation

$$\text{TVaR}_\alpha(X) = \inf_a \left\{ a + \frac{1}{1-\alpha} E(X - a)_+ \right\} \quad (2.56)$$

and the inf is attained for $a = \text{VaR}_\alpha(X)$ (see [21]). In consequence the upper dual bounds for the stop loss premia (or excess function) $\pi_X(a)$ as in Section 2.3 B) implies upper dual bounds for $\text{TVaR}_\alpha(X)$. Minimizing these upper bounds over the parameter a then gives the best dual bound obtainable by this method.

- b) **Model independent price bounds.** In a similar way the method of dual bounds in this chapter also applies to various other types of constraints. For robust model independent price bounds in recent years dual representations with martingale constraints have been developed (see [1] and [2]). These constraints are due to the fact that reasonable pricing measures have the martingale property. The dual method in this chapter can be extended to infinitely many constraints (as mentioned in the introduction of this paper) to deal with the problem of determining robust model independent price bounds i.e. with model independent price bounds based solely on the martingale constraint additional to the marginal structure over time.

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