Research Article

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A multivariate version of Williamson's theorem, ℓ^1 -symmetric survival functions, and generalized Archimedean copulas

<https://doi.org/10.1515/demo-2018-0020>

Received July 20, 2018; accepted November 27, 2018

Abstract: Williamson's integral representation of n -monotone functions on the half-line is generalized to several dimensions. This leads to a characterization of multivariate survival functions with multiply ℓ^1 -symmetry. We then introduce a new class of generalized Archimedean copulas, where in contrast to nested Archimedean copulas no extra compatibility conditions for their generators are required.

Keywords: Williamson's theorem, multivariate survival function, Archimedean copula, higher order monotonicity, monotone composition theorem

MSC: 60E05, 62H05, 26A48, 26B40

Introduction

One of the first non-trivial results on higher order monotonic functions was published in 1956 by Williamson; it states essentially that every non-negative n -monotone function on the half-line is a unique mixture of the special ones of the form $(1 - c \cdot s)_+^{n-1}$, with $c \geq 0$. This theorem has been cited and used very much in recent years, in particular in connection with the so-called Archimedean copulas. It is also intimately related to multivariate survival functions with a special symmetry.

Our aim in this paper is to prove a Williamson-type theorem for functions of several variables, relating it to “multiply ℓ^1 -symmetric” survival functions, and — perhaps more important — using it to create a new class of “generalized Archimedean copulas”, where the disadvantage of exchangeability (i.e. “too much symmetry”) in ordinary Archimedean copulas is overcome in a controlled way, without any “nesting conditions” which have to be fulfilled by hierarchical Archimedean copulas.

Higher order monotonicity plays naturally a major role in this connection — some pertaining basic notions and results will be presented first.

Notations used in this paper:

$\mathbf{R}_+ := [0, \infty[$, $\mathbf{N} := \{1, 2, 3, \dots\}$, $\mathbf{N}_0 := \{0, 1, 2, \dots\}$, $x \odot y := (x_1 y_1, x_2 y_2, \dots)$ for two vectors of equal length, $\frac{1}{x} := (\frac{1}{x_1}, \frac{1}{x_2}, \dots)$ for $x \in [0, \infty]^d$ with $\frac{1}{0} := \infty$, $\frac{1}{\infty} := 0$, $[d] := \{1, 2, \dots, d\}$,

$$|\mathbf{n}| := \sum_{i=1}^d n_i \text{ for } \mathbf{n} \in \mathbf{N}_0^d, \quad \mathbf{1}_d := (1, \dots, 1) \in \mathbf{N}_0^d, \quad \mathbf{2}_d := (2, \dots, 2)$$

$$\binom{\mathbf{p}}{\mathbf{q}} := \prod_{i=1}^d \binom{p_i}{q_i} \text{ for } \mathbf{p}, \mathbf{q} \in \mathbf{N}_0^d \text{ (usually } \mathbf{q} \leq \mathbf{p}), \quad (f \times g)(x, y) := (f(x), g(y)) \text{ for mappings } f, g$$

$$(f \otimes g)(x, y) := f(x) \cdot g(y) \text{ for real-valued } f, g, \quad \sigma_k(x_1, \dots, x_k) := \sum_{i=1}^k x_i$$

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$M_+(X)$ is the set of Radon measures on a locally compact space X , $M_+^1(X)$ the subset of probability Radon measures

$e_1, \dots, e_d$ are the usual unit vectors in $\mathbf{R}^d$

d.f. = distribution function

s.f. = survival function

co-s.f. = co-survival function

MCT = monotone composition theorem (see below)

1 Multivariate higher order monotonicity

Let $I_1, \dots, I_d \subseteq \mathbf{R}$ be non-degenerate intervals, $I := I_1 \times \dots \times I_d$, and let $f : I \rightarrow \mathbf{R}$ be any function. For $s \in I, h \in \mathbf{R}_+^d$ such that also $s + h \in I$ put

$$(E_h f)(s) := f(s + h)$$

and $\Delta_h := E_h - E_0$, i.e. $(\Delta_h f)(s) := f(s + h) - f(s)$, and $\nabla_h := -\Delta_h$.

Since $\{E_h\}$ is commutative (where defined), so are $\{\Delta_h\}$ and $\{\nabla_h\}$. In particular, with $e_1, \dots, e_d$ denoting the standard unit vectors in $\mathbf{R}^d$, $\Delta_{h_1 e_1}, \dots, \Delta_{h_d e_d}$ commute. As usual $\Delta_h^0 f := f =: \nabla_h^0 f$. For $\mathbf{n} = (n_1, \dots, n_d) \in \mathbf{N}_0^d$ and $h \in \mathbf{R}_+^d$ we put

$$\Delta_h^{\mathbf{n}} := \Delta_{h_1 e_1}^{n_1} \Delta_{h_2 e_2}^{n_2} \dots \Delta_{h_d e_d}^{n_d}$$

and similarly $\nabla_h^{\mathbf{n}}$. The multinomial theorem gives

$$(\Delta_h^{\mathbf{n}} f)(s) = \sum_{0 \leq \mathbf{q} \leq \mathbf{n}} (-1)^{|\mathbf{n}| - |\mathbf{q}|} \binom{\mathbf{n}}{\mathbf{q}} f(s + \mathbf{q} \odot h).$$

Note that $\Delta_h^{1_d} \neq \Delta_h$ for $d > 1$.

Definition. $f : I \rightarrow \mathbf{R}$ is called

(i) **$\mathbf{n}$ -increasing** (" $\mathbf{n}$ - $\uparrow$ ") if

$$(\Delta_h^{\mathbf{p}} f)(s) \geq 0$$

for all $s \in I, h \in \mathbf{R}_+^d, \mathbf{p} \in \mathbf{N}_0^d, 0 \neq \mathbf{p} \leq \mathbf{n}$ such that $s + \mathbf{p} \odot h \in I$

(ii) **$\mathbf{n}$ -decreasing** (" $\mathbf{n}$ - $\downarrow$ ") if instead

$$(\nabla_h^{\mathbf{p}} f)(s) \geq 0$$

(iii) **$\mathbf{n}$ -alternating** (" $\mathbf{n}$ - $\updownarrow$ ") if instead

$$(\nabla_h^{\mathbf{p}} f)(s) \leq 0.$$

It is easy to see that, using the notation $(f(-\cdot))(s) := f(-s)$,

$$f \text{ is } \mathbf{n}\text{-}\downarrow \text{ on } I \iff f(-\cdot) \text{ is } \mathbf{n}\text{-}\uparrow \text{ on } -I$$

$$f \text{ is } \mathbf{n}\text{-}\updownarrow \text{ on } I \iff -f(-\cdot) \text{ is } \mathbf{n}\text{-}\uparrow \text{ on } -I.$$

A specially important index of monotonicity is $\mathbf{n} = \mathbf{1}_d := (1, \dots, 1) \in \mathbf{N}^d$. Define for any measure μ on $[0, \infty]^d$ its

distribution function ("d.f.") $\hat{\mu}$ by $\hat{\mu}(x) := \mu([0, x])$,

survival function ("s.f.") $\check{\mu}$ by $\check{\mu}(x) := \mu([x, \infty])$,

co-survival function ("co-s.f.") μ^* by $\mu^*(x) := \mu([x, \infty]^c)$

(all of which of interest only where they are finite), then $\hat{\mu}$ is essentially characterized by being $\mathbf{1}_d \cdot \uparrow$, $\check{\mu}$ by the condition $\mathbf{1}_d \cdot \downarrow$, and μ^* by $\mathbf{1}_d \cdot \downarrow$. See [3], [4] and [5] for the original results, and [7] for a more detailed overview.

In the univariate case ($d = 1$), n -decreasing functions are usually called “ n -monotone”, and n -increasing ones “ n -absolutely monotone”. Williamson’s theorem is about n -decreasing (non-negative) functions, its multivariate extension is the central object of this paper.

An important notion in the following is that of a *Bauer simplex*; this is a compact convex subset K of some locally convex Hausdorff space X , for which its set of extreme points $\text{ex}(K)$ is closed (hence compact as well), and for which the integral representation given by the Krein-Milman theorem is unique. In other words, for each $x \in K$ there is a unique $\mu \in M_+^1(\text{ex}(K))$ such that

$$x = \int_{\text{ex}(K)} y \, d\mu(y)$$

in the sense of

$$\varphi(x) = \int_{\text{ex}(K)} \varphi(y) \, d\mu(y)$$

for each continuous linear functional $\varphi : X \rightarrow \mathbf{R}$. In our applications X will always be a suitable set of real-valued functions on some set T , with the topology of pointwise convergence. The point evaluations $\varphi_t(x) := x(t)$ for some $t \in T$ are then continuous linear functionals, and we obtain

$$x(t) = \int_{\text{ex}(K)} y(t) \, d\mu(y), \quad t \in T.$$

Two basic multivariate results on higher order monotonicity were shown in [6], and will be used in what follows. The first one is about the structure of $\mathbf{n} \cdot \uparrow$ functions on $[0, 1]^d$, $d \geq 2$; cf. [6], Theorem 2 and 5. Define the functions τ_a , $0 \leq a \leq 1$, on $[0, 1]$ by

$$\tau_a(t) := \frac{(t-a)_+}{1-a} \text{ for } 0 \leq a < 1, \text{ and } \tau_1 := 1_{\{1\}}; \quad (1)$$

note that $\tau_0(t) = t$. For $n \geq 2$ put

$$K_n := \{f : [0, 1] \rightarrow \mathbf{R}_+ \mid f \text{ is } \mathbf{n} \cdot \uparrow, f(1) = 1\}.$$

Then K_n is a Bauer simplex and

$$E_n := \text{ex}(K_n) = \{\tau_0^j \mid j = 0, \dots, n-2\} \cup \{\tau_a^{n-1} \mid 0 \leq a \leq 1\}.$$

For $\mathbf{n} \in \mathbf{N}^d$, $\mathbf{n} \geq \mathbf{2}_d$ we put

$$E_{\mathbf{n}} := E_{n_1} \otimes \dots \otimes E_{n_d},$$

the set of all tensor products $h_1 \otimes \dots \otimes h_d$ with $h_i \in E_{n_i}$ for all i . Then

$$K_{\mathbf{n}} := \{f : [0, 1]^d \rightarrow \mathbf{R}_+ \mid f \text{ is } \mathbf{n} \cdot \uparrow, f(\mathbf{1}_d) = 1\}$$

is again a Bauer simplex, with $E_{\mathbf{n}}$ as its set of extreme points.

The other result is the highly applicable **Monotone Composition Theorem (“MCT”)**:

Let $d \in \mathbf{N}$, $\mathbf{n} \in \mathbf{N}^d$, $\mathbf{m}^{(i)} \in \mathbf{N}^{n_i}$, $J_i \subseteq \mathbf{R}^{n_i}$ and $I_i \subseteq \mathbf{R}$ be non-degenerate intervals, $g_i : J_i \rightarrow I_i$, $1 \leq i \leq d$. Let further $f : I_1 \times \dots \times I_d \rightarrow \mathbf{R}$ be given. Then if g_i is $\mathbf{m}^{(i)} \cdot \uparrow$ for each $i \leq d$, and f is $(|\mathbf{m}^{(1)}|, \dots, |\mathbf{m}^{(d)}|) \cdot \uparrow$, the composed mapping $f \circ (g_1 \times \dots \times g_d)$ is $(\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(d)}) \cdot \uparrow$ on $J_1 \times \dots \times J_d$ (cf. [6], Theorem 12).

2 Wilkinson's theorem in higher dimensions

Let $d \in \mathbf{N}$, $\mathbf{n} \in \mathbf{N}^d$, $\mathbf{n} \geq \mathbf{2}_d$. We have already introduced

$$K_{\mathbf{n}} := \{f : [0, 1]^d \longrightarrow \mathbf{R}_+ \mid f \text{ is } \mathbf{n}\text{-}\uparrow, f(\mathbf{1}_d) = 1\},$$

a Bauer simplex with $\text{ex}(K_{\mathbf{n}}) = E_{\mathbf{n}} = E_{n_1} \otimes \dots \otimes E_{n_d}$. We define

$$L_{\mathbf{n}} := \{g : [0, 1]^d \longrightarrow \mathbf{R}_+ \mid g \text{ is } \mathbf{n}\text{-}\downarrow, g(0) = 1\}$$

and

$$H_{\mathbf{n}} := \{g : \mathbf{R}_+^d \longrightarrow \mathbf{R}_+ \mid g \text{ is } \mathbf{n}\text{-}\downarrow, g(0) = 1\}.$$

In the one-dimensional case it was shown in [6], Theorem 4 that L_k and H_k are Bauer simplices ($k \geq 2$). With

$$g_c(s) := (1 - cs)_+ \text{ for } c \in \mathbf{R}_+, \quad g_{\infty} := 1_{\{0\}}$$

it was shown that

$$\text{ex}(L_k) = \{g_1^j \mid j = 0, 1, \dots, k-2\} \cup \{g_c^{k-1} \mid c \in [1, \infty]\}$$

and

$$\text{ex}(H_k) = \{g_c^{k-1} \mid c \in [0, \infty]\}.$$

Our aim is to extend these results to the multivariate situation. This is easy for $L_{\mathbf{n}}$:

Lemma. $L_{\mathbf{n}}$ is a Bauer simplex, and

$$\text{ex}(L_{\mathbf{n}}) = \text{ex}(L_{n_1}) \otimes \dots \otimes \text{ex}(L_{n_d}).$$

That is, for each $g \in L_{\mathbf{n}}$ there is a unique $\nu \in M_+^1(\text{ex}(L_{\mathbf{n}}))$ such that

$$g = \int h_1 \otimes \dots \otimes h_d \, d\nu(h_1, \dots, h_d)$$

or

$$g(s) = \int \prod_{i=1}^d h_i(s_i) \, d\nu(h_1, \dots, h_d), \quad s \in [0, 1]^d.$$

Proof. The mapping $(\Phi(f))(s) := f(\mathbf{1}_d - s)$ is a bijection from $K_{\mathbf{n}}$ to $L_{\mathbf{n}}$, and even an affine homeomorphism. Therefore $L_{\mathbf{n}}$, too, is a Bauer simplex, and

$$\text{ex}(L_{\mathbf{n}}) = \Phi(\text{ex}(K_{\mathbf{n}})) = \Phi(E_{\mathbf{n}}).$$

Now any $f \in E_{\mathbf{n}}$ has the form $f = \varphi_1 \otimes \dots \otimes \varphi_d$ with $\varphi_i \in E_{n_i}$, $i \leq d$. We have

$$f(\mathbf{1}_d - s) = \varphi_1(1 - s_1) \cdot \dots \cdot \varphi_d(1 - s_d), \quad s \in [0, 1]^d,$$

and so it suffices to show $\varphi(1 - \cdot) \in \text{ex}(L_k)$ for $\varphi \in E_k$. If $\varphi = \tau_0^j$, i.e. $\varphi(t) = t^j$, where $0 \leq j \leq k-2$, then $\varphi(1 - \cdot) = g_1^j$; and for $\varphi = \tau_a^{k-1}$, with τ_a given by (1), we get

$$\varphi(1 - \cdot) = g_{c_a}^{k-1}, \quad c_a := 1/(1 - a),$$

for each $a \in [0, 1]$. □

The corresponding result for $H_{\mathbf{n}}$ is the main result of this paper. It was (partially) stated without proof in [6] as Theorem 9, with a loose remark on a possible proof, which finally turned out to be a lot more involved than anticipated at that time.

Theorem 1. Let $d \geq 2$, $\mathbf{n} \in \mathbf{N}^d$, $\mathbf{n} \geq \mathbf{2}_d$. Then

(i) H_n is a Bauer simplex, and

$$\text{ex}(H_n) = \text{ex}(H_{n_1}) \otimes \dots \otimes \text{ex}(H_{n_d}).$$

For each $f \in H_n$ there is a unique $\mu \in M_+^1(\text{ex}(H_n))$ such that

$$f(s) = \int_{[0, \infty]^d} \prod_{i=1}^d (1 - x_i s_i)_+^{n_i-1} d\mu(x), \quad s \in \mathbb{R}_+^d. \quad (2)$$

(ii) Every $f \in H_n$ is continuous on $]0, \infty[^d$. It is everywhere continuous iff it is continuous in 0 iff $\mu(\mathbb{R}_+^d) = 1$. We have $\mu([0, \infty]^d) = 1$ iff $\lim_{s_j \rightarrow \infty} f(s; e_j) = 0$ for each $j = 1, \dots, d$.

Proof. (i) Let $f \in H_n$ be given, put $f_k(s) := f(ks)$, for $k \in \mathbb{N}$ and $s \in [0, 1]^d$. Then $f_k \in L_n$ and has thus by the preceding Lemma the integral representation

$$f_k(s) = \int h_1(s_1) \cdot \dots \cdot h_d(s_d) dv_k(h_1, \dots, h_d)$$

where $v_k \in M_+^1(\text{ex}(L_n))$.

We put $[d] := \{1, \dots, d\}$, $J := \prod_{i=1}^d \{0, 1, \dots, n_i - 2\}$, and for $\emptyset \neq \alpha \subseteq [d]$ and $x \in \overline{\mathbb{R}}_+^d$ we need the restrictions $x_\alpha := (x_i, i \in \alpha)$. A general $h = h_1 \otimes \dots \otimes h_d \in \text{ex}(L_n)$ has the form

$$h(s) = \prod_{i \in \alpha^c} (1 - s_i)_+^{j_i} \cdot \prod_{i \in \alpha} (1 - x_i s_i)_+^{n_i-1}$$

where $\alpha \subseteq [d]$ and $\mathbf{j} = (j_1, \dots, j_d) \in J$, empty products being 1, of course. Splitting up the probability measure v_k we may write

$$f_k(s) = \sum_{\alpha \subseteq [d]} \sum_{\mathbf{j} \in J} \prod_{i \in \alpha^c} (1 - s_i)_+^{j_i} \int_{[1, \infty]^\alpha} \prod_{i \in \alpha} (1 - x_i s_i)_+^{n_i-1} dv_{\alpha, \mathbf{j}}^{(k)}(x_\alpha),$$

where $v_{\emptyset, \mathbf{j}}^{(k)} \geq 0$ are just constants, i.e. the weights

$$v_{\emptyset, \mathbf{j}}^{(k)} = v_k \left(\left\{ \prod_{i=1}^d (1 - s_i)_+^{j_i} \right\} \right), \quad \mathbf{j} \in J.$$

If $s \in [0, k]^d$ then $s/k \in [0, 1]^d$, and

$$f(s) = f_k \left(\frac{s}{k} \right) = \sum_{\alpha} \sum_{\mathbf{j}} \prod_{i \in \alpha^c} \left(1 - \frac{s_i}{k} \right)_+^{j_i} \int_{[1, \infty]^\alpha} \prod_{i \in \alpha} \left(1 - x_i \frac{s_i}{k} \right)_+^{n_i-1} dv_{\alpha, \mathbf{j}}^{(k)}(x_\alpha) \quad (3)$$

where the integrals may be rewritten as

$$\int_{[\frac{1}{k}, \infty]^\alpha} \prod_{i \in \alpha} (1 - x_i s_i)_+^{n_i-1} d\mu_{\alpha, \mathbf{j}}^{(k)}(x_\alpha)$$

with the obvious image measures under $x \mapsto x/k$. Under the canonical embedding

$$[0, \infty]^\alpha \ni x_\alpha \mapsto (x_\alpha, \mathbf{0}_{\alpha^c}) \in [0, \infty]^d$$

each $\mu_{\alpha, \mathbf{j}}^{(k)}$ may be considered as a measure on $[0, \infty]^d$ (it leads to the same integrals as above!), and the sequence

$$\mu_k := \sum_{\alpha \subseteq [d]} \sum_{\mathbf{j} \in J} \mu_{\alpha, \mathbf{j}}^{(k)} \in M_+^1([0, \infty]^d)$$

has a (weakly) convergent subsequence with limit μ . Without restriction we may assume $\mu_k \rightarrow \mu$; even $\mu_{\alpha,j}^{(k)} \rightarrow \mu_{\alpha,j}$ for each α and j may be assumed, by taking adequate subsequences. Taking the limit in (3) yields

$$\begin{aligned} f(s) &= \sum_{\alpha} \sum_j \int_{[0,\infty]^d} \prod_{\alpha} (1 - x_i s_i)_+^{n_i-1} d\mu_{\alpha,j}(x) \\ &= \sum_{\alpha} \int_{[0,\infty]^d} \prod_{\alpha} (1 - x_i s_i)_+^{n_i-1} d\mu_{\alpha}(x) \quad , \quad \mu_{\alpha} := \sum_{j \in J} \mu_{\alpha,j} \\ &= \int_{[0,\infty]^d} \prod_{i=1}^d (1 - x_i s_i)_+^{n_i-1} d\mu(x) . \end{aligned}$$

(Note that all integrands here are continuous on $[0, \infty]^d$!)

Having shown the existence of an integral representation for f , we now want to prove the uniqueness of a representing measure. So, let $\bar{\mu} \in M_+^1([0, \infty]^d)$ be another measure representing f , i.e. (2) also holds for $\bar{\mu}$; we want to show that $\bar{\mu} = \mu$. We shall use induction on d , knowing the result to be true in the one-dimensional case; cf. Theorem 4 in [6].

Obviously f (with these representations) is on $]0, \infty[^d$ $(n_i - 2)$ times partially differentiable w.r. to s_i , for $i = 1, \dots, d$, and

$$\frac{\partial^{|n|-2d} f}{\partial x_1^{n_1-2} \dots \partial x_d^{n_d-2}}(s) = (-1)^{|n|} \prod_{i=1}^d (n_i - 1)! \int_{[0,1/s]} \prod_{i=1}^d x_i^{n_i-2} \prod_{i=1}^d (1 - x_i s_i)_+ d\mu(x)$$

where $1/s := (1/s_1, \dots, 1/s_d)$, and the same holds for $\bar{\mu}$ instead of μ . Taking once more the (left!) derivative with respect to every s_i , we get

$$\int_{[0,1/s]} \prod_{i=1}^d x_i^{n_i-1} d\mu(x) = \int_{[0,1/s]} \prod_{i=1}^d x_i^{n_i-1} d\bar{\mu}(x)$$

and in fact these integrals are unchanged if we replace $[0, 1/s]$ by $]0, 1/s]$. Standard arguments lead to

$$\mu \mid]0, \infty[^d = \bar{\mu} \mid]0, \infty[^d .$$

We have

$$[0, \infty]^d \setminus]0, \infty[^d = \{x \mid \exists i \leq d : x_i \in \{0, \infty\}\} = \bigcup_{\substack{\beta, \gamma \subseteq [d] \\ \beta \cup \gamma \neq \emptyset}} X_{\beta, \gamma}$$

with $X_{\beta, \gamma} := \{x \in [0, \infty]^d \mid x_{\beta} = \mathbf{0}_{\beta}, x_{\gamma} = \infty_{\gamma}\}$ for $\beta \cap \gamma = \emptyset$.

Let $\emptyset \neq \alpha \subsetneq [d]$ and $\beta := \alpha^c$. Then

$$\lim_{m \rightarrow \infty} f(s_{\alpha}, (m)_{\beta}) = \int_{[0,\infty]^d} \prod_{i \in \alpha} (1 - x_i s_i)_+^{n_i-1} \cdot 1_{\{\mathbf{0}_{\beta}\}}(x_{\beta}) d\mu(x)$$

and

$$\lim_{m \rightarrow \infty} f\left(s_{\alpha}, \left(\frac{1}{m}\right)_{\beta}\right) = \int_{[0,\infty]^d} \prod_{i \in \alpha} (1 - x_i s_i)_+^{n_i-1} 1_{\{\infty_{\beta}\}}(x_{\beta}) d\mu(x)$$

so that by induction hypothesis

$$\mu \mid \{x \mid x_{\beta} = \mathbf{0}_{\beta}\} = \bar{\mu} \mid \{x \mid x_{\beta} = \mathbf{0}_{\beta}\}$$

as well as

$$\mu \mid \{x \mid x_{\beta} = \infty_{\beta}\} = \bar{\mu}(\{x \mid x_{\beta} = \infty_{\beta}\}) .$$

It follows $\mu \mid X_{\beta, \gamma} = \bar{\mu} \mid X_{\beta, \gamma}$ for $\emptyset \neq \beta \cup \gamma \subsetneq [d]$, including the special cases $\beta = \emptyset$ or $\gamma = \emptyset$. Two cases are left:

$$\beta = [d] : \quad \mu(\{\mathbf{0}_d\}) = \lim_n f(n, \dots, n) = \bar{\mu}(\{\mathbf{0}_d\}),$$

and $\gamma = [d]$: We know by now that μ and $\bar{\mu}$ coincide on $[0, \infty]^d \setminus \{\infty_d\}$; since both μ and $\bar{\mu}$ are probability measures, we also get finally $\mu(\{\infty_d\}) = \bar{\mu}(\{\infty_d\})$.

(ii) Since f has the representation (2) its continuity on $]0, \infty[^d$ follows immediately by dominated convergence. From

$$\lim_{\substack{s \rightarrow 0 \\ s > 0}} f(s) = \mu(\mathbf{R}_+^d)$$

and

$$\lim_{s_j \rightarrow \infty} f(s_j e_j) = \mu(\{x \in [0, \infty]^d \mid x_j = 0\})$$

we get the rest of what is claimed. $\square$

Functions in H_n are of course bounded. However, $\mathbf{n}$ -decreasing functions may well have a “singularity” at the origin, and still have a similar integral representation. This is made precise in the following

Theorem 2.

- (i) A function $f :]0, \infty[^d \rightarrow \mathbf{R}_+$ is $\mathbf{n}$ - $\downarrow$ iff f has the (unique) representation (2), however with a possibly unbounded $\mu \in M_+(\mathbf{R}_+^d)$. The function f is automatically continuous; it is bounded iff $\mu(\mathbf{R}_+^d) < \infty$, and has in this case a unique extension to a continuous $\mathbf{n}$ - $\downarrow$ function on $\mathbf{R}_+^d$, given by (2).
- (ii) If a continuous function $f : \mathbf{R}_+^d \setminus \{0\} \rightarrow \mathbf{R}_+$ is $\mathbf{n}$ - $\downarrow$, it has the unique representation (2) with $\mu \in M_+(\mathbf{R}_+^d)$, which may be unbounded, but has a finite co-survival function on $\mathbf{R}_+^d \setminus \{0\}$.

Proof. (i) Choose any sequence $a^{(m)} \in]0, \infty[^d$ that converges to 0. Then $f_m(s) := f(a^{(m)} + s)$ is $\mathbf{n}$ - $\downarrow$ and continuous on $\mathbf{R}_+^d$, and has therefore by (i) and (ii) the representation

$$f_m(s) = \int_{\mathbf{R}_+^d} \prod_{i=1}^d (1 - x_i s_i)_+^{n_i-1} d\mu_m(x)$$

where $\mu_m \in M_+(\mathbf{R}_+^d)$ is a finite measure. For $k \in \mathbf{N}$ we have

$$\begin{aligned} f\left(\frac{1}{k} \cdot \mathbf{1}_d\right) &\geq f\left(a^{(m)} + \frac{1}{k} \mathbf{1}_d\right) = f_m\left(\frac{1}{k} \mathbf{1}_d\right) \\ &\geq \int_{[0, k-1]^d} \prod_{i=1}^d \left(1 - \frac{x_i}{k}\right)_+^{n_i-1} d\mu_m(x) \geq \left(\frac{1}{k}\right)^{|\mathbf{n}|-d} \cdot \mu_m([0, k-1]^d), \end{aligned}$$

implying

$$\sup_m \mu_m([0, k-1]^d) \leq k^{|\mathbf{n}|-d} f\left(\frac{1}{k} \mathbf{1}_d\right) < \infty.$$

Therefore there is a vaguely converging subsequence of $(\mu_m)_{m \in \mathbf{N}}$ with a limit $\mu \in M_+(\mathbf{R}_+^d)$. Again it is no restriction to assume $\mu_m \rightarrow \mu$ vaguely. Since the function $x \mapsto \prod_{i \leq d} (1 - x_i s_i)_+^{n_i-1}$ is for any $s \in]0, \infty[^d$ continuous with compact support, we directly get

$$f_m(s) \rightarrow \int \prod_{i \leq d} (1 - x_i s_i)_+^{n_i-1} d\mu(x)$$

for any $s \in]0, \infty[^d$, but of course also

$$f_m(s) = f(a^{(m)} + s) \rightarrow f(s).$$

The unicity of a representing measure can be seen as before, by taking partial derivatives (no boundary has to be taken care of here).

- (ii) The first part follows from (i). In order to see that μ has a finite co-s.f. let μ^i be the i -th projection of μ . Then for $s_i, y_i \in]0, \infty[$

$$\begin{aligned} f(s_i e_i) &= \int_{\mathbb{R}_+} (1 - x_i s_i)_+^{n_i-1} d\mu^i(x_i) \\ &\geq \int_{[0, y_i]} \dots \\ &\geq (1 - y_i s_i)_+^{n_i-1} \cdot \mu^i([0, y_i]), \end{aligned}$$

so that with $s_i := 1/(2y_i)$

$$\mu^i([0, y_i]) \leq 2^{n_i-1} \cdot f\left(\frac{1}{2y_i} \cdot e_i\right) < \infty.$$

Hence for $y \in]0, \infty[^d$

$$\mu^*(y) = \mu([y, \infty[^c) \leq \sum_{i=1}^d \mu^i([0, y_i]) < \infty.$$

□

Part (i) for $d = 1$ is Williamson's theorem; see [8], where the author states that this result goes back to Schoenberg.

3 Multiple ℓ^1 -norm symmetric survival functions

We start with an elementary remark: an equilateral n -dimensional simplex with side-length $a > 0$ has a volume proportional to a^n , it is in fact given by

$$\frac{\sqrt{n+1}}{n! \cdot (\sqrt{2})^n} \cdot a^n.$$

For $d \geq 2$ and $s > 0$ the set $A_s^{(d)} := \{x \in \mathbb{R}_+^d \mid \|x\|_1 = s\}$ is a $(d-1)$ -dimensional equilateral simplex with side-length $s \cdot \sqrt{2}$, and for $x \in \mathbb{R}_+^d$ with $\|x\|_1 < s$ also the intersection

$$[x, \infty[\cap A_s^{(d)} = \text{conv}(\{x + (s - \|x\|_1) \cdot e_i \mid 1 \leq i \leq d\})$$

is of this type, with side-length $(s - \|x\|_1) \cdot \sqrt{2}$. Therefore, denoting $\lambda_s^{(d)}$ the normalized $(d-1)$ -dimensional Lebesgue measure on $A_s^{(d)}$, considered as a probability measure on $\mathbb{R}_+^d$, the s.f. of $\lambda_s^{(d)}$ is given by

$$\check{\lambda}_s^{(d)}(x) = \frac{(s - \|x\|_1)_+^{d-1}}{s^{d-1}} = \left(1 - \frac{\|x\|_1}{s}\right)_+^{d-1}, \quad x \in \mathbb{R}_+^d,$$

i.e. the s.f. of $\lambda_s^{(d)}$ depends via $\|x\|_1$ on x , and this obviously holds also for any mixture

$$\mu := \int_{\mathbb{R}_+} \lambda_s^{(d)} d\rho(s) \tag{4}$$

with $\rho \in M_+^1(\mathbb{R}_+)$, where $\lambda_0^{(d)}$ is the Dirac measure in $0 \in \mathbb{R}_+^d$, with $\check{\lambda}_0^{(d)} = 1_{\{0\}}$.

In [2], Proposition 2.2 the remarkable converse of this statement was shown: writing now $\sigma(x) := \|x\|_1$, every $\mu \in M_+^1(\mathbb{R}_+^d)$ such that $\check{\mu} = f \circ \sigma$ for some function $f: \mathbb{R}_+ \rightarrow \mathbb{R}$, is of the form (4), and f is d - $\downarrow$, with

$$f(t) = \int_{[0, \infty[} \left(1 - \frac{t}{s}\right)_+^{d-1} d\rho(s), \quad t \in \mathbb{R}_+,$$

for some $\rho \in M_+^1(\mathbf{R}_+)$. Here $(1 - \frac{t}{\theta})_+ := 1_{\{0\}}(t)$.

We are now aiming at a multivariate generalisation of this result.

Let $\{\alpha_1, \dots, \alpha_d\}$ be a partition of $[k] := \{1, \dots, k\}$ into disjoint subsets of cardinalities $n_i := |\alpha_i| \geq 2$, $1 \leq i \leq d$. The sum function (“ ℓ^1 -norm”) on $\mathbf{R}_+^{\alpha_i}$ will be denoted σ_{n_i} , $1 \leq i \leq d$, $\mathbf{n} := (n_1, \dots, n_d)$.

Using this notation we have the following

Theorem 3. Let $\mu \in M_+^1(\mathbf{R}_+^k)$ be “multiply ℓ^1 -norm symmetric” in the sense that

$$\check{\mu} = f \circ (\sigma_{n_1} \times \dots \times \sigma_{n_d})$$

for some function $f : \mathbf{R}_+^d \rightarrow \mathbf{R}$. Then f is $\mathbf{n}$ - $\downarrow$ and μ is a unique mixture of product measures

$$\mu = \int_{\mathbf{R}_+^d} \lambda_{s_1}^{(n_1)} \otimes \dots \otimes \lambda_{s_d}^{(n_d)} d\rho(s)$$

where $\rho \in M_+^1(\mathbf{R}_+^d)$. Obviously the converse holds as well.

Proof. By Theorem 5 in [7] we know that f is $\mathbf{n}$ - $\downarrow$ (taking into account that this is equivalent with $s \mapsto f(-s)$ being $\mathbf{n}$ - $\uparrow$ on $\mathbf{R}_+^d$). Theorem 1 (i) above ensures the representation (with $\kappa \in M_+^1([0, \infty]^d)$)

$$\begin{aligned} f(y) &= \int_{[0, \infty]^d} \prod_{i=1}^d (1 - y_i t_i)_+^{n_i-1} d\kappa(t) \quad , \quad y \in \mathbf{R}_+^d \\ &= \int_{[0, \infty]^d} \prod_{i=1}^d \left(1 - \frac{y_i}{s_i}\right)_+^{n_i-1} d\rho(s) \end{aligned}$$

where ρ is the image measure of κ under $t \mapsto \frac{1}{t}$. For any $j \leq d$ we have, since μ is concentrated on $\mathbf{R}_+^k$,

$$0 = \lim_{y_j \rightarrow \infty} f(y_j e_j) = \rho^j(\{\infty\}),$$

ρ^j being the j -th projection of ρ ; therefore $\rho(\mathbf{R}_+^d) = 1$. We get for $x = (x_{\alpha_1}, \dots, x_{\alpha_d}) \in \mathbf{R}_+^k$,

$$\begin{aligned} \check{\mu}(x) &= f(\sigma_{n_1}(x_{\alpha_1}), \dots, \sigma_{n_d}(x_{\alpha_d})) \\ &= \int_{\mathbf{R}_+^d} \prod_{i=1}^d \left(1 - \frac{\|x_{\alpha_i}\|_1}{s_i}\right)_+^{n_i-1} d\rho(s) \\ &= \int_{\mathbf{R}_+^d} \prod_{i=1}^d \check{\lambda}_{s_i}^{(n_i)}(x_{\alpha_i}) d\rho(s) \\ &= \left[\int_{\mathbf{R}_+^d} \lambda_{s_1}^{(n_1)} \otimes \dots \otimes \lambda_{s_d}^{(n_d)} d\rho(s) \right]^\vee (x) \end{aligned}$$

and so the claimed result. $\square$

Let Z_j be a random vector which is uniformly distributed on $A_1^{(j)} = \{x \in \mathbf{R}_+^j \mid \|x\|_1 = 1\}$, and $R = (R_1, \dots, R_d)$ any non-negative random vector, let $Z_{n_1}, \dots, Z_{n_d}$ and R be independent. Then the $|\mathbf{n}|$ -dimensional random vector

$$X := (R_1 Z_{n_1}, \dots, R_d Z_{n_d})$$

has the distribution μ of the above theorem. That is, conditioned on the d radii $s_1, \dots, s_d$ given by R , $X = (s_1 Z_{n_1}, \dots, s_d Z_{n_d})$ consists of independent uniform vectors on the " ℓ^1 -spheres" $A_{s_i}^{(n_i)}$. The components R_i may of course be highly dependent. Writing $X = (X^{(n_1)}, \dots, X^{(n_d)})$ the vector

$$(\|X^{(n_1)}\|_1, \dots, \|X^{(n_d)}\|_1) = R$$

has the distribution ρ in the above theorem.

Theorem 3 can be extended to some unbounded measures:

Theorem 3'. Let $\mu \in M_+(\mathbf{R}_+^k \setminus \{0\})$ have a finite s.f. $\check{\mu}$, i.e. $\check{\mu}(x) = \mu([x, \infty]) < \infty$ for every $x \in \mathbf{R}_+^k \setminus \{0\}$. Assume that

$$\check{\mu} = f \circ (\sigma_{n_1} \times \dots \times \sigma_{n_d})$$

for some continuous function f on $\mathbf{R}_+^d \setminus \{0\}$. Then f is $\mathbf{n}\text{-}\downarrow$ and μ has the representation

$$\mu = \int_{]0, \infty[^d} \lambda_{s_1}^{(n_1)} \otimes \dots \otimes \lambda_{s_d}^{(n_d)} d\rho(s)$$

with a unique $\rho \in M_+(]0, \infty[^d)$. The converse holds as well.

A similar result holds for $\mu \in M_+(]0, \infty[^k)$ with finite $\check{\mu}$. In this case the function f is automatically continuous.

Proof. We get as before that f is $\mathbf{n}\text{-}\downarrow$, and then by Theorem 2 (ii) (resp. (i)) the representation

$$f(s) = \int \prod_{i=1}^d (1 - x_i s_i)_+^{n_i-1} d\kappa(x)$$

with $\kappa \in M_+(\mathbf{R}_+^d)$. Again let ρ be the image of κ under $x \mapsto 1/x$, then $\rho \in M_+(]0, \infty[^d)$. Now κ is concentrated on $]0, \infty[^d$, since $f(y_j e_j) = \mu^j([y_j, \infty]) \rightarrow 0$ for $y_j \rightarrow \infty$, for each $j \leq d$; consequently also ρ concentrates on $]0, \infty[^d$. We then proceed as in the proof of Theorem 3. $\square$

4 Generalized Archimedean copulas

Let $\mathbf{n} \geq \mathbf{2}_d$ and $f \in H_{\mathbf{n}}$ be continuous (at the origin) tending to zero at infinity; more precisely we assume that $\lim_{s_j \rightarrow \infty} f(s_j e_j) = 0$ for each $j \leq d$. Then f has according to Theorem 1 (ii) the representation

$$f(s) = \int_{]0, \infty[^d} \prod (1 - x_i s_i)_+^{n_i-1} d\mu(x), \quad s \in \mathbf{R}_+^d$$

with $\mu \in M_+^1(]0, \infty[^d)$. The univariate marginal $f_j(t) := f(t e_j)$ on $\mathbf{R}_+$ is $n_j\text{-}\downarrow$, strictly so on $[0, c_j[$, where $c_j := \sup\{f_j > 0\} \in]0, \infty]$, and thus has an inverse on $]0, 1]$, complemented by $f_j^{-1}(0) := c_j$. Consider now the function $C_f : [0, 1]^{|\mathbf{n}|} \rightarrow [0, 1]$ defined by

$$C_f(x) := f\left(\sum_{i \in \alpha_1} f_1^{-1}(x_i), \dots, \sum_{i \in \alpha_d} f_d^{-1}(x_i)\right)$$

(with $\{\alpha_1, \dots, \alpha_d\}$ a partition of $\{1, \dots, |\mathbf{n}|\}$, $|\alpha_i| = n_i \forall i$). We claim that C is a copula: evidently C has uniform marginals; and since $f(-\cdot)$ is $\mathbf{n}\text{-}\uparrow$, and $-\sum_{i \in \alpha_j} f_j^{-1}(x_i)$ is $\mathbf{1}_{n_j}\text{-}\uparrow$, $j = 1, \dots, d$, the MCT shows C to be

$\mathbf{1}_{|n|} \uparrow$. Thus C_f is a copula, for which the name “Generalized Archimedean Copula” seems to be appropriate; f will be called its *generator*. Each “group marginal” $A_j(x_{\alpha_j})$ on $[0, 1]^{\alpha_j}$, where $x_i = 1 \ \forall i \in \bigcup_{k \neq j} \alpha_k$, is an Archimedean copula in the usual sense, with f_j as generator.

A converse statement holds as well: if $f : \mathbf{R}_+^d \rightarrow [0, 1]$ is given, $f(0) = 1$, and $g_1, \dots, g_d :]0, 1] \rightarrow [0, c_j[$ are strictly decreasing bijections ($0 < c_j \leq \infty$), such that

$$C(x) := f \left(\sum_{i \in \alpha_1} g_1(x_i), \dots, \sum_{i \in \alpha_d} g_d(x_i) \right)$$

is a copula, then $g_j = f_j^{-1}$ for each j and f is $\mathbf{n} \downarrow$; this follows easily from Theorem 5 in [7].

Clearly, each Laplace transform f of a probability measure on $]0, \infty[^d$ “qualifies” as a generator of this type of a copula, for each d in fact, since f is completely monotone, which is equivalent with f being $\mathbf{n} \downarrow \ \forall \mathbf{n} \in \mathbf{N}^d$.

The idea to use a multivariate Laplace transform in this connection is not totally new: in [1] the authors use a very special type of such transforms, for which they could prove that “bona fide copulas” emerge. In analytical terms f was supposed to be of the form

$$f(t) = \sum_{k=1}^{\infty} [\varphi_1(t_1) \cdots \varphi_d(t_d)]^k \cdot p_k$$

with one-dimensional Laplace transforms $\varphi_1, \dots, \varphi_d$, and $p_k \geq 0$ such that $\sum_{k=1}^{\infty} p_k = 1$, i.e. f corresponds to a particular type of “multivariate compound distribution”. These restrictions are of course unnecessary, as we saw.

It is not unlikely that a truly interesting new field of “copula constructions” is opened in this way. We shall give but two examples.

Example 1. (related to the Gumbel family)

Let $0 < \alpha_i \leq 1 \leq \vartheta$, $i = 1, \dots, d$, and consider

$$f(s) := \exp \left[- \left(\sum_{i=1}^d s_i^{\alpha_i} \right)^{1/\vartheta} \right], \quad s \in \mathbf{R}_+^d.$$

Basic properties of Bernstein functions show that f is a d -dimensional Laplace transform. The marginals are given by

$$f_i(s_i) = \exp \left(-s_i^{\alpha_i/\vartheta} \right) \\ f_i^{-1}(t_i) = (-\log t_i)^{\vartheta/\alpha_i},$$

they lead to Gumbel copulas with parameters ϑ/α_i . For $d = 2$ and $m, n \geq 2$ the generalized Archimedean copula C_f on $]0, 1]^{m+n}$ is given by

$$C_f(x, y) = f \left(\sum_{i=1}^m f_1^{-1}(x_i), \sum_{j=1}^n f_2^{-1}(y_j) \right) = \exp \left\{ - \left\langle \left[\sum_{i=1}^m (-\log x_i)^{\vartheta/\alpha_1} \right]^{\alpha_1} + \left[\sum_{j=1}^n (-\log y_j)^{\vartheta/\alpha_2} \right]^{\alpha_2} \right\rangle^{1/\vartheta} \right\}.$$

The marginal copula that considers one variable of each group is of particular interest:

$$C_f(x_1, 1, \dots, 1, y_1, 1, \dots, 1) = \exp \left\{ - \left[(-\log x_1)^{\vartheta} + (-\log y_1)^{\vartheta} \right]^{1/\vartheta} \right\}$$

Note that this is a Gumbel copula with parameter ϑ .

It may be instructive to compare C_f with the nested function

$$D := G_{\vartheta}^{(2)} \circ \left(G_{\alpha_1^{-1}}^{(m)} \times G_{\alpha_2^{-1}}^{(n)} \right),$$

where $G_{\vartheta}^{(k)}$ is the k -dimensional Gumbel copula with parameter ϑ . We get

$$D(x, y) = \exp \left\{ - \left\langle \left[\sum_i (-\log x_i)^{1/\alpha_1} \right]^{\alpha_1 \vartheta} + \left[\sum_j (-\log y_j)^{1/\alpha_2} \right]^{\alpha_2 \vartheta} \right\rangle^{1/\vartheta} \right\},$$

and here only the “nesting condition” $\alpha_i \vartheta \leq 1$ for $i = 1, 2$ would ensure that D is a bona fide copula. If this condition is violated, then in fact it isn't necessarily a copula: taking $\vartheta = 2$, $\alpha_1 = \alpha_2 = 1$ we have

$$D(x, y) = \exp \left[- \sqrt{\log^2 \left(\prod_i x_i \right) + \log^2 \left(\prod_j y_j \right)} \right]$$

and with $m = 2$, and y chosen such that $\log^2 \left(\prod_j y_j \right) = 1$ the resulting function of $x \in]0, 1]^2$

$$(x_1, x_2) \mapsto \exp \left(- \sqrt{\log^2(x_1 x_2) + 1} \right)$$

is not $\mathbf{1}_2 \uparrow$, as is easily seen.

No such nesting conditions are required for the generalized Archimedean copulas considered here.

Example 2. (related to the Clayton family)

Let $0 < \eta \leq 1$ and $\vartheta_i \geq 1, i = 1, \dots, d$, and consider

$$f(s) := \left[\sum_{i=1}^d (1 + \vartheta_i s_i)^{1/\vartheta_i} - (d-1) \right]^{-\eta}, \quad s \in \mathbf{R}_+^d.$$

Here $1/f$ is a “ d -dimensional Bernstein function”, i.e. completely alternating, equivalently $\mathbf{n}$ -alternating for each $\mathbf{n} \in \mathbf{N}^d$ (and all d); therefore f is completely monotone, continuous at 0 with $f(0) = 1$, and tending to 0 at infinity; hence f is a generator. The marginals $f_1, \dots, f_d$ and their inverses are

$$f_i(s_i) = (1 + \vartheta_i s_i)^{-\eta/\vartheta_i}, \quad f_i^{-1}(t_i) = \frac{1}{\vartheta_i} \left(t_i^{-\vartheta_i/\eta} - 1 \right),$$

i.e. the group marginals “are Clayton”.

For $d = 2$ and arbitrary $m, n \geq 2$ we get the generalized Archimedean copula

$$C_f(x, y) = f \left(\sum_{i=1}^m f_1^{-1}(x_i), \sum_{j=1}^n f_2^{-1}(y_j) \right) \left\{ \left[1 + \sum_{i=1}^m \left(x_i^{-\vartheta_1/\eta} - 1 \right) \right]^{1/\vartheta_1} + \left[1 + \sum_{j=1}^n \left(y_j^{-\vartheta_2/\eta} - 1 \right) \right]^{1/\vartheta_2} - 1 \right\}^{-\eta}.$$

The “inter group marginal” is given by

$$C_f(x_1, 1, \dots, 1, y_1, 1, \dots, 1) = \frac{x_1 y_1}{\left[x_1^{1/\eta} + y_1^{1/\eta} - (x_1 y_1)^{1/\eta} \right]^\eta}.$$

Acknowledgement: Many thanks are due to the referees, who gave numerous hints for improvement, resulting in a (hopefully) much more reader friendly version.

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