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Asymptotic normality of the relative error regression function estimator for censored and time series data

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Abstract: Consider a survival time study, where a sequence of possibly censored failure times is observed with d -dimensional covariate. The main goal of this article is to establish the asymptotic normality of the kernel estimator of the relative error regression function when the data exhibit some kind of dependency. The asymptotic variance is explicitly given. Some simulations are drawn to lend further support to our theoretical result and illustrate the good accuracy of the studied method. Furthermore, a real data example is treated to show the good quality of the prediction and that the true data are well inside in the confidence intervals.

Keywords: Asymptotic normality, censored data, kernel smoothing, probability consistency, regression function, relative error, strong mixing.

MSC: 62G05, 62G08, 62G30, 62N01, 62N02, 62P10

1 Introduction

Studying the relationship between two random variables is a very important issue in statistics. Let (T, X) be a random vector where T represents the response variable and X is a d -dimensional covariates. We assume that T is subject to random right censoring, i.e. instead of observing T we only observe the couple (Y, δ) , where $Y = \min(T, C)$, $\delta = \mathbb{1}_{\{T \leq C\}}$ where $\mathbb{1}_A$ denotes the indicator function of the set A and C represents the censoring time. The importance of studying this kind of data is highlighted by the application of the latter to real data, see among other references [1] and [22].

In the statistical literature it is very common to assume that (T, X) and C are independent. Under this assumption a lot of work has been done on the nonparametric estimation of the regression operator issue from $T = m(X) + \varepsilon$ where ε is the observation error term. For a review about nonparametric regression using a local linear technical, we refer the reader to [11].

However, in several practical situations it is not realistic to assume that the data are independent. Consider for example the situation where the patients from the same hospital have correlated survival times due to unmeasured variables like the quality of the hospital equipment. An example of such data can be found in [21]. In economic duration data, event times are often correlated and the observation of the event may be prevented by the occurrence of an earlier competing event called censoring. Another example is the observations on duration of unemployment, which may be right censored and are typically correlated. For real data, the reader can refer to [29], [4] in such a case the data are correlated. This was the reason that motivated us to consider in this paper the situation when the data exhibit some kind of dependency.

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Many types of dependency are considered in the literature. In this article, we consider the α -mixing condition, which is defined hereafter:

Definition 1. Let $(U_i)_{i \geq 1}$ denote a sequence of stationary random variables (rv)'s and consider the family of σ -fields $\mathcal{F}_j^k = \{U_i, j \leq i \leq k\}$, Given a positive integer n , set

$$\alpha(n) = \sup \left\{ |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{F}_1^k \text{ and } B \in \mathcal{F}_{k+n}^\infty, k \in \mathbb{N}^* \right\}.$$

The sequence $(U_i)_{i \geq 1}$ is said to be α -mixing (strong mixing) if the mixing coefficient $\alpha(n) \rightarrow 0$ as $n \rightarrow \infty$.

In what follows, we suppose that $(T_i)_{i \geq 1}$ and $(C_i)_{i \geq 1}$ are two independent sequences of stationary strong-mixing rv's with mixing coefficients $\alpha_1(n)$ and $\alpha_2(n)$ respectively. [5] showed that $(Y_i)_{i \geq 1}$ is strongly mixing with coefficient $\alpha(n) = 4 \max(\alpha_1(n), \alpha_2(n))$. Consequently, we suppose that $(Y_i, \delta_i, X_i)_{i \geq 1}$ is strong mixing with a mixing coefficient equal to α .

In the context of dependent observations, there is a large literature about the nonparametric estimation, [9, 10] considered the estimation of the mean regression and the quantile regression functions and [14, 15] showed respectively the uniform strong consistency and the asymptotic normality of the regression function. For more references concerning the nonparametric estimation with dependent data, see [12]. A censored real data example which satisfied the α -mixing condition can be found in [27].

When a regression model is used for prediction the size of the error is of interest. What about when the data contains outliers, we know that the mean function is very sensible to outliers. This kind of data are usually encountered in medical and financial follow up studies. Hence, we consider the mean square relative error as loss function defined for all $T > 0$ and $x \in \mathbb{R}^d$ by

$$\mathbb{E} \left[T^{-2} (T - m(x))^2 | X = x \right]. \quad (1)$$

[23] studied (1) and derive an expression for the predictor $m(x)$ in terms of conditional inverse moments

$$m(x) = \frac{\int t^{-1} f_{T|X}(t|x) dt}{\int t^{-2} f_{T|X}(t|x) dt} =: \frac{m_1(x)}{m_2(x)} \quad (2)$$

where $f_{T|X}(\cdot|\cdot)$ is the conditional density function of the couple (X, T) . The advantage of using the mean squared relative error as loss function is that is robust to outliers (see : [16], [18] and [23]).

Having in mind such kind of data, [20] extend the work of [13] to the case of a relative error regression (RER) model. They established the uniform strong consistency and the asymptotic normality of the RER function estimator. However, no simulation study has been done to comfort the theoretical result. We follow their approach, except that we assume that the data are dependent which is a very common case in practice.

Our major objective in this paper is to study the asymptotic propriety of $m(\cdot)$ uniformly in x under strong mixing condition. This study extends the result of [20] to dependent data and constitutes a continuation result of [3] dealing with the uniform almost sure consistency of the RER estimator studied here.

The purpose of this paper is twofold. Section 2 is devoted to introduce the RER kernel estimate under right censoring and strong mixing condition. Section 3 contains the assumptions and main results. The first contribution of this paper is a consistency in probability result. Then, we establish an asymptotic normality result. Section 4 is dedicated to a numerical study on generated and real data sets. The proofs and auxiliary results are given in Section 5.

2 Presentation of the model

Consider $\{T_i, 1 \leq i \leq n\}$ be a sequence of true survival times for n individuals. Let the survival time T_i be censored on the right by the censoring time C_i with unknown df $G(c) = \mathbb{P}(C_i \leq c)$. So, we observe only

$Y_i = \min(T_i, C_i)$ and $\delta_i = \mathbb{1}_{\{T_i \leq C_i\}}$ where δ_i the censoring index which takes a value of 0 if T_i is censored and a value of 1 if we observe a variable of interest. Now let $X = (X_1, \dots, X_d)$ be a vector of d -dimensional covariates with the marginal density function $f(x) := f_X(x)$ where $x = (x_1, \dots, x_d)$. The data consist of n replications (Y_i, δ_i, X_i) of the triplet (Y, δ, X) . Throughout this article, we assume that $(C_i)_i$ and $(X_i, T_i)_i$ are independent. Now, to consider the censoring effect, we define, for $\gamma = 1, 2$, the so-called synthetic data given by

$$\tilde{T}_i^{-\gamma} = \frac{\delta_i Y_i^{-\gamma}}{\bar{G}(Y_i)}, \quad 1 \leq i \leq n. \quad (3)$$

Using the conditional expectation property and the independence assumption, for $\gamma = 1, 2$ and for all $x \in \mathbb{R}^d$, we get

$$\begin{aligned} \mathbb{E} [\tilde{T}_1^{-\gamma} | X_1 = x] &= \mathbb{E} \left[\frac{\delta_1 Y_1^{-\gamma}}{\bar{G}(Y_1)} | X_1 = x \right] \\ &= \mathbb{E} \left[\frac{T_1^{-\gamma}}{\bar{G}(T_1)} \mathbb{E} [\mathbb{1}_{\{T_1 \leq C_1\}} | T_1] | X_1 = x \right] \\ &= m_\gamma(x). \end{aligned}$$

Consider now the "pseudo"-estimator of $m(\cdot)$ given by

$$\tilde{m}(x) =: \frac{\tilde{m}_1(x)}{\tilde{m}_2(x)}$$

with

$$\tilde{m}_\gamma(x) := \frac{\frac{1}{nh_n^d} \sum_{i=1}^n \tilde{T}_i^{-\gamma} K_d \left(\frac{x - X_i}{h_n} \right)}{\frac{1}{nh_n^d} \sum_{i=1}^n K_d \left(\frac{x - X_i}{h_n} \right)} =: \frac{\tilde{\mu}_\gamma(x)}{\hat{f}(x)}, \quad (4)$$

for $\gamma = 1, 2$, where the kernel $K_d(\cdot)$ is a d -variate density function and $h_n > 0$ is a smoothing parameter converging to zero as the sample size rises. Furthermore, $\hat{f}(\cdot)$ is the kernel density function estimator defined by [25].

In practice $\bar{G}(\cdot)$ is unknown. Therefore, in order to define a genuine estimator of $m(\cdot)$, we replace $\bar{G}(\cdot)$ by its [19] estimate $\bar{G}_n(\cdot)$ given by

$$\bar{G}_n(t) = \begin{cases} \prod_{i=1}^n \left(1 - \frac{1 - \delta_{(i)}}{n - i + 1} \right)^{\mathbb{1}_{\{Y_{(i)} \leq t\}}} & \text{if } t < Y_{(n)} \\ 0 & \text{otherwise,} \end{cases} \quad (5)$$

where $Y_{(1)} < Y_{(2)} < \dots < Y_{(n)}$ are the order statistics of the Y_i 's and $\delta_{(i)}$ are the concomitant indicator of non-censoring. Then, (3) becomes for $\gamma = 1, 2$

$$\hat{T}_i^{-\gamma} = \frac{\delta_i Y_i^{-\gamma}}{\bar{G}_n(Y_i)}, \quad 1 \leq i \leq n. \quad (6)$$

Therefore, replacing (6) in (4) we get the following estimator of (2):

$$\hat{m}(x) =: \frac{\hat{m}_1(x)}{\hat{m}_2(x)} \quad (7)$$

with

$$\hat{m}_\gamma(x) := \frac{\frac{1}{nh_n^d} \sum_{i=1}^n \hat{T}_i^{-\gamma} K_d \left(\frac{x - X_i}{h_n} \right)}{\frac{1}{nh_n^d} \sum_{i=1}^n K_d \left(\frac{x - X_i}{h_n} \right)} =: \frac{\hat{\mu}_\gamma(x)}{\hat{f}(x)}, \quad (8)$$

for $\gamma = 1, 2$. This is a proper representation of $m(x)$ in terms of estimable quantities. Note that this estimator is in the same spirit as the one proposed by [20], who worked under the independence and identically distributed data assumption. Therefore, if we take $\gamma = -1$ in formula (8), the estimator $\hat{m}_\gamma(x)$ reduce to the ones proposed in [14, 15]. Furthermore, when there is no censoring (all δ 's are equal to 1) the RER estimator in (7) reduces to [18] in the uni-dimensional case.

Remark 1. Among various mixing conditions used in the literature, α -mixing is reasonably weak and has many practical applications see [7] and [5, 6]. In particular, the stationary autoregressive-moving average (ARMA) processes, which are widely applied in time series analysis, are α -mixing with an exponential decreasing rate. When the innovations satisfying some moment conditions, the sequence is geometrically strongly-mixing (see : [17]).

3 Assumptions and main results

To formulate our assumptions, some additional notations are required. Let $\mathcal{S}$ be a compact set of $\mathcal{S}_0 := \{x \in \mathbb{R}^d \mid f(x) > 0\}$. For any df F Let $\tau_F = \sup\{x, \bar{F}(x) > 0\}$ be the upper bound of the support of F and we assume that $\tau < \tau_F < \infty$ and then $0 < \bar{G}(\tau_F) < \bar{G}(\tau)$.

Throughout the paper, $C, C_1, C_2, C_3, \dots$, denote some positive constants not depending on n , which may be different in various places. Let us denote by

$$\mu_\gamma(x) = \int t^{-\gamma} f_{X,T}(x, t) dt, \quad \gamma = \{1, 2\} \quad (9)$$

and

$$r_\ell(x) = \int \frac{t^{-\ell}}{\bar{G}(t)} f_{X,T}(x, t) dt, \quad \ell = \{2, 3, 4\} \quad (10)$$

where $f_{X,T}(\cdot, \cdot) =: f(\cdot, \cdot)$ is the joint density function of the couple (X, T) .

3.1 Assumptions

The assumptions used in this article are listed below. At this point, it is mentioned that all limits are taken as $n \rightarrow \infty$, unless otherwise specified.

(K) The kernel $K_d(\cdot)$ is a continuously differentiable compactly supported density function.

(i) $\int u_j K_d(u) du_j = 0$, for $1 \leq j \leq d$ and $u = (u_1, \dots, u_d)^t$.

(ii) $\int \|u\| K_d^2(u) du < \infty$ with $\|u\| = \sum_{i=1}^n |u_i|$.

(iii) $\int |u_1 u_2 \dots u_d| K_d(u) du < \infty$.

(D) The functions

(i) $\mu_\gamma(\cdot)$, for $\gamma = 1, 2$, are twice continuously differentiable.

(ii) $\mu_\gamma(\cdot, \cdot)$, for $\gamma = 1, 2$ exist and satisfy for all $(u, v) \in \mathbb{R}^{2d}$

$$\sup_{i,j} \sup_{u,v} |\mu_{i,j,\gamma}(u, v) - \mu_{i,\gamma}(u) \mu_{j,\gamma}(v)| \leq C < \infty.$$

(iii) $r_\ell(x)$, for $\ell = 2, 3, 4$, are continuously differentiable.

(iv) $f(\cdot)$ is continuously differentiable.

(M) $(T_n, X_n)_{n \geq 1}$ is a stationary α -mixing sequence of rv's, with $\alpha(n)$. Furthermore, the mixing coefficient satisfies:

- (i) $\alpha(n) = O(n^{-\nu})$ for some $\nu > 3$.
- (ii) $h^{-\frac{d}{a_1}} \sum_{s > v_n} \alpha^{\frac{1}{a_1}}(s) \leq \infty$ for $0 < a_1 < 1$ and $d > 1$.

where v_n is a sequence of positive integers going to infinity which will be defined later.

(H) The sequence h_n satisfies:

- (i) $\lim n h_n^{d+2} = 0$,
- (ii) $\lim h_n^d \log \log n = 0$.

(B) There exist integer sequences p_n and q_n going to ∞ along with n such that :

$$\frac{p_n}{q_n} \rightarrow \infty, \quad \frac{k(p_n + q_n)}{n} \rightarrow 1, \quad \frac{p_n}{\sqrt{n h_n^d}} \rightarrow 0 \quad \text{and} \quad k \alpha(q_n) \rightarrow 0.$$

Moreover, for $k := k_n = \left\lfloor \frac{n}{p_n + q_n} \right\rfloor$ (where $\lfloor \cdot \rfloor$ stands for the integer part function), we have

$$k q_n^{-\nu} \rightarrow 0.$$

3.2 Discussion on the assumptions:

First, the assumptions on the kernel (K) are standard in nonparametric estimation and assumptions (D) are regularity conditions which permit to get our result. The assumption (H) about the bandwidth are reduced to the strict minimum to make some terms negligible (see Lemma 1). These three assumptions intervene in Lemmas 1–3 and corollaries 2 and 3. The assumption (B) intervenes in Lemma 4, for using Doob's technique and established the asymptotic normality of the main part and makes the other terms negligible. The assumptions (M) are technical and concern the mixing coefficient α .

The following result states the convergence in probability of the RER estimator.

Theorem 1. Suppose that the assumptions (K), (D)(ii, iii) and (M)(i) holds. We have

$$\forall \varepsilon > 0, \quad \lim_{n \rightarrow \infty} \mathbb{P}(|\hat{m}(x) - m(x)| > \varepsilon) = 0.$$

The following result gives the asymptotic normality of the RER function estimator. Let $S^* = \{x \in S, r_2(x)\mu_2^2(x) - 2\mu_1(x)\mu_2(x)r_3(x) + \mu_1^2(x)r_4(x) \neq 0\}$.

Theorem 2. Under assumptions (B), (H), (M), (D) and (K), we have

$$\left(\frac{n h_n^d}{\sigma^2(x)} \right)^{1/2} (\hat{m}(x) - m(x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1), \quad \text{for all } x \in S^*$$

where

$$\sigma^2(x) = \frac{r_2(x)\mu_2^2(x) - 2\mu_1(x)\mu_2(x)r_3(x) + \mu_1^2(x)r_4(x)}{\mu_2^4(x)} \int_{\mathbb{R}^d} K_d^2(s) ds \quad (11)$$

with $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution.

3.3 Pointwise confidence intervals

In nonparametric estimation, the asymptotic variance depends on certain unknown functions. In our case, to determine the pointwise confidences intervals, we have to estimate the unknown quantity (10), that appears

in the asymptotic variance expression. Let us define a consistent estimator of $r_\ell(\cdot)$ for $\ell = 2, 3, 4$ and for all $x \in \mathbb{R}^d$ by

$$\hat{r}_\ell(x) = \frac{1}{nh_n^d} \sum_{i=1}^n \frac{\delta_i Y_i^{-\ell}}{\bar{G}_n^2(Y_i)} K_d\left(\frac{x - X_i}{h_n}\right). \quad (12)$$

Then, we replace (12) in (11) we get a calculable estimator :

$$\hat{\sigma}^2(x) = \frac{\hat{r}_2(x)\hat{\mu}_2^2(x) - 2\hat{\mu}_1(x)\hat{\mu}_2(x)\hat{r}_3(x) + \hat{\mu}_1^2(x)\hat{r}_4(x)}{\hat{\mu}_2^4(x)} \int_{\mathbb{R}^d} K_d^2(s) ds. \quad (13)$$

Finally, Theorem 2 becomes

Corollary 1. *Under the assumptions of Theorem 2, we have*

$$\left(\frac{nh_n^d}{\hat{\sigma}^2(x)}\right)^{1/2} (\hat{m}(x) - m(x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

The confidence intervals of level $0 < \beta < 1$ are giving by

$$\left[\hat{m}(x) - z_{1-\frac{\beta}{2}} \left(\frac{\hat{\sigma}^2(x)}{nh_n^d}\right)^{1/2}; \hat{m}(x) + z_{1-\frac{\beta}{2}} \left(\frac{\hat{\sigma}^2(x)}{nh_n^d}\right)^{1/2} \right]$$

where $z_{1-\frac{\beta}{2}}$ stands for the quantile of the standard normal distribution.

4 Simulations

The goal of this section is to examine the performance of $\hat{m}(\cdot)$. A generated data illustration of our RER estimator's asymptotic normality is done. We then compare the shape of their estimated density with normalized deviation with respect to the standard normal density. To this aim, two stationary strong mixing processes were considered in this study :

$$\text{Model 1 : } \begin{cases} X_i &= \rho_1 X_{i-1} + \sqrt{1 - \rho_1^2} \epsilon_i \\ T_i &= \frac{1}{2} X_{i+1} + 2 \end{cases} \quad i = 1, \dots, n.$$

and

$$\text{Model 2 : } \begin{cases} X_i &= \rho_1 X_{i-1} + \rho_2 X_{i-2} + \epsilon_i \\ T_i &= X_{i+1} + 4 \end{cases} \quad i = 1, \dots, n.$$

where $0 \leq \rho_1, \rho_2 \leq 1$ and ϵ_i is a white noise sequence and $X_0, X_1 \rightsquigarrow \mathcal{N}(0, 1)$. On the one hand, the model 1 is an (auto-regressive) AR(1) of theoretical function given by

$$m(x) = \frac{\rho_1}{2} x + 2. \quad (14)$$

On the other hand, the model 2 is an AR(2) process of theoretical function given by

$$m(x) = (\rho_1 + \rho_2)x + 4, \quad (15)$$

for all $x \in [1, 4]$. We simulate $(C_i)_i$ as an exponential law (i.e. with density $\lambda \exp(-\lambda x) \mathbb{1}_{\{x \geq 0\}}$) where the parameter λ is a varying constant which allows us to adapt the censoring rate (CR). The simulations are carried out for samples size $n = 100$, $n = 200$ and $n = 400$ and the results are obtained by using $L = 200$ replications. We calculate the following quantities:

$$\begin{aligned} \text{Model 1 : } \chi_j &:= \sqrt{\frac{nh_n}{\hat{\sigma}^2(0)}} (\hat{m}_j(0) - 2) \\ \text{Model 2 : } \omega_j &:= \sqrt{\frac{nh_n}{\hat{\sigma}^2(0)}} (\hat{m}_j(0) - 4), \quad \text{for } 1 \leq j \leq L. \end{aligned}$$

where $\hat{\sigma}^2(0)$ is computed from the formula (13) at $x = 0$. For the weights that appear in our estimator $\hat{m}(x)$, we use the standard Gaussian kernel function $\left(K(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)\right)$. In this study, three values of the parameters ρ_1 and ρ_2 are investigated, $\rho_i = 0.3, 0.6$ and 0.9 for $i = 1, 2$ (where 0.9 is a strong dependency). For each figure, we use a different bandwidth h_n that we select over a grid of 200 equidistant values of h_n in the grid 0.01 to 2 and we take the value that minimizes the cross-validation criterion (see the remark below). We construct for χ_j and ω_j where $1 \leq j \leq L$ a kernel estimator of the normal standard density, see [25]. It is well-known, that to estimate the density function, we choose $h^* = CL^{-0.2}$ where the constant C is appropriately chosen, see [26].

Remark 2. *The practical performance of a nonparametric regression model depend strongly on the smoothing parameter. Choosing an optimal h_n is often problematic. In order to obtain a satisfactory results, it is clear that an appropriate value of h_n is very important as it influence the behavior of the predictor. We use the cross-validation method to determine the optimal bandwidth. In our case of RER estimation, it remains to minimize the mean square of the relative prediction error. One may use the following criterion:*

$$CV_{h_n}(x) = \frac{1}{n-1} \sum_{i=1}^n \left(\frac{T_i - \hat{m}_{h_n}^{-i}(x)}{T_i} \right)^2$$

where $\hat{m}_{h_n}^{-i}(\cdot)$ is the RER estimator defined in (7) suppressing the triplet observation (X_i, Y_i, δ_i) , $i = 1, \dots, n$.

4.1 Model 1

4.1.1 Effect of sample size

From Fig. 1, we can see that the quality of fit becomes better when n increases. However, we have remarked that as the percentage of censoring increases, one needs a large sample to get better results.

Figure 1: $CR \approx 55\%$ and $\rho = 0.3$ for $n = 50, 200$ and 400 respectively.

4.1.2 Effect of dependency

From Fig. 2, as predicted when ρ_1 increases the quality of fit deteriorates a little bit. However, we see clearly that it remains well.

Figure 2: $n = 200$ and $CR \approx 55\%$ for $\rho_1 = 0.3, 0.6$ and 0.9 respectively.

4.1.3 Effect of censoring

As expected, we can see from Fig. 3 the quality of the estimator decreases with increasing censoring. This can be explained by the fact that we observe less true values.

Figure 3: $n = 200$ and $\rho = 0.6$ for $CR \approx 18, 54$ and 80% respectively.

4.2 Model 2

4.2.1 Effect of sample size

As claimed in Fig 1, the quality of fit to $\mathcal{N}(0, 1)$ curve becomes better when n rises. We can also remark that there is no significant difference in the performance between model 1 and model 2 (see Fig. 4).

Figure 4: $CR \approx 48\%$ and $(\rho_1, \rho_2) = (0.3, 0.3)$ for $n = 50, 200$ and 400 respectively.

4.2.2 Effect of censoring

From 5, we can observe that when the CR is small the theoretical and estimated curves are very close. However, when the CR is high (83%), the estimated curve deviates from the theoretical one taking into account that $\rho_2 = 0.8$ which is predictable.

Figure 5: $n = 200$ and $(\rho_1, \rho_2) = (0.1, 0.8)$ for $CR \approx 12, 42, 83\%$ respectively.

4.2.3 Effect of dependency

As we can see from Fig. 6, the Gaussian standard normality estimator is slightly affected by the degree of dependency in data even if we only observe almost half of the sample size. Generally, the instability increases as both ρ_1, ρ_2 tend 1.

So, we can conclude that the main variation in the quality of the proposed estimator is connected with strong censoring and dependency levels.

Figure 6: $n = 200$ and $CR \approx 46\%$ for $(\rho_1, \rho_2) = (0.1, 0.1), (0.2, 0.5)$ and $(0.3, 0.8)$ respectively.

4.3 Coverage probability

In the following table, we have the coverage probabilities for confidence levels $1 - \beta$ where $\beta = 0.01$ and 0.05 . We performed 100 replications of sample size $n = 50, 200$ and 400 with different censoring rates $CR \approx 10, 30$ and 50 and dependency levels $\rho_1 = 0.3$ and 0.9 . Moreover, as we stated that our estimator is robust to the presence of outliers in the sample. We generate outliers in the sample by multiplying a certain number of observed data Y_i for $\{i = 20 \times \iota \text{ with } 1 \leq \iota \leq \frac{n}{20}\}$ by a multiplying coefficient (MC) that we vary. In order to calculate the coverage probabilities, we observe the proportion of times that the confidence intervals contains the true regression function. We would like to mention that the model used here is model 1. After analyzing and comparing, we have the following remarks. Note that, for $n = 50$ and $\rho_1 = 0.3$ the coverage probabilities are equal to 1 expected two cases ($CR \approx 30\%$, $\beta = 0.05$, $MC = 80$) and ($CR \approx 50\%$, $\beta = 0.01$, $MC = 10$). The justification for this comes from the fact that for the first one, the MC is high and for the second one the CR is high. Regarding the dependence in data, by comparing for $\rho_1 = 0.3$ and $\rho_1 = 0.9$ (which means comparing between weak and strong dependency), we can see that ρ_1 has relative effect on the results depending on the other parameters. Furthermore, we can see that the MC has no significant influence on the coverage probabilities. This may be explained by the robustness of the relative regression to outliers. The latter remains an important element of table 1. It is clear that as the sample size increases, the probability that the interval contains the true value becomes small due to the fact that the width of the intervals reduces. Another interesting remark related to dependency is the fact that with $\rho_1 = 0.9$, the proportion of observed true values decreases which is normal because that data are in a strong dependence case. Obviously, the RER estimator behaves better under weak dependency.

Remark 3. Regarding the number of replications, it is not very high, this is due only for the duration of the calculations and the fact that there are many loops in the program. Furthermore, we see clearly, that as the sample size increases and in the strong dependency the recovery decreases. We point out that coverage remains stable with respect to high outliers, which is one of the advantages of the relative error method.

4.4 Pointwise confidence interval

In this subsection, we implement our methodology and see how good is the asymptotic normality in the determination of confidences bands. To this purpose, we consider the model 1 and construct a 95%-confidence curves given in corollary 1. On the four figures that follow, we display the theoretical function 14 and the RER function estimator $\hat{m}(x)$ for $x \in [1, 4]$. The kernel and the bandwidth are chosen as above.

Table 1: Coverage probabilities for 100 replications.

CR($\approx$ %)	β	MC	$n = 50$		$n = 200$		$n = 400$	
			$\rho_1 = 0.3$	$\rho_1 = 0.9$	$\rho_1 = 0.3$	$\rho_1 = 0.9$	$\rho_1 = 0.3$	$\rho_1 = 0.9$
10	0.05	10	1.0000	0.8321	0.9996	0.9148	0.8774	0.7749
		40	1.0000	0.8296	0.9975	0.9112	0.8742	0.7623
		80	1.0000	0.8141	0.9971	0.8939	0.8670	0.7485
	0.01	10	1.0000	0.8540	0.9995	0.9148	0.9077	0.8022
		40	1.0000	0.8499	0.9994	0.8905	0.8973	0.8009
		80	1.0000	0.8354	0.9992	0.8760	0.8932	0.7991
30	0.05	10	1.0000	0.8224	0.9989	0.9047	0.8915	0.7926
		40	1.0000	0.8013	0.9974	0.9025	0.9807	0.7813
		80	0.9995	0.7448	0.9961	0.8402	0.8779	0.7439
	0.01	10	1.0000	0.8327	0.9990	0.9355	0.9216	0.8130
		40	1.0000	0.8187	0.9993	0.9204	0.9155	0.8048
		80	1.0000	0.8172	0.9993	0.8875	0.8979	0.7757
50	0.05	10	1.0000	0.7770	0.9996	0.8620	0.9126	0.7850
		40	1.0000	0.7318	0.9994	0.8435	0.9090	0.6748
		80	1.0000	0.7251	0.9993	0.8327	0.9032	0.6665
	0.01	10	0.9970	0.7987	0.9998	0.9025	0.9497	0.7961
		40	1.0000	0.7687	0.9994	0.9019	0.9381	0.7566
		80	1.0000	0.6823	0.9993	0.8476	0.9349	0.7309

4.4.1 Weak dependency and censoring

Fig. 7 represents the "ideal" case where the data are weakly dependent and the censoring percentage is approximately 1/4 of the sample size. Obviously, the confidence bands width decreases when n increases which is predictable.

Figure 7: $\rho_1 = 0.3$ and $CR \approx 25\%$ for $n = 50, 200$ and 400 respectively.

4.4.2 Weak dependency and strong censoring

From Fig 8, we can see that for moderate dependency level ($\rho_1 = 0.3$) and a strong percentage of censoring (85%), the RER estimator behaves well. An interesting remark related to censoring is the fact that even with a high level of censoring the RER function estimator continues to track the theoretical curve.

Figure 8: $\rho_1 = 0.3$ with CR $\approx 85\%$ and $n = 50, 200$ and 400 respectively.

4.4.3 Strong dependency and weak censoring

Fig. 9 is the opposite case of Fig. 8. We can observe that the quality of adjustment becomes better when n rises as well as the intervals decrease. It can be seen that the dependency in the data influences mostly the right edge. We can say the fact that the RER estimator remains resistant to a strong dependence.

Figure 9: $\rho_1 = 0.9$ with CR $\approx 25\%$ and $n = 50, 200$ and 400 respectively.

4.4.4 Strong dependency and censoring

Fig. 10 displays the "worst" case which is a high levels of censoring (85%) and dependency ($\rho_1 = 0.9$). As we can see, the estimator deviates completely from the theoretical curve at the right edge. This is due to the high rate of censoring and dependence. This becomes more apparent as the intervals become smaller (when

$n = 400$). This can eventually be corrected by choosing the value of the risk β at 1% which allows us to increase the interval. However we preferred to keep the same value of beta risk to be able comparing the results.

Figure 10: $\rho_1 = 0.9$ with $CR \approx 85\%$ and $n = 50, 200$ and 400 respectively.

4.5 A real data study

In this subsection, we analyze a real data set to illustrate the efficiency of the RER predictor in the presence of censored dependent data. The data consists on patients diagnosed with acquired immunodeficiency syndrome (AIDS) in Australia before 1st July 1991. The source of data is Dr P.J. Solomon and the Australian National Center in HIV Epidemiology and Clinical Research. For more information about the data, see : [28]. A total of $N = 2843$ patients have been taken into account where about 1770 had died by the end of the study. The file contains the dates of first diagnosis; the date of death or end of observation; the sex of patients; age (in year) at diagnosis; the status ("A" for alive or "D" for dead) at the end of observation; as well as the state and the coded transmission category. The factors that are expected to be important is the times to death and diagnosis. After calculating the covariance matrix, we found a strong correlation between time to diagnosis and time to death, (the correlation coefficient is equal to 0.8312) which permit to claim that the data dependent. So, by using the partial correlation plot, it appears that our data are an auto-regressive process of order greater than 1. In the aim of prediction and pointwise 95%-confidence bounds, we split the N -sample into two subsets. The learning sample of size $n = 2400$ (i.e. $\{(Y_i, \delta_i, X_i), 1 \leq i \leq n\}$) will be our statistical sample for which our estimator is calculated. The test sample of size $n^* = 443$ allows us to evaluate our predictions. In addition, the kernel function and the optimal bandwidth are selected as for generated data. The censoring rate is approximately equal to 62% in N -sample. Note that, we eliminate the censoring data from the predicted values (i.e. if for any $n + 1 \leq i \leq N$ the observed variable $Y_i = C_i$, we remove the observation Y_i from the predicted values. Obviously, censored data cannot be predicted. We forecast only the interest variables). We point out, the CR in the N -sample is different of the CR of n^* -sample due that the selection is random. Then, only 13 values are predicted from the n^* -sample. For the left hand of Fig. 11 we show a scatter plot of the data where we distinguish between the censored and uncensored data. For the right hand, we plot true, predicted and confidence bands. It can be seen that all true value are very near to predicted value and between the upper and lower bands. This shows that our method is robust.

Figure 11: Australian AIDS survival scatter plot of censored / uncensored data. (left)
RER prediction of survival time versus time to diagnosis for AIDS patients with pointwise interval confidence. (right)

5 Appendix : Proofs

For $x \in \mathbb{R}^d$, we consider the following decomposition :

$$\hat{m}(x) - m(x) = \frac{1}{\hat{\mu}_2(x)} \left\{ \sum_{1 \leq k \leq 3} \mathcal{B}_{k,1}(x) - m(x) \sum_{1 \leq k \leq 3} \mathcal{B}_{k,2}(x) \right\}$$

where

$$\begin{aligned} \mathcal{B}_{1,\gamma}(x) &:= \hat{\mu}_\gamma(x) - \tilde{\mu}_\gamma(x), \\ \mathcal{B}_{2,\gamma}(x) &:= \tilde{\mu}_\gamma(x) - \mathbb{E}[\tilde{\mu}_\gamma(x)], \\ \mathcal{B}_{3,\gamma}(x) &:= \mathbb{E}[\tilde{\mu}_\gamma(x)] - \mu_\gamma(x), \end{aligned} \quad \text{for } \gamma = 1, 2. \quad (16)$$

Proof of Theorem 1. To establish the result, it suffices to show that each term in (16) tends to 0 in probability. For $\mathcal{B}_{1,\gamma}(x)$ and $\mathcal{B}_{3,\gamma}(x)$, we have from Lemma 1 and Lemma 2 in [3] that

$$\mathcal{B}_{1,\gamma}(x) = O_{a.s.} \left(\sqrt{\frac{\log \log n}{n}} \right)$$

and

$$\mathcal{B}_{3,\gamma}(x) = O(h_n).$$

We consider now the dominant term $\{\mathcal{B}_{2,\gamma}(x), \text{ for } \gamma = 1, 2\}$ and prove that

$$\mathcal{V} := \text{Var}(\tilde{\mu}_\gamma(x)) \rightarrow 0.$$

We have

$$\begin{aligned} \mathcal{V} &= \frac{1}{(nh_n^d)^2} \left\{ \sum_{i=1}^n \text{Var} \left(\tilde{T}_i^{-\gamma} K_d \left(\frac{x - X_i}{h_n} \right) \right) + \sum_{\substack{i,j=1 \\ i \neq j}}^n \text{Cov} \left(\tilde{T}_i^{-\gamma} K_d \left(\frac{x - X_i}{h_n} \right), \tilde{T}_j^{-\gamma} K_d \left(\frac{x - X_j}{h_n} \right) \right) \right\} \\ &=: \frac{1}{(nh_n^d)^2} \left\{ \sum_{i=1}^n \mathcal{V}_1 + \sum_{\substack{i,j=1 \\ i \neq j}}^n \mathcal{V}_2 \right\}. \end{aligned}$$

We have now to deal with $\mathcal{V}_1$; for this compute

$$\begin{aligned}\mathcal{V}_1 &= \mathbb{E} \left[\frac{\delta_1 Y_1^{-2\gamma}}{\bar{G}^2(Y_1)} K_d^2 \left(\frac{x - X_1}{h_n} \right) \right] - \mathbb{E}^2 \left[\tilde{T}_1^{-\gamma} K_d \left(\frac{x - X_1}{h_n} \right) \right] \\ &=: \mathcal{V}_{1,1} - \mathcal{V}_{1,2}.\end{aligned}$$

For $\mathcal{V}_{1,1}$, let denote by $\{\ell = 2\gamma \text{ where } \gamma = 1, 2\}$. Then, using the conditional expectation property and a change of variable, we get

$$\begin{aligned}\mathcal{V}_{1,1} &= \int K_d^2 \left(\frac{x-u}{h_n} \right) \int \frac{t^{-\ell}}{\bar{G}(t)} f(t|u) dt f(u) du \\ &= \int K_d^2 \left(\frac{x-u}{h_n} \right) \int \frac{t^{-\ell}}{\bar{G}(t)} f(u, t) dt du \\ &= h_n^d \int K_d^2(t) r_\ell(x - h_n t) dt.\end{aligned}$$

Using first order Taylor expansion for $\xi \in]x - h_n t, x[$, we have

$$r_\ell(x - h_n t) = r_\ell(x) - h_n \sum_{i=1}^d t_i \frac{\partial r_\ell(\xi)}{\partial x_i}. \quad (17)$$

Hence, under (D)(iii) and (K)(ii) we get

$$\begin{aligned}|\mathcal{V}_{1,1}| &= h_n^d \left| \int K_d^2(t) \left(r_\ell(x) - h_n \sum_{i=1}^d t_i \frac{\partial r_\ell(\xi)}{\partial x_i} \right) dt \right| \\ &\leq h_n^d \int K_d^2(t) \left(|r_\ell(x)| + h_n \left| \sum_{i=1}^d t_i \frac{\partial r_\ell(\xi)}{\partial x_i} \right| \right) dt \\ &= C_1 O(h_n^d). \quad (18)\end{aligned}$$

For $\mathcal{V}_{1,2}$, using the conditional expectation property, a change of variable and second order Taylor development of $\mu_\gamma(\cdot)$, for $\gamma = 1, 2$. Under (K)(ii, iii) and (D)(i) we get

$$\begin{aligned}|\mathcal{V}_{1,2}| &= \left| \left(\int K_d \left(\frac{x-u}{h_n} \right) \mu_\gamma(u) du \right)^2 \right| \\ &= h_n^{2d} \left| \left(\int K_d(t) \mu_\gamma(x - h_n t) dt \right)^2 \right| \\ &= h_n^{2d} \left| \left(\mu_\gamma(x) + \frac{h_n^2}{2} \int K_d(t) \left\{ \sum_{i=1}^d t_i^2 \frac{\partial^2 \mu_\gamma(\xi)}{\partial x_i^2} + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^d t_i t_j \frac{\partial^2 \mu_\gamma(\xi)}{\partial x_i \partial x_j} \right\} dt \right)^2 \right| \\ &\leq h_n^{2d} \left(|\mu_\gamma(x)| + \frac{h_n^2}{2} \left\{ \int K_d(t) \sum_{i=1}^d t_i^2 \left| \frac{\partial^2 \mu_\gamma(\xi)}{\partial x_i^2} \right| dt + 2 \int K_d(t) \sum_{\substack{i,j=1 \\ i \neq j}}^d |t_i t_j \frac{\partial^2 \mu_\gamma(\xi)}{\partial x_i \partial x_j}| dt \right\} \right)^2 \\ &= C_2 O(h_n^{2d}). \quad (19)\end{aligned}$$

Then, combining (18) and (19) give

$$\begin{aligned}|\mathcal{V}_1| &\leq |\mathcal{V}_{1,1}| + |\mathcal{V}_{1,2}| \\ &= C_1 O(h_n^d) + C_2 O(h_n^{2d}) \\ &= C_3 O(h_n^d).\end{aligned} \quad (20)$$

Next, we will calculate the second term $\mathcal{V}_2$. Using the conditional expectation property, a change of variable and under (K)(i) and (D)(i, ii) we obtain

$$\begin{aligned}
 |\mathcal{V}_2| &= \left| \mathbb{E} \left[\tilde{T}_1^{-\gamma} \tilde{T}_2^{-\gamma} K_d \left(\frac{x - X_1}{h_n} \right) K_d \left(\frac{x - X_2}{h_n} \right) \right] \right. \\
 &\quad \left. - \mathbb{E} \left[\tilde{T}_1^{-\gamma} K_d \left(\frac{x - X_1}{h_n} \right) \right] \mathbb{E} \left[\tilde{T}_2^{-\gamma} K_d \left(\frac{x - X_2}{h_n} \right) \right] \right| \\
 &= \left| \int \int K_d \left(\frac{x - u}{h_n} \right) K_d \left(\frac{x - v}{h_n} \right) \mu_\gamma(u, v) du dv \right. \\
 &\quad \left. - \int K_d \left(\frac{x - u}{h_n} \right) \mu_\gamma(u) du \int K_d \left(\frac{x - v}{h_n} \right) \mu_\gamma(v) dv \right| \\
 &\leq \int \int K_d \left(\frac{x - u}{h_n} \right) K_d \left(\frac{x - v}{h_n} \right) |\mu_\gamma(u, v) - \mu_\gamma(u) \mu_\gamma(v)| du dv \\
 &\leq h_n^{2d} \int \int K_d(t) K_d(s) |\mu_\gamma(x - h_n t, x - h_n s) - \mu_\gamma(x - h_n t) \mu_\gamma(x - h_n s)| dt ds \\
 &= C_4 O(h_n^{2d}).
 \end{aligned} \tag{21}$$

Let introduce a sequence of integers $v_n \rightarrow \infty$, so that

$$(nh_n^d)^{-2} \sum_{\substack{1 \leq i, j \leq n \\ i \neq j}} |\mathcal{V}_2| \leq (nh_n^d)^{-2} \sum_{1 \leq |i-j| \leq v_n} |\mathcal{V}_2| + (nh_n^d)^{-2} \sum_{v_n < |i-j| \leq n} |\mathcal{V}_2|. \tag{22}$$

On the one hand, let $[x]$ is the integer part of x , by choosing $v_n = [h_n^{-d}]$, we get from (21)

$$\begin{aligned}
 (nh_n^d)^{-2} \sum_{1 \leq |i-j| \leq v_n} |\mathcal{V}_2| &\leq C_5 (nh_n^d)^{-2} h_n^{2d} n v_n \\
 &= C_5 O((nh_n^d)^{-1}).
 \end{aligned} \tag{23}$$

On the other hand, from [2], we have $|\mathcal{V}_2| \leq C\alpha(|i-j|)$, then

$$\begin{aligned}
 (nh_n^d)^{-2} \sum_{v_n < |i-j| \leq n} |\mathcal{V}_2| &\leq C_5 (nh_n^d)^{-2} \sum_{v_n < |i-j| \leq n} \alpha(|i-j|) \\
 &\leq C_6 n^2 (nh_n^d)^{-2} \alpha(v_n) \\
 &\leq C_6 h_n^{d(v-2)} = o(1).
 \end{aligned} \tag{24}$$

In conclusion, from (20) we have $\mathcal{V}_1 \rightarrow 0$ and combining (23) and (24) in (22) we have $\mathcal{V}_2 \rightarrow 0$ yields $\mathcal{V} \rightarrow 0$.

Proof of Theorem 2. According to the decomposition (16), we establish the next lemmas that estimates the negligible terms $\mathcal{B}_{1,\gamma}(x)$ and $\mathcal{B}_{3,\gamma}(x)$, the other lemmas control the main term $\mathcal{B}_{2,\gamma}(x)$, by calculating the variance and covariance.

Lemma 1. Under assumptions (H), (K)(ii) and (D)(i) we have for $\gamma = 1, 2$ and n large enough

$$\begin{aligned}
 \sqrt{nh_n^d} |\mathcal{B}_{1,\gamma}(x)| &= o(1), \\
 \sqrt{nh_n^d} |\mathcal{B}_{3,\gamma}(x)| &= o(1).
 \end{aligned}$$

Proof of Lemma 1. Following the proof of lemmas 1 and 2 in [3], we get

$$\begin{aligned}
 \sqrt{nh_n^d} |\mathcal{B}_{1,\gamma}(x)| &= O\left(\sqrt{h_n^d \log \log n}\right) = o(1), \\
 \sqrt{nh_n^d} |\mathcal{B}_{3,\gamma}(x)| &= O\left(\sqrt{nh_n^{d+2}}\right) = o(1).
 \end{aligned}$$

which conclude the proof.

Lemma 2. Under assumptions (K) and (D) with $\{\ell = 2\gamma$, for $\gamma = 1, 2\}$, we have

$$nh_n^d \text{Var}(\mathcal{B}_{2,\gamma}(x)) \longrightarrow \sigma_\gamma(x) := r_\ell(x) \int_{\mathbb{R}^d} K_d^2(t) dt.$$

Proof of Lemma 2. Let for $1 \leq i \leq n$

$$U_{i,\gamma}(x) := \tilde{T}_i^{-\gamma} K_d \left(\frac{x - X_i}{h_n} \right) - \mathbb{E} \left[\tilde{T}_i^{-\gamma} K_d \left(\frac{x - X_i}{h_n} \right) \right], \quad \text{for } \gamma = 1, 2.$$

In terms of that quantity we have

$$\mathcal{B}_{2,\gamma}(x) = \frac{1}{nh_n^d} \sum_{i=1}^n U_{i,\gamma}(x).$$

For this, note that for $\gamma = 1, 2$

$$\begin{aligned} nh_n^d \text{Var}(\mathcal{B}_{2,\gamma}(x)) &= \frac{1}{nh_n^d} \text{Var} \left(\sum_{i=1}^n U_{i,\gamma}(x) \right) \\ &= \frac{1}{h_n^d} \mathbb{E} [U_{1,\gamma}^2(x)] + \frac{1}{nh_n^d} \sum_{\substack{i,j=1 \\ i \neq j}}^n \mathbb{E} [U_{i,\gamma}(x) U_{j,\gamma}(x)] \\ &=: \mathcal{A}_1 + \mathcal{A}_2. \end{aligned}$$

We shall first study $\mathcal{A}_1$ in greater detail. We have

$$\begin{aligned} \mathcal{A}_1 &= \frac{1}{h_n^d} \left\{ \mathbb{E} \left[\frac{\delta_1 Y^{-\ell}}{G^2(Y_1)} K_d^2 \left(\frac{x - X_1}{h_n} \right) \right] - \mathbb{E}^2 \left[\tilde{T}_1^{-\gamma} K_d \left(\frac{x - X_1}{h_n} \right) \right] \right\} \\ &=: \frac{1}{h_n^d} \{ \mathcal{A}_{1,1} - \mathcal{A}_{1,2} \}. \end{aligned}$$

As a consequence of (18), for $\ell = 2\gamma$ with $\gamma = 1, 2$, we get that

$$\mathcal{A}_{1,1} = h_n^d r_\ell(x) \int K_d^2(t) dt,$$

and from (19) we obtain

$$\mathcal{A}_{1,2} \leq O(h_n^{2d}).$$

Then, we join the two results obtained just before by

$$\begin{aligned} \mathcal{A}_1 &\leq h_n^{-d} \{ \mathcal{A}_{1,1} - \mathcal{A}_{1,2} \} \\ &\leq h_n^{-d} \left\{ h_n^d r_\ell(x) \int K_d^2(t) dt - h_n^{2d} \right\} \\ &= r_\ell(x) \int K_d^2(t) dt. \end{aligned} \tag{25}$$

From (21) we have

$$|\mathbb{E} [U_{i,\gamma}(x) U_{j,\gamma}(x)]| \leq Ch_n^{2d}. \tag{26}$$

Now, to deal with the term $\mathcal{A}_2$ we consider a sequence of integer v_n defined before, so that

$$\mathcal{A}_2 = \frac{1}{nh_n^d} \left\{ \sum_{1 \leq |i-j| \leq v_n} \mathbb{E} [U_{i,\gamma}(x) U_{j,\gamma}(x)] + \sum_{v_n < |i-j| \leq n} \mathbb{E} [U_{i,\gamma}(x) U_{j,\gamma}(x)] \right\}.$$

On the one hand, from (26) we have

$$\sum_{1 \leq |i-j| \leq v_n} |\mathbb{E} [U_{i,\gamma}(x) U_{j,\gamma}(x)]| \leq Ch_n^{2d} n v_n.$$

By choosing for $b > 1$, $v_n = [(h_n^{-d})^{1/b}]$, we get

$$\frac{1}{nh_n^d} \sum_{1 \leq |i-j| \leq v_n} |\mathbb{E}[U_{i,\gamma}(x)U_{j,\gamma}(x)]| \leq Ch_n^{(b-1)d/b} \rightarrow 0. \quad (27)$$

On the other hand, we use a version of the moment inequality due to [24]. Let a_1, a_2 and a_3 taking values in $\mathbb{R}^+$ and greater or equal to 1 such that: $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = 1$ and $b < a_1 < \frac{v}{2}$. We have

$$\begin{aligned} \frac{1}{nh_n^d} \sum_{v_n < |i-j| \leq n} |\mathbb{E}[U_{i,\gamma}(x)U_{j,\gamma}(x)]| &\leq (nh_n^d)^{-1} \sum_{v_n < |i-j| \leq n} C \alpha^{\frac{1}{a_1}}(|i-j|) \\ &\times \left(\mathbb{E} \left| K_d \left(\frac{x - X_i}{h_n} \right) \right|^{a_2} \right)^{\frac{1}{a_2}} \left(\mathbb{E} \left| K_d \left(\frac{x - X_j}{h_n} \right) \right|^{a_3} \right)^{\frac{1}{a_3}}. \end{aligned}$$

We have

$$\mathbb{E} \left| K_d \left(\frac{x - X_i}{h_n} \right) \right|^{a_2} \leq Ch_n^d.$$

So, under (M)(ii) and for $b < a < \frac{v}{2}$ we have

$$\begin{aligned} \frac{1}{nh_n^d} \sum_{v_n < |i-j| \leq n} \mathbb{E}[U_{i,\gamma}(x)U_{j,\gamma}(x)] &\leq \frac{C}{nh_n^d} \sum_{v_n < |i-j| \leq n} \alpha^{\frac{1}{a_1}}(|i-j|)(h_n^d)^{\frac{1}{a_2}}(h_n^d)^{\frac{1}{a_3}} \\ &\leq \frac{C}{nh_n^d} \sum_{v_n < |i-j| \leq n} \alpha^{\frac{1}{a_1}}(|i-j|)h_n^{\frac{d}{a_1}(a_1-1)} \\ &\leq \frac{C}{nh_n^{\frac{d}{a_1}}} \sum_{v_n < |i-j| \leq n} \alpha^{\frac{1}{a_1}}(|i-j|) \\ &\leq \frac{C}{nh_n^{\frac{d}{a_1}}} \sum_{v_n < |i-j| \leq n} \frac{|i-j|}{v_n} \alpha^{\frac{1}{a_1}}(|i-j|) \\ &\leq \frac{C}{nh_n^{\frac{d}{a_1}}} + \sum_{s > v_n} \frac{s}{h_n^{\frac{d}{b}}} \alpha^{\frac{1}{a_1}}(s) \\ &\leq \frac{C}{nh_n^{d(\frac{1}{a_1} - \frac{1}{b})}} \sum_{s > v_n} s \alpha^{\frac{1}{a_1}}(s) \rightarrow 0. \end{aligned}$$

Finally combining (27) and (28) we get the result of Lemma 2.

Lemma 3. Under assumptions (K) and (D)(i, iii), we have

$$\text{Cov} \left(\sqrt{nh_n^d} \mathcal{B}_{2,1}(x), \sqrt{nh_n^d} \mathcal{B}_{2,2}(x) \right) \rightarrow \sigma_3(x) := r_3(x) \int_{\mathbb{R}^d} K_d^2(t) dt.$$

Proof of Lemma 3. By definition we get

$$\begin{aligned} \sqrt{nh_n^d} \text{Cov}(\mathcal{B}_{2,1}(x), \mathcal{B}_{2,2}(x)) &= nh_n^d \{ \mathbb{E}[\mathcal{B}_{2,1}(x)\mathcal{B}_{2,2}(x)] - \mathbb{E}[\mathcal{B}_{2,1}(x)]\mathbb{E}[\mathcal{B}_{2,2}(x)] \} \\ &=: nh_n^d \{ \mathcal{S}_3 - \mathcal{S}_1\mathcal{S}_2 \}. \end{aligned}$$

On the one hand, we have

$$\begin{aligned} \mathcal{S}_3 &= \mathbb{E}[\tilde{\mu}_1(x)\tilde{\mu}_2(x)] - \mathbb{E}[\tilde{\mu}_1(x)]\mathbb{E}[\tilde{\mu}_2(x)] \\ &=: \mathcal{S}_{3,1} - \mathcal{S}_{3,2}. \end{aligned}$$

For $\mathcal{S}_{3,1}$ we have

$$\begin{aligned}\mathcal{S}_{3,1} &= \frac{1}{(nh_n^d)^2} \sum_{i,j=1}^n \mathbb{E} \left[\tilde{T}_i^{-1} \tilde{T}_j^{-2} K_d \left(\frac{x - X_i}{h_n} \right) K_d \left(\frac{x - X_j}{h_n} \right) \right] \\ &= \frac{1}{(nh_n^d)^2} \left\{ \sum_{i,j=1}^n \mathbb{E} \left[\frac{\delta_i Y_i^{-3}}{\bar{G}^2(Y_i)} K_d^2 \left(\frac{x - X_i}{h_n} \right) \right] \right. \\ &\quad \left. + \sum_{\substack{i,j=1 \\ i \neq j}}^n \mathbb{E} \left[\tilde{T}_i^{-1} \tilde{T}_j^{-2} K_d \left(\frac{x - X_i}{h_n} \right) K_d \left(\frac{x - X_j}{h_n} \right) \right] \right\} \\ &= \frac{1}{(nh_n^d)^2} \left\{ \sum_{i,j=1}^n \mathcal{S}_{3,1,1} + \sum_{\substack{i,j=1 \\ i \neq j}}^n \mathcal{S}_{3,1,2} \right\}.\end{aligned}$$

For $\mathcal{S}_{3,1,1}$, using the conditional expectation property, a change of variable and Taylor expansion in (17) for $\ell = 3$, we have

$$\begin{aligned}\mathcal{S}_{3,1,1} &= \int K_d^2 \left(\frac{x - u}{h_n} \right) \int \frac{t^{-3}}{\bar{G}(t)} f(t|u) dt f(u) du \\ &= \int K_d^2 \left(\frac{x - u}{h_n} \right) r_3(u) du \\ &= h_n^d \int K_d^2(t) r_3(x - h_n t) dt \\ &\leq h_n^d r_3(x) \int_{\mathbb{R}^d} K_d^2(t) dt.\end{aligned}$$

From (26), we have $|\mathbb{E}[U_{i,\gamma}(x)U_{j,\gamma}(x)]| \leq Ch_n^{2d}$ then

$$\begin{aligned}\mathcal{S}_{3,1} &\leq (nh_n^d)^{-2} \left\{ nh_n^d r_3(x) \int_{\mathbb{R}^d} K_d^2(t) dt + O((nh_n^d)^2) \right\} \\ &= (nh_n^d)^{-1} r_3(x) \int_{\mathbb{R}^d} K_d^2(t) dt + o(1)\end{aligned}$$

which conclude the proof of Lemma 3.

In order to prove the asymptotic normality of $\sqrt{nh_n^d} \mathcal{B}_{2,\gamma}(x)$ for $\gamma = 1, 2$, we use the Doob technique (see : [8], pages 228 to 232). The idea is to split the n -sample into large and small blocks : p -blocks and q -blocks respectively. For this purpose, let $p := p_n < n$ and $q := q_n < n$ satisfying the assumption (M).

For a given pair of real numbers (c_1, c_2) , define

$$\mathcal{B}_{2,n}(x) := \sqrt{nh_n^d} \sum_{\gamma=1}^2 c_\gamma \mathcal{B}_{2,\gamma}(x) = c_1 \sqrt{nh_n^d} \mathcal{B}_{2,1}(x) + c_2 \sqrt{nh_n^d} \mathcal{B}_{2,2}(x). \quad (28)$$

Lemma 4. Under assumptions (K), (B), (D) and (M) we have

$$\mathcal{B}_{2,n}(x) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_4^2(x))$$

where $\sigma^2(x) = \kappa c^t \Sigma_x c$ for $c = (c_1, c_2)^t$ and

$$\Sigma_x = \begin{pmatrix} r_2(x) & r_3(x) \\ r_3(x) & r_4(x) \end{pmatrix}.$$

Proof of Lemma 4. We have from (28) that

$$\begin{aligned} \text{Var}(\mathcal{B}_{2,n}(x)) &= \text{Var}\left(c_1 \sqrt{nh_n^d} \mathcal{B}_{1,2}(x) + c_2 \sqrt{nh_n^d} \mathcal{B}_{2,2}(x)\right) \\ &= c_1^2 nh_n^d \text{Var}(\mathcal{B}_{1,2}(x)) + c_2^2 nh_n^d \text{Var}(\mathcal{B}_{2,2}(x)) + 2c_1 c_2 nh_n^d \text{Cov}(\mathcal{B}_{1,2}(x), \mathcal{B}_{2,2}(x)) \end{aligned}$$

which the variance terms have been studied in Lemma 2 and the covariance term has been treated in Lemma 3. Now let $W_i(x) := c_1 U_{i,1}(x) + c_2 U_{i,2}(x)$, then

$$\mathcal{B}_{2,n}(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{W_i(x)}{\sqrt{h_n^d}} =: \frac{\tilde{W}_n(x)}{\sqrt{n}}. \quad (29)$$

Our main aim is to show that :

$$\frac{\tilde{W}_n(x)}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_4^2(x)).$$

Using Doob's technique, under (B) we divide the set into k large p -blocks and k small q -blocks. Let

$$\begin{aligned} w_{1,j} &= \sum_{i=(j-1)(p_n+q_n)+1}^{(j-1)(p_n+q_n)+p_n} \frac{W_i(x)}{\sqrt{h_n^d}}, \\ w_{2,j} &= \sum_{i=(j-1)(p_n+q_n)+p_n+1}^{j(p_n+q_n)} \frac{W_i(x)}{\sqrt{h_n^d}}, \\ w_{3,k} &= \sum_{i=k(p_n+q_n)+1}^n \frac{W_i(x)}{\sqrt{h_n^d}} \end{aligned}$$

and

$$W_{1,n} = \sum_{j=1}^k w_{1,j}, \quad W_{2,n} = \sum_{j=1}^k w_{2,j} \quad \text{and} \quad W_{3,n} = \sum_{j=1}^k w_{3,j}$$

where $W_n(x) = W_{1,n} + W_{2,n} + W_{3,n}$.

$$\frac{1}{\sqrt{n}} W_{1,n} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_4^2(x)), \quad (30)$$

and

$$\frac{1}{n} \mathbb{E}[W_{2,n}^2] + \frac{1}{n} \mathbb{E}[W_{3,n}^2] \rightarrow 0. \quad (31)$$

We start by proving (31)

$$\begin{aligned} \mathbb{E}[W_{2,n}^2] &= \mathbb{E}\left[\left(\sum_{j=1}^k w_{2,j}\right)^2\right] \\ &= \sum_{j=1}^k \text{Var}(w_{2,j}) + 2 \sum_{1 \leq j, l \leq k} \text{Cov}(w_{2,j}, w_{2,l}). \end{aligned} \quad (32)$$

On the one hand, we have

$$\begin{aligned} \text{Var}(w_{2,j}) &= \text{Var}\left(\sum_{i=(j-1)(p_n+q_n)+p_n+1}^{j(p_n+q_n)} \frac{W_i(x)}{\sqrt{h_n^d}}\right) \\ &= q_n \text{Var}\left(\frac{W_1(x)}{h_n^d}\right) + \sum_{0 \leq |i-j| \leq q_n} \text{Cov}\left(\frac{W_i(x)}{h_n^d}, \frac{W_l(x)}{h_n^d}\right) \\ &\leq q_n \text{Var}\left(\frac{W_1(x)}{\sqrt{h_n^d}}\right) + o(q_n). \end{aligned} \quad (33)$$

So,

$$\sum_{j=1}^k \text{Var}(w_{2,j}) = kq \text{Var}\left(\frac{W_1(x)}{\sqrt{h_n^d}}\right) + o(kq_n). \quad (34)$$

On the other hand, we have

$$\text{Var}\left(\frac{W_1(x)}{\sqrt{h_n^d}}\right) = c_1^2 \sigma_1^2(x) + c_2^2 \sigma_2^2(x) + 2c_1 c_2 \sigma_3(x) + o(1). \quad (35)$$

From assumption (B) we have that $kq_n = \frac{nq_n}{(p_n + q_n)} = o(n)$ then

$$\sum_{j=1}^k \text{Var}(w_{2,j}) = o(n). \quad (36)$$

Now, by stationary and assumption (M)(ii), we have

$$\begin{aligned} 2 \sum_{j,\ell=1}^k \text{Cov}(w_{2,j}, w_{2,\ell}) &\leq C \frac{kq_n}{h_n^d} \sum_{l=1}^{k-1} \sum_{m=1}^{q_n} h^{d(1-\frac{1}{a_1})} \alpha^{\frac{1}{a_1}}(l(p_n + q_n) + m) \\ &\leq C \frac{kq_n}{h_n^{\frac{d}{a}}} \sum_{j \geq p_n + q_n + 1} \alpha^{\frac{1}{a_1}}(j) = o(n). \end{aligned} \quad (37)$$

Then, from (32), (36) and (37), we deduce that

$$\frac{1}{n} \mathbb{E}[W_{2,n}^2] \xrightarrow{n \rightarrow \infty} 0. \quad (38)$$

Now, we treat the term $W_{3,n}$ in the same spirit of $W_{2,n}$, we have

$$\begin{aligned} \frac{1}{n} \mathbb{E}[W_{3,n}^2] &= \frac{1}{n} \left\{ (n - k(p_n + q_n)) \text{Var}\left(\frac{W_1(x)}{\sqrt{h_n^d}}\right) \right. \\ &\quad \left. + \sum_{0 \leq |i-j| \leq n-k(p_n+q_n)} \text{Cov}\left(\frac{W_i(x)}{\sqrt{h_n^d}}, \frac{W_j(x)}{\sqrt{h_n^d}}\right) \right\} \\ &\leq \left(1 - \frac{k(p_n + q_n)}{n}\right) \text{Var}\left(\frac{W_1(x)}{\sqrt{h_n^d}}\right) + o(1). \end{aligned}$$

Under assumption (B) we know that $\frac{k(p_n+q_n)}{n} \rightarrow 1$, then we get

$$\frac{1}{n} \mathbb{E}[W_{3,n}^2] \rightarrow 0. \quad (39)$$

So, (38) and (39) conclude the proof of (31).

Now concerning (30), let $\psi(s)$ and $\psi(j)$ be the characteristics function of $\frac{1}{\sqrt{n}} W_{1,n}$ and $w_{1,j}$ respectively. We show that

$$\lim \left| \psi(x) - \prod_{j=1}^k \psi_j(s) \right| = 0$$

which proves that the r.v.'s $w_{1,j}$ are asymptotically independent, and then, prove that

$$\prod_{j=1}^k \psi_j(s) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_4^2(x)).$$

For the first part, we have

$$\begin{aligned} \left| \psi(s) - \prod_{j=1}^k \psi_j(s) \right| &= \left| \mathbb{E} \left[e^{it \frac{W_{1,n}}{\sqrt{n}}} \right] - \prod_{j=1}^k \mathbb{E} \left[e^{it w_{1,j}} \right] \right| \\ &\leq 16ka(q_n + 1) \leq C ka(q_n) \rightarrow 0. \end{aligned}$$

For the second part, we have

$$\frac{1}{n} \sum_{j=1}^k \mathbb{E} \left[w_{1,j}^2 \right] \rightarrow \sigma_4^2(x) \quad (40)$$

and for all $\epsilon > 0$

$$\frac{1}{n} \sum_{j=1}^k \mathbb{E} \left[w_{1,j}^2 \mathbb{1}_{\{|w_{1,j}| > \epsilon \sqrt{n} \sigma_{4,n}(x)\}} \right] \rightarrow 0. \quad (41)$$

Furthermore

$$\begin{aligned} \text{Var}(w_{1,j}) &= \left\{ p_n \text{Var} \left(\frac{W_1(x)}{\sqrt{h_n^d}} \right) + \sum_{0 < |i-\ell| \leq p_n} \text{Cov} \left(\frac{W_i(x)}{\sqrt{h_n^d}}, \frac{W_\ell(x)}{\sqrt{h_n^d}} \right) \right\} \\ &\leq p_n \text{Var} \left(\frac{W_1(x)}{\sqrt{h_n^d}} \right) + o(p_n). \end{aligned}$$

So

$$\frac{1}{n} \sum_{j=1}^k \mathbb{E} \left[w_{1,j}^2 \right] = \frac{kp_n}{n} \text{Var} \left(\frac{W_1(x)}{\sqrt{h_n^d}} \right) + o \left(\frac{kp_n}{n} \right).$$

From (35) and knowing that $\frac{p_n k}{n} \rightarrow 1$ (40) is proved.

Now, to establish (41) we have that $\frac{W_i(x)}{\sqrt{h_n^d}} \leq \frac{C}{\sqrt{h_n^d}}$. Then under (B)

$$\frac{1}{\sqrt{h_n^d}} |w_{1,j}| \leq \frac{C p_n}{\sqrt{n h_n^d}} \rightarrow 0.$$

Then the set $\{|w_{1,j}| > \epsilon \sqrt{n} \sigma_{4,n}(x)\}$ is empty for n large enough and conclude the proof of (41).

Corollary 2. Under assumptions (K) and (D), for $\gamma = 1, 2$, we have

$$\sqrt{n h_n^d} \mathcal{B}_{2,\gamma}(x) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_\gamma^2(x)).$$

Proof of Corollary 2. If $\gamma = 1$ and we take $c_1 = 1$ and $c_2 = 0$ (If $\gamma = 2$ and we take $c_1 = 0$ and $c_2 = 1$ respectively) in (28), using the results in Lemma 1 and Lemma 4 we get the result.

Corollary 3. Under assumptions (K) and (D), we have

$$\sqrt{n h_n^d} (\mathcal{B}_{2,1}(x), \mathcal{B}_{2,2}(x)) \xrightarrow{\mathcal{D}} \mathcal{N} \left(0, \Sigma_x \int_{\mathbb{R}^d} K_d^2(t) dt \right)$$

where Σ_x is defined in Lemma 4.

Proof of Corollary 3. Using the results in Lemma 4 and Corollary 2 we get the result.

Corollary 4. Define the mapping $\zeta : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\zeta(x_1, x_2) = \frac{x_1}{x_2}$ with $x_2 \neq 0$, then

$$\begin{aligned} \hat{m}(x) - m(x) &= \frac{\hat{\mu}_1(x)}{\hat{\mu}_2(x)} - \frac{\mu_1(x)}{\mu_2(x)} \\ &= \zeta(\hat{\mu}_1(x), \hat{\mu}_2(x)) - \zeta(\mu_1(x), \mu_2(x)). \end{aligned}$$

By using the delta method we get

$$\sqrt{n h_n^d} (\hat{m}(x) - m(x)) \xrightarrow{\mathcal{D}} \mathcal{N} \left(0, \nabla \zeta^t \Sigma_x \nabla \zeta \int_{\mathbb{R}^d} K_d^2(t) dt \right)$$

where $\nabla \zeta^t = \left(\frac{\partial \zeta}{\partial x_1}, \frac{\partial \zeta}{\partial x_2} \right)$ is the gradient at point $(\mu_1(x), \mu_2(x))$. So an elementary calculus gives $\sigma^2(x)$ defined in Theorem 2.

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