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A tribute to Abe Sklar

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This paper gives an account of the life and works of the American mathematician Abe Sklar. Born in Chicago on November 17, 1925, Sklar completed his PhD at the California Institute of Technology in 1956. He then joined the Illinois Institute of Technology, where he taught mathematics until his retirement in 1995. With his close friend and lifelong collaborator Berthold Schweizer (1929–2010), he was a pioneer of the theory of probabilistic metric spaces, which were introduced in 1942 by the Austro-American mathematician Karl Menger (1902–85). Together, Schweizer and Sklar made important contributions to the algebra of functions, the study of t-norms, and distributional chaos. Sklar is also credited for the notion of copula and for showing that any multivariate distribution function can be expressed in terms of its univariate margins and a copula. This result, known as Sklar's representation theorem, is the bedrock of a widespread data analytical technique called copula modeling. Sklar passed away in Chicago on October 30, 2020.

By now, just about anyone who is conscious of the role of dependence in data analysis has heard of copulas as a powerful and flexible tool for modeling association and assessing its impact on inference, decision making, and risk management. This approach is rooted in a 3-page note, written in French, which appeared in 1959 in the *Publications de l'Institut de statistique de l'Université de Paris*. This paper [S6], attributed only to “M. Sklar” (M. for Mr.), without address or affiliation, claimed without proof that given any d -variate cumulative distribution function H with one-dimensional margins $F_1, \dots, F_d$, a function $C : [0, 1]^d \rightarrow [0, 1]$ having specific analytical properties can always be found such that

$$H = C(F_1, \dots, F_d). \quad (1)$$

The author, Abe Sklar, called C a ‘copula’ by analogy with the grammatical term for a word or expression that links a subject and a predicate [S57]. He also pointed out that any choice of copula C and marginal distributions $F_1, \dots, F_d$ in formula (1) constitutes a *bona fide* multivariate distribution function.

Eq. (1), now widely referred to as Sklar's representation theorem, is the corner stone of the popular copula approach to modeling. According to *MathSciNet*, Sklar's seminal paper [S6] has been referenced over 450 times since its publication; *Google Scholar* lists 10 times as many citations. What a success story for this note which stemmed from correspondence between Sklar, who was then a modest Assistant Professor of Mathematics in Chicago, and the famous French mathematician Maurice Fréchet!

This paper surveys the life and works of Abe Sklar (1925–2020), a talented and discreet man with a singular passion for mathematics. His biography is sketched in Section 1 and an overview of his scientific contributions is given in Section 2. His spirit continues to live through his students, whose own careers are outlined in Section 3. A few testimonies are collected in Section 4 and Appendix 1 contains a (hopefully complete) list of Sklar's publications. Finally, an annotated transcript of an interview that Sklar granted to Bonnie Schulman in 2006 is included in Appendix 2.

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1 Abe's biography

Abe Sklar was born in Chicago, Illinois, on November 17, 1925. His parents were Jewish immigrants of humble extraction from (modern-day) Ukraine. Abe, who was their only child, graduated from Von Steuben High School in Chicago on June 25, 1942 and entered the University of Chicago at the age of 16. He was mainly interested in physics but after two years, he decided to interrupt his studies and join the war effort. He enlisted in the US Navy on June 20, 1944, and proudly served as a radio technician in the Philippines until his honorable discharge on June 2, 1946; his final rank was RT2/c.

Upon returning home, now aged 21, Abe resumed his studies at the University of Chicago but switched to a major in mathematics. Having completed a master's degree in Fall 1948, he worked as a physics lab technician at the university while taking, on and off, additional courses in mathematics, science, and the arts until Fall 1951. Allen Strehler (1921–78), an educator [43] and later dean at the Carnegie Institute of Technology, kindled and sustained Abe's interest in number theory, leading to his first paper published in 1952 in the *Notices of the American Mathematical Society* [S1]. The same year, Abe registered for a PhD in mathematics at the California Institute of Technology.

In moving to California, Abe's initial intention was to contribute to the Bateman Project, a major effort at collation and encyclopedic compilation of the theory of special functions. However, the project was nearing completion and he wound up working under the guidance of analytic number theorist Tom Apostol (1923–2016), best known for his widely used textbooks. Abe was just two years younger than his supervisor and was one of his first PhD students. He earned his PhD in mathematics in 1956 with a thesis entitled "Summation Formulas Associated with a Class of Dirichlet Series" [S2].

In February 1956, Haim Reingold (1910–2003), who served for many years as Chair of the Department of Mathematics at the Illinois Institute of Technology (IIT), persuaded Abe to return to Chicago and join his department. Except for sabbatical leaves, Abe's entire academic career was spent at IIT. He was promoted to Assistant Professor in 1957, to Associate Professor in 1960, and to Professor of Mathematics in 1967. He retired in 1995 and was appointed Professor Emeritus in 1996. He passed away in Chicago on October 30, 2020, after a short illness.

Chicago Tribune, May 20, 1956

At IIT, Abe was greatly influenced by Karl Menger (1902–85), a prominent Austrian mathematician who had emigrated to the United States in 1937 to flee nazism and whose longest and last academic appointment was at IIT from 1946 to 1971. Menger is considered as one of the founders of distance geometry, which is the study of sets of points based only on given values of the distances between pairs. He is also well-known for Menger's theorem in graph theory and the so-called Menger sponge, which is a three-dimensional generalization of the classical Cantor set (have a look on *Wikipedia*!).

In 1942, Menger initiated a study of probabilistic metric spaces [24], in which distances are represented by distributions rather than by numbers. Abraham Wald (1902–50), a well-known statistician and decision theorist who had been Menger's student in Vienna, showed interest [46] and upon completing his own thesis with Menger at IIT, Bert Schweizer (1929–2010) also started investigating the topic.

Abe joined these efforts shortly after coming to IIT (see Figure 1). He nurtured a lifelong friendship and collaboration with Bert, with whom he developed the theory of probabilistic metric spaces, and contributed to ergodic theory as well as distributional chaos to much acclaim. Together they wrote over 40 papers and an influential monograph on probabilistic metric spaces [S44] published in 1983. For over 60 years, Abe used his vast knowledge of analysis and functional equation theory to explore these and other themes. In addition, he invested much time and effort in memorializing Menger and his work.

During his career, Abe enjoyed (at least) two sabbatical leaves. One was in Spring 1962, when he taught courses at the University of Arizona [S41] while collaborating with Bert, who was then on faculty there. He had a second leave during the 1982–83 academic year. He spent part of that year working with Luigi Paganoni at the Università di Milano, in Italy. Abe's paper [S62] with a coauthor of Paganoni, Jaroslav Smítal (1942–), also reflects research that he conducted on a subsequent visit there.

Figure 1: Karl Menger with some of his students* and colleagues[†] at IIT circa 1956. From left to right: Abe Sklar[†], Bert Schweizer^{**†}, Karl Menger, Tom Erber[†], and Michael McKiernan.^{**†}

2 An overview of Abe's work

In a paper published in 1980 in *The American Mathematical Monthly* [S41], Abe cited analytic number theory, functional equations, function systems, and probabilistic geometry as his research areas. At the 1993 conference on distributions with fixed marginals held in Seattle [S57], he explained how he actually became interested in probability through number theory, and how Bert convinced him to join his project to develop probabilistic metric spaces soon after the two met at IIT.

Motivated by his youthful encounters with Albert Einstein (1879–1955), who was then a neighbor, Bert had come to Chicago to explore probabilistic metric spaces after reading Menger's contribution to a book written in appreciation of Einstein's work [25]. In Menger's opinion, however, this topic was unsuitable for a thesis, and hence Bert could only delve into the matter after completing his degree [37].

Thus began a lifelong friendship and collaboration between Bert and Abe, who also remained close to Karl Menger. When the latter died in 1985, Abe was entrusted with his archives and, with the help of Bert and others, he published a series of volumes devoted to Menger's work [29, 31, 32] and arranged for Menger's calculus textbook [27] to be reissued by Dover in 2007.

Abe's extensive knowledge of Karl Menger's life and thinking is apparent from his interview with Bonnie Shulman [42] which is reproduced in Appendix 2. This discussion is included here in full, as it provides a faithful reflection of Abe's humility, wit, and good humor. It also illustrates his close relationship with both Bert and Karl Menger – so close, in fact, that it is difficult to talk about his research without describing also some of Bert's and Menger's contributions.

As a full review of Abe's work would be too ambitious, Sections 2.1–2.3 are limited to three major themes that occupied his mind over his long and fruitful career: probabilistic metric spaces, derivability issues and bounds, and distributional chaos. One distinct omission is the algebra of functions, another theme pioneered by Menger which was among Abe's finest achievements and about which a book project with Bert remained unfinished. Section 2.4 gives a glimpse of Abe's *modus operandi* as a researcher.

Inspiration for this section was drawn in part from the “personal look backward and forward” that Abe presented at the 1993 meeting in Seattle [S57]. Also useful were two limpid survey papers written by Bert: the first for the 1990 conference on distributions with fixed marginals held in Rome [38], and the second, for

Bert's talk in the series *Seminario Matematico e Fisico di Milano* [39]. For Section 2.3, I consulted the survey of distributional chaos and its measurement that Bert and Abe wrote with their Czech collaborator Jaroslav Smítal for *Real Analysis Exchange* in 2000 [S63].

For precise statements, additional details and further results, readers should consult these four papers and references therein. In particular, note that following an old European tradition, distribution functions were taken to be *left-continuous* in many of the papers cited below.

2.1 Probabilistic metric spaces

Consider a set S and a map $\mathcal{F} : S \times S \rightarrow \Delta_0$ such that, for every pair $(x, y) \in S \times S$, $\mathcal{F}(x, y) = F_{xy}$ is an element in the space Δ_0 of cumulative distribution functions that vanish on $(-\infty, 0)$. For any $F, G \in \Delta_0$, write $F \leq G$ whenever $F(t) \leq G(t)$ for all $t \in \mathbb{R} = (-\infty, \infty)$.

Following the authoritative book that Bert and Abe on probabilistic metric spaces [S44], a map $\tau : \Delta_0 \times \Delta_0 \rightarrow \Delta_0$ is called a *triangle function* if, for all $F, G, H, K \in \Delta_0$, (i) $\tau(F, G) = \tau(G, F)$; (ii) $\tau\{\tau(F, G), H\} = \tau\{F, \tau(G, H)\}$; (iii) $\tau(F, G) \leq \tau(H, K)$ whenever $F \leq H$ and $G \leq K$; (iv) $\tau(F, \delta_0) = F$, where δ_0 is the distribution function of a random variable which is identically equal to 0.

Because of its properties, the map τ can be viewed as a commutative and associative operation on Δ_0 . The pair $(S, \mathcal{F})$ is then said to form a *probabilistic metric space* with respect to τ if, for all $x, y, z \in S$, (i) $F_{xy} = F_{yx}$; (ii) $x = y \Leftrightarrow F_{xy} = \delta_0$; (iii) $F_{xz} \geq \tau(F_{xy}, F_{yz})$ holds point-wise, so that F_{xz} is stochastically smaller than $\tau(F_{xy}, F_{yz})$. Condition (iii) is called Menger's triangle inequality.

A natural choice of triangle function τ is convolution, as noted by Wald [46]. However, Menger [24] viewed this choice as too stringent and proposed to generate τ from a *triangular norm* or *t-norm* T , i.e., a binary algebraic operation such that for all $a, b, c, d \in [0, 1]$, (i) $T(a, 1) = a$; (ii) $T(a, b) = T(b, a)$; (iii) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$; (iv) $T\{T(a, b), c\} = T\{a, T(b, c)\}$. A triangle function τ is then induced by T upon setting, for all $F, G \in \Delta_0$ and $t \in [0, \infty)$,

$$\tau_T(F, G)(t) = \sup \{ T\{F(u), G(v)\} : u + v = t \}. \quad (2)$$

Simple examples of t-norms include T_W , T_M , and T_Π , respectively defined for all $a, b \in [0, 1]$, by $T_W(a, b) = \max(0, a + b - 1)$, $T_M(a, b) = \min(a, b)$, and $T_\Pi(a, b) = ab$. The first two are known in the copula literature as the Fréchet–Hoeffding bounds, and T_Π is the independence (or product) copula.

For nearly 20 years, Bert and Abe studied these and many other aspects of the theory of probabilistic metric spaces with their students, colleagues, and a few collaborators. They worked in relative isolation and at their own pace. In his 1991 review paper [38, p. 29], Bert stated that in hindsight, this...

“... let the ideas and techniques mature in an environment uninfected by the contentious competition that seems to surround so many ‘hot’ topics.”

In particular, the statistical community would only start noticing in the 1980s, after Bert and his PhD student Ed Wolff made a connection with measures of dependence in an *Annals of Statistics* paper [41].

One very fruitful line of investigation is exemplified by a 1961 paper in *Publicationes Mathematicae Debrecen* [S11], where Bert and Abe used results from the theory of functional equations to show that if a t-norm T is continuous and strictly increasing, it can be expressed, for all $a, b \in (0, 1]$, in the form

$$T(a, b) = \phi^{-1} \{ \phi(u) + \phi(v) \}. \quad (3)$$

for some continuous and strictly decreasing function $\phi : [0, 1] \rightarrow [0, \infty]$ with $\phi(1) = 0$ and quasi-inverse ϕ^{-1} . In the same paper, they also proved the familiar fact that T is then a copula if and only if ϕ is convex; see [10] or Theorem 4.1.4 in [34]. Abe's first PhD student, Cho-hsin (Joyce) Ling [23], later published in the same journal a proof that if a continuous t-norm T is Archimedean, i.e., $T(a, a) < a$ for all $a \in (0, 1)$, then T is necessarily of the form (3). And so were born *Archimedean copulas*.

Among many other things, this line of research led Bert and Abe to propose various one-parameter classes of strict t-norms, the most prominent of which is the *Schweizer–Sklar family* which includes what are now

known as Clayton copulas. In this context, the Gumbel copulas are Aczél–Alsina t-norms and the Hoeffding–Fréchet upper and lower bounds are called the Gödel and Łukasiewicz t-norm, respectively.

Also, because t-norms can be viewed as an extension of the usual two-valued logical conjunction, the work of Bert and Abe in this area bore fruit in fuzzy logic. When Lotfi Zadeh (1921–2017) first introduced fuzzy mathematics [47], back in 1965, max and min were the only (associative) operations used to perform generalized unions and intersections. Bert and Abe’s contributions eventually made it possible to replace these operations by t-norms and their duals called t-conorms. These concepts have since become essential tools to represent logical disjunction in fuzzy logic and union in fuzzy set theory.

Years later, new and fruitful definitions for a probabilistic normed space [S56] and a probabilistic inner product space [S58] were found by Bert and Abe in collaboration with Claudi Alsina (1952–), who was Bert’s postdoc at the University of Massachusetts in Amherst before joining Universitat Politècnica de Catalunya; Carlo Sempì (1948–), from the Università del Salento [13] also coauthored [S58]. For more about t-norms, refer to the monographs by Alsina et al. [1] and Klement et al. [20].

2.2 Derivability issues and bounds

Abe’s work with Bert Schweizer led them to look into operations on distribution functions that could be derived from operations on random variables. Formally, a binary operation $*$ on the class Δ of distributions is said to be derivable from a (Borel-measurable) function Ψ on random variables if for any $F, G \in \Delta$, random variables X and Y with these distributions can be constructed on a common probability space and $\Psi(X, Y)$ has distribution $F * G$. Convolution is an example of an operation which is derivable, as it corresponds to the addition of random variables.

In their remarkable 1974 paper [S34], Bert and Abe showed that if T is any continuous t-norm other than T_M , then the algebraic operation τ_T defined in Eq. (2) is not derivable from any operation on random variables. This led them to the insightful conclusion, stated in Section 7.6 of [S44], that

“the distinction between working directly with distribution functions... and working with them indirectly, via random variables, is intrinsic and not just a matter of taste.”

This observation has an analog in dependence modeling: notwithstanding Sklar’s representation theorem, models that are easily expressible in terms of copulas may not be interpretable in terms of operations on random variables, and vice versa. The two approaches are useful, and complementary.

In their search for classes of triangle functions, Bert and Abe further noticed very early that an extension of Eq. (2) would arise if addition, viz. $L(u, v) = u + v$, were replaced by any other two-place commutative and associative function L satisfying $L(0, 0) = 0$. In particular, for any (bivariate) copula C and marginal distributions $F, G \in \Delta_0$, one can set

$$\tau_{C,L}(F, G)(t) = \sup \{ C\{F(u), G(v)\} : L(u, v) = t \}$$

and

$$\rho_{C,L}(F, G)(t) = \inf \{ \bar{C}\{F(u), G(v)\} : L(u, v) = t \},$$

where $\bar{C}$ stands for the dual copula defined, for all $u, v \in [0, 1]$, by $\bar{C}(u, v) = u + v - C(u, v)$.

In the 1970s, one of Abe’s PhD students, Jerry Frank, investigated yet another class of binary operations defined, for every copula C and marginal distributions $F, G \in \Delta_0$, by

$$\sigma_{C,L}(F, G)(t) = \iint \mathbf{1}\{L(u, v) \leq t\} dC\{F(u), G(v)\},$$

where for any set A , $\mathbf{1}(A) = 1$ whenever A holds true, and 0 otherwise. The operation $\sigma_{C,L}$ reduces to standard convolution when L is the sum and C is the product copula. It is Frank who came to the surprising conclusion that in order for the latter operation to be associative, C must be what later came to be known as a Frank copula [7, 9]. This is not sufficient, however, as he later showed [8].

Figure 2: Abe Sklar (left) and Bert Schweizer (right) at the 35th International Symposium on Functional Equations held September 7–14, 1997 in Graz, Austria (participation in this meeting was by invitation only).

In joint work with a PhD student, Richard Moynihan [33], Bert and Abe were able to relate these different operations and showed in [S37] that for a large choice of functions L , the string of inequalities

$$\tau_{W,L} \leq \tau_{C,L} \leq \sigma_{C,L} \leq \rho_{C,L} \leq \rho_{W,L}$$

holds. The special case of the sum had first been treated by Abe alone [S32].

These inequalities, which are analogous to the Fréchet–Hoeffding bounds on any (bivariate) copula C , viz. $T_W \leq C \leq T_M$, are useful in exploring the geometric notion of betweenness in the context of a probabilistic metric space S with triangle function τ , where for any distinct elements x, y, z of S , y is said to lie between x and z if and only if $F_{xz} = \tau(F_{xy}, F_{yz})$.

2.3 Distributional chaos

The associativity of a t-norm $T : [0, 1] \rightarrow [0, 1]$ leads naturally to the consideration of serial iterates. The latter are defined recursively, for every integer $n \geq 1$ and scalars $a_1, a_2, \dots \in [0, 1]$, by

$$T^{n+1}(a_1, \dots, a_{n+2}) = T\{T^n(a_1, \dots, a_{n+1}), a_{n+2}\}, \quad (4)$$

where $T^1 = T$. It was Abe who first noted in [S32] that if T is a strict t-norm such that Eq. (4) yields an n -copula for every integer $n \geq 1$, then T must be of the form (3), i.e., an Archimedean copula with a completely monotonic generator ϕ^{-1} ; furthermore, $T \geq T_{II}$ point-wise, i.e., T is positive quadrant dependent [34]. Another of Abe's PhD students, Clark Kimberling, used this result to study sets of exchangeable random variables and give a probabilistic interpretation of complete monotonicity [19].

As recounted by Bert in the survey paper [39], the study of function iterates ultimately led Abe to study with him distributional chaos. To avoid technicalities, consider a compact, separable metric space (S, d) and a continuous function $f : S \rightarrow S$. For any $x, y \in S$ and every integer $n \geq 0$, set $\delta_{xy}(n) = d\{f^n(x), f^n(y)\}$, where

$f^0(x) = x$. The sequence $f^n(x)$ can be viewed as the trajectory of x in S , and for any real $t \in \mathbb{R}$, the proportion $F_{xy}^n(t)$ of points in the set $\{\delta_{xy}(0), \dots, \delta_{xy}(n-1)\}$ which are less than or equal to t defines a distribution function F_{xy}^n in Δ_0 . Next set, for all $t \in \mathbb{R}$,

$$F_{xy}(t) = \liminf_{n \rightarrow \infty} F_{xy}^n(t), \quad F_{xy}^*(t) = \limsup_{n \rightarrow \infty} F_{xy}^n(t).$$

It can be seen that both F_{xy} and F_{xy}^* are distribution functions, and that the former satisfies Menger's triangle inequality. Together with the map $\mathcal{F} : S \times S \rightarrow \Delta_0$ defined by $\mathcal{F}(x, y) = F_{xy}$, the set S then forms what is called a "transformation generated space."

Bert and Abe initiated a study of this construction in the early 1970s. To describe their main results, assume that f is measure preserving with respect to some (non-singular) probability measure P on S , i.e., $P\{f^{-1}(A)\} = P(A)$ for any (measurable) set $A \subseteq S$. In the paper [S31] published in 1973, they used the Birkhoff–Khinchin ergodic theorem to show that, for any such transformation f ,

$$F_{xy} = F_{xy}^* \tag{5}$$

for almost all pairs $(x, y) \in S \times S$ endowed with the measure $P \times P$. The same year [S30], they established with Tom Erber, a mathematical physicist at IIT (see again Figure 1), that the limiting distribution in Eq. (5) is independent of x and y when in addition to being measure preserving, the transformation f is mixing, so that for any (measurable) subsets A and B of S , $P\{f^{-n}(A) \cap B\} \rightarrow P(A)P(B)$ as $n \rightarrow \infty$. Abe carried out calculations in a special case with Tom Erber [S33] and he studied the recurrence and dispersive behavior of Čebyšev polynomials with another IIT physicist, Porter Johnson [S35].

These results allowed Abe, Bert and their collaborators to shed new light on a famous controversy between Ludwig Boltzmann (1844–1906) and Ernst Zermelo (1871–1953) about the relevance to physics of Poincaré's recurrence theorem. The latter result states that a broad class of systems will, after a finite time, return to a state arbitrarily close to their initial state. While this conclusion appears to contradict thermodynamic principles, it relies critically on the assumption that the state of the system is known exactly. The results of Erber et al. [S30] vindicate Boltzmann's intuition that due to the uncertainty induced by experimentally indistinguishable states, recurrence may actually fail to occur.

These considerations eventually led Bert and Abe to elaborate a theory of distributional chaos to which Jaroslav Smítal, Professor of Mathematics at Slezská Univerzita in Opava, Czechia, was associated from its inception in 1985; see [39] and Figure 3. Specifically, a pair $(x, y) \in S \times S$ is then said to exhibit distributional chaos if and only if there exists an interval I of positive length such that, for all $t \in I$, $F_{xy}(t) < F_{xy}^*(t)$. This leads in turn to a measure of the chaos induced by f , viz.

$$\mu_P(f) = \sup_{x, y \in S} \frac{1}{d_S} \int_0^\infty \{F_{xy}^*(t) - F_{xy}(t)\} dt,$$

where the normalizing factor d_S , which ensures that $\mu_P(f) \leq 1$, is assumed to be finite.

As shown by Schweizer and Smítal [40], this notion of distributional chaos implies chaos in the classical sense of Li and Yorke [22]. With their collaborators, Bert and Abe then continued to contribute to chaos theory, which studies dynamical systems whose apparently random states of disorder and irregularities are actually governed by underlying patterns and deterministic laws that are highly sensitive to initial conditions. Three of Abe's last five papers, published in the early 2000s, are concerned with the elucidation of this concept and the conditions on f required for distributional chaos to exist; see [S62], [S63], and [S65]. All of them were coauthored by his friend Jaroslav Smítal (see again Figure 3). Ultimately, the notion of distributional chaos introduced in this body of work makes it possible to clearly distinguish ergodic behavior from genuinely chaotic behavior.

2.4 Modus operandi

Judging from the meetings he attended regularly, Abe's scientific home was the community of functional equation specialists that was first brought together by János Aczél (1924–2020), Otto Haupt (1887–1988) and

Figure 3: Abe Sklar and Jaroslav Smítal featured in the Czech newspaper *Opavský a Hlučinský Deník*, June 22, 2008. The 46th International Symposium on Functional Equations was held in Opava June 22–29, 2008.

Alexander Ostrowski (1893–1986) at the *Mathematisches Forschungsinstitut Oberwolfach* in 1962. Annual meetings known as the *International Symposium on Functional Equations (ISFE)* have been held since then (with the exception of 1964). Participation is by invitation only. Abe was present in 1962 and remained a regular attendee and member of the scientific committee thereafter. He often used these meetings as an opportunity to have intensive working sessions with Bert. The picture shown in Figure 2 was taken on one such occasion. Reports of these meetings have appeared in *Aequationes Mathematicae* since this journal was founded by Aczél in 1968. Abstracts of Abe’s talks published therein provide a remarkable record of what was on his mind at any given time; see, e.g., Sklar [S39], cited in [S55].

3 Sklar’s students

Part of any researcher’s intellectual legacy is in the graduate students he/she supervised, and what they became. In this regard also, Abe’s career was remarkable. A list of his PhD students is given in Table 1. It is based in part on, but more complete than, the data provided by the *Mathematical Genealogy Project*.

Looking at Table 1, copula modelers will immediately recognize Jerry Frank (1942–), of Frank copula fame, who was already mentioned in Section 2. After graduation, he had a long career as a professor in, and for some time chair of, the Department of Mathematics at IIT. In addition to publishing a dozen papers on functional equations, distribution theory, and non-linear dynamics, he coauthored with Claudi Alsina and Bert Schweizer a monograph on t-norms and copulas [1]; their work, published in 2006, nicely complements the 2000 book on the same topic by Klement et al. [20].

Table 1: A list of Abe Sklar’s 14 PhD students at the Illinois Institute of Technology.

Name	Year	Name	Year
Ling, Cho-hsin (Joyce)	1964	Frank, Maurice J.	1972
Penner, Sidney	1964	Jurschak, Jerome	1974
Senechal, Marjorie	1965	Ranade, Mary S.	1974
Calabrese, Philip G.	1968	Thoresen, Lee L.	1978
Harkin, Joseph B.	1968	Terrana, Victor E.	1979
Marcus, Philip	1968	Krause, Gerianne M.	1981
Kimberling, Clark	1970	Ghebremeskel, Kuflu	1989

From the angle of dependence modeling, another name that stands out is Clark Kimberling (1942–), mentioned earlier for his probabilistic interpretation of complete monotonicity in Archimedean copulas [19]. He too had a successful career as a professor of mathematics, publishing extensively in number theory and geometry. He has been affiliated with the University of Evansville, Indiana, since earning his PhD. A man of multiple talents, he maintains an extensive website ([link](#)) that features, among others, the world's largest internet collection of triangle centers, facts and problems about integer sequences and arrays, but also many biographical studies (starting with [18]) and musical compositions of his own.

The name of Marjorie Senechal (1939–) is also intimately linked with geometry. A Professor Emerita in Mathematics and History of Science and Technology at Smith College, Northampton, Massachusetts, Senechal is well known for her work on tessellations and quasicrystals. She also taught a course on ancient inventions and published several books about silk. Moreover, she served as co-editor-in-chief (2005–13) and later sole editor-in-chief (2013–20) of *The Mathematical Intelligencer* (TMI) in addition to editing the column “Mathematical Communities” since 1997. A fascinating account of her life and career is given in her *Adventures of an Amateur Crystallographer*, which may be found on the website of the American Crystallographic Association ([link](#)). See also her interview in TMI ([link](#)).

Many other students of Abe had careers teaching mathematics in colleges and universities. Sidney Penner, raised in the Bronx, returned there after completing his PhD and teaching mathematics for a few years at SUNY Buffalo. Affiliated to the Bronx Community College at the time of his premature death in May 1980, he had extensive journal contributions to problems and problem-solving, and was Problem Editor of the *New York State Mathematics Teachers Journal* [2, 14].

Pursuing an idea sketched by Abe [S19], Joseph Harkin (1926–2006) wrote a thesis on “Sklar–Stieltjes integrals” [16] and then taught mathematics for 32 years (1971–2003) at SUNY Brockport. After graduation, Philip Marcus (1936–2020) was successively affiliated with Shimer College, Indiana University, the University of Kentucky, Christian Brothers College, and Eureka College. Similarly, Victor Terrana (1945–) taught mathematics at Indiana University Northwest, Queens University of Charlotte, North Carolina, and then at Newberry College, South Carolina, from 1995 to 2011. As for Gerianne Krause (1951–), she taught discrete mathematics at San Francisco State University for most of her career.

Other students of Abe's had careers in industry. For example, Jerry Jurschak (1948–) was a college teacher for three years in Chicago before holding various executive positions within the insurance brokerage, underwriting, and reinsurance business (e.g., Marsh McLennan, CNA Insurance) in Chicago and in New York City. Now in retirement, he is running a charitable foundation with his wife out of Katonah, New York. Another example is Mary Ranade (1941–), who taught college level math and computer/information science part and full time at several colleges and universities in Illinois, North Carolina, and Maryland. Most of her career was devoted to the design, administration, and evaluation of large database systems; she also developed software tools for relational database administrators.

Two of Abe's students had a major spiritual part to their lives. Philip Calabrese (1941–) initially taught mathematics in California (Naval Postgraduate School, Monterey; California State University, Bakersfield; Humboldt State University, Arcata). He then had an aerospace career, founded a consulting firm called Data Synthesis, and retired in 2004 as senior scientist at the Space and Naval Warfare Systems Center, San Diego. His work synthesizing logic and conditional probability culminated with his 2017 book [4]. Follow this [link](#) for a 2-hour interview with him about his “axiom of free will vs. mechanism.”

Kuflu Ghebremeskel returned to Eritrea after completing his PhD. He was a mathematics lecturer at the University of Asmara for 10 years [S61]. He then became a full-time pastor and a leading figure of the Full Gospel Church of Eritrea before he was arrested in May 2004 because of his religious beliefs. He has been detained incommunicado since then and is presumed dead.

Finally, Huse Fatkić, a professor of mathematics at Univerzitet u Sarajevu, in Bosnia and Herzegovina, is also listed on the *Mathematical Genealogy Project* as having been supervised by Bert and Abe. As explained in his thesis [6], they mentored him and inspired his research topic while he visited the University of Massachusetts at Amherst on a fellowship in 1985–86. Although neither Bert nor Abe appears to have been his supervisor in a formal sense, Abe's last paper was with Fatkić [S68].

Alas, I was unable to retrace two of Abe's PhD students: Cho-hsin (Joyce) Ling and Lee L. Thoresen.

4 Personal testimonies

Abe, who was single, had a modest life at home but a passion for traveling which he lived out every summer. We crossed paths a number of times over the years. We were introduced in April 1990 at the inaugural *Symposium on Distributions With Given Marginals* held in Rome, and we met again at two subsequent conferences in this series: Barcelona (2000), and Québec (2004). I last saw him in 2009 at the meeting organized in Lecce, Italy, to mark the 50th anniversary of his seminal 1959 paper on copulas.

In our encounters, Abe never revealed much about his private life but besides his love of mathematics and his broad interest for science and philosophy, it was obvious that he enjoyed Italian cuisine and the opera. He also had a predilection for words and languages, and he was a strong advocate of Esperanto as a means of international communication and as an instrument of peace. However, Abe's command of English, coupled with his wry sense of humor, did not always go down well with people. Some initially frowned at his choice of the word 'copula' for distribution functions with uniform margins. He often proudly referred to it as a *sexy name*, a fact on which Jock MacKay and I later capitalized [11].

In preparation for this memorial article, I interviewed several of Abe's colleagues, students, and friends, all of whom are gratefully thanked in the Acknowledgments. One of them is Ty Olsen (1942–), a 1983 PhD graduate of IIT who taught mathematics there before retiring in 1995. Ty, who had a first career as a lawyer, was named executor of Abe's estate in his will. He dug out many facts about Abe's youth for me by sifting through boxes of papers, but he could find no curriculum vitae. He wrote to me:

"Besides being a first-rate mathematician, Abe was a wonderful colleague and mentor, the backbone of our department at IIT for many years. He had a tremendous breadth of knowledge that he was happy to share."

Arthur Lubin, a current Professor of Applied Mathematics at IIT, also commented as follows:

"Abe gladly gave advice and guidance to people working on outgrowths of his work. He could immediately cite relevant references, often providing a copy of the papers the next day. He was happy to spend time helping students, graduate or undergraduate, embarking on research projects. He was highly principled, and exceedingly modest. Many of his strongest papers appeared in somewhat obscure journals, due to his modesty and abnegation of fame."

Another longtime colleague and friend of Abe's, Tom Erber (1930–), Distinguished Professor Emeritus of Physics and Mathematics at IIT (see again Figure 1), had this to say:

"Abe's strong abnegation to prominence and 'fame' meant that he scattered many basic results in somewhat obscure mathematics journals. His one contact with 'fame' occurred as a consequence of having invented 'copulas'... Beneath Abe's placid exterior there was a hard unyielding commitment to certain principles. For instance, when IIT resurrected its mathematical activities within the framework of an 'Applied Mathematics Department', Abe promptly resigned because he refused to carry the title of Professor of *Applied Mathematics*."

A surprising aspect of Abe's career is that except for Bert and his PhD students, he had next to no collaborator outside IIT. Claudi Alsina, a prolific functional equation researcher and mathematical popularizer, who is one of Abe's rare foreign coauthors (with Smítal), offered the following explanation:

"While Abe liked to think about math and scribble notes in pencil on small pieces of paper, he did not enjoy writing formal papers or books. This is why, in particular, it took him so many years to publish his proof of what is known today as Sklar's representation theorem about copulas. In that regard, Abe was lucky to have an accomplished expositor and a perfectionist like Bert as his coauthor; they complemented one another very well. A problem with Abe was that it was easier to meet him anywhere in the world than to receive a letter from him. Bert really laughed when I once said that IIT is like a black hole: you may send anything there, but nothing ever comes out of it."

Claudi also pointed out to me that Abe was proud to have an Erdős number of 2. This means that Abe never wrote a paper with the renowned Hungarian mathematician Paul Erdős (1913–96), who was one of the most prolific mathematicians of the 20th century, but that he wrote a paper with someone who did, namely his PhD advisor Tom Apostol (see [S3] but I could not locate the Apostol–Erdős paper).

On the subject of fame, which does have its drawbacks, it is perhaps fitting to conclude this tribute with an anecdote recounted by Tom Erber. To give a bit of context first, the monthly American magazine *Wired* ran a cover story entitled “Recipe for disaster: The formula that killed Wall Street” on February 23, 2009. Later reissued in *Significance* [36], this award-winning paper, written by the financial journalist Felix Salmon (1972–), described how the unwise use of the Gaussian copula by some quants played a role in the onset of the 2007–08 financial crisis. As a result of this negative publicity, indiscriminating eyes suddenly came to blame copulas and copula modeling for this global crisis that brought about a great deal of suffering. Erber reported as follows on the collateral damage this inflicted upon Abe:

“Unbeknownst to Abe, some ‘quants’ adapted copulas to predicting the behavior of financial markets. Since big money was involved some subset of quants became very angry with Abe, and one day a group of vociferous demonstrators showed up at Abe’s office at IIT. It took some time to quiet the situation down, prevent physical harm, and assure the crowd that although Abe had invented copulas, he had no idea about the financial ramifications.”

More discussion on copulas and the financial crisis can be found in the interviews of Paul Embrechts and David Li in this journal [5, 35], and in Paul’s interview in *International Statistical Review* [12].

Appendix 1: A partial list of Abe Sklar’s scientific writings

Below is a possibly incomplete list of publications authored or coauthored by Abe Sklar. Lacking a curriculum vitae, I constructed this list to the best of my ability using mainly, but not exclusively, *Google Scholar*, *MathSciNet*, *Research Gate*, and *Web of Science*. I regret any involuntary omission.

The list comprises 71 items, grouped in yearly blocks to give a visual clue of Abe’s research productivity over the course of his career. It excludes the works of Karl Menger [29, 31, 32] which were coedited by Abe and published many years after Menger’s death.

In [29], Menger reminisced about the celebrated *Wiener Kreis* (Vienna Circle), a group of philosophers and scientists chaired by Moritz Schlick (1882–1936) that met regularly from 1924 to 1936 at Universität Wien. Abe edited this book with Louise Ahrndt Golland (1942–), an independent scholar in the history of mathematics, and the famous Wittgenstein scholar Brian McGuinness (1927–2019).

The other two posthumous books by Menger [31, 32] represent his collected works, coedited by seven scholars, including Bert and Abe who were also the driving force behind a new release by Dover of Menger’s calculus [27], for which they wrote a foreword.

Abe Sklar’s scientific writings in chronological order

- [S1] Sklar, A. (1952). On the factorization of squarefree integers. *Proc. Amer. Math. Soc.* 3(5), 701–705.
- [S2] Sklar, A. (1956). *Summation Formulas Associated with a Class of Dirichlet Series*. PhD thesis, California Institute of Technology, Pasadena CA.
- [S3] Apostol, T.M. and A. Sklar (1957). The approximate functional equation of Hecke’s Dirichlet series. *Trans. Amer. Math. Soc.* 86(2), 446–462.
- [S4] Schweizer, B. and A. Sklar (1958). Espaces métriques aléatoires. *C. R. Acad. Sci. Paris* 247(2), 2092–2094.
- [S5] Menger, K., B. Schweizer, and A. Sklar (1959). On probabilistic metrics and numerical metrics with probability 1. *Czechoslovak Math. J.* 9(3), 459–466.
- [S6] Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris* 8, 229–231.
- [S7] Schweizer, B. and A. Sklar (1960). Statistical metric spaces. *Pacific J. Math.* 10(1), 313–334.

- [S8] Schweizer, B. and A. Sklar (1960). The algebra of functions. *Math. Ann.* 139(5), 366–382.
- [S9] Schweizer, B., A. Sklar, and E. Thorp (1960). The metrization of statistical metric spaces. *Pacific J. Math.* 10(2), 673–675.
- [S10] Sklar, A. (1960). On the definition of the Riemann integral. *Amer. Math. Monthly* 67(9), 897–900.
- [S11] Schweizer, B. and A. Sklar (1961). Associative functions and statistical triangle inequalities. *Publ. Math. Debrecen* 8(1), 169–186.
- [S12] Schweizer, B. and A. Sklar (1961). The algebra of functions. II. *Math. Ann.* 143(5), 440–447.
- [S13] Schweizer, B. and A. Sklar (1961). Topology and Tchebycheff. *Amer. Math. Monthly* 68(8), 760–762.
- [S14] Schweizer, B. and A. Sklar (1962). A mapping-algebra with infinitely many operations. *Colloq. Math.* 9(1), 33–38.
- [S15] Schweizer, B. and A. Sklar (1962). Statistical metric spaces arising from sets of random variables in Euclidean n -space. *Teor. Veroyatnost. i Primenen.* 7(4), 456–465.
- [S16] Schweizer, B. and A. Sklar (1963). Associative functions and abstract semigroups. *Publ. Math. Debrecen* 10(1), 69–81.
- [S17] Schweizer, B. and A. Sklar (1963). Triangle inequalities in a class of statistical metric spaces. *J. London Math. Soc.* 38(1), 401–406.
- [S18] Schweizer, B. and A. Sklar (1963). Inequalities for the confluent hypergeometric function. *J. Math. and Phys.* 42, 329–330.
- [S19] Sklar, A. (1964). Uniform Stieltjes integrals. *Notices Amer. Math. Soc.* 11(3), 342–343.
- [S20] Sklar, A. (1964). On some exact formulæ in analytic number theory. In Marsch, D.C.B. (Ed.), *Report of the Institute in the Theory of Numbers*, pp. 104–110, University of Colorado, Boulder CO.
- [S21] Schweizer, B. and A. Sklar (1965). The algebra of functions. III. *Math. Ann.* 161(3), 171–196.
- [S22] Schweizer, B. and A. Sklar (1967). Function systems. *Math. Ann.* 172(1), 1–16.
- [S23] Schweizer, B. and A. Sklar (1967). The algebra of multiplace vector-valued functions. *Bull. Amer. Math. Soc.* 73(4), 510–515.
- [S24] Schweizer, B. and A. Sklar (1968). A grammar of functions. I. *Aequationes Math.* 2(1), 62–85.
- [S25] Schweizer, B. and A. Sklar (1969). A grammar of functions. II. *Aequationes Math.* 3(1–2), 15–43.
- [S26] Schweizer, B. and A. Sklar (1969). Mesures aléatoires de l'information. *C. R. Acad. Sci. Paris Sér. A-B* 269, A721–A723.
- [S27] Sklar, A. (1969). Canonical decompositions, stable functions, and fractional iterates. *Aequationes Math.* 3(1–2), 118–129.
- [S28] Erber, T., Schweizer, B., and A. Sklar (1971). Probabilistic metric spaces and hysteresis systems. *Comm. Math. Phys.* 20(3), 205–219.
- [S29] Schweizer, B. and A. Sklar (1971). Mesure aléatoire de l'information et mesure de l'information par un ensemble d'observateurs. *C. R. Acad. Sci. Paris Sér. A-B* 272, A149–A152.
- [S30] Erber, T., B. Schweizer, and A. Sklar (1973). Mixing transformations on metric spaces. *Comm. Math. Phys.* 29(4), 311–317.
- [S31] Schweizer, B. and A. Sklar (1973). Probabilistic metric spaces determined by measure preserving transformations. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 26(3), 235–239.
- [S32] Sklar, A. (1973). Random variables, joint distribution functions, and copulas. *Kybernetika* 9(6), 449–460.

- [S33] Erber, T. and A. Sklar (1974). Macroscopic irreversibility as a manifestation of micro-instabilities. In Gal-Or, B. (Ed.), *Modern Developments in Thermodynamics*, pp. 281–301. Israel Universities Press, Jerusalem.
- [S34] Schweizer, B. and A. Sklar (1974). Operations on distribution functions not derivable from operations on random variables. *Studia Math.* 52(1), 43–52.
- [S35] Johnson, P. and A. Sklar (1976). Recurrence and dispersion under iteration of Čebyšev polynomials. *J. Math. Anal. Appl.* 54(3), 752–771.
- [S36] Schweizer, B. and A. Sklar (1977). The axiomatic characterization of functions. *Z. Math. Logik Grundlagen Math.* 23(5), 373–382.
- [S37] Moynihan, R., B. Schweizer, and A. Sklar (1978). Inequalities among operations on probability distribution functions. In Beckenbach, E.F. (Ed.), *General Inequalities 1 / Allgemeine Ungleichungen 1* (Proc. First Internat. Conf., Math. Res. Inst., Oberwolfach, 1976), I, pp. 133–149. Birkhäuser, Basel.
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- [S39] Sklar, A. (1978). The embedding of functions in flows. *Aequationes Math.* 17(1), 367.
- [S40] Sklar, A. and B. Schweizer (1978). Unsolvability of the general “problème réciproque” for probabilistic and random information spaces. *Information Theory* (Proc. Internat. CNRS Colloq., Cachan, 1977), pp. 495–501, Colloq. Internat. CNRS, 276, CNRS, Paris.
- [S41] Rice, R.E., B. Schweizer, and A. Sklar (1980). When is $f(f(z)) = az^2 + bz + c$? *Amer. Math. Monthly* 87(4), 252–263.
- [S42] Schweizer, B. and A. Sklar (1980). How to derive all L_p -metrics from a single probabilistic metric. In Beckenbach, E.F. (Ed.), *General Inequalities, 2* (Proc. Second Internat. Conf., Oberwolfach, 1978), pp. 429–434, Birkhäuser, Basel.
- [S43] Erber, T., T.M. Rynne, and A. Sklar (1981). Quantum mechanics and mixing transformations. *Acta Phys. Austriaca* 53(3), 145–155.
- [S44] Schweizer, B. and A. Sklar (1983). *Probabilistic Metric Spaces*. North-Holland Publishing Co., New York.
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- [S46] Sklar, A. (1984). Confluence and stability of orbits of quadratic polynomials. *C. R. Math. Rep. Acad. Sci. Canada* 6(5), 291–296.
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- [S57] Sklar, A. (1996). Random variables, distribution functions, and copulas – a personal look backward and forward. In Rüschendorf, L., B. Schweizer, and M.D. Taylor (Eds.), *Distributions With Fixed Marginals and Related Topics*, pp. 1–14. Inst. Math. Statist., Hayward CA.
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- [S67] Schweizer, B. and A. Sklar (2008). More on j (Letter to the Editor). *Math. Intell.* 30, 4.
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[Also published in Bay, O.F. and I. Saritas (Eds.), *Proceedings of the International Conference on Advanced Technology & Sciences (ICAT’14)*, pp. 772–777.]

Other professional writings by Abe Sklar

- [S69] Sklar, A. (1964). On category overlapping in taxonomy. In Gregg, J.R. and F.T.C. Harris (Eds.), *Form and Strategy in Science*, pp. 395–401. Reidel, Dordrecht.
- [S70] Sklar, A. (1983). Book review of *Topics in Iteration Theory*, by György Targonski, *Studia Mathematica*, Skript 6, Vandenhoeck & Ruprecht, Göttingen and Zürich, 1981, 292 pp., (kart. DM 45, – –), ISBN 3-5244-0126-9. *Bull. Amer. Math. Soc.* 9(3), 345–348.
- [S71] Kass, S., B. Schweizer, A. Sklar, and A. Alt (1997). Interview of Franz Alt, New York, May 17, 1997.

Appendix 2: Sklar’s interview with Bonnie Shulman

Below is a transcription of an interview of Abe Sklar conducted and taped in August 2006 by Bonnie Shulman, now a Professor Emerita of Mathematics at Bates College in Lewiston, Maine. This document, stored in the Duke University archives [42], is published here with her permission. The footnotes, which contain additional information and clarifications, are mine.

This is Bonnie Shulman (BS). It’s August 25, 2006. I’m talking with Abe Sklar (AS) in Chicago, IL.

BS: So you worked after you got out of the army, you went to school at the University of Chicago, and from your major in physics you moved on to a major in mathematics.

AS: Mmm, hmm, and got a master’s degree and then it turned out there was nobody on the faculty who was doing things that really interested me and I was not by any appearances going to be a mathematical genius, so nobody on the faculty particularly wanted me as a graduate student. So I hung around for a while and then we had a visiting professor who was interested in number theory. He and I got along very well and one result of that was that I had my first published paper. It was in number theory and appeared the *Notices of the AMS*.¹ And then although he left, another visiting professor came in and this was a rather famous person, Professor V.J. Rajagopal from India who at that time was the head of the Ramanujan Institute.² He and I got along very well and he encouraged me, and then he had again a very well-known colleague drop in for a short time – Professor Chowla³ – and Chowla asked me about my interests and my work, and I told him of another result that I had – different from the published one, and he said “oh, that’s very beautiful.”

Anyway, so as it happened one of the more senior graduate students had gotten his PhD and had gone to Cal Tech and was working on the so-called Bateman Project. He wrote me and said, would you be interested in coming down to work on the Bateman Project? I said yes and so, anyway, I went to Mac Lane⁴ and said there’s this possibility and he essentially said “go for it” and I’m pretty sure he wrote me a good recommendation – if only to get rid of me!

So anyway, I went to Cal Tech and as it turned out they were in the process really of finishing off the Bateman Project so they didn’t really need anyone at the time. So instead I got a – actually I guess it is called a fellowship, a research fellowship. So I didn’t have teaching duties and I was working with Tom Apostol and I was one of his first two PhD students.⁵ Of course he was not that much older than I was.⁶ I got my PhD with

¹ This is paper [S1] in Sklar’s bibliography; see Appendix 1.

² The Ramanujan Institute of Mathematics was founded in 1950. Here, Sklar presumably refers to C.T. – not V.J. – Rajagopal (1903–78), who joined the institute in 1951 but only became its director in 1955.

³ Sarvadaman D.S. Chowla (1907–95) was a British-born Indian American number theorist.

⁴ Saunders Mac Lane (1909–2005) was an American mathematician who co-founded category theory with Samuel Eilenberg. He was Chair of the Department of Mathematics at the University of Chicago from 1952 to 1957.

⁵ The other was Basil Gordon (1931–2012), a professor at UCLA, specializing in number theory and combinatorics.

⁶ Apostol was born August 20, 1923, which made him 820 days older than Abe Sklar.

him and of course had the chance to meet and interact with some really top notch mathematicians. Well, Cal Tech is a small place and the math faculty was small.

BS: And your first job was at IIT?

AS: My family, both my parents were in Chicago and at that time it was very easy to get jobs.

BS: Those were the days!

AS: So I went to IIT and actually I had contact with IIT before, because a lot of graduate students at Chicago were hired as teaching assistants at IIT. Not at Chicago, of course.

BS: So you began there in 1956 and that was the year Menger had his first thesis student, Berthold Schweizer?

AS: Well of course he'd had some in Vienna and he had students at Notre Dame.⁷ I believe Joe Landin⁸ was one of his students at Notre Dame. Rufus Isaacs⁹ may have been another. He was at Notre Dame and certainly had contact with Menger. He's listed in what is now a quite famous paper.¹⁰ He answered a question that Menger had raised, presumably either in class or in a general lecture, namely to find – if such a thing exists – a function, real function, whose second iterate was the negative of the identity.

BS: Kind of like a i , like a square root of -1 or something.

AS: Well if you do it in the complex plane, there's no problem, but if you want a real function...

BS: Right, that's what's so hard.

AS: Well, it turns out that there are in fact lots and lots of real functions that do it. None of them, of course, are continuous or monotonic. You can get piecewise continuous, but not – you have to have discontinuities.

BS: So he answered that.

AS: Yes.

BS: So, I am most interested in, or I'm going to try and focus on Menger in America. And I have this quote from a biography that van Dalen wrote of Brouwer.¹¹ There are two volumes.¹² The second volume just came out and I reviewed it. I reviewed both volumes, and I've had some correspondence with van Dalen about this. And there's a short paragraph I wanted to read you on Menger from that biography:

"Menger's fate was typical of the middle European scientist of the pre-War era, highly successful, versatile, cultured, with many contacts, who through a forced emigration suddenly became separated from his scientific cultural background and lived on in relative obscurity. He would never reach the heights of his European period again."

That's van Dalen speaking.

AS: Mmm, hmm, mmm, hmm.

BS: So a question that I have for you is, I think it's not to be disputed that although he may have done – and I think he did do much fine mathematical and other kinds of work while here – he never did recapture the – I don't know what to call them – the glory days in the Vienna Circle. He didn't achieve the same prominence here that say, von Neumann...¹³

AS: No?

BS: ... or Teller¹⁴ or you know, some of the other émigrés did.

AS: Mmm, hmm.

⁷ The University of Notre Dame is a famous private Catholic research university located near South Bend, Indiana.

⁸ Joseph Landin (1913–2000) was a professor of mathematics at the University of Illinois at Chicago who is best known for his textbooks on algebraic structures and set theory. He completed his PhD under Karl Menger's tutelage in 1946.

⁹ Rufus Isaacs (1914–1981) was a game theorist, prominent in the 1950s and 1960s with his work on differential games.

¹⁰ The paper in question appears to be [17].

¹¹ L.E.J. Brouwer (1881–1966) was a Dutch mathematician and philosopher, who is known as the founder of modern topology; a famous fixed-point theorem bears his name.

¹² See entries [44, 45] in the bibliography.

¹³ John von Neumann (1903–57) was a Hungarian-American mathematician, physicist, computer scientist, engineer and polymath. He is generally regarded as the foremost mathematician of his time.

¹⁴ Edward Teller (Hungarian: Teller Ede), born in 1908, deceased in 2003, was a Hungarian-American theoretical physicist who is known colloquially as "the father of the hydrogen bomb."

BS: And that was true for some other émigrés as well. But I just wonder what you had to say about that, and why you think that might be the case.

AS: Well I think it's basically as van Dalen says, that he had lost a very favorable milieu and of course it was a time and a place where many exciting things were going on and in fact of course before the Nazi induced immigration, the United States just didn't compare with Europe in the sciences in general, and I'd say in mathematics in particular, although there were some very brilliant people in the United States. In fact in a sense Menger adjusted to the United States much better than probably most of the European émigrés. He liked the United States, I think he liked South Bend,¹⁵ although he certainly liked Chicago more.

I know one very well-known mathematician came to the States from Frankfurt, of course a full professor, and went to Urbana,¹⁶ and OK so he was professor at Urbana, but being professor at Urbana was not the same thing as being full professor in Frankfurt. And in particular he really resented the fact that he was asked to teach trigonometry. Whereas Menger started writing about his experiences at IIT and he says then that when he came to IIT, I think after the first semester he was asked – very reluctantly –

BS: To teach at night school?

AS: Yes!

BS: I've read some of those reminiscences from his papers.

AS: Aha!

BS: And nobody else wanted to do it, and he in fact embraced it; he enjoyed that population.

AS: Yes he did. And he kept on enjoying it. And teaching so-called elementary courses.

BS: So do you think that some of this was that he put his heart and soul and energy into the teaching and put a lot of thought – somewhat maybe like Lagrange¹⁷ when he ended up writing his book based on his experience trying to teach young cadets? Do you think there's maybe some analogy there, with how Menger put his creative energies, his thinking about mathematics, influenced by his heavy teaching load? Didn't he train a lot of Navy cadets?

AS: During the war, yes. B-12 cadets at Notre Dame. Very large classes, presumably more than one of them. And the idea was to teach them quickly, cause they really weren't supposed to spend the entire duration of the war in college. Anyway, but one thing was that it would have made a difference, say if he had gone from Europe to Harvard, Yale, or Princeton.

BS: As von Neumann, for instance, did.

AS: Yes. But as it was, he was at Notre Dame, and then at IIT, where – at Notre Dame of course he tried to revive the Colloquium, not the Circle, but the –

BS: The Mathematics Colloquium.

AS: Yes.

BS: He published –

AS: Yes. But of course it just wasn't the same. And of course as Bert Schweizer says, "how could it have been?"

BS: How did he make that transition? Was he disappointed? Did he regret that for the rest of his life? Did he move on with that? How was that for him?

AS: I don't get the feeling that he regretted that.¹⁸ He was still of course in contact with a great many people, particularly at Notre Dame. After all, one of the Notre Dame lecture series was the original place for Artin's Galois Theory.¹⁹ He was never one for doing what was fashionable. So he just did things that he wanted to do.

¹⁵ South Bend, Indiana, where Menger worked at the University of Notre Dame from 1937 to 1946.

¹⁶ Urbana is a town in the state of Illinois which shares the campus of the University of Illinois at Urbana-Champaign with its sister city of Champaign.

¹⁷ Joseph-Louis Lagrange (1736–1813), famous Italo-French mathematician and astronomer.

¹⁸ Nevertheless, Bert Schweizer (private comm.) reported that after the war, Menger asked Universität Wien to be reinstated. As he had resigned of his own, however, his request was turned down, which deeply disappointed him.

¹⁹ Reference to the book by Emil Artin [3] published in the Notre Dame Lecture Series.

BS: And he dug into the work that was in front of him.

AS: Yes. If he'd had enough students of the caliber that he would have had, had he gone to Harvard or Princeton, say, then it might have been a different story, because he would really have nurtured the sort of students that he had had in Vienna.

BS: But it doesn't sound like he begrudged –

AS: No.

BS: He dealt with the students he had.

AS: He never complained. He never begrudged anything. Unlike the professor I mentioned –

BS: Who shall remain nameless.

AS: Oh I think you could find out easily. In fact you probably already know.

BS: I suspect.

BS: So, what was he like as a colleague? So you came there, not young chronologically, but a young – newly minted – PhD. How close were you in age to him, how much senior to you was he?

AS: Well, basically, he was 24 years older.²⁰

BS: I haven't done the math, is why I asked. So yes, he was quite a bit older – was he a mentor to you?

AS: Not when I just came to IIT because for a while, I did work in number theory. And in fact I did write a paper.²¹ But actually what happened was that I'd come to IIT, I was getting settled in, when one of the other instructors came, and that was Bert Schweizer,²² and he was looking for people to work with him on the subject of what we later called probabilistic metric spaces – at the time it was statistical metric spaces – which was something that Menger had started actually. And anyway, the subject intrigued me, so we started working together. And it was through Bert that I really got in contact with Menger. And then since...

(END OF TAPE 1, SIDE 1)

[Bert] left Chicago basically for healthier climates that turned out not to really be not really that much healthier. Let's see; he went to Los Angeles.²³ Anyway, so I remained the one person physically closest to Menger. So of course we kept up contact and...

BS: So, what was he like? I heard him described when he was younger as various things, from brilliant, ambitious, introverted, some mixture... some of those seem contradictory to each other.

AS: Yes, in fact I think those characterizations are from van Dalen. Van Dalen corresponded with Bert while writing the second volume, essentially because of the connection of Menger through Brouwer. So I got actually to read some of the biography before it was published. And I remember that characterization, which Bert particularly said wouldn't have called – nor would I have called Menger introverted. Actually he was, if anything, quite the opposite.

BS: Could this have been different when he was younger?

AS: Uh, I don't think so. I got the impression that he was a rather brash young man.

BS: Yes, that came through in van Dalen's treatment of Brouwer as well. They clashed.

AS: Yes. Because after all he was the son of a quite famous person²⁴ and –

BS: And he was smart.

AS: Yes. And he would have had enough self-confidence and self-assurance to be able as a beginning college student to go and talk to a professor after the first lecture. So no, I wouldn't call him introverted at all.

BS: It sounds like he was very kind.

AS: He could be very kind. He could also get angry and be very angry, but it didn't last.

²⁰ Karl Menger was born on January 13, 1902, and Abe on November 17, 1925; so their age difference is exactly 23 years, 10 months, 4 days (or 8709 days in total) excluding the end date.

²¹ This is entry [S3] in Sklar's bibliography.

²² After completing his PhD under Menger's tutelage, Bert Schweizer was an instructor at IIT in 1956–57.

²³ Bert Schweizer was an Assistant Professor at San Diego State College in 1957–58 and at UCLA from 1958 to 1960.

²⁴ Karl Menger's father was the famous Austrian economist Carl Menger (1840–1921), who is regarded as the founder of the Austrian school of economics.

BS: So, as well as not begrudging, he also didn't hold a grudge?

AS: Right. Olga Taussky,²⁵ who of course had known Menger very well, once remarked that with him "everything is either plus infinity or minus infinity."

BS: So he was effusive. And he was a man maybe of extreme passions or emotions, felt strongly about issues.

AS: Yes he did.

BS: And did he change his mind, was he willing to change his mind about things? Was he rigid?

AS: He could be, I'd say if it was in connection with something that he himself had, say, originated. But if he wasn't really –

BS: – invested, his self wasn't invested in it –

AS: Yes.

BS: So you could imagine his fight with Brouwer,²⁶ how that –

AS: Yes, yes. Well they were, let me find another adjective – I'd say both Brouwer and Menger were very touchy. That was an adjective I first heard applied to Menger by John Dawson,²⁷ who edited the Gödel papers, and who had interviewed Menger in connection with Gödel.²⁸

BS: So I imagine the reaction to his calculus textbook²⁹ must have disappointed him – did it anger him? I mean this was something he was very much invested in –

AS: Very much so.

BS: And it never really took off.

AS: Right. Not only of course did he himself put a lot of work into it, but he got his children involved.³⁰

BS: In helping proofread it?

AS: Oh yes.

BS: Right, no word processors in those days.

AS: And in actual physical production because the first editions were mimeographed.

BS: I've seen one of those copies, yes.

AS: And you can visualize the kids doing that. Also the – I don't know who did secretarial work for him. Because although he would type himself, he was not a particularly good typist and often he would type something and then change it, sometimes make handwritten changes, or sometimes take typed pages and cut and paste. This meant that actually converting something like that into a decent manuscript took a great deal of work. So although probably the department secretary did part of it, I wouldn't be surprised if the kids also got in on that.

BS: I'll have to ask them. So you did say you thought it was going to be reissued by Dover?

AS: Yes.

BS: That's interesting. Who's responsible for that happening?

AS: Well, it's basically again Bert who – Bert and I of course had published a book on probabilistic metric spaces in '83 and it went out of print.³¹ Then a few years ago Bert thought what the hell, I'll see if I can get Dover to reprint it. Which he did, it's been reprinted.³² Anyway, apparently Dover likes it, one reason being that of course we added material to it and that was done by a – the actual production, of course everything was photocopied by Dover – the actual production of the notes and the supplementary bibliography and the

²⁵ Olga Taussky-Todd (1906–95) was a Czech-American mathematician, famous for her abundant work in algebraic number theory, integral matrices, and matrices in algebra and analysis. She completed her PhD at Universität Wien in 1930 under the supervision of Philipp Furtwängler (1869–1940). Menger was a professor there as of 1927.

²⁶ For additional information about the relationship between Brouwer and Menger from the latter's perspective, see [21], pp. 5–6.

²⁷ John W. Dawson Jr. (1944–), Professor of Mathematics, Emeritus at Pennsylvania State University at York.

²⁸ Reference to the monumental work of the logician, mathematician, and philosopher Kurt Gödel (1906–78).

²⁹ See entries [26, 27] in the bibliography.

³⁰ In 1935 Karl Menger had married Hilda Axamit, an actuarial student. They had four children. Karl Jr., born in 1936, Rosemary and Fred, twins born in 1937, and Eve, born in 1942.

³¹ This is reference [S44] in the bibliography.

³² It appeared in 2005.

preface and so on were done by a lady in California – very very good – in fact so good that Dover has used her in other projects. So anyway, Bert finally had the idea of approaching the editor at Dover and – or maybe it was really the other way around, that the editor said, you know “I looked at Menger’s thing and thought that it might be a worthwhile thing for Dover to reprint,” and whichever way it went. So there is an agreement although contracts have yet to be – probably will be in a month or so.³³

BS: How do you think it will fare in this current market? Do you think it will kind of be a specialty item? Do you anticipate anyone actually using it as a calculus text?

AS: Probably not, although it’s been done.

BS: I noticed in his papers, for instance, there were classes being taught at IIT, and there was a notice of where to obtain it at the bookstore. So it was clear that at least he – and I don’t know if any other instructors – had actually used it, there.

AS: Yes, yes. He used it there, and there are people, in particular one of my PhD students, who used it as a text in other places. Unfortunately, he died recently, so he could have told you more about [how] it worked as a text.³⁴ He thought it worked very well. If you were –

BS: – on board with the program.

AS: Yes. The thing with that as a text is that of course it is different enough so that using it as a text, it would be very hard to deviate from it. And of course it would be just as hard, maybe harder, to use it as a supplement to a standard text.

BS: So it’s not clear who the audience would be.

AS: I would say the audience resides in, besides people interested in it as a historical document, would be some future writers of calculus textbooks.

BS: It would be – I think he had some good ideas.

AS: Very much so.

BS: It would be nice to see them get back into the literature.

AS: Yes.

BS: What role do you think his earlier – the impact of the intellectual issues in the Vienna Circle, you know the emphasis on clarity of language, clarity of thought, and so on, had on him. I know he also critiqued many of the discussions in the Vienna Circle, but how the emphasis on clear thought and expression, how did that play out in his later work, perhaps in the calculus text and in other ways?

AS: I think there was a very definite influence. As you say, striving to be clear, which means in mathematics in particular, that you say essentially clearly, what you are assuming and taking for granted and if you – if you’re doing something like writing a text – you’re saying well, as essentially he does with the mean value theorem, this is an important notion, but at this point you don’t really have enough background to do it properly.

BS: So it’s kind of an honesty.

AS: Yes. Probably honesty is a better term than clarity. Yes. I mean, of course as far as clarity goes, don’t confuse notions which are different.

BS: So, making distinctions – I mean that’s what language is really about, categorizing, naming things. Thinking of Wittgenstein’s influence perhaps there.³⁵ But that it’s very important that you say what you mean and that you aren’t fuzzy about things that are really different.

AS: But of course besides this there are things in the calculus book, aside from the general mode of presentation, which are very beautiful mathematical things that you don’t find in other places. The graphical composition.

BS: So there might be parts of this [that] could be used out of the context of the entire book.

³³ The Dover edition, published in 2007, features a new preface and guide to further reading by Bert and Abe.

³⁴ This is Joe Harkin (1926–2006), who taught mathematics for 32 years at SUNY Brockport; see Section 3.

³⁵ Ludwig Wittgenstein (1889–1951) was an Austrian-British philosopher who worked primarily in logic, the philosophy of mathematics, language and mind. He is considered to be one of the greatest philosophers of the 20th century.

AS: Oh yes, very much so. One thing that he did get from the Circle, but of course also to some extent from the general philosophic milieu of the time, applies more to his attitude towards science than mathematics as such – was a positivistic attitude, where you try not to deal with things that are not concrete. And this of course is out of favor in science.

BS: Today, you think?

AS: Yes and of course part of this comes from Mach.³⁶ Menger had great – well, the whole Circle did – great respect for Mach. But you could say this got Mach into trouble, scientifically. Mach, until his dying day, refused to believe in such things as atoms.

BS: So I know metaphysics was a bad word.

AS: Yes, in the Circle.

BS: And Wittgenstein saying, you know, if we can talk about it, it's not important. You know, the really important stuff is what we can't even talk about.

AS: Of course Menger wouldn't agree with that.

BS: Yes, that's right he definitely did not agree with that. Given that, though, I don't know if this is a relevant question or not, but would you say that Menger had a spiritual life? Did he have any part of his life that was intangible, not concrete?

AS: Well, inasmuch as (laughs) inasmuch as mathematics is intangible? But otherwise, neither he nor the family were say religious in any sense. He was interested in theology.

BS: From a philosophical point of view?

AS: Yes, I would say so. And in philosophy he had read Kant,³⁷ and although I don't think he agreed with most of what Kant said, but he had read him.

BS: How about Spinoza, did he read Spinoza?³⁸

AS: I don't remember him ever mentioning Spinoza. He may very well have.

BS: The only reason I'm thinking of that is, well first of all I recently read Rebecca Goldstein's book *Betraying Spinoza*³⁹ but also his – Spinoza's Ethics where he talks about –

AS: Oh, I'm sorry, yes, in fact he – a very long – he had read Spinoza and in fact he criticizes Spinoza, and in particular his attempt to treat ethics –

BS: – as Euclid's postulates. Is that in his Morality – I might have read that in –

AS: I don't think it's there. Let me see, I (rummages in bag). Do you know this?

BS: Yes, I have a copy of that, the *Reminiscences*.⁴⁰ So maybe that's where he talks about it?

AS: He talks about Spinoza at some length, I don't know if it's here.

BS: Where else might it be? Published, something published?

AS: Yes, p. 117.

BS: OK, so it's in the *Reminiscences*.

(END OF SIDE 2, TAPE 1)

BS: So how did you come – were you the last person to have his papers. And you worked on editing some of his work. How did that come about?

AS: The person who essentially had a complete collection was Bert Schweitzer. And Menger had talked to Bert, oh, probably something like 10 years before his death, picking out the papers that he would like to see in a *Selecta*. And as I say this was some time before his death. And then he did go ahead and publish the *Selected*

³⁶ Ernst Mach (1838–1916) was an Austrian physicist and philosopher who studied shock waves, inter alia, and who influenced logical positivism. His criticism of Newton's theories foreshadowed Einstein's theory of relativity.

³⁷ Immanuel Kant (1724–1804) was a German philosopher and one of the central Enlightenment thinkers. He is considered one of the most influential figures in modern Western philosophy.

³⁸ Baruch (de) Spinoza (1632–77) was a Dutch philosopher of Portuguese Sephardi origin. He is regarded as one of the great rationalists of 17th-century philosophy.

³⁹ The book's full title is *Betraying Spinoza: The Renegade Jew Who Gave Us Modernity* [15].

⁴⁰ See entry [29] in the bibliography.

Papers, what do you call it, in logic, didactics, and so on...⁴¹ So this was a selection from the *Selection*. So he did that. And then in his will he mentioned some things that he wanted to have published, and one was a revised and expanded version of *Calculus*, one was a *Selecta*, and one was a biography of his father which he started but never got very far into and the last one was, I could look it up, but I don't remember at the moment. Of course he would have liked to have had those things published within 10 years of his death. But that was quite impossible. But of course now the *Selecta* has appeared and I think very well done. Because we got some really top notch people to do commentaries, including van Dalen.

BS: So I don't know if I'm aware of this volume...

AS: Two volumes, published by Springer.

BS: Called?

AS: *Selecta Mathematica*. Karl Menger: *Selecta Mathematica*. And for Springer books, well, I'll show you, for Springer books, they're not astronomically expensive.

BS: Great!

AS: Which is maybe why Springer hasn't pushed them as much as we think they should.

BS: So you were instrumental in seeing that?

AS: Yeah, actually the most instrumental person was Professor Karl Sigmund⁴² in Vienna. But he and Bert and I basically were the editors. We got the other people to write the commentaries.

BS: What about the biography of his father, what's up with that?

AS: He started it, as I say, he couldn't have gotten very far, if really anything still exists, it would probably be with Rosemary or Eve who would have it.⁴³ Of course he also started an autobiography, and again he didn't get too far into that, but I have a copy of that as far as it goes. He doesn't really get out of his childhood.

BS: So, you were close to him in later years. You guys developed a relationship over time?

AS: Yeah. He of course had developed a heart problem in his life and in '83 I got a phone call from him, I think it was close to midnight.⁴⁴ So I went up to the place and we talked and finally convinced him that he should go to a hospital. And I took him to Swedish Covenant.⁴⁵ So he went into the Emergency Room, I think it was about 3 in the morning. And he was there for an hour or two. Finally they officially admitted him. I left the hospital and in the morning I called Rosemary.⁴⁶ Anyway he was in the hospital for something like a week, then was released and he went to stay with Rosemary and Dick in Highland Park; he was there for two weeks or so and apparently was doing all right, and then died in his sleep.

BS: So you were the first person that he called...

AS: Yes. Well, I was close.

BS: Physically, geographically close. But also... yeah. I guess, I mean I could ask you a hundred more questions, but those are pretty much the ones I had written down. What kinds of materials did you bring along?

AS: Basically just that book (*Reminiscences*).

BS: And we did consult it!

AS: And a reminder of the other books that I have related to Menger. Let's see, well beside the *Selecta* and the *Selected Papers*, there is the *Ergebnisse Kolloquiums*⁴⁷ which has been republished by Springer and that was essentially Sigmund's doing. That is very worthwhile. In connection with that I know mathematicians in Graz⁴⁸ (laughs) I have another thing to say about Graz – but anyway, one of the leading mathematicians there found out that I had copies of the *Ergebnisse eines Mathematischen Kolloquiums* and he told me that they didn't have anything in Graz, because apparently the Nazis had taken care of that. So I sent him a set and when

⁴¹ The exact title is *Selected Papers in Logic and Foundations, Didactics, Economics*; see [28].

⁴² Karl Sigmund (1945–) is a Professor of Mathematics at Universität Wien and a pioneer of evolutionary game theory.

⁴³ These are Menger's two daughters already mentioned in footnote no 30.

⁴⁴ Here, Sklar actually means 1985 as Karl Menger died on October 5, 1985.

⁴⁵ Swedish Hospital (formerly Swedish Covenant Hospital) is a teaching hospital located on the north side of Chicago.

⁴⁶ This is Karl Menger's first daughter; her husband Dick Gilmore is mentioned in the subsequent sentence.

⁴⁷ See entry [30] in the bibliography.

⁴⁸ Graz is the second largest city in Austria after Vienna.

he got it he wrote back and he said “we were very pleased to get this.” He said, “Vienna doesn’t have one.” So obviously that’s one reason why Sigmund (who’s at Vienna) wanted to get the *Ergebnisse* republished, which they have done, very nicely.

I mentioned Graz – I remember one thing he said in connection with Graz, and that was about Schrödinger.⁴⁹ Schrödinger had been at Cambridge⁵⁰ and ultimately had to leave Cambridge basically because of the two families.⁵¹ So he went back to Austria, but not to Vienna, he went to Graz. Again, apparently because Vienna may have – this was in the middle 30s – basically again because Vienna was a bit puritanical. So he went to Graz and Menger said that – who apparently wasn’t aware of Schrödinger’s –

BS: – unconventional arrangements?

AS: Unconventional arrangements. He said it was strange that he went to Graz because although Vienna at the time was 80% Nazi, Graz was 100% Nazi. And of course after the Anschluß, Schrödinger was in very deep trouble in Austria and he was essentially rescued out of Austria by de Valera in Ireland,⁵² who set up the Institute for Advanced Study in Dublin, especially for Schrödinger.⁵³

BS: So, was Menger charismatic? I mean he seems to have inspired devotion from a number of people, seeing that his legacy continues, and in his lifetime as well. What was it about him that inspired that?

AS: I think one reason for that is that he was always interested in what you were doing. And it didn’t matter, it could be – of course this is in connection primarily with mathematics – but in other things as well. And it didn’t matter whether you were a student or professor or whatever. When he’d meet someone, even after only a day, he’d say, “Well, what’s the news?” As if to say, well what fabulous discoveries have you made? And of course he could be very charming. When he got angry, of course, he could be very un-charming.

BS: And did you ever have an altercation with him? Were you ever the recipient of his anger?

AS: Yes. I think everybody at some point. But they were rare.

BS: And it would flare up and then subside.

AS: Yeah, yeah.

BS: Well, thank you.

AS: Oh, you’re very welcome.

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⁴⁹ Erwin Schrödinger (1887–1961) was a Nobel Prize-winning Austrian physicist who developed fundamental results in quantum theory: the Schrödinger equation gives a way of computing a system’s wave function and its dynamic change.

⁵⁰ Here, Sklar seems to have mixed up Cambridge and Oxford. Schrödinger had a position at Oxford in 1933–34.

⁵¹ Schrödinger shared living quarters with his wife and his mistress, a fact that did not meet with acceptance.

⁵² Éamon de Valera (1882–1975), who was trained as a mathematician, is an Irish statesman who was, inter alia, President of the Republic of Ireland.

⁵³ The Dublin Institute for Advanced Studies is a statutory, independent research institute which was established in 1940 by Éamon de Valera, who was then the “Taoiseach,” i.e., the Prime Minister of Ireland.

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