DE GRUYTER Depend. Model. 2021; 9:225–242

Research Article
Special Issue in memory of Abe Sklar Open Access

Claude Lefèvre*

On partially Schur-constant models and their associated copulas

https://doi.org/10.1515/demo-2021-0111
Received March 25, 2021; accepted May 10, 2021

Abstract: Schur-constant vectors are used to model duration phenomena in various areas of economics and statistics. They form a particular class of exchangeable vectors and, as such, rely on a strong property of symmetry. To broaden the field of applications, partially Schur-constant vectors are introduced which correspond to partially exchangeable vectors. First, their copulas of survival, said to be partially Archimedean, are explicitly obtained and analyzed. Next, much attention is devoted to the construction of different partially Schur-constant models with two groups of exchangeable variables. Finally, partial Schur-constancy is briefly extended to the modeling of nested and multi-level dependencies.

Keywords: Schur-constant model, Archimedean copula, partial exchangeability, multivariate monotonicity, bivariate survival functions

MSC: 60G09, 62H05, 62H10

1 Introduction

Schur-constant vectors play a central role in modeling lifetime data in actuarial science, reliability and survival analysis. The case traditionally considered is that where lifetimes are absolutely continuous random variables with values in $\mathbb{R}_+$. Continuous Schur-constant models have been discussed by many authors including [4, 6, 13, 35, 38, 46, 47]. Recently, Schur-constancy for discrete lifetimes with values in $\mathbb{N}_0$ has been studied by [8] and then in [7, 21, 28, 36].

By definition, a positive random vector $\mathbf{X} = (X_1, \dots, X_n)$ is Schur-constant if its joint survival function $\Pr(X_1 \geq x_1, \dots, X_n \geq x_n)$ depends on the vector $(x_1, \dots, x_n)$ only through its sum $x_1 + \dots + x_n$, via a univariate function (generator) S. Thus, the distribution of $\mathbf{X}$ is of the exchangeable type (in the sense of [15]) with a particular expression. By exchangeability, the n variables X_i have the same distribution and the correlations between any pair of variables (X_i, X_j) are all equal. Such a property of symmetry is very restrictive, and often unrealistic, for a large number of real situations. For example, in insurance or finance, a Schur-constant portfolio would necessarily consist of n contracts or assets that are similar in all respects.

To break this symmetry, it is natural to widen the Schur-constant model by making it only partially exchangeable (as defined by de Finetti [16]). This leads us to introduce a generalized more flexible dependency model called partially Schur-constant. More precisely, the vector $\mathbf{X}$ is now partitioned into $m \geq 2$ groups $\mathbf{X}_1$ of size n_1, …, $\mathbf{X}_m$ of size n_m. Its survival function $P(\mathbf{X}_1 \geq \mathbf{x}_1, \dots, \mathbf{X}_m \geq \mathbf{x}_m)$ is then assumed to depend on the vector $(\mathbf{x}_1, \dots, \mathbf{x}_m)$ only through the vector of corresponding sums $(|\mathbf{x}_1|, \dots, |\mathbf{x}_m|)$, via an m-variates generator S. In this way, we create a new model composed of m groups of variables which are homogeneous with different Schur-constant structures and which can be interdependent with different intensity levels. A discrete version of such a vector has been proposed and applied to risk management by Castañer et al. [9]. In

***Corresponding Author: Claude Lefèvre:** Université Libre de Bruxelles, Département de Mathématique, Campus de la Plaine C.P. 210, B-1050 Bruxelles, Belgium, E-mail: Claude.Lefevre@ulb.be

what follows, we focus on the continuous version, that is when $\mathbf{X}$ is an absolutely continuous random vector over $\mathbb{R}_+^n$, as well as the associated copula.

It is well-known that continuous Schur-constant models are closely linked to Archimedean copulas (see e.g. [13, 29, 38]). In the current context, we will establish a connection between continuous partially Schur-constant vectors and an associated family of copulas, said to be partially Archimedean. These copulas have the practical property of having a partially exchangeable distribution instead of being simply exchangeable. We point out that this family of copulas was recently derived by Ressel [42] following his comprehensive papers on the multiple monotonicity of multivariate functions. At the beginning, we will refer moreover to several important results that he obtained. Our approach, which is different, comes from our desire to generalize the property of symmetry in a Schur-constant model. It is therefore intrinsically probabilistic by nature. We also mention a related article by [14] who discussed an Archimedean copula of similar type based on a particular model of multivariate compound distributions.

The paper is organized as follows. In Section 2, we introduce the class of partially Schur-constant models and we present their main properties. In Section 3, we define a family of survival copulas, called partially Archimedean, and we show their close links with the partially Schur-constant vectors. The following four sections concern the construction of different partially Schur-constant models with two groups of exchangeable variables. This step will lead us to consider various classical bivariate survival functions. In Section 4, we choose as generator a survival function which corresponds to a univariate Laplace transform. Several notable bivariate distributions are discussed in detail. In Section 5, this time we take as generator the Laplace transform of a bivariate random vector. Special attention is paid here to a bivariate distribution of gamma type. In Section 6, we use as generator the Gumbel bivariate exponential survival function. It is a more complex case because the survival does not correspond to a Laplace transform. In Section 7, we define a generator from the bivariate Williamsom transform. This construction is however often little convenient as illustrated by two particular models. Finally, we return in Section 8 to the general partial Schur-constancy to briefly show that this modeling covers the usual nested Archimedean copulas and how it can be extended to multi-level dependencies.

2 Partially Schur-constant vectors

In this Section, we introduce the notion of partially Schur-constant (continuous) vector and we present its main properties. Let us start by recalling the simple Schur-constancy (see e.g. Nelsen [38]).

Schur-constant model. A vector $\mathbf{X} = (X_1, \dots, X_n)$ on $\mathbb{R}_+^n$ is Schur-constant if its joint survival function can be written in the form

$$P(X_1 > x_1, \dots, X_n > x_n) = S(x_1 + \dots + x_n), \quad x_1, \dots, x_n \geq 0, \tag{2.1}$$

for some univariate function $S(x) : \mathbb{R}_+ \to [0, 1]$, called generator.

From (2.1), we directly notice that the vector $(X_1, \dots, X_n)$ has an exchangeable distribution of particular structure. As a result, the n variables X_i have the same marginal distributions and their interdependencies are identical. Such properties can considerably limit the practical field of application of Schur-constant vectors. To remedy this, we introduce a more general model, called partially Schur-constant, which is based on the property of partial exchangeability of a random vector ([16]).

Partially Schur-constant model. Suppose that the vector $\mathbf{X} = (X_1, \dots, X_n)$ can be partitoned into $m \geq 2$ sub-vectors $\mathbf{X}_j = (X_{j,1}, \dots, X_{j,n_j})$, $1 \leq j \leq m$, with $n_j \geq 1$ and $n_1 + \dots + n_m = n$.

Definition 2.1. *A vector* $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_m)$ *on* $\mathbb{R}_+^n$ *is partially Schur-constant if its joint survival function can be written in the form*

$$P(\mathbf{X}_1 \geq \mathbf{x}_1, \dots, \mathbf{X}_m \geq \mathbf{x}_m) = S\left(|\mathbf{x}_1|, \dots, |\mathbf{x}_m|\right), \quad \textit{for all } \mathbf{x}_j \in \mathbb{R}_+^{n_j}, \tag{2.2}$$

for some m-variates generator $S(x_1, \dots, x_m) : \mathbb{R}_+^m \to [0, 1]$, *and using the notation*

$$|\mathbf{x}_j| = x_{j,1} + \dots + x_{j,n_j}, \quad 1 \le j \le m.$$

Clearly, $\mathbf{X}$ is a partially exchangeable vector such that the sub-vectors $\mathbf{X}_j$, $1 \le j \le m$, are Schur-constant of univariate generator

$$S_j(x_j) = S(0, \dots, 0, x_j, 0, \dots, 0), \quad x_j \in \mathbb{R}_+. \tag{2.3}$$

In particular, inside each $\mathbf{X}_j$, the variables $X_{j,1}, \dots, X_{j,n_j}$ have the same mean μ_j, variance σ_j^2 and correlation coefficient ρ_j (if it exists). They can be computed from the survival function S_j since

$$\mu_j = E(X_{j,1}) = \int_0^\infty S_j(x_j)dx, \quad E(X_{j,1}^2) = 2\int_0^\infty x_j S_j(x_j)dx_j,$$
$$E(X_{j,1}X_{j,2}) = \int_0^\infty \int_0^\infty S_j(x_j + y_j)dx_j dy_j = \int_0^\infty t_j S_j(t_j)dt_j = E(X_{j,1}^2)/2, \tag{2.4}$$

so that, after some calculations,

$$\rho_j = (v_j^2 - 1)/2v_j^2,$$

where $v_j = \sigma_j/\mu_j$ is the usual coefficient of variation. Note that $\rho_j > 0$ is equivalent to $v_j < 1$ (the standard deviation of $X_{j,1}$ does not exceed its mean). As for the dependence between two different groups $j < k$, all the pairs of variables have the same correlation coefficient $\rho_{j,k}$ (if it exists), positive or not. Considering the bivariate survival function

$$S_{j,k}(x_j, x_k) = S(0, \dots, 0, x_j, 0, \dots, 0, x_k, 0, \dots, 0), \quad x_j, x_k \in \mathbb{R}_+,$$

we can obtain $\rho_{j,k}$ by calculating the expectation

$$E(X_{j,1}X_{k,1}) = \int_0^\infty \int_0^\infty S_{j,k}(x_j, x_k)dx_j dx_k. \tag{2.5}$$

A central problem is how to construct a partially Schur-constant vector. For this, we will use a standard vectorial notation. Let $\mathbf{k} = (k_1, \dots, k_m)$, $\mathbf{l} = (l_1, \dots, l_m) \in \mathbb{N}_0^m$, and $\mathbf{x} = (x_1, \dots, x_m)$, $\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{R}_+^m$. We set $\mathbf{k}! = k_1! \dots k_m!$, $\mathbf{t}^{\mathbf{k}} = t_1^{k_1} \dots t_m^{k_m}$, $\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{l}} = \sum_{k_1=0}^{l_1} \dots \sum_{k_m=0}^{l_m}$ and $\mathbf{k} \le (\lneqq) \mathbf{l}$ if $k_j \le l_j$ for all j (with $k_j < l_j$ for at least one j). We also write $S^{(\mathbf{k})}(\mathbf{x}) = \partial^{k_1+\dots+k_m} S(x_1, \dots, x_m)/\partial x_1^{k_1} \dots \partial x_m^{k_m}$.

First, the generator S is characterized through the property of multiple monotonicity for a multivariate function. A thorough study of this property is provided by [40, 41]. Let f be any function f on $\mathbb{R}_+^m$ and, for simplicity, suppose that it is $\mathbf{n}$ times differentiable. Then, f is said to be $\mathbf{n}$-monotone if for all $\mathbf{x} \in \mathbb{R}_+^m$,

$$(-1)^{|\mathbf{k}|} S^{(\mathbf{k})}(\mathbf{x}) \ge 0, \quad \mathbf{0} \lneqq \mathbf{k} \le \mathbf{n}. \tag{2.6}$$

Note that monotonicity is a standard property in the univariate case; see e.g. [27, 34, 49].

From the definition (2.2), the generator S is an m-variates function which corresponds to a survival function on $\mathbb{R}_+^m$. Theorem 10 of [40] states the following characterization of such a function S.

Proposition 2.2. *(Monotonicity of S)*
A survival function $S : \mathbb{R}_+^m \to [0, 1]$ *may generate a partially Schur-constant vector (2.2) if and only if the function S is* $\mathbf{n} = (n_1, \dots, n_m)$*-monotone on* $\mathbb{R}_+^m$.

Now, a vector that is partially Schur-constant can be represented in two different ways. Both representations are determined from the m sums of all the variables inside the m sub-vectors $\mathbf{X}_1, \dots, \mathbf{X}_m$. So, denote this vector by $\mathbf{T} = (T_1, \dots, T_m)$ with $T_j = X_{j,1} + \dots + X_{j,n_j}$, $1 \le j \le m$. Given its importance, we derive below the distribution of $\mathbf{T}$.

Proposition 2.3. *(Distribution of* **T***)*
The density function of **T** *is given by*

$$f_{\mathbf{T}}(\mathbf{t}) = (-1)^{|\mathbf{n}|}\, S^{(\mathbf{n})}(\mathbf{t}) \frac{\mathbf{t}^{\mathbf{n}-\mathbf{1}}}{(\mathbf{n}-\mathbf{1})!}, \tag{2.7}$$

and the survival function by

$$P(\mathbf{T} > \mathbf{t}) = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} (-1)^{|\mathbf{k}|+m}\, S^{(\mathbf{k})}(\mathbf{t}) \frac{\mathbf{t}^{\mathbf{k}}}{\mathbf{k}!}, \quad \mathbf{t} \in \mathbb{R}_+^{\mathrm{m}}.$$

Proof. Consider all the possible partial sums $T_{j,k_j} = X_{j,1} + \ldots + X_{j,k_j}$, for $k_j = 1, \ldots, n_j$ and $j = 1, \ldots, m$. From (2.1), we see that their joint density function is given by

$$\begin{aligned} &f_{(T_{1,1},\ldots,T_{1,n_1};\ldots;T_{m,1},\ldots,T_{m,n_m})}(t_{1,1}, \ldots, t_{1,n_1}; \ldots; t_{m,1}, \ldots, t_{m,n_m}) \\ &= (-1)^{n_1+\ldots+n_m} S^{(n_1,\ldots,n_m)}(t_{1,n_1}, \ldots, t_{m,n_m}), \quad 0 \le t_{1,1} \le \ldots \le t_{1,n_1}; \ldots; 0 \le t_{m,1} \le \ldots \le t_{m,n_m}. \end{aligned}$$

Integrating over the range of $(t_{1,1}, \ldots, t_{1,n_1-1}; \ldots; t_{m,1}, \ldots, t_{m,n_m-1})$, we then get

$$\begin{aligned} &f_{(T_{1,n_1},\ldots,T_{m,n_m})}(t_{1,n_1}, \ldots, t_{m,n_m}) \\ &= (-1)^{n_1+\ldots+n_m} S^{(n_1,\ldots,n_m)}(t_{1,n_1}, \ldots, t_{m,n_m}) \frac{t_{1,n_1}^{n_1-1} \ldots t_{m,n_m}^{n_m-1}}{(n_1-1)! \ldots (n_m-1)!}, \quad t_{1,n_1}, \ldots, t_{m,n_m} \ge 0. \end{aligned}$$

As each T_{j,n_j} corresponds to T_j, this gives precisely the formula (2.7) for the density function of **T**. The associated survival function follows easily by integration. ⋄

An important theorem of Williamson [49] states that any n-monotone function on $\mathbb{R}_+$ can be expressed as a mixture of functions of the form $(1 - rx)_+^{n-1}$ with $r > 0$. A generalization of this theorem with positive multivariate functions is established by [42], Theorem 3. Applied to the generator S, it reads as follows.

Proposition 2.4. *(Representation of S)*
A function $S : \mathbb{R}_+^{\mathrm{m}} \to [0, 1]$ is the generator of a partially Schur-constant vector (2.2) if and only if it admits the integral representation

$$S(\mathbf{x}) = E\left[\prod_{j=1}^{m} \left(1 - \frac{x_j}{Z_j}\right)_+^{n_j-1}\right], \quad \mathbf{x} \in \mathbb{R}_+^{\mathrm{m}}, \tag{2.8}$$

where the random vector $(Z_1, \ldots, Z_m)$ is distributed as $\mathbf{T} = (T_1, \ldots, T_m)$ (of density (2.7)).

Using (2.8), it can be seen that a Schur-constant vector **X** has the remarkable distributional representation (2.9), (2.10) below.

Proposition 2.5. *(Representation of* **X***)*
A Schur-constant vector **X** *can be represented in distribution as*

$$(\mathbf{X}_1, \ldots, \mathbf{X}_m) =_d [Z_1(U_{1,1}, \ldots, U_{1,n_1}), \ldots, Z_m(U_{m,1}, \ldots, U_{m,n_m})], \tag{2.9}$$

where the random vector $(Z_1, \ldots, Z_m)$ is distributed as $\mathbf{T} = (T_1, \ldots, T_m)$ (of density (2.7)), and the random vectors $(U_{j,1}, \ldots, U_{j,n_j})$, $1 \le j \le m$, are independent of each other and of $(Z_1, \ldots, Z_m)$ and each forms a Schur-constant vector of joint survival function

$$P(U_{j,1} > u_{j,1}, \ldots, U_{j,n_j} > u_{j,n_j}) = [1 - (u_{j,1} + \ldots + u_{j,n_j})]_+^{n_j-1}, \quad u_{j,1}, \ldots, u_{j,n_j} \in (0, 1). \tag{2.10}$$

Note that when if $n_j = 1$, i.e. when the group j contains a single variable $X_{j,1}$, then (2.8) or (2.9) gives $Z_j =_d X_{j,1}$. From (2.9) and (2.10), a partially Schur-constant model can be constructed in two steps: first a random vector $(Z_1, \ldots, Z_m)$ gives the total sums of variables in the m groups, and then each sum Z_j is independently distributed on the standard simplex using the vector $(U_{j,1}, \ldots, U_{j,n_j})$.

From (2.9), the moments and correlations of **X** can be directly expressed as a function of those of the vector **Z** ($=_d$ **T**). Inside each group j, we get, in obvious notation,

$$\mu_j = \mu_{Z_j}/n_j, \quad \sigma_j^2 = 2\sigma_{Z_j}^2/n_j(n_j+1) + (n_j-1)\mu_{Z_j}^2/n_j^2(n_j+1),$$
$$\rho_j = (n_j v_{Z_j}^2 - 1)/(2n_j v_{Z_j}^2 + n_j - 1). \tag{2.11}$$

Between two different groups $j \neq k$, we have

$$\rho_{j,k} = \rho_{Z_j,Z_k}(v_{Z_j} v_{Z_j}/v_j v_k), \tag{2.12}$$

and as $v_j \geq v_{Z_j}$, this implies that $\rho_{j,k} \leq \rho_{Z_j,Z_k}$, which is in agreement with intuition.

To conclude, we recall that a product ZU with Z positive and U uniform on $(0, 1)$ is the definition due to Khintchine [22] for the unimodality on $\mathbb{R}_+$ with mode at 0. So, each Schur-constant vector $Z_j(U_{j,1}, \ldots, U_{j,n_j})$ could be considered as defining a unimodal distribution on $\mathbb{R}_+^{n_j}$. By (2.9), a partially Schur-constant vector would then represent a kind of multimodal distribution on $\mathbb{R}_+^{\mathbf{n}}$.

3 Associated survival copulas

As a corollary, we will now build an associated family of survival copulas. These are called partially Archimedean because they generalize classical Archimedean copulas by assuming the partial exchangeability of the uniform vector involved. Next, we will show that a partially Schur-constant vector has a survival copula which is precisely partially Archimedean with the same generator, and conversely. Let us start by recalling the definition of an Archimedean copula and, before that, of a copula in general (see e.g. [20, 39]).

Let $\mathbf{U} = (U_1, \ldots, U_n)$, $n \geq 2$, be a random vector over the unit cube $[0, 1]^n$ with uniform marginals. Its survival copula $C : [0, 1]^n \to [0, 1]$ is defined by

$$C(u_1, \ldots, u_n) = P(U_1 > 1 - u_1, \ldots, U_n > 1 - u_n), \quad u_1, \ldots, u_n \in [0, 1]. \tag{3.1}$$

Thus, the survival function of $(U_1, \ldots, U_n)$ is given by $C(1 - u_1, \ldots, 1 - u_n)$. Now, consider a continuous random vector $(X_1, \ldots, X_n)$. Its distribution can be represented in terms of its marginals and a copula: this is the famous Sklar theorem [45]. Specifically, let $\bar{F}_i(x_i) = P(X_i > x_i)$ be the marginal survival functions, $1 \leq i \leq n$. Then, there exists a unique survival copula C such that the survival function of $(X_1, \ldots, X_n)$ is expressed as

$$P(X_1 > x_1, \ldots, X_n > x_n) = C[\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)], \quad x_1, \ldots, x_n \in \mathbb{R}. \tag{3.2}$$

Archimedean copula. A survival copula C is Archimedean if (3.1) can be simplified in the form

$$C(u_1, \ldots, u_n) = \psi\left[\psi^{-1}(u_1) + \ldots + \psi^{-1}(u_n)\right], \quad u_1, \ldots, u_n \in [0, 1], \tag{3.3}$$

for some univariate function $\psi(x) : \mathbb{R}_+ \to [0, 1]$, called generator, which satisfies $\lim_{x\to\infty} \psi(x) = 0$ and $\psi(0) = 1$, and where ψ^{-1} denotes the generalized inverse of ψ (i.e. $\psi^{-1}(u) = \inf\{x : \psi(x) \leq u\}$).

From (3.3), the vector $(U_1, \ldots, U_n)$ has again a particular exchangeable distribution. Arguing as in Section 2, we break the symmetry of this distribution by making this vector only partially exchangeable.

Partially Archimedean copula. Let us partition the vector $\mathbf{U} = (U_1, \ldots, U_n)$ into $m \geq 2$ sub-vectors $\mathbf{U}_j = (U_{j,1}, \ldots, U_{j,n_j})$, $1 \leq j \leq m$, with $n_j \geq 1$ and $n_1 + \ldots + n_m = n$. We then introduce an m-variates function $\psi(x_1, \ldots, x_m) : \mathbb{R}_+^m \to [0, 1]$ which satisfies the border conditions $\psi(0, \ldots, 0) = 1$ and $\psi(x_1, \ldots, x_m) \to 0$ when any $x_j \to \infty$.

Definition 3.1. *A survival copula $C(\mathbf{u}) = C(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ is partially Archimedean of generator ψ if (3.1) can be written in the form*

$$C(\mathbf{u}_1, \ldots, \mathbf{u}_m) = \psi\left[|\psi_1^{-1}(\mathbf{u}_1)|, \ldots, |\psi_m^{-1}(\mathbf{u}_m)|\right], \quad \textit{for all } \mathbf{u}_j \in [0, 1]^{n_j}, \tag{3.4}$$

where the univariate functions $\psi_j : \mathbb{R}_+ \to [0,1]$, $1 \le j \le m$, are the m marginal projections of ψ given by

$$\psi_j(x_j) = \psi(0, \ldots, 0, x_j, 0, \ldots, 0), \quad x_j \in \mathbb{R}_+, \tag{3.5}$$

and using the notation

$$|\psi_j^{-1}(\mathbf{u}_j)| = \psi_j^{-1}(u_{j,1}) + \ldots + \psi_j^{-1}(u_{j,n_j}).$$

The idea of defining a generalized Archimedean copula by the formula (3.4) is not entirely new and was recently proposed by [42] in a different context. In Section 4 of that paper, it is also proved that the generator ψ can be characterized as in the partially Schur-constant model.

Proposition 3.2. *(Characterization of ψ)*
A survival function $\psi : \mathbb{R}_+^{\mathrm{m}} \to [0,1]$ may generate a partially Archimedean copula (3.4) if and only if the function ψ is $\mathbf{n} = (n_1, \ldots, n_m)$-monotone on $\mathbb{R}^{\mathrm{m}}$. In that case, ψ can be expressed as the Williamson $\mathbf{n}$-transform given by (2.8).

Schur-constant models and Archimedean copulas are known to be closely related (see e.g. [13, 29, 38]). We show below that a similar link is also valid in the partially exchangeable case.

Proposition 3.3. *(Correspondence $S \leftrightarrow \psi$)*
(i) Let $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_m)$ be a partially Schur-constant vector of generator S. Then, the associated survival copula is a partially Archimedean copula of generator ψ with $\psi = S$.
(ii) Let $\mathbf{U} = (\mathbf{U}_1, \ldots, \mathbf{U}_m)$ be a vector over $[0,1]^n$ with uniform marginals whose survival copula is partially Archimedean of generator ψ with marginal projections ψ_j, $1 \le j \le m$. Then, the vector $[\psi_1^{-1}(1-\mathbf{U}_1), \ldots, \psi_m^{-1}(1-\mathbf{U}_m)]$ formed by the m sub-vectors

$$\psi_j^{-1}(1-\mathbf{U}_j) = [\psi_j^{-1}(1-U_{j,1}), \ldots, \psi_j^{-1}(1-U_{j,n_j}],$$

is a partially Schur-constant model of generator S with $S = \psi$.

Proof. The reasoning is roughly similar to that followed in the simple Schur-constant model. For this case, a complete proof is provided via Proposition 4.5 of [29]. For simplicity, we assume here that the functions S_j andψ_j have an ordinary inverse function S_j^{-1} and ψ_j^{-1}. Otherwise, the proof method requires working with the generalized inverses.
Let $\mathbf{X}$ be partially Schur-constant of generator S. Its distribution function is given by (2.2). From (3.2), we also know that

$$P(\mathbf{X}_1 \ge \mathbf{x}_1, \ldots, \mathbf{X}_m \ge \mathbf{x}_m) = C\left\{[S_j(x_{j,1}), \ldots, S_j(x_{j,n_j})],\ 1 \le j \le m\right\}.$$

We thus get the identity

$$S(|\mathbf{x}_1|, \ldots, |\mathbf{x}_m|) = C\left\{[S_j(x_{j,1}), \ldots, S_j(x_{j,n_j})],\ 1 \le j \le m\right\}. \tag{3.6}$$

In each group j, we write $x_{j,i} = S_j^{-1}(u_{j,i})$ with $u_{j,i} \in (0,1)$, $1 \le i \le n_j$. Then, (3.6) becomes

$$S\left[S_j^{-1}(u_{j,1}) + \ldots + S_j^{-1}(u_{j,n_j}),\ 1 \le j \le m\right] = C\left[(u_{j,1}, \ldots, u_{j,n_j}),\ 1 \le j \le m\right].$$

By virtue of (3.4), this means that the copula C is partially Archimedean of generator S.
Consider now the vector $[\psi_1^{-1}(1-\mathbf{U}_1), \ldots, \psi_m^{-1}(1-\mathbf{U}_m)]$. Its distribution function is

$$\begin{aligned} P&\left\{[\psi_j^{-1}(1-U_{j,1}) > x_{j,1}, \ldots, \psi_j^{-1}(1-U_{j,n_j}) > x_{j,n_j}],\ 1 \le j \le m\right\} \\ &= P\left\{[U_{j,1} > 1-\psi_j(x_{j,1}), \ldots, U_{j,n_j} > 1-\psi_j(x_{j,n_j})],\ 1 \le j \le m\right\}. \end{aligned} \tag{3.7}$$

By assumption, the copula of $\mathbf{U}$ is partially Archimedean of generator ψ. From (3.1) and (3.4), we can thus express the r.h.s. of (3.7) as

$$\begin{aligned} P\left\{[U_{j,1} > 1-\psi_j(x_{j,1}),\ldots,U_{j,n_j} > 1-\psi_j(x_{j,n_j})],\ 1\le j\le m\right\} \\ = \psi\left[\psi_j^{-1}(\psi_j(x_{j,1}))+\ldots+\psi_j^{-1}(\psi_j(x_{j,n_j})),\ 1\le j\le m\right] \\ = \psi\left(x_{j,1}+\ldots+x_{j,n_j},\ 1\le j\le m\right). \end{aligned} \tag{3.8}$$

From (3.7), (3.8) and (2.2), we deduce that $[\psi_1^{-1}(1-\mathbf{U}_1),\ldots,\psi_m^{-1}(1-\mathbf{U}_m)]$ is partially Schur-contant of generator ψ. ⋄

In the next four sections, we focus on partially Schur-constant models built for two groups of variables. So, $\mathbf{X} = [\mathbf{X}_1 = (X_{1,1},\ldots,X_{1,n_1}), \mathbf{X}_2 = (X_{2,1},\ldots,X_{2,n_2})]$, and from (2.2), there is a bivariate generator $S(x_1,x_2) : \mathbb{R}_+^2 \to [0,1]$ such that

$$\begin{aligned} P[(X_{1,1} > x_{1,1},\ldots,X_{1,n_1} > x_{1,n_1}),(X_{2,1} > x_{2,1},\ldots,X_{2,n_2} > x_{2,n_2})] \\ = S(x_{1,1}+\ldots+x_{1,n_1}, x_{2,1}+\ldots+x_{2,n_2}), \quad \text{for all } x_{j,i}\ge 0. \end{aligned} \tag{3.9}$$

This generator S is a bivariate survival function and by Proposition 2.2, it must be (n_1,n_2)-monotone. Our goal is to present different possible functions for S and to specify, if possible, which are the corresponding probability distributions (for a vector (X_1,X_2) say).

4 From univariate Laplace transforms

We begin with the simplest case where $S(x_1,x_2)$ is the Laplace transform of a positive variable Λ of argument $\zeta_1x_1+\zeta_2x_2$ with $\zeta_1,\zeta_2>0$. So, $S(x_1,x_2)$ is of the form

$$S(x_1,x_2) = E(e^{-\Lambda(\zeta_1x_1+\zeta_2x_2)}), \quad x_1,x_2\ge 0. \tag{4.1}$$

Of course, $(\zeta_1X_1,\zeta_2X_2) =_d (Y_1/\Lambda, Y_2/\Lambda)$ where Y_1,Y_2 are two independent exponentials of parameter 1. In fact, (ζ_1X_1,ζ_2X_2) is Schur-constant and is distributed as the vector $(Z/\Lambda)(U,1-U)$ where U is uniform on $(0,1)$, and Z is distributed as Y_1+Y_2 independently of U, i.e. Z has an Erlang distribution of density $x\exp(-x)$, $x\ge 0$.

Obviously, S defined by (4.1) is infinitely monotone in x_1 and x_2. From (3.9), the associated partially Schur-constant vector $\mathbf{X}$ is of survival function

$$P(\mathbf{X}_1\ge \mathbf{x}_1, \mathbf{X}_2\ge \mathbf{x}_2) = E(e^{-\Lambda(\zeta_1|\mathbf{x}_1|+\zeta_2|\mathbf{x}_2|)}). \tag{4.2}$$

For the main moments of the model, we get using (2.4), (2.5) that inside each group j, $\mu_j = E(X_{j,1}) = (1/\zeta_j)E(1/\Lambda)$, $E(X_{j,1}^2) = (2/\zeta_j^2)E(1/\Lambda^2)$, $E(X_{j,1}X_{j,2}) = E(X_{j,1}^2)/2$, and between the two groups, $E(X_{1,1}X_{2,1}) = (1/\zeta_1\zeta_2)E(1/\Lambda^2)$. Thus, $\rho_j = \rho_{1,2} \equiv \rho$ with

$$\rho = \frac{E(1/\Lambda^2)-[E(1/\Lambda)]^2}{2E(1/\Lambda^2)-[E(1/\Lambda)]^2},$$

i.e., all the correlations within and between the groups are identical and valued in $(0,1/2)$, independently of ζ_1,ζ_2. In other words, the generator (4.1) produces two Schur-constant groups with different distributions but keeps the correlations all equal.

The corresponding partially Archimedean copula (3.4) shows that the dependency structure is not modified by (4.1). Indeed, $C(\mathbf{U})$ in this case is reduced to

$$C(\mathbf{u}_1,\mathbf{u}_2) = \mathcal{L}_\Lambda[|\mathcal{L}_\Lambda^{-1}(\mathbf{u}_1)|+|\mathcal{L}_\Lambda^{-1}(\mathbf{u}_2)|], \quad u_{j,i}\in[0,1], \tag{4.3}$$

where $|\mathcal{L}_\Lambda^{-1}(\mathbf{u}_j)| = \mathcal{L}_\Lambda^{-1}(u_{j,1})+\ldots+\mathcal{L}_\Lambda^{-1}(u_{j,n_j})$ and $\mathcal{L}_\Lambda(x) = E(e^{-\Lambda x})$ is the Laplace transform of Λ of parameter $x\ge 0$. The formula (4.3) is simply that of a Schur-constant vector of dimension $|\mathbf{n}| = n_1+n_2$ and of generator $\mathcal{L}_\Lambda$. For illustration, we consider several particular distributions for Λ.

(1) *Λ has a gamma distribution with positive parameters* $(1/\theta, 1/\beta)$. Its Laplace transform is $\mathcal{L}_\Lambda(x) = 1/(1+\beta x)^{1/\theta}$, so that from (4.1),

$$S(x_1, x_2) = \frac{1}{(1+\beta_1 x_1 + \beta_2 x_2)^{1/\theta}}, \quad x_1, x_2 \geq 0, \tag{4.4}$$

where $\beta_j = \beta\zeta_j$, $j = 1, 2$. Thus, (X_1, X_2) has a bivariate Lomax (Pareto Type II) distribution discussed by [31] and [37] (see also e.g. [50]).

Consider a partially Schur-constant vector $\mathbf{X}$ built from (4.4). Its survival function (4.2) then becomes

$$P(\mathbf{X}_1 \geq \mathbf{x}_1, \mathbf{X}_2 \geq \mathbf{x}_2) = \frac{1}{(1+\beta_1|\mathbf{x}_1| + \beta_2|\mathbf{x}_2|)^{1/\theta}}, \quad x_{j,i} \geq 0.$$

If $\theta < 1/2$, we have $\mu_j = (1/\beta_j)\theta/(1-\theta)$, $\sigma_j^2 = (1/\beta_j^2)\theta^2/(1-\theta)^2(1-2\theta)$, and $\rho = \theta$ which increases with θ from 0 to 1/2. The survival copula (4.3) is the Clayton copula

$$C(\mathbf{u}_1, \mathbf{u}_2) = \frac{1}{(|\mathbf{u}_1^{-\theta}| + |\mathbf{u}_2^{-\theta}| - n_1 - n_2 + 1)_+^{1/\theta}}, \quad u_{j,i} \in [0, 1],$$

after denoting $|\mathbf{u}_j^{-\theta}| = u_{j,1}^{-\theta} + \ldots + u_{j,n_j}^{-\theta}$.

(2) *Λ has a one-sided stable distribution with parameter* $\theta \geq 1$. Its Laplace transform is $\mathcal{L}_\Lambda(x) = \exp(-x^{1/\theta})$, so that from (4.1),

$$S(x_1, x_2) = e^{-(\zeta_1 x_1 + \zeta_2 x_2)^{1/\theta}}, \quad x_1, x_2 \geq 0. \tag{4.5}$$

Thus, (X_1, X_2) has a bivariate Weibull distribution discussed in Section 5 of [25] (see also e.g. [19] and [32]).

A partially Schur-constant vector $\mathbf{X}$ built from (4.5) is of survival function

$$P(\mathbf{X}_1 \geq \mathbf{x}_1, \mathbf{X}_2 \geq \mathbf{x}_2) = e^{-(\zeta_1|\mathbf{x}_1| + \zeta_2|\mathbf{x}_2|)^{1/\theta}}, \quad x_{j,i} \geq 0.$$

We get $\mu_j = (1/\zeta_j)\,\theta\Gamma(\theta)$, $\sigma_j^2 = (1/\zeta_j^2)[2\theta\Gamma(2\theta) - \theta^2\Gamma^2(\theta)]$, and

$$\rho = \frac{\theta\Gamma(2\theta) - \theta^2\Gamma^2(\theta)}{2\theta\Gamma(2\theta) - \theta^2\Gamma^2(\theta)} = 1 - \frac{1}{2 - \theta\Gamma^2(\theta)/\Gamma(2\theta)}.$$

which increases with θ from 0 for $\theta = 1$ to 1/2 as $\theta \to \infty$. The associated survival copula is the Gumbel copula

$$C(\mathbf{u}_1, \mathbf{u}_2) = e^{-\left\{[-\ln(|\mathbf{u}_1|)]^\theta + [-\ln(|\mathbf{u}_2|)]^\theta\right\}^{1/\theta}}, \quad u_{j,i} \in [0, 1],$$

where $[-\ln(|\mathbf{u}_j|)]^\theta$ denotes $[-\ln(u_{j,1})]^\theta + \ldots + [-\ln(u_{j,n_j})]^\theta$.

(3) *Λ has a shifted geometric distribution of parameter* $\theta \in [0, 1)$. Its Laplace transform is $\mathcal{L}_\Lambda(x) = (1-\theta)/(\exp(x) - \theta)$, so that from (4.1),

$$S(x_1, x_2) = \frac{1-\theta}{\exp(\zeta_1 x_1 + \zeta_2 x_2) - \theta}, \quad x_1, x_2 \geq 0, \tag{4.6}$$

This survival function does not seem to be explicitly referenced in the literature.

A partially Schur-constant vector $\mathbf{X}$ built from (4.6) is such that

$$P(\mathbf{X}_1 \geq \mathbf{x}_1, \mathbf{X}_2 \geq \mathbf{x}_2) = \frac{1-\theta}{\exp(\zeta_1|\mathbf{x}_1| + \zeta_2|\mathbf{x}_2|) - \theta}, \quad x_{j,i} \geq 0.$$

We find that $\mu_j = [(1-\theta)/\theta\zeta_j]Li_1(\theta)$, $\sigma_j^2 = [2(1-\theta)/\theta\zeta_j^2]Li_2(\theta) - \mu_j^2$, and

$$\rho = \frac{\theta Li_2(\theta) - (1-\theta)[Li_1(\theta)]^2}{2\theta Li_2(\theta) - (1-\theta)[Li_1(\theta)]^2} = 1 - \frac{1}{2 - (1-\theta)[Li_1(\theta)]^2/\theta Li_2(\theta)},$$

in which the function $Li_s(\theta) = \sum_{i=1}^{\infty} \theta^i/i^s$ is the polylogarithm of order $s = 1, 2$. Recall that for $s = 1$, $Li_1(\theta) = -\ln(1-\theta)$, and for $s \geq 2$, $\theta d[Li_s(\theta)]/d\theta = Li_{s-1}(\theta)$. It can be proved that the correlation increases with θ from 0 to 1/2. The associated survival copula is the Ali-Mikhail-Haq copula

$$C(\mathbf{u}_1, \mathbf{u}_2) = \frac{\prod_{i=1}^{n_1} u_{1,i} \prod_{i=1}^{n_2} u_{2,i}}{1 - \theta \prod_{i=1}^{n_1}(1-u_{1,i}) \prod_{i=1}^{n_2}(1-u_{2,i})}, \quad u_{j,i} \in [0, 1].$$

(4) Λ *has a logarithmic distribution of parameter* $p \in (0, 1)$. Its Laplace transform is $\mathcal{L}_\Lambda(x) = \ln(1 - pe^{-x})/\ln(1-p)$, so that from (4.1),

$$S(x_1, x_2) = \ln(1 - pe^{-\zeta_1 x_1 - \zeta_2 x_2})/\ln(1-p), \quad x_1, x_2 \geq 0, \tag{4.7}$$

which does not seem to be standard either, to our knowledge.

A partially Schur-constant vector $\mathbf{X}$ built from (4.7) is of survival function

$$P(\mathbf{X}_1 \geq \mathbf{x}_1, \mathbf{X}_2 \geq \mathbf{x}_2) = \ln(1 - pe^{-\zeta_1|\mathbf{x}_1| - \zeta_2|\mathbf{x}_2|})/\ln(1-p), \quad x_{j,i} \geq 0.$$

We get $\mu_j = (1/\zeta_j)Li_2(p)/Li_1(p)$, $\sigma_j^2 = (2/\zeta_j^2)Li_3(p)/Li_1(p) - \mu_j^2$, and

$$\rho = \frac{Li_1(p)Li_3(p) - [Li_2(p)]^2}{2Li_1(p)Li_3(p) - [Li_2(p)]^2} = 1 - \frac{1}{2 - [Li_2(p)]^2/Li_1(p)Li_3(p)},$$

which increases with p from 0 to 1/2 (numerical check). Setting $p = 1 - e^{-\theta}$ ($\theta > 0$), the associated survival copula is the Frank copula

$$C(\mathbf{u}_1, \mathbf{u}_2) = -(1/\theta)\ln[1 - (1 - e^{-\theta|\mathbf{u}_1|})(1 - e^{-\theta|\mathbf{u}_2|})/(1 - e^{-\theta})], \quad u_{j,i} \in [0, 1].$$

5 From bivariate Laplace transforms

We pass here to the interesting case, more complex, where $S(x_1, x_2)$ is the Laplace transform of a positive random vector (Λ_1, Λ_2) of arguments (x_1, x_2). So, $S(x_1, x_2)$ is of the form

$$S(x_1, x_2) = E(e^{-\Lambda_1 x_1 - \Lambda_2 x_2}), \quad x_1, x_2 \geq 0, \tag{5.1}$$

The function S defined by (5.1) is again infinitely monotone in x_1 and x_2. From (3.9), the partially Schur-constant vector $\mathbf{X}$ is of survival function

$$P(\mathbf{X}_1 \geq \mathbf{x}_1, \mathbf{X}_2 \geq \mathbf{x}_2) = E(e^{-\Lambda_1|\mathbf{x}_1| - \Lambda_2|\mathbf{x}_2|}). \tag{5.2}$$

For the main moments, we get from (2.4), (2.5) that inside each group j, $\mu_j = E(X_{j,1}) = E(1/\Lambda_j)$, $E(X_{j,1}^2) = 2E(1/\Lambda_j^2)$ and $E(X_{j,1}X_{j,2}) = E(X_{j,1}^2)/2$, and between the two groups, $E(X_{1,1}X_{2,1}) = E(1/\Lambda_1\Lambda_2)$. Thus, ρ_j has values in (0, 1/2) and is equal to

$$\rho_j = \frac{E(1/\Lambda_j^2) - [E(1/\Lambda_j)]^2}{2E(1/\Lambda_j^2) - [E(1/\Lambda_j)]^2},$$

while $\rho_{1,2} \equiv \rho(X_{1,1}, X_{2,1})$ can be positive or not and is given by

$$\rho_{1,2} = \frac{E(1/\Lambda_1\Lambda_2) - E(1/\Lambda_1)E(1/\Lambda_2)}{\{2E(1/\Lambda_1^2) - [E(1/\Lambda_1)]^2\}^{1/2}\{2E(1/\Lambda_2^2) - [E(1/\Lambda_2)]^2\}^{1/2}}.$$

The associated partially Archimedean copula (3.4) is reduced to

$$C(\mathbf{u}_1, \mathbf{u}_2) = \mathcal{L}_{(\Lambda_1,\Lambda_2)}[|\mathcal{L}_{\Lambda_1}^{-1}(\mathbf{u}_1)| + |\mathcal{L}_{\Lambda_2}^{-1}(\mathbf{u}_2)|], \quad u_{j,i} \in [0, 1], \tag{5.3}$$

where $\mathcal{L}_{\Lambda_1}(x_1)$, $\mathcal{L}_{\Lambda_2}(x_2)$, $\mathcal{L}_{(\Lambda_1,\Lambda_2)}(x_1, x_2)$ are the Laplace transforms of $\Lambda_1, \Lambda_2, (\Lambda_1, \Lambda_2)$ of parameters $x_1, x_2 \geq 0$. For illustration, we consider several particular distributions for (Λ_1, Λ_2).

(1) (Λ_1, Λ_2) *has a bivariate gamma distribution of positive parameters* $(\alpha, \zeta_1, \zeta_2, \zeta_3)$ *with* $\zeta_3 \leq \zeta_1\zeta_2$. Following [10, 12, 30], this means that the Laplace transform of (Λ_1, Λ_2) is of the form

$$E(e^{-\Lambda_1 x_1 - \Lambda_2 x_2}) = \frac{1}{(1 + \zeta_1 x_1 + \zeta_2 x_2 + \zeta_3 x_1 x_2)^\alpha}, \quad x_1, x_2 \geq 0. \tag{5.4}$$

As proved in [10], getting an effective Laplace transform requires that $\zeta_3 \leq \zeta_1\zeta_2$. Note that each Λ_j has a univariate gamma distribution of parameters (α, ζ_j), and they are independent if $\zeta_3 = \zeta_1\zeta_2$. In the literature, the distribution of (Λ_1, Λ_2) is often called the Kibble and Moran distribution (see e.g. the books [3] and [23]). We see that $E(\Lambda_j) = \alpha\zeta_j$, $\sigma^2(\Lambda_j) = \alpha\zeta_j^2$ (the two coefficients of variation are identical and equal to $1/\sqrt{\alpha}$), and $\rho(\Lambda_1, \Lambda_2) = 1 - \zeta_3/\zeta_1\zeta_2 \geq 0$ and independent of α. As with the bivariate Normal distribution, the two variables are independent when their correlation is 0.

Let us insert (5.4) in the survival function $S(x_1, x_2)$ defined by (5.1). The corresponding vector (X_1, X_2) has a bivariate Lomax (Pareto Type II) distribution of survival function

$$S(x_1, x_2) = \frac{1}{(1 + \zeta_1 x_1 + \zeta_2 x_2 + \zeta_3 x_1 x_2)^{\alpha}}, \quad x_1, x_2 \geq 0. \tag{5.5}$$

The condition on the parameters for the existence of a Lomax distribution is $\zeta_3 \leq (1+\alpha)\zeta_1\zeta_2$. Recall that this distribution comes from a Laplace transform under the more restrictive constraint $\zeta_3 \leq \zeta_1\zeta_2$. This bivariate Lomax distribution was proposed by [17] in the case where $\zeta_1 = \zeta_2 = 1$. The general form was examined by [44] and revisited by [43] as a bivariate Pareto model; see also e.g. [2], formula (16). Obviously, if (X_1, X_2) has this Lomax distribution of parameters $(\zeta_1, \zeta_2, \zeta_3, \alpha)$, then $(\zeta_1 X_1, \zeta_2 X_2)$ has the Lomax distribution of parameters $(1, 1, \zeta_3/\zeta_1\zeta_2, \alpha)$ (considered by [17]). Note that each X_j has a univariate Lomax distribution of parameters (α, μ_j), and they are independent if $\zeta_3 = \zeta_1\zeta_2$.

A partially Schur-constant vector $\mathbf{X}$ built from (5.5) is such that

$$P(\mathbf{X}_1 \geq \mathbf{x}_1, \mathbf{X}_2 \geq \mathbf{x}_2) = \frac{1}{(1 + \zeta_1|\mathbf{x}_1| + \zeta_2|\mathbf{x}_2| + \zeta_3|\mathbf{x}_1||\mathbf{x}_2|)^{\alpha}}, \quad x_{j,i} \geq 0.$$

For each group, if $\alpha > 2$, we get $\mu_j = 1/\zeta_j(\alpha - 1)$, $\sigma_j^2 = \alpha/\zeta_j^2(\alpha-1)^2(\alpha-2)$, and $\rho_j = 1/\alpha$, the same for the two groups, which decreases with α from $1/2$ to 0. Moreover, we see that if $\zeta_3 \leq (\geq)\, \zeta_1\zeta_2$, then $S(x_1, x_2) \geq (\leq) S(x_1, 0)S(0, x_2)$ and thus $\rho_{1,2} \geq (\leq)\, 0$, which is in agreement with Figure 1 in [43]. In the present case, however, we assumed $\zeta_3 \leq \zeta_1\zeta_2$, so that $\rho_{1,2} \geq 0$ (as $\rho(\Lambda_1, \Lambda_2)$). In [24], it is shown that when $\alpha > 2$,

$$\rho_{1,2} = [(1-\zeta)(\alpha-2)/\alpha^2]\, F(1, 2; \alpha+1; 1-\zeta),$$

where $\zeta = \zeta_3/\zeta_1\zeta_2$ (≤ 1 in our case) and $F(a, b; c; z)$ is the Gauss hypergeometric function (see e.g. Chapter 15 of [1]). Thus, $\rho_{1,2}$ decreases with ζ, from $1/\alpha$ when $\zeta = 0$, i.e. $\zeta_3 = 0$ (since $F(1, 2; \alpha+1; 1) = \alpha/(\alpha-2)$) to 0 when $\zeta = 1$, i.e. $\zeta_3 = \zeta_1\zeta_2$. The associated survival copula (5.3) is

$$C(\mathbf{u}_1, \mathbf{u}_2) = \frac{1}{[|\mathbf{u}_1^{-1/\alpha}| + |\mathbf{u}_2^{-1/\alpha}| - n_1 - n_2 + 1 + (\zeta_3/\zeta_1\zeta_2)(|\mathbf{u}_1^{-1/\alpha}| - n_1)(|\mathbf{u}_2^{-1/\alpha}| - n_2)]_+^{\alpha}}, \quad u_{j,i} \in [0, 1].$$

(2) (Λ_1, Λ_2) *has a bifactor gamma distribution of positive parameters* $(\alpha_1, \alpha_2, \zeta_1, \zeta_2, \zeta_3)$ *with* $\zeta_3 \leq \zeta_1\zeta_2$ *and* $\alpha_1 \leq \alpha_2$. Following [5, 11], this means that $(\Lambda_1, \Lambda_2) =_d (\Lambda_{[1]}, \Lambda_{[2]} + \Lambda)$ with $(\Lambda_{[1]}, \Lambda_{[2]})$ has a bivariate gamma distribution of positive parameters $\alpha_1, \zeta_1, \zeta_2, \zeta_3 \leq \zeta_1\zeta_2$ and Λ has an independent gamma distribution of positive parameters $(\alpha_2 - \alpha_1, \zeta_2)$. Equivalently, the Laplace transform of (Λ_1, Λ_2) is given by

$$E(e^{-\Lambda_1 x_1 - \Lambda_2 x_2}) = \frac{1}{(1 + \zeta_1 x_1 + \zeta_2 x_2 + \zeta_3 x_1 x_2)^{\alpha_1}} \frac{1}{(1 + \zeta_2 x_2)^{\alpha_2 - \alpha_1}}, \quad x_1, x_2 \geq 0. \tag{5.6}$$

Note that each Λ_j has a univariate gamma distribution of parameters (α_j, ζ_j), i.e. the parameters α_j can also be different here. We see that $\rho(\Lambda_1, \Lambda_2) = \rho(\Lambda_{[1]}, \Lambda_{[2]})\, \sigma(\Lambda_{[2]})/\sigma(\Lambda_2)$ with $\sigma^2(\Lambda_2) = \sigma^2(\Lambda_{[2]}) + \sigma^2(\Lambda)$, hence $\rho(\Lambda_1, \Lambda_2) = (1 - \zeta_3/\zeta_1\zeta_2)\sqrt{\alpha_1/\alpha_2} \in [0, \sqrt{\alpha_1/\alpha_2}]$.

Combining (5.1), (5.6) leads us to define a vector (X_1, X_2) of survival function

$$S(x_1, x_2) = \frac{1}{(1 + \zeta_1 x_1 + \zeta_2 x_2 + \zeta_3 x_1 x_2)^{\alpha_1}} \frac{1}{(1 + \zeta_2 x_2)^{\alpha_2 - \alpha_1}}, \quad x_1, x_2 \geq 0. \tag{5.7}$$

This implies that $(X_1, X_2) =_d [X_{[1]}, \min(X_{[2]}, X)]$ where $(X_{[1]}, X_{[2]})$ is a bivariate Lomax vector of survival function (5.5) with α_1 substitued for α, and X is an independent Lomax variable of parameters $(\alpha_2 - \alpha_1, \zeta_2)$. Indeed, thanks to the assumptions made, we have

$$P[X_{[1]} > x_1, \min(X_{[2]}, X) > x_2] = P(X_{[1]} > x_1, X_{[2]} > x_2)P(X > x_2) = S(x_1, x_2).$$

Note that X_2 is a Lomax variable of parameters (α_2, ζ_2). The survival function (5.7) does not seem to be referenced in the literature.

We associate with (5.7) a partially Schur-constant vector $\mathbf{X}$ defined by

$$P(\mathbf{X}_1 \geq \mathbf{x}_1, \mathbf{X}_2 \geq \mathbf{x}_2) = \frac{1}{(1+\zeta_1|\mathbf{x}_1| + \zeta_2|\mathbf{x}_2| + \zeta_3|\mathbf{x}_1||\mathbf{x}_2|)^{\alpha_1}} \frac{1}{(1+\zeta_2|\mathbf{x}_2|)^{\alpha_2-\alpha_1}}, \quad x_{j,i} \geq 0.$$

If $\alpha_1, \alpha_2 > 2$, we have $\mu_j = 1/\zeta_j(\alpha_j - 1)$, $\sigma_j^2 = \alpha_j/\zeta_j^2(\alpha_j-1)^2(\alpha_j-2)$, and $\rho_j = 1/\alpha_j$, different for the two groups. Moreover, $\rho_{1,2} = \rho(X_{[1]}, X_{[2]})\,\sigma(X_{[2]})/\sigma(X_2)$ with $\sigma^2(X_2) = \sigma_2^2$ and $\sigma^2(X_{[2]}) = \alpha_1/\zeta_2^2(\alpha_1-1)^2(\alpha_1-2)$, hence

$$\rho_{1,2} = \rho(X_{[1]}, X_{[2]})\sqrt{\alpha_1(\alpha_2-1)^2(\alpha_2-2)/\alpha_2(\alpha_1-1)^2(\alpha_1-2)}.$$

This time, the survival copula (5.3) is

$$C(\mathbf{u}_1,\mathbf{u}_2) = \frac{1}{[|\mathbf{u}_1^{-1/\alpha_1}| + |\mathbf{u}_2^{-1/\alpha_2}| - n_1 - n_2 + 1 + (\zeta_3/\zeta_1\zeta_2)(|\mathbf{u}_1^{-1/\alpha_1}| - n_1)(|\mathbf{u}_2^{-1/\alpha_2}| - n_2)]_+^{\alpha_1}}$$
$$\frac{1}{[|\mathbf{u}_2^{-1/\alpha_2}| - n_2 + 1]_+^{\alpha_2-\alpha_1}}, \quad u_{j,i} \in [0,1].$$

(3) *Using an exponential transform of positive powers* $\alpha_1, \alpha_2 \leq 1$. Suppose that the vector $(\zeta_1 X_1, \zeta_2 X_2)$, $\zeta_1, \zeta_2 > 0$, is distributed as the vector $[(Y_1/\Lambda)^{1/\alpha_1}, (Y_2/\Lambda)^{1/\alpha_2}]$, $\alpha_1, \alpha_2 > 0$, where Λ is some positive variable and Y_1, Y_2 are two independent exponentials of parameter 1. Note that the model (4.1) corresponds to the particular case where $\alpha_1 = \alpha_2 = 1$. Then, $S(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$ is given by

$$S(x_1, x_2) = E(e^{-\Lambda[(\zeta_1 x_1)^{\alpha_1} + (\zeta_2 x_2)^{\alpha_2}]}), \quad x_1, x_2 \geq 0. \tag{5.8}$$

This function is a bivariate Laplace transform if it is infinitely monotone in x_1 and x_2. We observe that this is verified with the additional conditions on the parameters $\alpha_1, \alpha_2 \leq 1$.

For example, if Λ as a one-sided stable distribution with parameter $\theta \geq 1$ (as for (4.5)), we get

$$S(x_1, x_2) = e^{-[(\zeta_1 x_1)^{\alpha_1} + (\zeta_2 x_2)^{\alpha_2}]^{1/\theta}}, \quad x_1, x_2 \geq 0. \tag{5.9}$$

Thus, (X_1, X_2) has a bivariate Weibull distribution discussed by [26] (see also e.g. [32]). For $\zeta_1 = \zeta_2 = 1$, this distribution is chosen by [42] as an example of a bivariate Laplace transform under the conditions $\alpha_1, \alpha_2 \leq 1$.

If Λ has a gamma distribution with positive parameters $(1/\theta, 1)$ (as for (4.4)),

$$S(x_1, x_2) = \frac{1}{[1 + (\zeta_1 x_1)^{\alpha_1} + (\zeta_2 x_2)^{\alpha_2}]^{1/\theta}}, \quad x_1, x_2 \geq 0. \tag{5.10}$$

Thus, (X_1, X_2) has a bivariate Burr (Pareto Type IV) distribution defined by [48] (see also e.g. [23]).

The partially Schur-constant vector built from (5.9) is of survival function

$$P(\mathbf{X}_1 \geq \mathbf{x}_1, \mathbf{X}_2 \geq \mathbf{x}_2) = E(e^{-\Lambda[(\zeta_1|\mathbf{x}_1|)^{\alpha_1} + (\zeta_2|\mathbf{x}_2|)^{\alpha_2}]}), \tag{5.11}$$

For the moments, since

$$\int_0^\infty x^m e^{-\beta x^\alpha}\,dx = \frac{1}{\alpha\beta^{(m+1)/\alpha}}\Gamma[(m+1)/\alpha], \quad m \geq 0, \alpha, \beta > 0,$$

we find that $\mu_j = E(X_{j,1}) = (1/\zeta_j\alpha_j)\Gamma(1/\alpha_j)E(1/\Lambda^{1/\alpha_j})$, $E(X_{j,1}^2) = (2/\zeta_j^2\alpha_j)\Gamma(2/\alpha_j)E(1/\Lambda^{2/\alpha_j})$, $E(X_{j,1}X_{j,2}) = E(X_{j,1}^2)/2$, and $E(X_{1,1}X_{2,1}) = (1/\zeta_1\alpha_1\zeta_2\alpha_2)\Gamma(1/\alpha_1)\Gamma(1/\alpha_2)E(1/\Lambda^{1/\alpha_1+1/\alpha_2})$. A formula for ρ_j and $\rho_{1,2}$ follows directly. For the associated survival copula, since $[\mathcal{L}_\Lambda(x_j^{\alpha_j})]^{-1} = [\mathcal{L}_\Lambda^{-1}(x_j)]^{1/\alpha_j}$, we get

$$C(\mathbf{u}_1, \mathbf{u}_2) = \mathcal{L}_\Lambda\{[(\mathcal{L}_\Lambda^{-1}(u_{1,1}))^{1/\alpha_1} + \ldots + (\mathcal{L}_\Lambda^{-1}(u_{1,n_1}))^{1/\alpha_1}]^{\alpha_1}$$
$$+ [(\mathcal{L}_\Lambda^{-1}(u_{2,1}))^{1/\alpha_2} + \ldots + (\mathcal{L}_\Lambda^{-1}(u_{2,n_2}))^{1/\alpha_2}]^{\alpha_2}\}, \quad u_{j,i} \in [0,1].$$

6 A finitely monotone survival function

So far, we have presented examples of survival functions $S(x_1, x_2)$ which correspond to Laplace transforms and are therefore infinitely monotone in x_1 and x_2. In this Section, we want to put forward a particular classical distribution for which $S(x_1, x_2)$ is not a Laplace transform.

Among various possible cases, we choose the bivariate exponential distribution of Gumbel [17] with positive parameters $(\zeta_1, \zeta_2, \zeta_3)$. Its survival function is given by

$$S(x_1, x_2) = e^{-\zeta_1 x_1 - \zeta_2 x_2 - \zeta_3 x_1 x_2}, \quad x_1, x_2 \geq 0, \tag{6.1}$$

and as the density shows, it is actually a distribution if $0 \leq \zeta_3 \leq \zeta_1\zeta_2$. Obviously, $(\zeta_1 X_1, \zeta_2 X_2)$ has a bivariate exponential distribution of parameters $(1, 1, \zeta)$ where $0 \leq \zeta \equiv \zeta_3/\zeta_1\zeta_2 \leq 1$.

Let us consider (6.1) as a possible generator for a partially Schur-constant model (3.9). According to Proposition 2.2, we must verify that $S(x_1, x_2)$ is indeed a (n_1, n_2)-monotone function. The proposition below illustrates that this property depends on the value of the parameters and the size of the groups.

Proposition 6.1. *$S(x_1, x_2)$ defined by (6.1) is $(1, n)$-monotone if and only if*

$$\zeta_3 \leq (1/n)\zeta_1\zeta_2, \quad n \geq 1. \tag{6.2}$$

To be $(2, n)$-monotone ($n \geq 2$), a sufficient condition is $\zeta_3 \leq [1-(1-1/n)^{1/2}]\zeta_1\zeta_2$. To be $(3, n)$-monotone ($n \geq 3$), it suffices that $\zeta_3 \leq [1 - (1 - 1/n(n-1)(n-2))^{1/3}]\zeta_1\zeta_2$.

Proof. From (2.6), the function S is $(1, n)$-monotone when

$$(-1)^{i+j} S^{(i,j)}(x_1, x_2) \geq 0, \quad \text{for } (i, j) = (0, 1), \ldots (0, n); (1, 0), \ldots, (1, n), \tag{6.3}$$

and all $x_1, x_2 \geq 0$. For $i = 0$, we have

$$S^{(0,j)}(x_1, x_2) = (-1)^j (\zeta_2 + \zeta_3 x_1)^j S(x_1, x_2),$$

hence (6.3) is satisfied for all j. For $i = 1$, we get

$$S^{(1,j)}(x_1, x_2) = (-1)^j (\zeta_1 + \zeta_3 x_2)^{j-1} [j\zeta_3 - (\zeta_1 + \zeta_3 x_2)(\zeta_2 + \zeta_3 x_1)] S(x_1, x_2),$$

so that (6.3) is satisfied for any given j if and only if

$$j\zeta_3 \leq (\zeta_1 + \zeta_3 x_2)(\zeta_2 + \zeta_3 x_1), \quad x_1, x_2 \geq 0,$$

which is equivalent to $\zeta_3 \leq (1/j)\zeta_1\zeta_2$. Since it must be true for $1 \leq j \leq n$, the global condition becomes (6.2).
The $(2, n)$-monotonicity of S, $n \geq 2$, requires that the condition (6.3) is also satisfied when $(i, j) = (2, 0), \ldots, (2, n)$. First, consider the case $j \geq 2$. Then, we have

$$\begin{aligned} S^{(2,j)}(x_1, x_2) = {} & (-1)^j (\zeta_2 + \zeta_3 x_1)^{j-2} \\ & [j(j-1)\zeta_3^2 - 2j\zeta_3(\zeta_1 + \zeta_3 x_2)(\zeta_2 + \zeta_3 x_1) + (\zeta_1 + \zeta_3 x_2)^2 (\zeta_2 + \zeta_3 x_1)^2] S(x_1, x_2), \end{aligned}$$

so that (6.3) for any given j is fulfilled if and only if

$$j(j-1)\zeta_3^2 - 2j\zeta_3(\zeta_1 + \zeta_3 x_2)(\zeta_2 + \zeta_3 x_1) + (\zeta_1 + \zeta_3 x_2)^2 (\zeta_2 + \zeta_3 x_1)^2 \geq 0,$$

which can be rewritten as

$$j[(\zeta_1 + \zeta_3 x_2)(\zeta_2 + \zeta_3 x_1) - \zeta_3]^2 + (j^2 - 2j)\zeta_3^2 - (j-1)(\zeta_1 + \zeta_3 x_2)^2 (\zeta_2 + \zeta_3 x_1)^2 \geq 0.$$

Since $j^2 - 2j \geq 0$ (because $j \geq 2$), a sufficient condition is

$$j[(\zeta_1 + \zeta_3 x_2)(\zeta_2 + \zeta_3 x_1) - \zeta_3]^2 \geq (j-1)(\zeta_1 + \zeta_3 x_2)^2 (\zeta_2 + \zeta_3 x_1)^2, \quad x_1, x_2 \geq 0,$$

and taking the square root then leads to the condition $\zeta_3 \le [1-(1-1/j)^{1/2}]\zeta_1\zeta_2$. Since the upper bound is decreasing with j, the overall condition for $2 \le j \le n$ becomes $\zeta_3 \le [1-(1-1/n)^{1/2}]\zeta_1\zeta_2$. Now, consider the cases $j = 0$ and $j = 1$. For $(2,0)$, there is no condition on the parameters. For $(2,1)$, the condition is, as for $(1,2)$, $\zeta_3 \le (1/2)\zeta_1\zeta_2$, and since $1/2 > 1-(1-1/n)^{1/2}$ (because $n \ge 2$), the condition $\zeta_3 \le [1-(1-1/n)^{1/2}]\zeta_1\zeta_2$ still applies. Finally, it remains to combine this result with the two other cases $i = 0$ and $i = 1$. As $1-(1-1/n)^{1/2} \le 1/n$, we keep the same sufficient condition.

For the $(3,n)$-monotonicity of S, $n \ge 3$, we obtain the sufficient condition announced using a similar argument. Details are omitted. ◇

If $S(x_1, x_2)$ is (n_1, n_2)-monotone, we can build a partially Schur-constant vector of survival function

$$P(\mathbf{X}_1 \ge \mathbf{x}_1, \mathbf{X}_2 \ge \mathbf{x}_2) = e^{-\zeta_1|\mathbf{x}_1|-\zeta_2|\mathbf{x}_2|-\zeta_3|\mathbf{x}_1||\mathbf{x}_2|}, \quad x_{j,i} \ge 0. \tag{6.4}$$

For example, when $(n_1, n_2) = (1, n)$, we know from Proposition 6.1 that $S(x_1, x_2)$ is $(1,n)$-monotone iff $\zeta_3 \le (1/n)\zeta_1\zeta_2$. Under this condition, the vector $\mathbf{X} = [X_1, \mathbf{X}_2 = (X_{2,1}, \ldots, X_{2,n})]$ is $(1,n)$-partially Schur-constant with

$$P(X_1 \ge x_1, \mathbf{X}_2 \ge \mathbf{x}_2) = e^{-\zeta_1 x_1 - (\zeta_2+\zeta_3 x_1)(x_{2,1}+\ldots+x_{2,n})}, \quad x_1, x_{2,i} \ge 0.$$

Each vector $\mathbf{X}_j$ is formed wih n_j independent exponentials of parameter ζ_j, so that $\mu_j = 1/\zeta_j$, $\sigma_j^2 = 1/\zeta_j^2$ and $\rho_j = 0$. The correlation $\rho_{1,2} \equiv \rho_{1,2}(\zeta)$ is given by the formula (2.12) of [17] in tems of the so-called logarithmic integral. Using the related exponential integral $E_1(x) = \int_x^\infty e^{-t}/t dt$ (see e.g. Chapter 5.1 in [1]), it becomes

$$\rho_{1,2}(\zeta) = -1 + (1/\zeta)e^{1/\zeta}E_1(1/\zeta),$$

which shows that $\rho_{1,2}(\zeta) \le 0$ and decreases with ζ from $\rho_{1,2}(0) = 0$ to $\rho_{1,2}(1) = -0.4036527$. Furthermore, we can use a known approximation for the function $xe^x E_1(x)$ (formula 5.1.54 in Chapter 5) to obtain

$$\rho_{1,2}(\zeta) \approx -1 + \frac{1/\zeta^2 + a_1/\zeta + a_2}{1/\zeta^2 + b_1/\zeta + b_2}, \quad \text{with an error of } 5/10^5,$$

where $a_1 = 2.334733$, $a_2 = 0.250621$, $b_1 = 3.330657$, $b_2 = 1.681534$. If the model (6.4) is well defined, the associated survival copula is given by

$$C(\mathbf{u}_1, \mathbf{u}_2) = \prod_{i=1}^{n_1} u_{1,i} \, (\prod_{i=1}^{n_2} u_{2,i})^{1-\zeta\sum_{i=1}^{n_1}\ln(u_{1,i})}, \quad u_{j,i} \in [0,1].$$

Other cases of finitely monotone survival functions are provided by some distributions presented before, but this time asking that $S(x_1, x_2)$ be only (n_1, n_2)-monotone (instead of infinitely monotone). So, we have seen that the bivariate Lomax survival function exists if $\zeta \le 1+\alpha$ and coresponds to a bivariate Laplace transform if $\zeta \le 1$. Now, it can be shown, for example, that $S(x_1, x_2)$ defined by (5.5) is $(1,n)$-monotone if and only if

$$\zeta \le 1 + \alpha/n, \quad n \ge 1,$$

while it is $(2,n)$-monotone ($n \ge 2$) under the stronger condition $\zeta \le \sqrt{1+\alpha/n}$.

7 From bivariate Williamson transforms

By (2.8), a survival function $S(x_1, x_2)$ which is (n_1, n_2)-monotone can be represented as

$$S(x_1, x_2) = E\left[(1-x_1/Z_1)_+^{n_1-1}\,(1-x_2/Z_2)_+^{n_2-1}\right], \quad x_1, x_2 \ge 0, \tag{7.1}$$

where Z_j is distributed as the sum of the variables in group j. Thus, it suffices to choose any distribution for (Z_1, Z_2) to obtain a partially Schur-constant model.

In practice, however, the formula (7.1) turns out to be less convenient than it looks. Let us give two simple illustrations.

(1) Suppose that (Z_1, Z_2) is a continuous Schur-constant vector of univariate generator $\hat{S}(x)$. This means that (Z_1, Z_2) is distributed as the vector $Z(U, 1-U)$ where U is a uniform variable on $(0, 1)$ and Z is an independent variable distributed as $Z_1 + Z_2$ (with density $f_Z(z) = z\hat{S}^{(2)}(z)$) (see e.g. [13]). By insertion in (7.1), we get

$$S(x_1, x_2) = \int_{u=0}^{1} du \int_{z=g_u(x_1,x_2)}^{\infty} (1 - x_1/zu)^{n_1-1} [1 - x_2/z(1-u)]^{n_2-1} f_Z(z)dz,$$

where $g_u(x_1, x_2) = \max\{x_1/u, x_2/(1-u)\}$. Applying the binomial rule then leads to

$$\begin{aligned} S(x_1, x_2) &= \int_{u=0}^{1} du \int_{z=g_u(x_1,x_2)}^{\infty} \sum_{i=0}^{n_1-1} \binom{n_1-1}{i} (\frac{-x_1}{zu})^i \sum_{j=0}^{n_2-1} \binom{n_2-1}{j} (\frac{-x_2}{z(1-u)})^j f_Z(z)dz \\ &= \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \binom{n_1-1}{i} \binom{n_2-1}{j} (-x_1)^i (-x_2)^j \int_{u=0}^{1} \frac{1}{u^i(1-u)^j} du \int_{z=g_u(x_1,x_2)}^{\infty} z^{-i-j} f_Z(z)dz. \end{aligned} \tag{7.2}$$

For example, choose $\hat{S}(x) = \exp(-\lambda x)$, $\lambda > 0$. Thus, Z has an Erlang distribution of parameters $(2, \lambda)$ with density $f_Z(z) = \lambda^2 z \exp(-\lambda z)$. In that case, the last integral in (7.2) can be rewritten as

$$\lambda^2 \int_{z=g_u(x_1,x_2)}^{\infty} z^{-i-j+1} e^{-\lambda z} dz = (\frac{1}{\lambda})^{i+j} \Gamma[-i-j+2, \lambda g_u(x_1, x_2)],$$

where $\Gamma(a, x)$ is the upper incomplete gamma function ($= \int_x^\infty t^{a-1} e^{-t} dt,$ for $x > 0$, $a \in \mathbb{R}$). Substituting this in formula (7.2) then gives us the expression of $S(x_1, x_2)$.

As another example, choose $1/Z$ of gamma distribution with parameter $1/\theta$, $\theta > 0$. Thus, Z is inverse gamma with density $f_Z(z) = [1/\Gamma(\theta)] z^{-\theta-1} \exp(-1/z)$. The last integral in (7.2) then becomes

$$\begin{aligned} \frac{1}{\Gamma(\theta)} \int_{z=g_u(x_1,x_2)}^{\infty} z^{-i-j-\theta-1} e^{-1/z} dz &= \frac{1}{\Gamma(\theta)} \int_{t=0}^{1/g_u(x_1,x_2)} t^{i+j+\theta-1} e^{-t} dt \\ &= \frac{1}{\Gamma(\theta)} \gamma[i + j + \theta, 1/g_u(x_1, x_2)], \end{aligned}$$

where $\gamma(a, x)$ is the lower incomplete gamma function ($= \int_0^x t^{a-1} e^{-t} dt,$ for $x > 0$, $a > 0$).

Due to the complexity of the function S, the partially Schur-constant vector itself is difficult to use. Its moments and correlations, however, can be easily calculated. First, concerning the two variables Z_j, we apply (2.11) with $n_j = 2$ to get μ_{Z_j}, $\sigma^2_{Z_j}$, ρ_{Z_1,Z_2}; for example, $\mu_{Z_j} = \mu_Z/2$. Then, passing to the variables $X_{j,i}$ of the model, we use (2.11), (2.12), this time with (n_1, n_2), to determine μ_j, σ_j^2, ρ_j and $\rho_{1,2}$; for example, $\mu_j = \mu_{Z_j}/n_j = \mu_Z/2n_j$.

(2) Suppose that (Z_1, Z_2) is distributed as the bivariate gamma vector (Λ_1, Λ_2) of Laplace transform (5.4). From (7.1), we have

$$\begin{aligned} S(x_1, x_2) &= \int_{z_1=0}^{\infty} \int_{z_2=0}^{\infty} (1 - x_1/z_1)_+^{n_1-1} (1 - x_2/z_2)_+^{n_2-1} f_{(Z_1,Z_2)}(z_1, z_2) dz_1 dz_2 \\ &= \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} \binom{n_1-1}{i} \binom{n_2-1}{j} x_1^i x_2^j \int_{z_1=x_1}^{\infty} \int_{z_2=x_2}^{\infty} z_1^{-i} z_2^{-j} f_{(Z_1,Z_2)}(z_1, z_2) dz_1 dz_2, \end{aligned} \tag{7.3}$$

in which $f_{(Z_1,Z_2)}$ is the corresponding density and is given by

$$f_{(Z_1,Z_2)}(z_1, z_2) = [1/\zeta_3^\alpha \Gamma(\alpha)] (z_1 z_2)^{\alpha-1} e^{-(\zeta_2/\zeta_3)z_1 - (\zeta_1/\zeta_3)z_2} f_\alpha(c z_1 z_2),$$

where $c = (\zeta_1\zeta_2 - \zeta_3)/\zeta_3^2$ and $f_\alpha(z) = \sum_{k=0}^{\infty} z^k/\Gamma(\alpha + k)k!$ (see [10]). The double integral $\int_{z_1=x_1}^{\infty}\int_{z_2=x_2}^{\infty}$ in (7.3) is explicitly written as

$$\frac{1}{\zeta_3^\alpha\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{c^k}{\Gamma(\alpha+k)k!}\int\limits_{z_1=x_1}^{\infty}\int\limits_{z_2=x_2}^{\infty} z_1^{\alpha+k-i-1}z_2^{\alpha+k-j-1}e^{-(\zeta_2/\zeta_3)z_1-(\zeta_1/\zeta_3)z_2}dz_1dz_2,$$

and the two integrals above can be factorized and then expressed as

$$(\frac{\zeta_3}{\zeta_2})^{\alpha+k-i}(\frac{\zeta_3}{\zeta_1})^{\alpha+k-j}\Gamma[\alpha+k-i,(\zeta_2/\zeta_3)x_1]\Gamma[\alpha+k-j,(\zeta_1/\zeta_3)x_2].$$

By inserting this in (7.3), we obtain the survival function $S(x_1, x_2)$.

The partially Schur-constant model is again little convenient. Still, as previously indicated, the formulas for μ_{Z_j}, $\sigma_{Z_j}^2$, ρ_{Z_1,Z_2} are particularly simple in the case of a bivariate gamma vector (Z_1, Z_2). Thus, for the variables $X_{j,i}$ of the model, we easily obtain from (2.11), (2.12) the parameters μ_j, σ_j^2, ρ_j and $\rho_{1,2}$; for example, $\mu_j = \mu_{Z_j}/n_j = \alpha\zeta_j/n_j$.

8 Building nested and multi-level models

Partially exchangeable models provide a flexible framework for representing the dependence between groups of random variables. We briefly show below that this approach covers the well-known nested Archimedean copulas, and that it can be easily generalized to model dependence on several levels.

Nested Schur-constant models. Hierarchical Archimedean copulas are an alternative to overcome the symmetry present in simple Archimedean copulas. A popular hierarchical method consists in nesting several Archimedean copulas one inside the other, with a certain nesting constraint in order to obtain a proper copula. It was originally developed by Joe [20] and studied in detail by e.g. [18] and [33]. We show below that this approach, although different, has links with the partially Archimedean copulas discussed here.

Let us partition the vector $\mathbf{U} = (U_1, \dots, U_n)$ as in Section 3, i.e.

$$\mathbf{U} = (\mathbf{U}_1, \dots, \mathbf{U}_m), \quad \text{where } \mathbf{U}_j = (U_{j,1}, \dots, U_{j,n_j}), \quad 1 \le j \le m.$$

A nested Archimedean copula with 1 level and m children is defined by

$$\begin{aligned} C(\mathbf{u}_1, \dots, \mathbf{u}_m) &= C_0[C_1(\mathbf{u}_1), \dots, C_m(\mathbf{u}_m)] \\ &= \psi_0\left\{\psi_0^{-1}\left[\psi_1\left(|\psi_1^{-1}(\mathbf{u}_1)|\right)\right] + \dots + \psi_0^{-1}\left[\psi_m\left(|\psi_m^{-1}(\mathbf{u}_m)|\right)\right]\right\}, \end{aligned} \tag{8.1}$$

where ψ_0 is the generator of the parent copula C_0, and $\psi_1, \dots, \psi_m$ are the generators of the m child copulas $C_1, \dots, C_m$.

Now, consider a partially Archimedean copula (3.4) of generator $\psi(x_1, \dots, x_m)$, and let $\psi_j(x_j)$, $1 \le j \le m$, be the m marginals of ψ.

Proposition 8.1. *If ψ can be expressed in the particular form*

$$\psi(x_1, \dots, x_m) = \psi_0[\psi_0^{-1}(\psi_1(x_1)) + \dots + \psi_0^{-1}(\psi_m(x_m))], \quad x_1, \dots, x_m \ge 0, \tag{8.2}$$

where the function $\psi_0(x) : \mathbb{R}_+ \to [0, 1]$ satisfies $\psi_0^{-1}(\psi(0, \dots, 0)) = 0$, then the copula (3.4) corresponds to the nested Archimedean copula (8.1).

Proof. By (3.4), the partially Archimedean copula is defined as $\psi[|\psi_1^{-1}(\mathbf{u}_1)|, \dots, |\psi_m^{-1}(\mathbf{u}_m)|]$. Inserting the expression (8.2) for ψ then gives the copula (8.1). Note that $\psi_0^{-1}(\psi(0, \dots, 0)) = 0$ guarantees that ψ_j, $1 \le j \le m$, are the marginals of ψ as it was supposed. ◇

We emphasize that (8.2) is a special form for ψ, not the general form. In addition, since a partially Archimedean generator is necessarily $(n_1, \dots, n_m)$-monotone, the function ψ_0 in (8.2) must be such that this condition is effectively fulfilled, which is not an easy task to verify. Partially Archimedean copulas seem to be more convenient to handle.

By Proposition 3.3 (ii), the partially Schur-constant model $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_m)$ whose generator $S = \psi$ is given by (8.2) is provided by the vector

$$[\mathbf{X}_1 = \psi_1^{-1}(1 - \mathbf{U}_1), \dots, \mathbf{X}_m = \psi_m^{-1}(1 - \mathbf{U}_m)]. \tag{8.3}$$

In other words, (8.3) represents the Schur-constant model associated with the nested Archimedean copula (8.1).

Multi-level Schur-constant models. Partially Schur-constant models can be generalized to describe a form of dependence on several levels. This leads us to the introduction of a notion of partial exchangeability at several levels which, to our knowledge, is not standard in the literature.

Specifically, consider 2 levels. The vector $\mathbf{X}$ is partitioned into l groups which are partially exchangeable (instead of simply exchangeable). Each group k, $1 \le k \le l$, is formed of m_k subgroups (k, j) of $n_{k,j}$ random variables, hence the notation

$$(\mathbf{X}_{k,1}, \dots, \mathbf{X}_{k,m_k}) \text{ where } \mathbf{X}_{k,j} = (X_{k,j,1}, \dots, X_{k,j,n_{k,j}}), \quad 1 \le j \le m_k.$$

The whole vector $\mathbf{X} = (X_1, \dots, X_n)$ is thus represented as

$$\mathbf{X} = [(\mathbf{X}_{1,1}, \dots, \mathbf{X}_{1,m_1}), \dots, (\mathbf{X}_{l,1}, \dots, \mathbf{X}_{l,m_l})],$$

where, of course, $n = n_{1,1} + \dots + n_{1,m_1} + \dots + n_{l,1} + \dots + n_{l,m_l}$.

Let us introduce a multivariate survival function $S[(x_{1,1}, \dots, x_{1,m_1}), \dots, (x_{l,1}, \dots, x_{l,m_l})] : \mathbb{R}_+^{m_1+\dots+m_l} \to [0, 1]$. We define the joint survival function of $\mathbf{X}$ by

$$\begin{aligned} &P[(\mathbf{X}_{1,1} > \mathbf{x}_{1,1}, \dots, \mathbf{X}_{1,m_1} > \mathbf{x}_{1,m_1}), \dots, (\mathbf{X}_{l,1} > \mathbf{x}_{l,1}, \dots, \mathbf{X}_{l,m_l} > \mathbf{x}_{l,m_l})] \\ &\quad = S\left[(|\mathbf{x}_{1,1}|, \dots, |\mathbf{x}_{1,m_1}|), \dots, (|\mathbf{x}_{l,1}|, \dots, |\mathbf{x}_{l,m_l}|)\right], \end{aligned} \tag{8.4}$$

using the previous notation

$$|\mathbf{x}_{k,j}| = x_{k,j,1} + \dots + x_{k,j,n_{k,j}}, \quad 1 \le k \le l, 1 \le j \le m_k.$$

We also consider the marginals of the l groups in the function S, with the notation

$$S_k(x_{k,1}, \dots, x_{k,m_k}) = S[(0, \dots, 0), \dots, (0, \dots, 0), (x_{k,1}, \dots, x_{k,m_k}), (0, \dots, 0), \dots, (0, \dots, 0)].$$

Then, we have

$$P(\mathbf{X}_{k,1} > \mathbf{x}_{k,1}, \dots, \mathbf{X}_{k,m_k} > \mathbf{x}_{k,m_k}) = S_k(|\mathbf{x}_{k,1}|, \dots, |\mathbf{x}_{k,m_k}|), \quad 1 \le k \le l,$$

which is asked to be a partially Schur-constant vector. Thus, each function $S_k : \mathbb{R}_+^{m_k} \to [0, 1]$ is $(n_{k,1}, \dots, n_{k,m_k})$-monotone and admits the representation (2.8).

Now, we construct a partially Archimedean survival copula associated with the model (8.4). For that, we introduce again a function $\psi[(x_{1,1}, \dots, x_{1,m_1}), \dots, (x_{l,1}, \dots, x_{l,m_l})] : \mathbb{R}_+^{m_1+\dots+m_l} \to [0, 1]$ whose marginals are denoted by

$$\psi_{k,j}(x_{k,j}) = \psi(0, \dots, 0, x_{k,j}, 0, \dots, 0),$$

and the marginals of the l groups by

$$\psi_k(x_{k,1}, \dots, x_{k,m_k}) = \psi[(0, \dots, 0), \dots, (0, \dots, 0), (x_{k,1}, \dots, x_{k,m_k}), (0, \dots, 0), \dots, (0, \dots, 0)].$$

We partition the vector $\mathbf{U} = (U_1, \dots, U_n)$ as $\mathbf{X}$ into l groups formed of m_k subgroups (k, j) of sizes $n_{k,j}$. For 2 levels, a partially Archimedean copula $C(\mathbf{u})$ is then defined by

$$\begin{aligned} &C[(\mathbf{u}_{1,1}, \dots, \mathbf{u}_{1,m_1}), \dots, (\mathbf{u}_{l,1}, \dots, \mathbf{u}_{l,m_l})] \\ &\quad = \psi\left[\left(|\psi_{1,1}^{-1}(\mathbf{u}_{1,1})|, \dots, |\psi_{1,m_1}^{-1}(\mathbf{u}_{1,m_1})|\right), \dots, \left(|\psi_{l,1}^{-1}(\mathbf{u}_{l,1})|, \dots, |\psi_{l,m_l}^{-1}(\mathbf{u}_{l,m_l})|\right)\right], \end{aligned} \tag{8.5}$$

where

$$|\psi_{k,j}^{-1}(\mathbf{u}_{k,j})| = \psi_{k,j}^{-1}(u_{k,j,1}) + \ldots + \psi_{k,j}^{-1}(u_{k,j,n_{k,j}}), \quad 1 \le k \le l, \ 1 \le j \le m_k.$$

When all the $\mathbf{u}_{i,j} = 0$ except in group k, $C(\mathbf{u})$ is reduced to

$$C_k(\mathbf{u}_{k,1}, \ldots, \mathbf{u}_{k,m_k}) = \psi_k \left[|\psi_{k,1}^{-1}(\mathbf{u}_{k,1})|, \ldots, |\psi_{k,m_k}^{-1}(\mathbf{u}_{k,m_k})|\right], \quad 1 \le k \le l,$$

in agreement with the definition (3.4).

Finally, compare with a nested Archimedean copula for 2 levels defined by

$$\begin{aligned} C[(\mathbf{u}_{1,1}, \ldots, \mathbf{u}_{1,m_1}), \ldots, (\mathbf{u}_{l,1}, \ldots, \mathbf{u}_{l,m_l})] \\ = \psi_0\{\psi_0^{-1}[C_1(\mathbf{u}_{1,1}, \ldots, \mathbf{u}_{1,m_1})] + \ldots + \psi_0^{-1}[C_l(\mathbf{u}_{l,1}), \ldots, C_{l,m_l}(\mathbf{u}_{l,m_l})]\}, \end{aligned} \tag{8.6}$$

in which $\psi_0 : \mathbb{R}_+ \to [0, 1]$ is the generator of the copula for the l groups, and each of them is of copula defined as in (8.1) by

$$\begin{aligned} C_k(\mathbf{u}_{k,1}, \ldots, \mathbf{u}_{k,m_k}) &= C_{k,0}\left[C_{k,1}(\mathbf{u}_{k,1}), \ldots, C_{k,m_k}(\mathbf{u}_{k,m_k})\right] \\ &= \psi_{k,0}\left\{\psi_{k,0}^{-1}\left[\psi_{k,1}(|\psi_{k,1}^{-1}(\mathbf{u}_{k,1})|)\right] + \ldots + \psi_{k,0}^{-1}\left[\psi_{k,m_k}(|\psi_{k,m_k}^{-1}(\mathbf{u}_{k,m_k})|)\right]\right\}, \quad 1 \le k \le l, \end{aligned}$$

where $\psi_{k,0} : \mathbb{R}_+^{m_k} \to [0, 1]$ and $\psi_{k,j} : \mathbb{R}_+ \to [0, 1]$, $j = 1, \ldots, m_k$, are the generators of the parent and child copulas, respectively. It is easily seen that Proposition 8.1 can be extended to 2 levels. In other words, the nested Archimedean copula (8.6) is a partially Archimedean copula (8.5) for which the generator ψ has a particular expression generalizing (8.2).

Acknowledgements: I thank the editor and the referees for their valuable comments. I am grateful to M.M. Claramunt (Universitat de Barcelona) and S. Loisel (ISFA, Université de Lyon) for interesting and stimulating discussions. This research received partial support via the *DIALog* Chair sponsored by CNP Assurances.

Conflict of interest statement: Author states no conflict of interest.

References

[1] Abramowitz, M. and I. A. Stegun (1964). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, New York.

[2] Arnold, B. C. (1990). A flexible family of multivariate Pareto distributions. *J. Statist. Plann. Inference 24*(2), 249–258.

[3] Balakrishnan, N. and C.-D. Lai (2009). *Continuous Bivariate Distributions*. Second edition. Springer, New York.

[4] Barlow, R. E. and M. B. Mendel (1993). Similarity as a probabilistic characteristic of aging. In R. E. Barlow, C. A. Clarotti, and F. Spizzichino (Eds.), *Reliability and Decision Making*, pp. 233–244. Chapman & Hall, London.

[5] Bernardoff, P. (2018). Marshall-Olkin Laplace transform copulas of multivariate gamma distributions. *Comm. Statist. Theory Methods 47*(3), 655–670.

[6] Caramellino, L. and F. Spizzichino (1994). Dependence and aging properties of lifetimes wih Schur-constant survival functions. *Probab. Engrg. Inform. Sci. 8*(1), 103–111.

[7] Castañer, A. and M. M. Claramunt (2019). Equilibrium distributions and discrete Schur-constant models. *Methodol. Comput. Appl. Probab. 21*, 449–459.

[8] Castañer, A., M. M. Claramunt, C. Lefèvre, and S. Loisel (2015). Discrete Schur-constant models. *J. Multivariate Anal. 140*, 343–362.

[9] Castañer, A., M. M. Claramunt, C. Lefèvre, and S. Loisel (2019). Partially Schur-constant models. *J. Multivariate Anal. 172*, 47–58.

[10] Chatelain, F. (2007). *Lois Gamma Multivariées pour le Traitement d'Images Radar*. PhD thesis, Institut National Polytechnique de Toulouse, France.

[11] Chatelain, F., J.-Y. Tourneret, and J. Inglada (2008). Change of detection in multisensor SAR images using bivariate gamma distributions. *IEEE Trans. Image Process. 17*(3), 249–258.

[12] Chatelain, F., J.-Y. Tourneret, J. Inglada, and A. Ferrari (2007). Bivariate gamma distributions for image registration and change detection. *IEEE Trans. Image Process. 16*(7), 1796–1806.

[13] Chi, Y., J. Yang, and Y. Qi (2009). Decompositon of a Schur-constant model and its applications. *Insurance Math. Econom. 44*(3), 398–408.

[14] Cossette, H., S.-P. Gadoury, E. Marceau, and I. Mtalai (2017). Hierarchical Archimedean copulas through multivariate compound distributions. *Insurance Math. Econom. 76*, 1–13.
[15] de Finetti, B. (1930). Funzione caratteristica di un fenomeno aleatorio. *Mem. R. Accad. Naz. Lincei 4*, 86–133.
[16] de Finetti, B. (1938). Sur la condition d'équivalence partielle. In *Actualités Scientifiques et Industrielles 739*, 5–18. Hermann & Cie, Paris.
[17] Gumbel, E. J. (1960). Bivariate exponential distributions. *J. Amer. Statist. Assoc. 55*(292), 698–707.
[18] Hofert, M. (2011). Efficiently sampling nested Archimedean copulas. *Comput. Statist. Data Anal. 55*(1), 57–70.
[19] Hougaard, P. (1986). A class of multivariate failure time distributions. *Biometrika 73*(3), 671–678.
[20] Joe, H. (1997). *Multivariate Models and Dependence Concepts*. Chapman & Hall, London.
[21] Jones, M. C. and É. Marchand (2019). Multivariate discrete distributions via sums and shares. *J. Multivariate Anal. 171*, 83–93.
[22] Khintchine, A. Y. (1938). On unimodal distributions. *Izv. Nauch. Inst. Mat. Mekh. 2*, 1–7 (in russian).
[23] Kotz, S., N. Balakrishnan, and N. L. Johnson (2000). *Continuous Multivariate Distributions*. Second edition. John Wiley & Sons, New York.
[24] Lai, C. D., M. Xie, and I. G. Bairamov (2001). Dependence and ageing properties of bivariate Lomax distribution. In Y. Hayakawa, T. Irony, and M. Xie (Eds.), *System and Bayesian Reliability*, pp. 243–256. World Scientific, Singapore.
[25] Lee, L. (1979). Multivariate distributions having Weibull properties. *J. Multivariate Anal. 9*(2), 267–277.
[26] Lee, C. and M.-J. Wen (2009). A multivariate Weibull distribution. *Pakistan J. Statist. Oper. Res. 5*(2), 55–66.
[27] Lefèvre, C. and S. Loisel (2013). On multiply monotone distributions, continuous or discrete, with applications. *J. Appl. Probab. 50*(3), 827–847.
[28] Lefèvre, C., S. Loisel, and S. Utev (2018). Markov property in discrete Schur-constant models. *Methodol. Comput. Appl. Probab. 20*, 1003–1012.
[29] Lefèvre, C. and M. Simon (2021). Schur-constant and related dependence models, with application to ruin probabilities. *Methodol. Comput. Appl. Probab. 23*, 317–339.
[30] Letac, G., J.-Y. Tourneret, and F. Chatelain (2007). The bivariate gamma distribution with Laplace transform $(1+as+bt+cst)^{-q}$: history, characterizations, estimation. Available at http://www.math.univ-toulouse.fr/~letac/bivgamma.pdf.
[31] Lindley, D. V. and N. D. Singpurwalla (1986). Multivariate distributions for the life lengths of components of a system sharing a common environment. *J. Appl. Probab. 23*(2), 418–431.
[32] Lu, J.-C. and G. K. Bhattacharyya (1990). Some new constructions of bivariate Weibull models. *Ann. Inst. Statist. Math. 42*, 543–559.
[33] McNeil, A. J. (2008). Sampling nested Archimedean copulas. *J. Stat. Comput. Simul. 78*(6), 567–581.
[34] McNeil, A. J. and J. Nešlehová (2009). Multivariate Archimedean copulas, d-monotone functions and l_1-norm symmetric distributions. *Ann. Statist. 37*(5B), 3059–3097.
[35] Nair, N. U. and P. G. Sankaran (2014). Modelling lifetimes with bivariate Schur-constant equilibrium distributions from renewal theory. *Metron 72*, 331–349.
[36] Nair, N. U, P. G. Sankaran, and N. Balakrishnan (2018). *Reliability Modelling and Analysis in Discrete Time*. Academic, San Diego.
[37] Nayak, T. K. (1987). Multivariate Lomax distribution: properties and usefulness in reliability theory. *J. Appl. Probab. 24*(1), 170–177.
[38] Nelsen, R. B. (2005). Some properties of Schur-constant survival models and their copulas. *Braz. J. Probab. Stat. 19*(2), 179–190.
[39] Nelsen, R. B. (2006). *An Introduction to Copulas*. Second edition. Springer, New York.
[40] Ressel, P. (2011). Monotonicity properties of multivariate distribution and survival functions - with an application to Lévy-frailty copulas. *J. Multivariate Anal. 102*(3), 393–404.
[41] Ressel, P. (2014). Higher order monotonic functions of several variables. *Positivity 18*, 257–285.
[42] Ressel, P. (2018). A multivariate version of Williamson's theorem, l^1-symmetric survival functions, and generalized Archimedean copulas. *Depend. Model. 6*, 356–368.
[43] Sankaran, P. G. and D. Kundu (2014). On a bivariate Pareto model. *Statistics 48*(2), 241–255.
[44] Sankaran, P. G. and N. U. Nair (1993). A bivariate Pareto model and its applications to reliability. *Naval Res. Logist. 40*(7), 1013–1020.
[45] Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris 8*, 229–231.
[46] Ta, B. Q., D. S. Le, M. B. Ha, and X. D. Tran (2018). On characterizations of bivariate Schur-constant models and applications. In L. H. Anh, L. S. Dong, V. Kreinovich, and N. N. Thach (Eds.), *Econometrics for Financial Applications*, pp. 890–901. Springer, Cham.
[47] Ta, B. Q. and C. P. Van (2017). Some properties of bivariate Schur-constant distributions. *Statist. Probab. Lett. 124*, 69–76.
[48] Takahasi, K. (1965). Note on multivariate Burr's distribution. *Ann. Inst. Statist. Math. 17*, 257–260.
[49] Williamson, R. E. (1956). Multiply monotone functions and their Laplace transforms. *Duke Math. J. 23*(2), 189–207.
[50] Yeh, H.-C. (2004). Some properties and characterizations for generalized multivariate Pareto distributions. *J. Multivariate Anal. 88*(1), 47–60.