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Piotr Jaworski* and Marcin Krzywda

On copulas of self-similar Ito processes

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Abstract: We characterize the cumulative distribution functions and copulas of two-dimensional self-similar Ito processes, with randomly correlated Wiener margins, as solutions of certain elliptic partial differential equations.

Keywords: copula, Ito diffusion, stochastic differential equations, self-similar processes, elliptic partial differential equations

MSC: 62H05, 60G18, 60H10, 60E05, 60J60

1 Introduction

The paper deals with the borderline of the copula theory and stochastic processes. It concerns the vector valued stochastic processes $X_t = (X_t^1, \dots, X_t^n)$, $t \in T$. The goal is to describe the evolution of interdependences between $X_t^1, \dots, X_t^n$ in terms of copulas and copula processes. In a way it is a continuation of papers by Sempi [31], Choe etal. [7], Jaworski and Krzywda [16] and Jaworski [15]. Sempi in ([31]) was studying the possibility of coupling two Wiener processes by using a given copula. In [16] we started to investigate copulas of self-similar processes. In [7] the partial differential equation for copulas of the Ito processes is derived but under very tight technical assumptions. In [15] it is achieved but in a general case. It is shown that the copula process is a weak solution of a parabolic partial differential equation.

In this paper we study the evolution of the dependence between two Wiener processes with random quadratic covariation. Specifically we extend the research on two-dimensional 1/2-self-similar Ito diffusions whose margins are Wiener processes and their interdependencies at every time moment are described by a given copula, that we have initiated in [16]. We drop the assumption of differentiability of cumulative distribution functions and corresponding copulas. We replace the classical derivatives by weak derivatives in Sobolev sense, so called distributional derivatives, as in [15]. This allows us to present new and more general results. We consider 1/2-self-similar processes as solutions of certain stochastic differential equations (SDE) and show how to construct 1/2-self-similar solution basing on a non-self-similar solution.

In more details, we consider a pair of stochastic processes $(X_t, Y_t)_{t\geq 0}$, where $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are one dimensional Wiener processes and their quadratic covariation is homogeneous for $t \geq 1$

$$d\langle X, Y\rangle_t = A\left(\frac{X_t}{\sqrt{t}}, \frac{Y_t}{\sqrt{t}}\right) dt, \tag{1.1}$$

where the real valued function A is defined on the real plane and $|A|$ is a bounded by 1. Such processes arise as solutions of the stochastic differential equations (SDE) with homogeneous coefficients

$$\begin{aligned} dX_t &= dW_t^1, \\ dY_t &= A\left(\frac{X_t}{\sqrt{t}}, \frac{Y_t}{\sqrt{t}}\right) dW_t^1 + B\left(\frac{X_t}{\sqrt{t}}, \frac{Y_t}{\sqrt{t}}\right) dW_t^2, \end{aligned} \tag{1.2}$$

***Corresponding Author: Piotr Jaworski:** Institute of Mathematics, University of Warsaw, Warszawa, Poland, ORCID: 0000-0001-8540-2698, E-mail: P.Jaworski@mimuw.edu.pl
Marcin Krzywda: Institute of Mathematics, Jagiellonian University, Kraków, Poland, E-mail: marcin@krzywda.net

where $B(x,y) = \sqrt{1 - A(x,y)^2}$.

We begin with showing that for A and B Lipschitz, any pair of Wiener processes fulfilling 1.2 gives rise to 1/2-self similar solutions. They appear as limits of time-rescaled (time-shifted) processes $(X_{t\tau}/\sqrt{\tau}, Y_{t\tau}/\sqrt{\tau})_{t\geq 1}$ when τ tends to infinity or as weighted generalized mixtures of time-rescaled processes. Then we derive a partial differential equation (PDE) (satisfied in a weak sense) that describes the copula of the initial value, i.e. the copula C of the random pair (X_1, Y_1), giving rise to 1/2-self similar solution.

$$\begin{aligned}&\varphi(\Phi^{-1}(u))^2 D^{2,0}C(u,v) + \varphi(\Phi^{-1}(v))^2 D^{0,2}C(u,v)\\&+2A(\Phi^{-1}(u),\Phi^{-1}(v))\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v))D^{1,1}C(u,v) = 0,\end{aligned} \tag{1.3}$$

where Φ and φ denote the cumulative distribution function and the density of the standard normal distribution $N(0,1)$ and $D^{i,j}$ denote the partial derivatives. We also present the opposite result by which if any copula C fulfills the previously obtained PDE in a weak sense with coefficients satisfying certain regularity conditions then there exists a 1/2-self-similar process $(X_t, Y_t)_{t\geq 1}$ such that the copula of $(X_t, Y_t)_{t\geq 1}$ is equal to C. Furthermore basing on the maximum principle for the elliptic PDE we show that the point-wise dominance of the quadratic covariations A implies the concordance dominance of the corresponding copulas.

The structure of the paper is as follows: On the start we shortly recall basics on copulas, self-similarity and weak derivatives (section 2). In section 3 we introduce the underlying stochastic processes X_t and Y_t and state the results concerning their self-similarity. First concerning the existence of 1/2-self-similar solutions among solutions of certain stochastic differential equations, next concerning their differential characterization. The proofs are provided in section 4. In the last section we discuss some examples based on Gaussian, FGM and Archimedean copulas.

2 Notation

We recall the basic concepts.

A (bivariate) *copula* is a restriction to $[0,1]^2$ of a distribution function whose univariate margins are uniformly distributed on $[0,1]$. Specifically, $C\colon [0,1]^2 \to [0,1]$ is a copula if it satisfies the following properties:
(C1) $C(x,0) = C(0,x) = 0$ for every $x \in [0,1]$,
(C2) $C(x,1) = C(1,x) = x$ for every $x \in [0,1]$,
(C3) C is 2-*increasing*, that is, the C-volume V_C of any rectangle $[x_1,x_2] \times [y_1,y_2]$ of $[0,1]^2$ is positive, i.e.

$$V_C([x_1,x_2]\times[y_1,y_2]) = C(x_1,y_1) + C(x_2,y_2) - C(x_1,y_2) - C(x_2,y_1) \geq 0.$$

Due to the celebrated *Sklar's Theorem*, the joint distribution function F of any pair (X,Y) of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ can be written as a composition of a copula C and the univariate marginals F_1 and F_2, i.e. for all $(x,y) \in \mathbb{R}^2$, $F(x,y) = C(F_1(x), F_2(y))$. Moreover, if X, Y are random variables with continuous cumulative distribution functions, then the copula C is uniquely determined.
We will denote the set of all two-dimensional copulas by $\mathcal{C}^2$. Note, that $\mathcal{C}^2$ is a bounded, compact, convex subset of the Banach space of the continuous functions on the unit square endowed with the supremum metric. For copulas C_1 and C_2 we put

$$||C_2 - C_1||_\infty = \sup\{|C_2(u,v) - C_1(u,v)| : (u,v) \in [0,1]^2\}. \tag{2.1}$$

Specifically any sequence of copulas contains a convergent subsequence, i.e.

$$\forall (C_n)_{n=1}^\infty \quad \exists C_\star \quad \exists (n_k)_{k=1}^\infty \quad n_k \to \infty, \quad ||C_{n_k} - C_\star||_\infty \to 0. \tag{2.2}$$

The copulas $C_\star$, being the limits of subsequences, are referred to as *cluster points* of the sequence C_n. For more details about copula theory and some of its applications, we refer to [6, 9, 17–20, 26].

A random process is self-similar if its distributions scale. Specifically (X_t, Y_t), $t \geq t_0 \geq 0$, is called H-self-similar (with $H \geq 0$) when

$$(X_{at}, Y_{at}) \sim a^H(X_t, Y_t) \qquad \text{for all } a \geq 1,$$

where $\sim$ denotes equality of joint distributions. We call H the exponent of self-similarity of the process. For example standard Brownian Motion is $1/2$-self-similar. For the details on self-similarity we refer to [10] and [33]. Please note that our definition is a bit weaker than the common literature definition, ie. we do not require the equality of distributions of stochastic processes but only those of random vectors at each time $t \geq t_0$.

We can re-write the definition of self-similarity as follows

$$F_t(x, y) = F(t^{-H}x, t^{-H}y),$$

where F_t is a cumulative distribution function of the pair (X_t, Y_t) and $F = F_{t_0}$. By Sklar's theorem applied to F

$$F_t(x, y) = C(\Phi(t^{-H}x), \Psi(t^{-H}y)),$$

where we use $\Phi = \Phi_{t_0}$ and $\Psi = \Psi_{t_0}$ to denote univariate cumulative distribution functions of X_{t_0} and Y_{t_0} which we assume to be continuous. Since the processes X_t and Y_t also are self-similar analogously we may write

$$\Phi_t(x) = \Phi(t^{-H}x), \qquad \Psi_t(y) = \Psi(t^{-H}y).$$

Once again by Sklar's theorem for each $t \geq t_0$ we obtain a copula C_t such that

$$F_t(x, y) = C_t(\Phi_t(x), \Psi_t(y)).$$

Therefore, by the uniqueness of copula for random variables with continuous distribution functions, we have $C_t(x, y) = C_{t_0}(x, y)$ for each $t \geq t_0$.

We conclude that, when X_t and Y_t have continuous distribution functions for each $t \geq t_0$, then the process $(X_t, Y_t)_{t \geq t_0}$ is self-similar if and only if its copula is constant and both $(X_t)_{t \geq t_0}$ and $(Y_t)_{t \geq t_0}$ are self-similar.

In the following we shall deal with $H = \frac{1}{2}$.

To characterize the copulas of 1/2-self-similar processes we will need to substantially weaken the notion of partial derivatives (see [4, 11, 13]).

Definition 2.1. *Suppose that U is an open subset of $\mathbb{R}^n$, functions u and v are locally integrable on U and $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index.*

We say that v is α^{th} weak partial derivative of u, written

$$D^\alpha u = v,$$

provided that

$$\int_U u(x) \frac{\partial^{|\alpha|} h(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} dx_1 \dots dx_n = (-1)^{|\alpha|} \int_U v(x) h(x) dx_1 \dots dx_n$$

for all test functions $h \in C^\infty(U)$ with compact support and $|\alpha| = \alpha_1 + \dots + \alpha_n$.

Weak partial derivatives, if they exist, are unique up to a set of measure zero. Note that, since copulas are Lipschitz functions, they are weakly differentiable. As an example of weak partial derivatives of a copula $C(u, v)$ one may consider one of the Dini derivatives ([8, 24]), say right-side upper one, applied respectively to both variables

$$D^{1,0}\, C(u,v) = \limsup_{h\to 0^+} \frac{C(u+h,v)-C(u,v)}{h}, \quad \text{for} \quad u\in[0,1), v\in[0,1], \tag{2.3}$$

$$D^{0,1}\, C(u,v) = \limsup_{h\to 0^+} \frac{C(u,v+h)-C(u,v)}{h}, \quad \text{for} \quad u\in[0,1], v\in[0,1). \tag{2.4}$$

The choice of the version of the weak derivative is related to the choice of the version of the conditional probability. In more details, if C is a copula of the random variables X and Y, then the formula

$$F_{X|Y=y}(x) = \lim_{\xi\to 0^+} D^{0,1} C(F_X(\xi), F_Y(y)), \tag{2.5}$$

expressing the conditional distribution function of X in terms of the right continuous modification of the weak derivative (2.4), determines the version of the conditional probability $P(\cdot|\sigma(Y))$ on the σ-field $\sigma(X)$ (compare [2, 14]). Since the modifications occur only at "jumps" of $D^{0,1}C(F_X(\cdot), F_Y(y))$, we have for fixed y, such that $F_Y(y) < 1$,

$$F_{X|Y=y}(x) \overset{ae}{=} D^{0,1} C(F_X(x), F_Y(y)). \tag{2.6}$$

3 Main results

We consider the solutions $(X_t, Y_t)_{t\geq 1}$ of the following system of stochastic differential equations:

$$\begin{aligned} dX_t &= dW_t^1, \\ dY_t &= A\left(\frac{X_t}{\sqrt{t}}, \frac{Y_t}{\sqrt{t}}\right) dW_t^1 + B\left(\frac{X_t}{\sqrt{t}}, \frac{Y_t}{\sqrt{t}}\right) dW_t^2, \end{aligned} \tag{3.1}$$

where

A1. W^1 and W^2 are two independent Wiener processes defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
A2. the functions $A, B : \mathbb{R}^2 \to \mathbb{R}$ are Lipschitz,
A3. for all $(x,y)\in\mathbb{R}^2$ we have $B(x,y) = \sqrt{1-A(x,y)^2}$ and $|A(x,y)| < 1$,
A4. A is differentiable with respect to the second variable.

By M_A and L_A we shall denote respectively the supremum of the modulus and the Lipschitz coefficient of A with respect to the euclidean distance

$$M_A = \sup\{|A(x,y)| : (x,y)\in\mathbb{R}^2\} \tag{3.2}$$

$$L_A = \sup\left\{ \frac{|A(x_2,y_2)-A(x_1,y_1)|}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} : (x_1,y_1)\neq(x_2,y_2)\right\} \tag{3.3}$$

Furthermore we denote by L the square of the Lipschitz coefficient of the vector (A, B)

$$L = \sup\left\{ \frac{|A(x_1,y_1)-A(x_2,y_2)|^2 + |B(x_1,y_1)-B(x_2,y_2)|^2}{(x_2-x_1)^2+(y_2-y_1)^2} : (x_1,y_1)\neq(x_2,y_2)\right\} \tag{3.4}$$

Note that when $M_A < 1$, we have a bound

$$L \leq \frac{L_A^2}{1-M_A^2}. \tag{3.5}$$

In fact we understand (3.1) as an integral equation:

$$\begin{aligned} X_t &= X_1 + W_t^1 - W_1^1, \\ Y_t &= Y_1 + \int_1^t A\left(\frac{X_s}{\sqrt{s}}, \frac{Y_s}{\sqrt{s}}\right) dW_s^1 + \int_1^t B\left(\frac{X_s}{\sqrt{s}}, \frac{Y_s}{\sqrt{s}}\right) dW_s^2, \end{aligned} \tag{3.6}$$

where the initial values, i.e. the random pair (X_1, Y_1), and the two-dimensional Wiener process $(W_t^1 - W_1^1, W_t^2 - W_1^2)_{t\geq 1}$ are independent. The process $(X_t, Y_t)_{t\geq 1}$ is adapted to the filtration defined by the Wiener processes W^1 and W^2 and the initial values

$$\mathcal{F}_t = \sigma(X_1, Y_1, W_s^1 - W_1^1, W_s^2 - W_1^2, s \in [1, t]). \tag{3.7}$$

Assumption **A1.** ensures that the Ito integrals in (3.6) are well defined and the quadratic covariation of any solution of SDE (3.1) $(X, Y) = (X_t, Y_t)_{t\geq 1}$ fulfills

$$d\langle X, Y\rangle_t = A\left(\frac{X_t}{\sqrt{t}}, \frac{Y_t}{\sqrt{t}}\right) dt. \tag{3.8}$$

The second one, **A2.**, implies the existence and uniqueness of the solution of SDE (3.1). Due to **A3.**, the quadratic variation of $Y = (Y_t)_{t\geq 1}$ is given by

$$d\langle Y, Y\rangle_t = (A^2 + B^2)dt = 1 \cdot dt, \tag{3.9}$$

which implies that $(Y_{t+1} - Y_1)_{t\geq 0}$ is a Wiener process.

Our goal is to study the existence and properties of $1/2$-self-similar solutions of SDE (3.1). Note that the self-similarity of the process $(X_t, Y_t)_{t\geq 1}$ does not depend on the choice of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ neither on the choice of the Wiener processes. It depends only on the joint distribution of the of the pair (X_1, Y_1). It is a corollary from the uniqueness of the weak solutions of the SDE, see [5]. In more details, when the seven tuple

$$\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, (\widetilde{X_1}, \widetilde{Y_1}), (\widetilde{W_t^1}, \widetilde{W_t^2})_{t\geq 1}, (\widetilde{X}_t, \widetilde{Y}_t)_{t\geq 1}\right)$$

is a weak solution of equations (3.1) and

$$(\widetilde{X_1}, \widetilde{Y_1}) \sim (X_1, Y_1),$$

then

$$(\widetilde{X}_t, \widetilde{Y}_t) \sim (X_t, Y_t).$$

Please also take note of the fact that the margins $(X_t)_{t\geq 1}$ and $(Y_t)_{t\geq 1}$ are $1/2$-self-similar if and only if they coincide with one-dimensional Wiener processes.

Proposition 3.1. *The processes $(X_t)_{t\geq 1}$ and $(Y_t)_{t\geq 1}$ are $1/2$-self-similar if and only if the beginning values X_1 and Y_1 have the standard normal distribution*

$$X_1 \sim Y_1 \sim N(0, 1).$$

The proof is elementary, we refer to section 4.1. Throughout the rest of this paper we denote by Φ and φ the cumulative distribution function and density of the standard normal distribution $N(0, 1)$.

On the other hand, as it is shown in section 4.3, the set of $1/2$-self-similar solutions is not empty. Indeed for any functions A and B fulfilling **A2** and **A3** there exits a selfsimilar solution.

Theorem 3.2. *Assume **A1, A2** and **A3**, then for properly chosen initial values the SDE 3.1 have a $1/2$-self-similar solution.*

Such $1/2$-self-similar solutions arise as a weighted generalized mixtures of τ-rescaled processes $(X_{\tau t}/\sqrt{\tau}, Y_{\tau t}/\sqrt{\tau})_{t\geq 1}$, $\tau > 1$.

When the functions A and $B = \sqrt{1 - A^2}$, i.e. the volatility of the solutions of (3.1), are varying in a moderate way then the probability law of "properly chosen" initial values can be described just as a limit with respect to convergence in distribution. Let C_t^C denote a copula of the pair (X_t, Y_t) for $t \geq 1$, where the process $(X_t, Y_t)_{t\geq 1}$ is a solution of eq. (3.1), with initial values X_1 and Y_1 having the standard normal distribution $X_1 \sim Y_1 \sim N(0, 1)$ and linked by a copula C.

Proposition 3.3. *If L, the square of the Lipschitz coefficient of (A, B), is smaller than 1, then for any copula C a solution of eq. (3.1), with initial values having a cumulative bivariate distribution $C^{\star}(\Phi(x), \Phi(y))$, where*

$$C^{\star}(u, v) = \lim_{t\to\infty} C_t^C(u, v), \tag{3.10}$$

is $1/2$*-self-similar.*

The proof is provided in section 4.3. Note that $L < 1$ when for example $L_A^2 + A_M^2 < 1$. Under assumption $L < 1$, Proposition 3.3 implies, that the rescalings (shiftings) of any solution of eq. (3.1) are converging in distribution to a self similar solution. This has a certain practical impact. When we observe some phenomena driven by a solution of SDE (3.1), with $L < 1$, for a sufficiently long time we may approximate the "true" solution by a 1/2-self-similar one. In the last part of the section 5.1 we provide an illustration of this effect.

After this short discussion on existence we state our results on differential characterization of copulas of $1/2$-selfsimilar solutions.

Theorem 3.4. *Assume A1,A2, A3 and A4, then:*
If the process $(X_t, Y_t)_{t\geq 1}$ is a $1/2$-self-similar solution of eq. (3.1), then the copula of (X_1, Y_1) which we denote by $C(u, v)$ is twice differentiable in a weak sense in $(0, 1)^2$ and almost everywhere in $(0, 1)^2$ fulfills the equation

$$\begin{aligned} &D^{2,0}C(u, v)\varphi(\Phi^{-1}(u))^2 + D^{0,2}C(u, v)\varphi(\Phi^{-1}(v))^2 \\ &+2A(\Phi^{-1}(u), \Phi^{-1}(v))D^{1,1}C(u, v)\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v)) = 0. \end{aligned} \tag{3.11}$$

the cumulative distribution function of (X_1, Y_1) which we denote by $F(x, y)$ is twice differentiable in a weak sense and fulfills the equation

$$\begin{aligned} &D^{2,0}F(x, y) + D^{0,2}F(x, y) + 2A(x, y)D^{1,1}F(x, y) \\ &+xD^{1,0}F(x, y) + yD^{0,1}F(x, y) = 0 \end{aligned} \tag{3.12}$$

almost everywhere in $\mathbb{R}^2$.

As a consequence of the maximum principle for elliptic PDE, we get:

Theorem 3.5. *Let A_i, $i = 1, 2$, be bounded Lipschitz functions, $|A_i| < 1$. If copulas C_1 and C_2 fulfill PDE (3.11) with A equals respectively to A_1 and A_2 and*

$$\forall (x, y) \in \mathbb{R}^2 \quad A_1(x, y) \geq A_2(x, y),$$

then copula C_1 dominates in concordance ordering

$$\forall (u, v) \in [0, 1]^2 \quad C_1(u, v) \geq C_2(u, v),$$

Theorem 3.5 implies two important corollaries.

Corollary 3.6. *Any two $1/2$-self-similar solutions of eq. (3.1), with coefficients fulfilling **A2**, **A3** and **A4**, are law equivalent. Specifically, the copulas of the initial values of any two such $1/2$-self-similar solutions of eq. (3.1) are equal to each other.*

Corollary 3.7. *Let a twice weakly differentiable copula C be a solution of PDE (3.11) with coefficient A and Ga_ρ, $\rho \in (-1, 1)$, be a Gaussian copula with correlation coefficient ρ.*
• *If $A(x, y) \leq \rho$ for all $(x, y) \in \mathbb{R}^2$ then*

$$\forall (u, v) \in [0, 1]^2 \quad C(u, v) \leq Ga_\rho(u, v);$$

• *If $A(x, y) \geq \rho$ for all $(x, y) \in \mathbb{R}^2$ then*

$$\forall (u, v) \in [0, 1]^2 \quad C(u, v) \geq Ga_\rho(u, v);$$

• *If $A(x, y)$ is nonnegative then the copula C is PQD*

$$\forall (u, v) \in [0, 1]^2 \quad C(u, v) \geq \Pi(u, v) = uv;$$

• *If $A(x, y)$ is nonpositive then the copula C is NPQD*

$$\forall (u, v) \in [0, 1]^2 \quad C(u, v) \leq \Pi(u, v) = uv;$$

The "existence" Theorem 3.2 and the "uniqueness" Corollary 3.6 imply the following theorem.

Theorem 3.8. *If a copula $C(x, y)$ is twice differentiable in a weak sense and fulfills the equation (3.11) with $A(x, y)$ bounded, $|A| < 1$, A and $\sqrt{1 - A^2}$ Lipschitz and A differentiable with respect to the second variable, then there exists a $1/2$-self-similar solution $(X_t, Y_t)_{t \geq 1}$ of eq. (3.1), such that C is the copula of (X_t, Y_t) for all $t \geq 1$.*

The "uniqueness" Corollary 3.6 allows us to restate Theorem 3.2 in a more effective way.

Theorem 3.9. *Assume A1,A2, A3 and A4, then for any copula C, a solution of eq. (3.1), with initial values having a cumulative bivariate distribution $C^\star(\Phi(x), \Phi(y))$, where*

$$C^\star(u, v) = \lim_{n \to \infty} \frac{1}{n} \int_1^{e^n} \frac{1}{s} C_s^C(u, v) ds, \tag{3.13}$$

is $1/2$-self-similar.

The proofs of the subsequent theorems and corollaries stated above are provided in sections 4.5, and 4.6. With the exception of Corollary 3.7 which is based on Section 5.1

The weak derivatives proved also to be very useful in a study concerning dynamics of copulas of more general Ito processes, see [15].

In section 5, we will show that the set of solutions of equations (3.11) contains Gaussian copulas with $\rho^2 < 1$, FGM copulas with $\alpha^2 < 1$ and some but not all Frank copulas and does not contain Clayton copulas. In the first case A is constant, it equals to correlation coefficient. In the second and third A may vary.

4 Proofs and auxiliary results

4.1 Margins - proof of Proposition 3.1

Let $X_t = X_1 + W_t^1 - W_1^1$, where X_1 and $W_t^1 - W_1^1$ are independent. If $X_1 \sim N(0, 1)$, then X_t, for $t > 1$, has normal distribution, with zero mean, as well. Furthermore

$$\sigma^2(X_t) = \sigma^2(X_1) + \sigma^2(W_t - W_1) = 1 + (t - 1) = t.$$

Hence $(X_t)_{t \geq 1}$ is $1/2$-self-similar, ie,: $X_t \sim \sqrt{t} X_1$.

Now let $(X_t)_{t \geq 1}$ be $1/2$-self-similar. We need to show that $X_1 \sim N(0, 1)$. We will base on the properties of characteristic functions (see [3] §26). Since

$$X_1 \sim \frac{1}{\sqrt{t}} X_t = \frac{1}{\sqrt{t}} X_1 + \frac{1}{\sqrt{t}} (W_t - W_1),$$

we get for fixed s

$$\mathbb{E}(\exp(isX_1)) = \mathbb{E}\left(\exp\left(is\frac{X_1}{\sqrt{t}}\right)\right)\exp\left(-\frac{t-1}{t}\frac{s^2}{2}\right).$$

Since $\frac{X_1}{\sqrt{t}}$ converges to 0 when $t \to +\infty$, we obtain by Lebesgue dominated convergence Theorem

$$\mathbb{E}(\exp(isX_1)) = \lim_{t\to\infty}\mathbb{E}\left(\exp\left(is\frac{X_1}{\sqrt{t}}\right)\right)\exp\left(-\frac{t-1}{t}\frac{s^2}{2}\right) = \exp\left(-\frac{s^2}{2}\right).$$

Hence X_1 has standard normal distribution.

When it comes to the second variable, the stochastic process

$$W'_t = \int_1^t A\left(\frac{X_s}{\sqrt{s}}, \frac{Y_s}{\sqrt{s}}\right) dW_s^1 + \int_1^t B\left(\frac{X_s}{\sqrt{s}}, \frac{Y_s}{\sqrt{s}}\right) dW_s^2,$$

where $B(x, y) = \sqrt{1 - A(x, y)^2}$, is a (time-rescaled) Brownian Motion, therefore re-writing $Y_t = Y_1 + W'_t$, we can see that by repeating the above reasoning we may conclude that Y_t is 1/2-self-similar if and only $Y_1 \sim N(0, 1)$.

4.2 Semigroup property

Let C_t^C denote a copula of the pair (X_t, Y_t) for $t \geq 1$, where the process $(X_t, Y_t)_{t\geq 1}$ is a solution of eq. (3.1), with initial values X_1 and Y_1 having the standard normal distribution $X_1 \sim Y_1 \sim N(0, 1)$ and linked by a copula C. Basing on Proposition 4.2 we show that the mapping

$$H : [1, \infty) \times \mathcal{C}^2 \longrightarrow \mathcal{C}^2, \quad H(t, C) = C_t^C, \tag{4.1}$$

is a representation of a multiplicative semigroup $\big([1, \infty), \cdot, 1\big)$ (see section 4.3 for more details). Observe that copula of a 1/2-self-similar process is a fixed point of H. It shows that such copulas can be obtained as generalized weighted averages along the orbit of H.

We start with the following auxiliary proposition which will be essential in the course of proving Theorem 3.2. We show that the rescaling (shifting) of a strong solution of SDE (3.1) give rise to a weak solution of the same equations.

Proposition 4.1. *If a stochastic process $(X_t, Y_t)_{t\geq 1}$ is a solution of the set of equations (3.1) with initial values (X_1, Y_1) then for any $\tau > 1$ the 7-tuple*

$$\left(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_{t\tau}\}_{t\geq 1}, \left(\frac{X_\tau}{\tau^{1/2}}, \frac{Y_\tau}{\tau^{1/2}}\right), \left(\frac{W_{t\tau}^1}{\tau^{1/2}}, \frac{W_{t\tau}^2}{\tau^{1/2}}\right)_{t\geq 1}, \left(\frac{X_{t\tau}}{\tau^{1/2}}, \frac{Y_{t\tau}}{\tau^{1/2}}\right)_{t\geq 1}\right)$$

is a (weak) solution of (3.1).

Proof.

First we observe that for $\tau > 1$

$$\begin{aligned} X_{t\tau} &= X_\tau + W_{t\tau}^1 - W_\tau^1, \\ Y_{t\tau} &= Y_\tau + \int_\tau^{t\tau} A\left(\frac{X_s}{\sqrt{s}}, \frac{Y_s}{\sqrt{s}}\right) dW_s^1 + \int_\tau^{t\tau} B\left(\frac{X_s}{\sqrt{s}}, \frac{Y_s}{\sqrt{s}}\right) dW_s^2, \end{aligned}$$

where $B(x, y) = \sqrt{1 - A(x, y)^2}$.

Next by the change of variables ($s := u\tau$) we get

$$\begin{aligned}\frac{X_{t\tau}}{\sqrt{\tau}} &= \frac{X_\tau}{\sqrt{\tau}} + \frac{1}{\sqrt{\tau}}W^1_{t\tau} - \frac{1}{\sqrt{\tau}}W^1_\tau,\\ \frac{Y_{t\tau}}{\sqrt{\tau}} &= \frac{Y_\tau}{\sqrt{\tau}} + \int_1^t A\left(\frac{X_{u\tau}}{\sqrt{u\tau}}, \frac{Y_{u\tau}}{\sqrt{u\tau}}\right)\frac{1}{\sqrt{\tau}}dW^1_{u\tau} + \int_1^t B\left(\frac{X_{u\tau}}{\sqrt{u\tau}}, \frac{Y_{u\tau}}{\sqrt{u\tau}}\right)\frac{1}{\sqrt{\tau}}dW^2_{u\tau}.\end{aligned}$$

Since for fixed τ

$$\tilde{W}^1_t = \frac{1}{\sqrt{\tau}}W^1_{t\tau} \quad \text{and} \quad \tilde{W}^2_t = \frac{1}{\sqrt{\tau}}W^2_{t\tau}$$

are independent Wiener processes (defined on the same probability space as W^1 and W^2), we conclude that indeed we obtained a solution of the stochastic differential equations (3.1) with another pair of Wiener processes

$$\begin{aligned}dX_t &= d\tilde{W}^1_t,\\ dY_t &= A\left(\frac{X_t}{\sqrt{t}}, \frac{Y_t}{\sqrt{t}}\right)d\tilde{W}^1_t + B\left(\frac{X_t}{\sqrt{t}}, \frac{Y_t}{\sqrt{t}}\right)d\tilde{W}^2_t.\end{aligned}$$

with initial values $(\tau^{-1/2}X_\tau, \tau^{-1/2}Y_\tau)$. □

Due to the uniqueness of the weak solutions of the SDE (see [5]) Proposition 4.1 implies Proposition 4.2.

Proposition 4.2. *We assume **A1, A2** and **A3**. Then the solution $(X'_t, Y'_t)_{t\geq 1}$ of SDE (3.1) with initial values law equivalent to $(X_\tau/\sqrt{\tau}, Y_\tau/\sqrt{\tau})$, $\tau > 1$,*

$$(X'_1, Y'_1) \sim \left(\frac{X_\tau}{\sqrt{\tau}}, \frac{Y_\tau}{\sqrt{\tau}}\right),$$

is law equivalent to the τ-rescaled process

$$(X'_t, Y'_t)_{t\geq 1} \sim \left(\frac{X_{\tau t}}{\sqrt{\tau}}, \frac{Y_{\tau t}}{\sqrt{\tau}}\right)_{t\geq 1}.$$

4.3 Existence

We assume **A1, A2** and **A3** throughout this section.

First we establish some estimates which imply the continuity of solutions of equations (3.1) in L^2 norm.

In more details, since both $(X_t)_{t\geq 1}$ and $(Y_t)_{t\geq 1}$ coincide with Wiener processes we have for $s, t \geq 1$

$$\begin{aligned}||X_t - X_s||^2_{L^2} &= \mathbb{E}((X_t - X_s)^2) = |t-s|,\\ ||Y_t - Y_s||^2_{L^2} &= \mathbb{E}((Y_t - Y_s)^2) = |t-s|. \end{aligned} \tag{4.2}$$

Let us consider two processes $(X'_t, Y'_t)_{t\geq 1}$ and $(X''_t, Y''_t)_{t\geq 1}$ which solve equation (3.1) with different initial values:

$$X'_t = X'_1 + W^1_t - W^1_1, \tag{4.3}$$

$$X''_t = X''_1 + W^1_t - W^1_1, \tag{4.4}$$

$$Y'_t = Y'_1 + \int_1^t A\left(\frac{X'_s}{\sqrt{s}}, \frac{Y'_s}{\sqrt{s}}\right)dW^1_s + \int_1^t B\left(\frac{X'_s}{\sqrt{s}}, \frac{Y'_s}{\sqrt{s}}\right)dW^2_s, \tag{4.5}$$

$$Y''_t = Y''_1 + \int_1^t A\left(\frac{X''_s}{\sqrt{s}}, \frac{Y''_s}{\sqrt{s}}\right)dW^1_s + \int_1^t B\left(\frac{X''_s}{\sqrt{s}}, \frac{Y''_s}{\sqrt{s}}\right)dW^2_s, \tag{4.6}$$

where $B(x,y) = \sqrt{1 - A(x,y)^2}$.

By the following lemma the distances between $(X'_t)_{t\geq 1}$ and $(X''_t)_{t\geq 1}$ and also between $(Y'_t)_{t\geq 1}$ and $(Y''_t)_{t\geq 1}$ are linear functions of the distances between the initial values.

Lemma 4.3. *For any* $T \geq 1$*:*

$$||X'_T - X''_T||_{L^2} = ||X'_1 - X''_1||_{L^2}, \tag{4.7}$$

$$\begin{aligned} \left|\left|Y'_T - Y''_T\right|\right|^2_{L^2} &\leq \left(1 + T^L \ln(T)\right) \left|\left|Y'_1 - Y''_1\right|\right|^2_{L^2} \\ &+ L\left(\ln(T) + \frac{T^L}{2}\ln^2(T)\right) \left|\left|X'_1 - X''_1\right|\right|^2_{L^2}, \end{aligned} \tag{4.8}$$

where L is a sum of squares of Lipschitz coefficients for A and B,
$\forall (x',y'), (x'',y'') \in \mathbb{R}^2$

$$|A(x',y') - A(x'',y'')|^2 + |B(x',y') - B(x'',y'')|^2 \leq L\left(|x' - x''|^2 + |y' - y''|^2\right).$$

Proof.

Equality (4.7) follows readily since

$$X'_t - X''_t = X'_1 - X''_1. \tag{4.9}$$

For the proof of (4.8), we apply subsequently the Ito formula, Lipschitz inequality and (4.9). Finally we get for $t \in [0, T-1]$

$$\begin{aligned} \mathbb{E}\left|Y'_{1+t} - Y''_{1+t}\right|^2 &= \mathbb{E}\left|Y'_1 - Y''_1\right|^2 \\ &+ \mathbb{E}\int_1^{1+t} \left(\left(A\left(\frac{X'_s}{\sqrt{s}}, \frac{Y'_s}{\sqrt{s}}\right) - A\left(\frac{X''_s}{\sqrt{s}}, \frac{Y''_s}{\sqrt{s}}\right)\right)^2 \right. \\ &\quad \left. + \left(B\left(\frac{X'_s}{\sqrt{s}}, \frac{Y'_s}{\sqrt{s}}\right) - B\left(\frac{X''_s}{\sqrt{s}}, \frac{Y''_s}{\sqrt{s}}\right)\right)^2\right) ds \\ &\leq \mathbb{E}\left|Y'_1 - Y''_1\right|^2 + \int_1^{1+t} \frac{L}{s}\left(\mathbb{E}\left|Y'_s - Y''_s\right|^2 + \mathbb{E}\left|X'_s - X''_s\right|^2\right) ds \\ &\leq \mathbb{E}\left|Y'_1 - Y''_1\right|^2 + L\ln(1+t)\mathbb{E}\left|X'_1 - X''_1\right|^2 \\ &\quad + \int_1^{1+t} \frac{L}{s}\mathbb{E}\left|Y'_s - Y''_s\right|^2 ds. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{1+t}\mathbb{E}\left|Y'_{1+t} - Y''_{1+t}\right|^2 &\leq \frac{1}{1+t}\mathbb{E}\left|Y'_1 - Y''_1\right|^2 + \frac{L\ln(1+t)}{1+t}\mathbb{E}\left|X'_1 - X''_1\right|^2 \\ &+ \frac{L}{1+t}\int_1^{1+t} \frac{1}{s}\mathbb{E}\left|Y'_s - Y''_s\right|^2 ds. \end{aligned} \tag{4.10}$$

The Gronwall Lemma (see [11] Appendix B.j) applied for a function

$$\eta(t) = \int_1^{1+t} \frac{1}{s}\mathbb{E}\left|Y'_s - Y''_s\right|^2 ds \tag{4.11}$$

yields for $t \in [0, T-1]$

$$\eta(t) \leq (1+t)^L \left(\ln(1+t)\mathbb{E}\left|Y'_1 - Y''_1\right|^2 + \frac{L}{2}\ln^2(1+t)\mathbb{E}\left|X'_1 - X''_1\right|^2\right). \tag{4.12}$$

Subsequently

$$\begin{aligned} ||Y'_{1+t} - Y''_{1+t}||^2_{L^2} &\leq \left(1 + (1+t)^L \ln(1+t)\right) ||Y'_1 - Y''_1||^2_{L^2} \\ &+ L\left(\ln(1+t) + \frac{(1+t)^L}{2}\ln^2(1+t)\right) ||X'_1 - X''_1||^2_{L^2} . \end{aligned} \tag{4.13}$$

Having substituted $t = T - 1$ we conclude the proof of Lemma. □

To prove the existence of self-similar solutions, we follow the approach of Khasminskii (§2.2 [21]) and construct a self-similar solution basing on the averages of a process.

In more details, let C be any copula. We shall consider the solutions $(X_t, Y_t)_{t\geq 1}$ of the set of equations (3.1) with initial values (X_1, Y_1), for which we assume that both X_1 and Y_1 are standard normal $N(0, 1)$ and the copula of their joint distribution is C. By $H(t, C)$ we denote the copula of (X_t, Y_t). Since X_t and Y_t have normal distribution with mean 0 and variance equal t ($N(0, t)$), the joint cumulative distribution of the pair (X_t, Y_t) is given by

$$F_t(x, y) = H(t, C)\left(\Phi(\frac{x}{\sqrt{t}}), \Phi(\frac{y}{\sqrt{t}})\right). \tag{4.14}$$

Since for any copula C there exists a pair of random variables with joint distribution function $C(\Phi(x), \Phi(y))$, the function H is well defined on $[1, \infty) \times \mathcal{C}^2$ (compare [21] Th. 3.4).

For the solution $(X_t, Y_t)_{t\geq 1}$ of (3.1) to be a 1/2-self-similar process we need to show that $H(t, C) = C$ for all $t \geq 1$, therefore we are in fact interested in the existence of fixed points of H.

In the following proposition we list the basic properties of H.

Proposition 4.4. *1. H defines a representation of the multiplicative semigroup $([1, \infty), \cdot, 1)$. For any $t, s \geq 1$ and copula C*

$$H(t, H(s, C)) = H(ts, C).$$

2. H is continuous with respect to the metric

$$d((t, C), (s, D)) = |t - s| + \sup\{|C(u, v) - D(u, v)| : (u, v) \in [0, 1]^2\},$$

where C and D are copulas and s and t are real numbers.
3. H commutes with the mixture of copulas, ie. if $C_\theta(u, v)$, $\theta \in \Theta \subset \mathbb{R}^k$ is a measurable family of copulas then for any probabilistic measure μ on Θ and any $t \geq 1$

$$H\left(t, \int_\Theta C_\theta d\mu(\theta)\right) = \int_\Theta H(t, C_\theta) d\mu(\theta).$$

Proof.

Point 1. Follows directly from Proposition 4.1.

Point 2. Follows from the continuity of solutions of equations (3.1) in L^2 norm. For the proof we shall apply Skorohod representation Theorem (see [3] Th. 25.6).

Let $\{C_n\}_{n\in\mathbb{N}_+}$ be any sequence of copulas convergent to copula C_∞. Then, by the Skorohod Theorem, there exists a probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, a sequence of random pairs $\{(Z^1_n, Z^2_n)\}_{n\in\mathbb{N}_+}$ and a random vector (Z^1_∞, Z^2_∞) (all defined on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$), such that:
1. $C_n(\Phi(x), \Phi(y))$ is a distribution function of (Z^1_n, Z^2_n), $n = 1, 2, \ldots, \infty$;
2. (Z^1_n, Z^2_n) almost surely converges to (Z^1_∞, Z^2_∞).

Let us consider the following product of probability spaces

$$(\Omega', \mathcal{F}', \mathbb{P}') = (\Omega \times \Omega_1, \mathcal{F} \times \mathcal{F}_1, \mathbb{P} \times \mathbb{P}_1).$$

Since Wiener processes W^1, W^2 have been defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and the initial values Z_n^i on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ we extend them onto the product space by putting

$$W_t^i(\omega, \omega_1) = W_t^i(\omega), \quad Z_n^i(\omega, \omega_1) = Z_n^i(\omega_1).$$

Next we analyze the stochastic equations (3.1) on the previously defined product space, denoting by $(X_{n,t}, Y_{n,t})$ their solutions with initial values (Z_n^1, Z_n^2), for $n = 1, 2, \dots, \infty$.

Obviously $H(t, C_n)$ are copulas of $(X_{n,t}, Y_{n,t})$.

Note that due to Lebesgue dominated convergence Theorem the sequence $\{(Z_n^1, Z_n^2)\}_{n\in\mathbb{N}_+}$ converges in L^2 norm

$$\mathbb{E}\Big((Z_n^1 - Z_\infty^1)^2 + (Z_n^2 - Z_\infty^2)^2\Big) \overset{n\to\infty}{\longrightarrow} 0.$$

Therefore, by Lemma 4.3 and equalities (4.2), for any convergent sequence of indices $\{s_n\}_{n\in\mathbb{N}_+} \subset [1, \infty)$ the sequence of random pairs $\{(X_{n,s_n}, Y_{n,s_n})\}_{n\in\mathbb{N}_+}$ converges in L^2 to $(X_{\infty,s_\infty}, Y_{\infty,s_\infty})$, where s_∞ denotes the limit of s_n.

By the following lemma, convergence in L^2 and convergence in distribution are closely related. To simplify the notation we put for two random pairs $V_i = (X_i, Y_i)$, $i = 1, 2$

$$||V_2 - V_1||_{L^2}^2 = \mathbb{E}\Big((X_2 - X_1)^2\Big) + \mathbb{E}\Big((Y_2 - Y_1)^2\Big). \tag{4.15}$$

Lemma 4.5. *Let V_1 and V_2 be two random pairs with standard normal margins and copulas respectively C^1 and C^2, defined on the same probability space. Then*

$$||C^2 - C^1||_\infty \le \frac{3}{\sqrt[3]{2\pi}} ||V_1 - V_2||_{L^2}^{2/3}.$$

Proof.
Let (x, y) be any point of the real plane and ε a positive constant. Applying the elementary set theory and Markov inequality (see [3] formula (21.12)) we get the following estimate

$$\begin{aligned} &C^2(\Phi(x), \Phi(y)) - C^1(\Phi(x+\varepsilon), \Phi(y+\varepsilon)) \\ &= \mathbb{P}(X_1 \le x, Y_1 \le y) - \mathbb{P}(X_2 \le x+\varepsilon, Y_2 \le y+\varepsilon) \\ &\le \mathbb{P}\Big(\{\omega : X_1(\omega) \le x \wedge Y_1(\omega) \le y\} \setminus \{\omega : X_2(\omega) \le x+\varepsilon \wedge Y_2(\omega) \le y+\varepsilon\}\Big) \\ &= \mathbb{P}\Big(\{\omega : X_1(\omega) \le x \wedge Y_1(\omega) \le y \wedge \Big(X_2(\omega) > x+\varepsilon \vee Y_2(\omega) > y+\varepsilon\Big)\}\Big) \\ &= \mathbb{P}\Big(\{\omega : \Big(X_1(\omega) \le x \wedge Y_1(\omega) \le y \wedge X_2(\omega) > x+\varepsilon\Big) \\ &\qquad \vee \Big(X_1(\omega) \le x \wedge Y_1(\omega) \le y \wedge Y_2(\omega) > y+\varepsilon\Big)\}\Big) \\ &\le \mathbb{P}\Big(\{\omega : X_1(\omega) \le x \wedge X_2(\omega) > x+\varepsilon\}\Big) + \mathbb{P}\Big(\{\omega : Y_1(\omega) \le y \wedge Y_2(\omega) > y+\varepsilon\}\Big) \\ &\le \mathbb{P}\Big(|X_1 - X_2| \ge \varepsilon\Big) + \mathbb{P}\Big(|Y_1 - Y_2| \ge \varepsilon\Big) \\ &\le \frac{1}{\varepsilon^2}\Big(\mathbb{E}\Big(|X_1 - X_2|^2\Big) + \mathbb{E}\Big(|Y_1 - Y_2|^2\Big)\Big) = \frac{1}{\varepsilon^2}||V_1 - V_2||_{L^2}^2. \end{aligned} \tag{4.16}$$

Since copulas are Lipschitz functions (with Lipschitz constant 1) and Φ is also Lipschitz function (with Lipschitz constant $\varphi(0) = \frac{1}{\sqrt{2\pi}}$) we further estimate

$$\begin{aligned} &C^2(\Phi(x), \Phi(y)) - C^1(\Phi(x), \Phi(y)) \\ &\le C^1(\Phi(x+\varepsilon), \Phi(y+\varepsilon)) - C^1(\Phi(x), \Phi(y)) + \frac{1}{\varepsilon^2}||V_1 - V_2||_{L^2}^2 \\ &= \frac{\sqrt{2}}{\sqrt{\pi}}\varepsilon + \frac{1}{\varepsilon^2}||V_1 - V_2||_{L^2}^2. \end{aligned} \tag{4.17}$$

Since the bound is valid for all positive ε, we substitute

$$\varepsilon = \sqrt[6]{2\pi}||V_1 - V_2||_{L^2}^{2/3},$$

which minimizes the estimate.

$$C^2(\Phi(x), \Phi(y)) - C^1(\Phi(x), \Phi(y)) \leq 3\frac{||V_1 - V_2||_{L^2}^{2/3}}{\sqrt[3]{2\pi}}.$$

The above bound remains valid when we replace C^2 and C^1. Moreover it is valid for all points $(x, y) \in \mathbb{R}^2$. Therefore

$$\begin{aligned} ||C^2 - C^1||_\infty &= \sup\{|C^2(u, v) - C^1(u, v)| : (u, v) \in [0, 1]^2\} \\ &= \sup\{|C^2(\Phi(x), \Phi(y)) - C^1(\Phi(x), \Phi(y))| : (x, y) \in \mathbb{R}^2\} \\ &\leq \frac{3}{\sqrt[3]{2\pi}}||V_1 - V_2||_{L^2}^{2/3}. \end{aligned} \tag{4.18}$$

□

Thus convergence in L^2 implies convergence in distribution and the joint cumulative distribution functions of (X_{n,s_n}, Y_{n,s_n}) are converging to the cumulative distribution functions of $(X_{\infty,s_\infty}, Y_{\infty,s_\infty})$

$$\lim_{n\to\infty} H(s_n, C_n)(\Phi(x), \Phi(y)) = H(s_\infty, C_\infty)(\Phi(x), \Phi(y)). \tag{4.19}$$

Hence the copulas $H(s_n, C_n)$ converge to $H(s_\infty, C_\infty)$

$$\lim_{n\to\infty} H(s_n, C_n) = H\left(\lim_{n\to\infty} s_n, \lim_{n\to\infty} C_n\right). \tag{4.20}$$

□

Point 3. follows from the fact that a solution of equations (3.1) with random initial values is a mixture of solutions with deterministic initial values (compare [21] Theorem 3.4 point 2). Thus we have to show the associativity of iterated mixtures.

We recall that a random pair Z_μ is a mixture of random pairs Z_θ, $\theta \in \Theta$, with respect to the probabilistic measure μ on Θ, when for every bounded Borel function f on $\mathbb{R}^2$

$$E(f(Z_\mu)) = \int_\Theta E(f(Z_\theta))d\mu. \tag{4.21}$$

We assume that the random pairs Z_μ and Z_θ, $\theta \in \Theta$, and the two-dimensional Wiener process $(W_t^1 - W_1^1, W_t^2 - W_1^2)_{t\geq 1}$ are independent. We denote by (X_t^z, Y_t^z) (respectively by (X_t^θ, Y_t^θ) and by (X_t^μ, Y_t^μ)) a solution of (3.1) with deterministic initial values $(X_1, Y_1) = z$, $z \in \mathbb{R}^2$ (respectively with random initial values $(X_1, Y_1) = Z_\theta$ and $(X_1, Y_1) = Z_\mu$). Thanks to the assumptions **A1, A2** and **A3** such objects exist and are unique (Theorem 3.4 [21]). Let f be a bounded Borel function on $\mathbb{R}^2$. We put

$$u(t, z) = E(f(X_t^z, Y_t^z)). \tag{4.22}$$

We get

$$E(f(X_t^\theta, Y_t^\theta)) = E(E(f(X_t^\theta, Y_t^\theta)|X_1^\theta, Y_1^\theta) = E(u(t, Z^\theta)) \tag{4.23}$$

and similarly

$$E(f(X_t^\mu, Y_t^\mu)) = E(u(t, Z^\mu)). \tag{4.24}$$

Applying (4.21) to the right side, we obtain

$$E(f(X_t^\mu, Y_t^\mu)) = \int_\Theta E(u(t, Z^\theta))d\mu = \int_\Theta E(f(X_t^\theta, Y_t^\theta))d\mu. \tag{4.25}$$

Thus for each $t \geq 1$ (X_t^μ, Y_t^μ) is a mixture of (X_t^θ, Y_t^θ).

To conclude the proof of point 3 it is enough to select

$$Z^{\theta} \sim C_{\theta}(\Phi(x), \Phi(y)), \text{ and } Z^{\mu} \sim C_{\mu}(\Phi(x), \Phi(y)),$$

where

$$C_{\mu}(x, y) = \int_{\Theta} C_{\theta}(x, y) d\mu(\theta).$$

□

Next we show that the set of fixed points of the semigroup H is not empty. We select a copula C and as a fixed point take the generalized average of the trajectory of C. In more details.

Proposition 4.6. *For any copula C the set of cluster points of the sequence $(C^k)_{k\geq 1}$, given by*

$$C^k(u, v) = \int_1^{e^k} \frac{1}{ks} H(s, C)(u, v) ds$$

is not empty. Furthermore any such cluster point is a fixed point of H.

Proof.
Due to Ascoli theorem there exists a subsequence of C^k having a limit, i.e. a cluster point of the sequence. We denote this limit copula by $C^{\star}$

$$C^{k_n} \longrightarrow C^{\star}.$$

Since H is continuous and commutes with the mixtures we get

$$\begin{aligned} & H(t, C^{\star}) \\ = \;& H(t, \lim_{n\to\infty} C^{k_n}) = \lim_{n\to\infty} H(t, C^{k_n}) = \lim_{n\to\infty} H(t, \int_1^{e^{k_n}} \frac{1}{k_n s} H(s, C) ds) \\ = \;& \lim_{n\to\infty} \int_1^{e^{k_n}} \frac{1}{k_n s} H(t, H(s, C)) ds = \lim_{n\to\infty} \int_1^{e^{k_n}} \frac{1}{k_n s} H(ts, C) ds \\ = \;& \lim_{n\to\infty} \int_t^{te^{k_n}} \frac{1}{k_n s} H(s, C) ds. \end{aligned} \tag{4.26}$$

Therefore, since copulas are bounded by 1, we get for $t \geq 1$

$$\begin{aligned} & |H(t, C^{\star})(u, v) - C^{\star}(u, v)| \\ = \;& \lim_{n\to\infty} |H(t, C^{k_n})(u, v) - C^{k_n}(u, v)| \\ = \;& \lim_{n\to\infty} \left| \int_t^{te^{k_n}} \frac{1}{k_n s} H(s, C)(u, v) ds - \int_1^{e^{k_n}} \frac{1}{k_n s} H(s, C)(u, v) ds \right| \\ = \;& \lim_{n\to\infty} \left| \int_{e^{k_n}}^{te^{k_n}} \frac{1}{k_n s} H(s, C)(u, v) ds - \int_1^{t} \frac{1}{k_n s} H(s, C)(u, v) ds \right| \\ \leq \;& \lim_{n\to\infty} \left(\int_{e^{k_n}}^{te^{k_n}} \frac{1}{k_n s} ds + \int_1^{t} \frac{1}{k_n s} ds \right) \\ = \;& \lim_{n\to\infty} \frac{1}{k_n} \left(\ln(t) + \ln(e^{k_n}) - \ln(e^{k_n}) + \ln(t) - \ln(1) \right) = \lim_{n\to\infty} \frac{2\ln(t)}{k_n} = 0. \end{aligned} \tag{4.27}$$

□

The following corollary concludes the proof of Theorem 3.2.

Corollary 4.7. *Let $C^\star$ be a fixed point of H. Then the solution of equations (3.1) with initial values*

$$X_1 = W_1^1, \quad Y_1 = \Phi^{-1}\Psi(\Phi(W_1^1), \Phi(W_1^2)),$$

where Ψ is a generalized inverse of the weak derivative of $C^\star$

$$\Psi(u, v) = \inf\{w : D^{1,0}C^\star(u, w) \geq v\}.$$

is (1/2)-self-similar.

Proof.
Let $F_t(x, y)$ be a cumulative distribution function of the solution (X_t, Y_t). Since

$$F_1(x, y) = C^\star(\Phi(x), \Phi(y)),$$

we get from Proposition 4.4

$$\begin{aligned} F_t(x, y) &= H(t, C^\star)\left(\Phi\left(\frac{x}{\sqrt{t}}\right), \Phi\left(\frac{y}{\sqrt{t}}\right)\right) = C^\star\left(\Phi\left(\frac{x}{\sqrt{t}}\right), \Phi\left(\frac{y}{\sqrt{t}}\right)\right) \\ &= F_1\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right). \end{aligned}$$

□

4.4 Proof of Proposition 3.3

In this section we assume that the sum of squares of Lipschitz coefficients of A and B, denoted by L, is smaller than 1. Let $C^\star$ be a fixed point of the semigroup H from Proposition 4.6 and $(X_t, Y_t)_{t\geq 1}$ be a 1/2-self-similar solution of equations (3.1) from Corollary 4.7. Its distribution functions are equal to

$$F_t(x, y) = C^\star\left(\Phi\left(\frac{x}{\sqrt{t}}\right), \Phi\left(\frac{y}{\sqrt{t}}\right)\right).$$

We compare it with an arbitrary solution of equations (3.1) $(X'_t, Y'_t)_{t\geq 1}$. Let

$$F'_t(x, y) = C_t\left(\Phi\left(\frac{x}{\sqrt{t}}\right), \Phi\left(\frac{y}{\sqrt{t}}\right)\right).$$

be its distribution functions. Due to Lemmas 4.3 and 4.5 we have for every $t \geq 1$

$$\begin{aligned} ||C_\star - C_t||_\infty^3 &\leq \frac{3^3}{2\pi}\left|\left|\frac{1}{\sqrt{t}}(X_t - X'_t, Y_t - Y'_t)\right|\right|_{L^2}^2 \\ &= \frac{27}{2\pi}\frac{1}{t}\left(||X_t - X'_t||_{L^2}^2 + ||Y_t - Y'_t||_{L^2}^2\right) \\ &\leq \frac{27}{2\pi}\frac{1}{t}\left(\left(1 + t^L\ln(t)\right)||Y'_1 - Y''_1||_{L^2}^2\right. \\ &\quad + \left.\left(1 + L\ln(t) + L\frac{t^L}{2}\ln^2(t)\right)||X'_1 - X''_1||_{L^2}^2\right), \end{aligned} \tag{4.28}$$

Since X_1, X'_1, Y_1 and Y'_1 have mean 0 and variance 1 we may bound

$$||X_t - X'_t||_{L^2}^2 \leq 4, \quad ||Y_t - Y'_t||_{L^2}^2 \leq 4.$$

Hence

$$||C_\star - C_t||_\infty^3 \le \frac{54}{\pi}\left(\frac{2 + L\ln(t)}{t} + t^{L-1}\left(\ln(t) + \frac{L}{2}\ln^2(t)\right)\right). \tag{4.29}$$

Since L is smaller than 1,

$$\lim_{t\to\infty} ||C_t - C_\star||_\infty = 0. \tag{4.30}$$

Which concludes the proof of Proposition 3.3.

4.5 Generalized solutions of PDEs

In this section we begin the proof of Theorem 3.4.

We show that the copula process C_t is a "weak generalized" solution of PDE (3.12) (we follow the naming used in [13] and [11], see also [15]). We assume **A1** – **A4**. By H^1 we denote the Hilbert space of weakly differentiable functions, which together with their derivatives are square integrable.

Proposition 4.8. *For any $h \in H^1(\mathbb{R}^2)$ and $t \ge 0$*

$$\frac{d}{dt}\int_{[0,1]^2} h(u,v)C_t(u,v)dudv = -B_t(h, C_t), \tag{4.31}$$

where for fixed $t \ge 0$ and fixed copula C, $B_t(\cdot, C)$ is a continuous linear functional on $H^1(\mathbb{R}^2)$, given by the formula

$$B_t(h, C) = \frac{1}{2t}\int_{[0,1]^2} D^{1,0}\left(h(u,v)\varphi(\Phi^{-1}(u))^2\right)D^{1,0}C(u,v)dudv \tag{4.32}$$

$$+ \frac{1}{2t}\int_{[0,1]^2} D^{0,1}\left(h(u,v)\varphi(\Phi^{-1}(v))^2\right)D^{0,1}C(u,v)dudv$$

$$+ \frac{1}{t}\int_{[0,1]^2} D^{0,1}\left(h(u,v)\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v))A(\Phi^{-1}(u),\Phi^{-1}(v))\right)D^{1,0}C(u,v)dudv.$$

Proof.
We base on results from [15]. We shift time t by 1, put $\sigma_i = 1$ and $\mu_i = 0$ and apply Theorem 4.1 from [15]. □

Since for selfsimilar processes the copula process C_t is constant, say $C_t = C$ for $t \ge 1$, we get

$$B_1(h, C) = 0. \tag{4.33}$$

To continue the proof of theorem 3.4 we have to improve the regularity of C. We apply Theorem 8.8 of [13]. In the notation used in [13], the divergence form of B_1 looks like

$$B_1(h, C) = \int_{[0,1]^2} a^{1,1}D^{1,0}CD^{1,0}h + a^{2,2}D^{0,1}CD^{0,1}h + a^{2,1}D^{1,0}CD^{0,1}h \tag{4.34}$$

$$+ \left(c^1 D^{1,0}C + c^2 D^{0,1}C\right) h dudv,$$

where

$$a^{1,1}(u,v) = \frac{1}{2}\varphi(\Phi^{-1}(u))^2, \tag{4.35}$$

$$a^{2,2}(u,v) = \frac{1}{2}\varphi(\Phi^{-1}(v))^2, \tag{4.36}$$

$$a^{2,1}(u,v) = \varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v))A(\Phi^{-1}(u),\Phi^{-1}(v)), \tag{4.37}$$

$$c^1(u,v) = -\varphi(\Phi^{-1}(u))\phi^{-1}(u) - \varphi(\Phi^{-1}(u))\Phi^{-1}(v)A(\Phi^{-1}(u),\Phi^{-1}(v)) \tag{4.38}$$

$$+ \varphi(\Phi^{-1}(u))D^{0,1}A(\Phi^{-1}(u),\Phi^{-1}(v)),$$

$$c^2(u,v) = -\varphi(\Phi^{-1}(u))\phi^{-1}(v). \tag{4.39}$$

We restrict the domain to the smaller square $\Omega_r = (\Phi(-r), \Phi(r))^2$, $r > 0$. We observe that, for any $r > 0$, on Ω_r the coefficients $a^{1,1}$, $a^{2,2}$, $a^{2,1}$, c^1 and c^2 are bounded. Furthermore $a^{1,1}$, $a^{2,2}$, $a^{2,1}$ are Lipschitz and B is strongly elliptic

$$a^{1,1}(u,v)a^{2,2}(u,v) - \frac{1}{4}a^{2,1}(u,v)^2 = \frac{1}{4}\varphi(\Phi^{-1}(u))^2\varphi(\Phi^{-1}(v))^2(1 - A(\Phi^{-1}(u), \Phi^{-1}(v))^2) > 0. \tag{4.40}$$

Therefore Theorem 8.8 of [13] implies that $C(u,v)$ belongs to $H^2(\Omega_r)$ for any $r > 0$, thus it is weakly twice differentiable on $(0,1)^2$ and fulfills almost everywhere equation (3.11).

$$\begin{aligned} &D^{2,0}C(u,v)\varphi(\Phi^{-1}(u))^2 + D^{0,2}C(u,v)\varphi(\Phi^{-1}(v))^2 \\ &+2A(\Phi^{-1}(u), \Phi^{-1}(v))D^{1,1}C(u,v)\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v)) = 0. \end{aligned} \tag{4.41}$$

When C is weakly twice differentiable so is the cumulative distribution function $F(x,y) = C(\Phi(x), \Phi(y))$. Furthermore

$$\begin{aligned} D^{1,0}F(x,y) &= D^{1,0}C(\Phi(x), \Phi(y))\varphi(x), \qquad (4.42)\\ D^{0,1}F(x,y) &= D^{0,1}C(\Phi(x), \Phi(y))\varphi(y), \\ D^{1,1}F(x,y) &= D^{1,1}C(\Phi(x), \Phi(y))\varphi(x)\varphi(y), \\ D^{2,0}F(x,y) &= D^{2,0}C(\Phi(x), \Phi(y))\varphi(x)^2 - D^{1,0}C(\Phi(x), \Phi(y))x\varphi(x), \\ D^{0,2}F(x,y) &= D^{0,2}C(\Phi(x), \Phi(y))\varphi(y)^2 - D^{0.1}C(\Phi(x), \Phi(y))y\varphi(y). \end{aligned}$$

If C fulfills almost everywhere equation (4.41), then F fulfills almost everywhere on $\mathbb{R}^2$ the equation (3.12)

$$D^{2,0}F + D^{0,2}F + 2A(x,y)D^{1,1}F + xD^{1,0}F + yD^{0,1}F = 0 \tag{4.43}$$

In such a way we conclude the proof of Theorem 3.4.

Remark 4.1. *1. Equation (3.11) can be derived from the Fokker-Planck equation (see [7]) But this requires the existence and twice-differentiability of the density of the process* (X_t, Y_t).
2. Equations (3.11) and (3.12) can be derived without the assumption ***A4****, see [22].*

4.6 Dominance and uniqueness

Theorem 4.9. *Let* A_i, $i = 1, 2$, *be bounded Lipschitz functions,* $|A_i| < 1$. *If copulas* C_1 *and* C_2 *fulfill PDE (3.11) with* A *equal respectively to* A_1 *and* A_2 *and*

$$\forall (x,y) \in \mathbb{R}^2 \quad A_1(x,y) \geq A_2(x,y),$$

then copula C_1 *dominates in concordance ordering*

$$\forall (u,v) \in [0,1]^2 \quad C_1(u,v) \geq C_2(u,v),$$

The proof of Theorem 3.5 follows from the maximum principle for elliptic partial differential equations, see [13] or [11]. In more details: If copulas C_1 and C_2 fulfill PDE (3.11) with A equals respectively to A_1 and A_2 and A_1 dominates

$$\forall (x,y) \in \mathbb{R}^2 \quad A_1(x,y) \geq A_2(x,y),$$

then a function $U(u,v) = C_1(u,v) - C_2(u,v)$ is a sub-solution of the equation (3.11) with A equals to A_2 with zero boundary condition. Indeed, since C_1 is a copula its mixed derivative is almost everywhere nonnegative and for almost all $(u,v) \in (o,1)^2$ we get

$$\begin{aligned} &D^{2,0}U(u,v)\varphi(\Phi^{-1}(u))^2 + D^{0,2}U(u,v)\varphi(\Phi^{-1}(v))^2 \qquad (4.44)\\ &+2A_2(\Phi^{-1}(u), \Phi^{-1}(v))D^{1,1}U(u,v)\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v)) \\ =\ &D^{2,0}C_1(u,v)\varphi(\Phi^{-1}(u))^2 + D^{0,2}C_1(u,v)\varphi(\Phi^{-1}(v))^2 \\ &+2A_2(\Phi^{-1}(u), \Phi^{-1}(v))D^{1,1}C_1(u,v)\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v)) \\ =\ &2\big(A_2(\Phi^{-1}(u), \Phi^{-1}(v)) - A_1(\Phi^{-1}(u), \Phi^{-1}(v))\big)D^{1,1}C_1(u,v)\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v)) \leq 0. \end{aligned}$$

Let δ be a minimum of U and let it be attained at the point (u_0, v_0)

$$\delta = \min\{U(u, v) : (u, v) \in [0, 1]^2\} = U(u_0, v_0).$$

We show that δ must be 0. If it happened that $\delta < 0$ then knowing that U is a Lipschitz function vanishing on the border of the unit square we would be able to select an enough big r that:
1. the point (u_0, v_0) would belong to $\Omega_r = (\Phi(-r), \Phi(r))^2$;
2. the infimum of U on the complement of Ω_r would be greater than $\delta/2$.
Next basing on the same arguments as in the previous section, we would apply Theorem 8.1 from [13] and would get

$$0 > \delta = U(u_0, v_0) = \inf\{U(u, v) : (u, v) \in \Omega_r \geq \inf\{U(u, v) : (u, v) \in \partial\Omega_r\} > \frac{\delta}{2},$$

Dividing both sides by negative δ we would get $1 < 1/2$, a contradiction.

Hence $U = C_1 - C_2$ must be nonnegative, i.e. copula C_1 dominates in concordance ordering.
This concludes the proof of Theorem 3.5.
To prove Corollary 3.6 we consider two copulas $C(u, v)$ and $D(u, v)$ which fulfill PDE (3.11) with the same A. Due to proved above Theorem 3.5 C dominates D and D dominates C and they have to be equal each other.
Theorem 3.8 follows from the "existence" Theorem 3.2, "PDE characterization" Theorem 3.4 and the "uniqueness" Corollary 3.6.

Let a copula $C(x, y)$ be twice differentiable in a weak sense and fulfill the equation (3.11) with $A(x, y)$ bounded, $|A| < 1$, A and $\sqrt{1 - A^2}$ Lipschitz and A differentiable with respect to the second variable. Due to Theorem 3.2 there exits a $1/2$-self-similar solution $(X_t, Y_t)_{t\geq 1}$ of eq. (3.1) with coefficients A and $B = \sqrt{1 - A^2}$. Let D be the copula of (X_t, Y_t) for all $t \geq 1$. Due to Theorem 3.4 copula D is a solution of the equation (3.11) with coefficient $A(x, y)$. Hence Corollary 3.6 implies that $C = D$. Which means that C is a copula of the mentioned above a $1/2$-self-similar process $(X_t, Y_t)_{t\geq 1}$.
Theorem 3.9 follows from Proposition 4.6, Corollary 4.7, "PDE characterization" Theorem 3.4 and the "uniqueness" Corollary 3.6.
Proposition 4.6 and Corollary 4.7 imply that every cluster point (cluster copula) of the sequence

$$C^n = \frac{1}{n} \int_1^{e^n} \frac{1}{s} C_s^C, \quad n = 1, 2, \ldots$$

is a copula of a 1/2-selfsimilar process. If we assume condition **A4** then the cluster copulas fulfill the same PDE. Hence due to Corollary 3.6 they coincide. Which implies that the limit of the sequence C_n exists.

5 Examples

5.1 Gaussian copula

Let us recall that the Gaussian copula is defined as follows:

$$Ga_\rho(u, v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)), \quad \text{for } \rho \in (-1, 1)$$

where Φ_ρ is the joint distribution function of a bi-dimensional standard normal vector, with linear correlation coefficient ρ. For more details see [25] or [20] §4.3.1.

Taking derivatives of Ga_ρ yields

$$\begin{aligned}
\frac{\partial^2 Ga_\rho}{\partial u^2}(u,v) &= \frac{-\rho}{\sqrt{1-\rho^2}}\phi\left(\frac{\Phi^{-1}(v)-\rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right)\frac{1}{\phi(\Phi^{-1}(u))},\\
\frac{\partial^2 Ga_\rho}{\partial v^2}(u,v) &= \frac{-\rho}{\sqrt{1-\rho^2}}\phi\left(\frac{\Phi^{-1}(u)-\rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right)\frac{1}{\phi(\Phi^{-1}(v))},\\
\frac{\partial^2 Ga_\rho}{\partial u\partial v}(u,v) &= \frac{1}{\sqrt{1-\rho^2}}\exp\left(\frac{\Phi^{-1}(u)^2+\Phi^{-1}(v)^2}{2}\right)\times\\
&\quad\times\exp\left(\frac{2\rho\Phi^{-1}(u)\Phi^{-1}(v)-\Phi^{-1}(u)^2-\Phi^{-1}(v)^2}{2(1-\rho^2)}\right).
\end{aligned}$$

We may substitute into eq. (3.11) and obtain

$$\begin{aligned}
&\frac{\partial^2 Ga_\rho}{\partial u^2}(u,v)\varphi(\Phi^{-1}(u))^2+\frac{\partial^2 Ga_\rho}{\partial v^2}(u,v)\varphi(\Phi^{-1}(v))^2\\
&+2\rho\frac{\partial^2 Ga_\rho}{\partial u\partial v}(u,v)\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v))=0
\end{aligned}$$

thus $A(x,y)=\rho$.

By the use of Theorem 3.8 we conclude that for any initial conditions X_1 and Y_1 having standard normal distribution and independent with the increments of the underlying Wiener process there exists a 1/2-self-similar solution of eq. (3.1), with coefficients $A(x,y)=\rho, B(x,y)=\sqrt{1-\rho^2}$, where $|\rho|<1$, such that the copula of its initial values is Gaussian with linear correlation coefficient ρ.

Since Gaussian copulas are solutions of PDE (3.11) with constant coefficient A equal to ρ, Theorem 3.5 implies that when the quadratic covariation A is bounded from 1 or -1 than the corresponding copula C is dominated by a Gaussian copula or dominates such. The above implies Corollary 3.7.

To conclude the Gaussian example, we consider the solutions of eq. (3.1), with coefficients $A(x,y)=\rho, B(x,y)=\sqrt{1-\rho^2}$ when the initial values (X_1,Y_1) having bivariate normal distribution with standard margins and independent with the increments of the underlying Wiener process, have the linear correlation coefficient ρ_1 different than ρ, $|\rho|<1$ and $|\rho_1|<1$.
We have

$$\begin{aligned}
X_t &= X_1+W_t^1-W_1^1,\\
Y_t &= Y_1+\rho(W_t^1-W_1^1)+\sqrt{1-\rho^2}(W_t^2-W_1^2).
\end{aligned}$$

The joint distribution of the pair (X_t,Y_t) is normal and the linear correlation coefficient equals

$$Corr(X_t,Y_t)=\rho+\frac{1}{t}(\rho_1-\rho).$$

For τ rescaled solution we get

$$Corr\left(\frac{1}{\sqrt{\tau}}X_{t\tau},\frac{1}{\sqrt{\tau}}Y_{t\tau}\right)=\rho+\frac{1}{t\tau}(\rho_1-\rho)\overset{\tau\to\infty}{\longrightarrow}\rho,$$

which implies that the rescaled processes when $\tau\to\infty$ are converging in distribution to the self-similar solution.
Basing on [25], formula 2.6, we get the following uniform bound for corresponding copulas

$$\begin{aligned}
\left|Ga_{\rho+(\rho_1-\rho)/t/\tau}(u,v)-Ga_\rho(u,v)\right| &= \left|\int_\rho^{\rho+(\rho_1-\rho)/t/\tau}\phi_2(\Phi^{-1}(u),\Phi^{-1}(v),r)dr\right|\\
&\le \frac{1}{2\pi}\max\left(\frac{1}{\sqrt{1-\rho^2}},\frac{1}{\sqrt{1-\rho_1^2}}\right)\frac{1}{t\tau}|\rho_1-\rho|\\
&\le \frac{1}{\tau}\frac{1}{2\pi}\frac{1}{\sqrt{1-\max(\rho^2,\rho_1^2)}}|\rho_1-\rho|,
\end{aligned}$$

where $t \geq 1$ and $\phi_2(x, y, r)$ is a density of a bivariate normal distribution with correlation r and standard margins. Note that ϕ_2 is bounded. For all $(x, y) \in \mathbb{R}^2$ and $r \in (-1, 1)$

$$\phi_2(x, y, r) \leq \frac{1}{2\pi}\frac{1}{\sqrt{1-r^2}}.$$

5.2 FGM copula

The Farlie-Gumbel-Morgenstern copula is defined as

$$C(u, v) = uv\left(1 + a(1-u)(1-v)\right), \quad \text{for } a \in [-1, 1]$$

The partial derivatives of C are given by

$$\begin{aligned} \frac{\partial^2 C}{\partial u^2}(u, v) &= 2av(v-1), \qquad \frac{\partial^2 C}{\partial v^2}(u, v) = 2au(u-1), \\ \frac{\partial^2 C}{\partial u \partial v}(u, v) &= 1 + a(1-2u)(1-2v). \end{aligned}$$

After substituting into eq. (3.11) we obtain

$$\begin{aligned} &2av(v-1)\varphi(\Phi^{-1}(u))^2 + 2au(u-1)\varphi(\Phi^{-1}(v))^2 \\ &+2A(\Phi^{-1}(u), \Phi^{-1}(v))(1 + a(1-2u)(1-2v))\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v)) = 0 \end{aligned}$$

where

$$A(x, y) = \frac{a\Phi(y)(1-\Phi(y))e^{\frac{y^2-x^2}{2}} + a\Phi(x)(1-\Phi(x))e^{\frac{x^2-y^2}{2}}}{1 + a(1-2\Phi(x))(1-2\Phi(y))}. \tag{5.1}$$

Note that

$$A(0, 0) = \frac{2a \cdot \frac{1}{4}}{1 + a \cdot 0} = \frac{a}{2}.$$

We show that $|A|$ is bounded by $|a|/2$ for any $a \in [-1, 1]$, so the maximum of $|A|$ is attained to the point $(0, 0)$ (compare [16] Section 3.4). Indeed, we have an estimate for any $x, y \in \mathbb{R}$ and $a \in [-1, 1]$

$$|A(x, y)| \leq |a|\frac{\Phi(y)(1-\Phi(y))e^{\frac{y^2-x^2}{2}} + \Phi(x)(1-\Phi(x))e^{\frac{x^2-y^2}{2}}}{1 - |(1-2\Phi(x))(1-2\Phi(y))|}. \tag{5.2}$$

Since the right side of (5.2) is invariant with respect to change of sign of x, change of sign of y and symmetry $(x, y) \to (y, x)$ it is enough to show that for $y \geq x \geq 0$, a function

$$\chi(x, y) = 2\Phi(y)\Phi(-y)e^{\frac{1}{2}(y^2-x^2)} + 2\Phi(x)\Phi(-x)e^{\frac{1}{2}(x^2-y^2)} + (1-2\Phi(x))(1-2\Phi(y))$$

is bounded by 1. We observe that the directional derivative

$$y\frac{\partial \chi(x, y)}{\partial x} + x\frac{\partial \chi(x, y)}{\partial y} = 2(y-x)\left[\phi(x)\left(2\Phi(y)-1\right) - \phi(y)\left(2\Phi(x)-1\right)\right]$$

is nonnegative for $y \geq x \geq 0$. Hence χ is nondecreasing along the hyperbola

$$y^2 = r^2 + x^2, \quad y \geq x \geq 0.$$

Since moreover $\chi(x, y)$ is positive for $y \geq x \geq 0$, we get

$$\begin{aligned} 0 \leq \chi(x, y) &\leq \sup\{\chi(\xi, \sqrt{r^2 + \xi^2}) : \xi > x\} \\ &= \lim_{\xi \to +\infty} \chi(\xi, \sqrt{r^2 + \xi^2}) = 1 \end{aligned} \tag{5.3}$$

Furthermore, since both nominator and denominator have bounded derivatives and for $|a| < 1$ denominator is bounded from zero, A is differentiable with bounded derivatives. Hence A and $B(x, y) = \sqrt{1 - A(x, y)^2}$ are Lipschitz. Therefore by Theorem 3.4 point 3 there exists a $1/2$-self-similar solution of eq. (3.1), with coefficients A as above and $B(x, y) = \sqrt{1 - A(x, y)^2}$, such that the copula of its initial values is FGM with parameter $a \in (-1, 1)$.

5.3 Archimedean copulas

We recall that by a strict Archimedean copula we mean a copula that takes the following form

$$C_g(u,v) = g^{-1}(g(u)+g(v)),$$

where a function $g\colon [0,1] \to [0,+\infty]$ is continuous, strictly decreasing, convex and

$$g(1) = 1, \quad g(0) = +\infty.$$

The function g is called a generator of the copula C_g. For more details concerning Archimedean copulas, both strict and nonstrict, the reader is referred to [9, 19, 20, 26].

Due to convexity, g is differentiable at all but at most countably many points and the derivative may have jumps. Therefore, for simplicity, we restrict ourselves to twice differentiable generators g. Then we have for $t \in (0,1)$

$$g'(t) < 0, \quad g''(t) \geq 0. \tag{5.4}$$

We start with calculating the partial derivatives of C. In order to simplify the notation let us introduce auxiliary variable $z = g^{-1}(g(u)+g(v))$.

$$\frac{\partial C(u,v)}{\partial u} = \frac{g'(u)}{g'(z)}, \tag{5.5}$$

$$\frac{\partial C(u,v)}{\partial v} = \frac{g'(v)}{g'(z)}, \tag{5.6}$$

$$\begin{aligned}\frac{\partial^2 C(u,v)}{\partial u^2} &= \frac{g''(u)g'(z) - g'(u)g''(z)\frac{g'(u)}{g'(z)}}{\left(g'(z)\right)^2}\\ &= \frac{g''(u)}{g'(z)} - \left(g'(u)\right)^2 \frac{g''(z)}{\left(g'(z)\right)^3},\end{aligned} \tag{5.7}$$

$$\begin{aligned}\frac{\partial^2 C(u,v)}{\partial v^2} &= \frac{g''(v)g'(z) - g'(v)g''(z)\frac{g'(v)}{g'(z)}}{\left(g'(z)\right)^2}\\ &= \frac{g''(v)}{g'(z)} - \left(g'(v)\right)^2 \frac{g''(z)}{\left(g'(z)\right)^3},\end{aligned} \tag{5.8}$$

$$\begin{aligned}\frac{\partial C(u,v)}{\partial u \partial v} &= g'(u)\cdot(-1)\cdot\left(g'(z)\right)^{-2} g''(z)\frac{g'(v)}{g'(z)}\\ &= -\frac{g''(z)g'(u)g'(v)}{\left(g'(z)\right)^3}.\end{aligned} \tag{5.9}$$

If we further assume that $g''(t) \neq$ for all $t \in (0,1)$, then substituting into eq. (3.11) we obtain

$$\begin{aligned}A(x,y) &= -\frac{1}{2}\left[\frac{g'(\Phi(x))}{g'(\Phi(y))} - \frac{g''(\Phi(x))\left(g'(C_g(\Phi(x),\Phi(y)))\right)^2}{g''(C_g(\Phi(x),\Phi(y)))g'(\Phi(x))g'(\Phi(y))}\right] e^{\frac{y^2-x^2}{2}}\\ &\quad -\frac{1}{2}\left[\frac{g'(\Phi(y))}{g'(\Phi(x))} - \frac{g''(\Phi(y))\left(g'(C_g(\Phi(x),\Phi(y)))\right)^2}{g''(C_g(\Phi(x),\Phi(y)))g'(\Phi(x))g'(\Phi(y))}\right] e^{\frac{x^2-y^2}{2}}.\end{aligned}$$

As we show in next subsections for some generators A fulfill assumptions **A2** and **A3**, for some not.

5.3.1 Clayton copula

As a specific example let us consider the following generator

$$g(t) = \frac{1}{\alpha}\left(t^{-\alpha} - 1\right), \qquad \alpha > 0. \tag{5.10}$$

This way we in fact obtain a member of the Clayton family of copulas (compare [9, 20, 26]),

$$C_g(u, v) = \left(u^{-\alpha} + v^{-\alpha} - 1\right)^{-\frac{1}{\alpha}}, \quad \alpha > 0. \tag{5.11}$$

Since

$$g'(t) = -t^{-\alpha-1} \quad \text{and} \quad g''(t) = (\alpha + 1)t^{-\alpha-2},$$

we get

$$A(x, y) = \frac{1}{2}\left((1 - \Phi(y)^{\alpha})\frac{\Phi(y)\phi(x)}{\Phi(x)\phi(y)} + (1 - \Phi(x)^{\alpha})\frac{\Phi(x)\phi(y)}{\Phi(y)\phi(x)}\right). \tag{5.12}$$

We show that A is not bounded by 1. We apply the limit

$$\lim_{z\to-\infty} \frac{|z|\Phi(z)}{\phi(z)} = 1, \tag{5.13}$$

which follows from the de l'Hospital rule:

$$\lim_{z\to-\infty} \frac{\Phi(z)}{z^{-1}\phi(z)} \overset{H}{=} \lim_{z\to-\infty} \frac{\phi(z)}{-\phi(z) - z^{-2}\phi(z)} = -1. \tag{5.14}$$

We substitute $x = z$ and $y = 2z$ and take the limit as z tends to $-\infty$.

$$\begin{aligned} & \lim_{z\to-\infty} A(z, 2z) \\ = & \lim_{z\to-\infty} \frac{1}{2}\left((1 - \Phi(2z)^{\alpha})\frac{1}{2}\frac{2|z|\Phi(2z)}{\phi(2z)}\frac{\phi(z)}{|z|\Phi(z)} + (1 - \Phi(z)^{\alpha}) \cdot 2\frac{|z|\Phi(z)}{\phi(z)}\frac{\phi(2z)}{2|z|\Phi(2z)}\right) \\ = & \frac{1}{2}\left(\frac{1}{2} + 2\right) = 1{,}25. \end{aligned} \tag{5.15}$$

5.3.2 Frank copula

As a next specific example let us consider the following generator

$$g(t) = -\ln\frac{e^{-\alpha t} - 1}{e^{-\alpha} - 1}, \qquad \alpha \neq 0. \tag{5.16}$$

This way we in fact obtain a member of the Frank family of copulas (compare [9, 20, 26]),

$$C_g(u, v) = -\frac{1}{\alpha}\ln\left(1 + \frac{(e^{-\alpha u} - 1)(e^{-\alpha v} - 1)}{e^{-\alpha} - 1}\right), \quad \alpha \neq 0. \tag{5.17}$$

We have

$$g'(t) = \frac{\alpha e^{-\alpha t}}{e^{-\alpha t} - 1} = \frac{\alpha}{1 - e^{\alpha t}}, \tag{5.18}$$

$$g''(t) = \frac{\alpha^2 e^{-\alpha t}}{(e^{-\alpha t} - 1)^2} = \frac{\alpha g'(t)}{e^{-\alpha t} - 1} = e^{\alpha t}(g'(t))^2 \tag{5.19}$$

and

$$\begin{aligned}\frac{g'(u)}{g'(v)} - \frac{g''(u)g'(z)^2}{g''(z)g'(u)g'(v)} &= \frac{g'(u)}{g'(v)} - \frac{e^{\alpha u}g'(u)}{e^{\alpha z}g'(v)} \\ &= \frac{g'(u)}{g'(v)}\left(1 - e^{-\alpha z}e^{\alpha u}\right) \\ &= \frac{g'(u)}{g'(v)}\left(1 - \left(1 + \frac{(e^{-\alpha u}-1)(e^{-\alpha v}-1)}{e^{-\alpha}-1}\right)e^{\alpha u}\right) \\ &= \frac{1-e^{\alpha v}}{1-e^{\alpha u}} \\ &\times \frac{(e^{-\alpha}-1)(1-e^{\alpha u}) - (e^{-\alpha u}-1)(e^{-\alpha v}-1)e^{\alpha u}}{e^{-\alpha}-1} \\ &= \frac{e^{-\alpha}-e^{-\alpha v}}{e^{-\alpha}-1}\cdot\left(1-e^{\alpha v}\right) \\ &= -\frac{1+e^{-\alpha}-e^{-\alpha v}-e^{-\alpha(1-v)}}{1-e^{-\alpha}}\end{aligned} \tag{5.20}$$

Finally the quadratic covariation is given by

$$A_\alpha(x,y) = \frac{1}{2}\left(\chi_\alpha(\Phi(y))\frac{\phi(x)}{\phi(y)} + \chi_\alpha(\Phi(x))\frac{\phi(y)}{\phi(x)}\right), \tag{5.21}$$

where

$$\begin{aligned}\chi_\alpha(t) &= \frac{1-e^{-\alpha t}+e^{-\alpha}-e^{-\alpha(1-t)}}{1-e^{-\alpha}} \\ &= \frac{1-e^{\alpha t}+e^{\alpha}-e^{\alpha(1-t)}}{e^{\alpha}-1}.\end{aligned} \tag{5.22}$$

Since for any $t \in (0,1)$

$$\lim_{\alpha\to\pm\infty}\chi_\alpha(t) = \pm 1,$$

we get for $x^2 \neq y^2$

$$\lim_{\alpha\to\pm\infty}|A_\alpha(x,y)| = \frac{1}{2}\left(\frac{\phi(x)}{\phi(y)} + \frac{\phi(y)}{\phi(x)}\right) > 1.$$

Hence for sufficiently big $|\alpha|$ $|A_\alpha|$ is not bounded by 1.
But for sufficiently small $|\alpha|$ $|A_\alpha|$ is bounded by 1. Specifically it is bounded for $|\alpha| \le 2$. Indeed, since the functions $|\chi_\alpha(t)|$ and $\phi(\Phi^{-1}(t))$ are concave, we get estimates for $t \in (0,1)$

$$|\chi_\alpha(t)| < |\alpha|\min(t, 1-t) \le \frac{|\alpha|}{2\phi(0)}\phi(\Phi^{-1}(t)). \tag{5.23}$$

Hence for all $x, y \in \mathbb{R}$

$$\begin{aligned}|A_\alpha(x,y)| &\le \frac{1}{2}\left(\frac{|\chi_\alpha(\Phi(y))|}{\phi(y)}\phi(x) + \frac{|\chi_\alpha(\Phi(x))|}{\phi(x)}\phi(y)\right) \\ &< \frac{1}{2}\left(\frac{|\alpha|}{2\phi(0)}\phi(0) + \frac{|\alpha|}{2\phi(0)}\phi(0)\right) = \frac{|\alpha|}{2}.\end{aligned} \tag{5.24}$$

Moreover $A(x,y)$ is Lipschitz continuous since derivatives of the ratio $\chi_\alpha(\Phi(\xi))/\phi(\xi)$ are bounded. Hence for $|\alpha| < 2$ Theorem 3.4 point 3 is applicable.

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