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Sklar's theorem, copula products, and ordering results in factor models

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Abstract: We consider a completely specified factor model for a risk vector $X = (X_1, \ldots, X_d)$, where the joint distributions of the components of X with a risk factor Z and the conditional distributions of X given Z are specified. We extend the notion of $*$-product of d-copulas as introduced for $d = 2$ and continuous factor distribution in Darsow et al. [6] and Durante et al. [8] to the multivariate and discontinuous case. We give a Sklar-type representation theorem for factor models showing that these $*$-products determine the copula of a completely specified factor model. We investigate in detail approximation, transformation, and ordering properties of $*$-products and, based on them, derive general orthant ordering results for completely specified factor models in dependence on their specifications. The paper generalizes previously known ordering results for the worst case partially specified risk factor models to some general classes of positive or negative dependent risk factor models. In particular, it develops some tools to derive sharp worst case dependence bounds in subclasses of completely specified factor models.

Keywords: componentwise convex copulas, concordance order, conditional distribution function, conditional independence, factor model, product of copulas

MSC: 60E15, 60E05, 28A50

1 Introduction

A relevant class of distributions for modeling dependencies are factor models where each component of the underlying random vector $X = (X_1, \ldots, X_d)$ is supposed to depend on some common random factor Z through

$$X_i = f_i(Z, \varepsilon_i), \quad 1 \leq i \leq d,$$

for some functions f_i and a random vector $(\varepsilon_1, \ldots, \varepsilon_d)$ that is independent of Z. In this paper, we consider the case where Z is a real-valued random variable. If the bivariate distribution of (X_i, Z) is specified and the distribution of $X|Z = z$ is known for all i and z, then the distribution of X is fully specified. We denote this setting a completely specified factor model (CSFM).

For applications to risk modeling, partially specified factor models (PSFMs) are introduced in Bernard et al. [5]. In these models, the distributions of (X_i, Z) are specified. The joint distribution of $(\varepsilon_1, \ldots, \varepsilon_d)$ is, however, not prescribed. This means, that only the distributions of X_i and Z as well as the copula $D^i = C_{X_i,Z}$ of (X_i, Z) are given. Then, the worst case distribution in the PSFM is determined by the conditionally comonotonic random vector $X_Z^c = (F^{-1}_{X_1|Z}(U), \ldots, F^{-1}_{X_d|Z}(U))$, where $U \sim U(0, 1)$ is independent of Z, assuming generally a non-atomic underlying probability space $(\Omega, \mathcal{A}, P)$. If Z has a continuous distribution, the copula of X_Z^c is given by the upper product of the bivariate copulas D^i, see [2].

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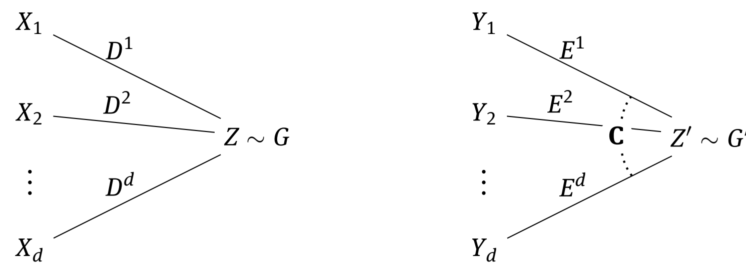

Figure 1 On the left: a partially specified factor model with dependence specifications $D^1, \dots, D^d$ and risk factor distribution function G . On the right: a completely specified factor model with dependence specifications $E^1, \dots, E^d$, conditional copula family **C** and factor distribution function G' .

In standard factor models, the individual factors $\varepsilon_1, \dots, \varepsilon_d$ are assumed to be independent. Then, the distribution of X is completely specified and the components of X are conditionally independent given $Z = z$ for all z . Further, the copula of X is then given by the conditional independence product of the bivariate specifications D^i , which is an extension of the bivariate copula product introduced in Darsow et al. [6] to arbitrary dimension, see [15].

In this paper, we introduce and study the $*$-product of copulas as an extension of the bivariate copula product considered in [8] to the multivariate case and to general factor distribution functions in order to model the copula of $X = (f_i(Z, \varepsilon_i))_i$ for general dependence structures among $(\varepsilon_1, \dots, \varepsilon_d)$ and also discontinuous Z . We provide a simple representation of a conditional distribution function by the corresponding univariate distribution functions and a generalized derivative of the associated copula. Then, we derive a Sklar-type theorem implying that the dependence structure of X is determined by the $*$-product of the dependence specifications in the CSFM. Further, we establish a general continuity result for the $*$-product in dependence on all its arguments which is useful for corresponding approximation results. We study transformation properties of the $*$-product and introduce, as a counterpart of the upper product, the lower product of bivariate copulas in the two- and three-dimensional case.

In Section 3, we derive general lower and upper orthant ordering results for the $*$-product in dependence on the copula specifications. This requires the consideration of integral inequalities like the rearrangement results of Lorentz [16] and Fan and Lorentz [11]. We extend and strengthen several recent results on the lower and upper orthant ordering of upper products to general $*$-products. In particular, we show that componentwise convexity of the conditional copulas plays an important role for the ordering of the $*$-products. We introduce the $\leq_{\partial_2 S}$-ordering on the set of bivariate copulas based on the Schur-ordering of copula derivatives allowing to derive a meaningful comparison criterion. We show that many well-known copula families satisfy this ordering.

Finally, in Section 4, we combine the $*$-product ordering results with the ordering of marginal distributions and obtain several general ordering results in CSFMs. As a consequence, this yields maximum elements and, thus, sharp bounds w.r.t. the lower and upper orthant ordering for classes of PSFMs as well as for classes of CSFMs with the conditional independence assumption.

2 The $*$-product of copulas in completely specified factor models

A d-copula is a distribution function $C\colon [0,1]^d \to [0,1]$ with uniform univariate marginal distribution functions. Due to Sklar's theorem, every d-dimensional distribution function F can be decomposed into a composition of a d-copula C and the univariate marginal distribution functions $F_1, \dots, F_d$ of F , i.e.,

$$F(x) = C\left(F_1(x_1), \dots, F_d(x_d)\right) \tag{1}$$

for all $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. The copula C is uniquely determined on the Cartesian product $\times_{i=1}^d \mathrm{Ran}(F_i)$ of the ranges of F_i, $1 \le i \le d$. Further, for every d-copula and for all distribution functions $F_1, \dots, F_d$, the right-hand side in (1) defines a d-variate distribution function, see the original papers of Sklar [29] and Schweizer and Sklar [27], see also Nelsen [22], Rüschendorf [25], and Durante and Sempi [10]. Denote by $\mathcal{C}_d$ the set of all d-copulas and by $\mathcal{F}^d$ ($\mathcal{F}_c^d$) the set of (continuous) d-dimensional distribution functions.

In the setting of a completely specified factor model, the distribution function F of $X = (X_1, \dots, X_d)$ can be decomposed into

$$F(x_1, \dots, x_d) = \int_{-\infty}^{\infty} F_z(x_1, \dots, x_d)\, \mathrm{d}G(z) = \int_{-\infty}^{\infty} C_z\left(F_{1|z}(x_1), \dots, F_{d|z}(x_d)\right)\, \mathrm{d}G(z),$$

where F_z is the conditional distribution function of $(X_1, \dots, X_d)|Z = z$ with univariate marginal conditional distribution functions $F_{i|z}$ and conditional copula $C_z \in \mathcal{C}_d$. Each $F_{i|z}$ depends via

$$D^i\left(F_i(x), G(z)\right) = F_{X_i,Z}(x, z) = \int_{-\infty}^{z} F_{i|s}(x)\, \mathrm{d}G(s)$$

only on the dependence specification $D^i = C_{X_i,Z}$ and the marginal distribution functions F_i and G, where $G = F_Z$ denotes the distribution function of Z.

Altogether, this motivates to introduce the $*$-product of copulas as a product of the specifications $D^1, \dots, D^d \in \mathcal{C}_2$, of the conditional copulas $(C_z)_z$, $C_z \in \mathcal{C}_d$, and of the risk factor distribution function $G \in \mathcal{F}^1$. In a Sklar-type theorem, we show that the $*$-product is a copula that describes the dependence structure of the risk vector X in the CSFM. We give the basic properties of the $*$-products that are used in the following sections to develop several ordering results for $*$-products and, thus, ordering results for CSFMs.

Our results extend the bivariate $*$-product considered in Durante et al. [8] and the bivariate conditional independence product introduced in Darsow et al. [6]. A discussion of some properties of bivariate $*$-products is given in Durante and Sempi [10, Section 5.5]. An important particular case of the $*$-product in the present paper is the multivariate conditional independence product which describes the dependence structure of the commonly used factor models with conditional independence assumption, compare Krupskii and Joe [15]. The particular case of upper products that corresponds to upper risk bounds in partially specified factor models has been investigated in [2]. As a counterpart of upper products, we introduce the lower product of bivariate copulas that describes best case bounds in the two-, respectively, three-dimensional PSFM.

2.1 Definition of $*$-products

The consideration of general factor distributions needs the following notion of generalized differentiation. For $G \in \mathcal{F}^1$ denote by

$$\iota_G \colon [0,1] \to \mathrm{Ran}(G), \qquad t \mapsto G \circ G^{-1}(t),$$
$$\iota_G^- \colon [0,1] \to \mathrm{Ran}(G^-), \qquad t \mapsto G^- \circ G^{-1}(t),$$

the transformation of the identity w.r.t. to G, resp. G^-, where $G^{-1} \colon [0,1] \to \mathbb{R} \cup \{\pm\infty\}$ given by $G^{-1}(u) := \inf\{x \mid G(x) \ge u\}$, $\inf \emptyset = \infty$, is the generalized inverse of G, and G^- is the left-continuous version of G. Several properties of the transformations ι_G and ι_G^- are given in Lemma A.1 in the Appendix, see also Figure 2.

Define for a function $f \colon [0,1] \to \mathbb{R}$ the generalized differential operator ∂^G by the left-hand limit

$$\partial^G f(t_0) := \lim_{t \nearrow t_0} \frac{f(\iota_G(t_0)) - f(\iota_G^-(t))}{\iota_G(t_0) - \iota_G^-(t)}, \tag{2}$$

$t_0 \in (0,1]$, if the limit exists. As usual, denote by ∂_i^G the operator ∂^G which is applied to the i-th coordinate of a function of several arguments.

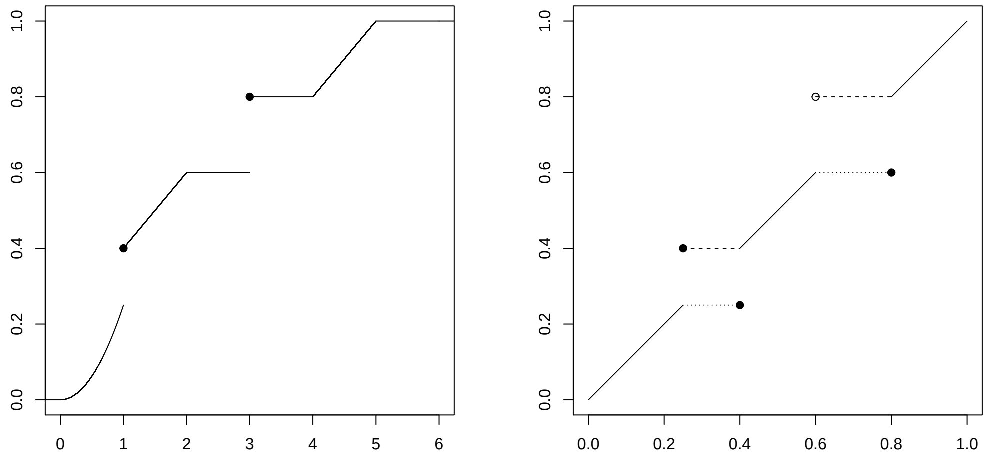

Figure 2 On the left: a distribution function G ; on the right: its corresponding transformations ι_G (dashed and solid line) and ι_G^- (dotted and solid line) which both coincide with the identity function on the interior of the range of G . Note that ι_G^- is left-continuous, and ι_G is neither left- nor right-continuous, see Lemma A.1.

Remark 2.1. *(a) The denominator in* (2) *is positive for all* $0 \le t < t_0 \le 1$ *because* $\iota_G(t_0) \ge t_0 > t \ge \iota_G^-(t)$ *by Lemma A.1*(iv).

(b) If f is left-continuous and if the (ordinary) left-hand derivative $f'_-(t_0) := \lim_{t \nearrow t_0} \frac{f(t_0)-f(t)}{t_0-t}$ *exists, then* $\partial^G f(t_0)$ *exists for all* $G \in \mathcal{F}^1$ *. To see this, we know by* (a) *that* $\iota_G(t_0) \ge t_0 \ge \lim_{t \nearrow t_0} \iota_G^-(t)$ *. Hence, if* $\iota_G(t_0) = \lim_{t \nearrow t_0} \iota_G^-(t) = t_0$ *, then* $\partial^G f(t_0) = f'_-(t_0)$ *. If* $\iota_G(t_0) > \lim_{t \nearrow t_0} \iota_G^-(t)$ *, then* $\partial^G f(t_0)$ *exists since f and ι_G are left-continuous, see Lemma A.1*(vi).

(c) A useful transformation property of ∂^G is that

$$\partial^G f(t) = \partial^G f(\iota_G(t)) = \partial^G f(G(x)) \quad \text{for all } G \in \mathcal{F}^1 \text{ and for Lebesgue-almost all } t\,, \tag{3}$$

where $x = G^{-1}(t)$ *. This is a consequence of Lemma A.1*(v) *considering the cases where G is continuous at x or has a jump discontinuity at x , compare equations* (38) *and* (39) *in the proof of Theorem 2.2.*

The following result gives the representation of a conditional distribution function by the univariate marginals and the generalized partial derivative of the corresponding copula.

Theorem 2.2 (Representation of conditional distribution functions)**.**
For $F, G \in \mathcal{F}^1$ *, let* $X \sim F$ *and* $Z \sim G$ *be real random variables with copula* $C \in \mathcal{C}_2$ *, i.e.,* $C = C_{X,Z}$ *. Then, the following statements hold true:*

(i) For all $x \in \mathbb{R}$ *, there exists a G-null set* N_x *such that the conditional distribution function of X given $Z = z$ evaluated at x is represented by*

$$F_{X|Z=z}(x) = \lim_{h \searrow 0} \frac{C(F(x), G(z)) - C(F(x), G(z-h))}{G(z) - G(z-h)} = \partial_2^G C(F(x), G(z)) \tag{4}$$

for all $z \in N_x^c$ *.*

(ii) There exists a G-null set N such that

$$F_{X|Z=z}(x) = \lim_{w \searrow x} \partial_2^G C(F(w), G(z)) \tag{5}$$

for all $x \in \mathbb{R}$ *and for all* $z \in N^c$ *.*

The proof is given in the Appendix.

Remark 2.3. *(a) For the representation of the conditional distribution function in* (4) *and* (5)*, we make use of the left-hand limit in the definition of the generalized differential operator given by* (2)*. If G has a discontinuity at z, then the operator ∂_2^G is the difference quotient operator w.r.t. the second component of C between $G(z)$ and $G^-(z)$. If G is continuous at z, the operator ∂_2^G reduces to the ∂_2^--operator denoting the left-hand partial derivative with respect to the second variable. Hence, if G is continuous for all z, then it holds that $\partial_2^G = \partial_2^-$. Denote by ∂_2 the operator which takes the partial derivative w.r.t. the second component of a multivariate function. Since copulas are almost surely partially differentiable, see Nelsen [22, Theorem 2.2.7], it holds for all u, that $\partial_2^- C(u, v) = \partial_2 C(u, v)$ for almost all v.*

(b) We point out that the right-hand expression in (4) *is not necessarily right-continuous in x, and, thus, it does not generally define a distribution function in x. However, in the following definition of the $*$-product as well as in most results of the paper, we integrate over the conditioning variable and, then, this representation of the conditional distribution function is appropriate.*

In the following definition, we extend the $*$-product introduced by Darsow et al. [6] for Markov structures, and, for arbitrary conditional dependencies, by Durante et al. [8] (for $d = 2$) and [2] (for $d \geq 2$) to $G \in \mathcal{F}^1$ allowing also discontinuous factor distribution functions.

We need a measurability assumption which is implicitly assumed in the above mentioned literature by the definition of the corresponding integrals. We call a family $\mathbf{B} = (B_t)_{t\in[0,1]}$ of d-copulas *measurable* if the mapping $(t, u) \mapsto B_t(u)$, $(t, u) \in [0, 1] \times [0, 1]^d$, is measurable.

The $*$-product of bivariate copulas is defined in dependence on a measurable family $\mathbf{B} = (B_t)_{t\in[0,1]}$ of d-dimensional copulas and on a distribution function $G \in \mathcal{F}^1$.

Definition 2.4 ($*$-product of copulas)**.**

(i) Let $\mathbf{B} := (B_t)_{t\in[0,1]}$ be measurable, $B_t \in \mathcal{C}_d$, $0 \leq t \leq 1$, and $G \in \mathcal{F}^1$. For bivariate copulas $D^1, \ldots, D^d \in \mathcal{C}_2$, the ($d$-dimensional) $*$-product *of $D^1, \ldots, D^d$ w.r.t. $\mathbf{B}$ and G is defined by*

$$\circledast_{\mathbf{B},G} D^i(u) := \circledast_{i=1,\mathbf{B},G}^d D^i(u) := \int_0^1 B_t^G\left(\partial_2^G D^1(u_1, t), \ldots, \partial_2^G D^d(u_d, t)\right) \mathrm{d}t \tag{6}$$

for $u = (u_1, \ldots, u_d) \in [0, 1]^d$ where B_t^G is defined by

$$B_t^G := \begin{cases} B_t\,, & \text{if } \iota_G^-(t) = \iota_G(t)\,, \\ \frac{1}{\iota_G(t)-\iota_G^-(t)} \int_{\iota_G^-(t)}^{\iota_G(t)} B_s \,\mathrm{d}s\,, & \text{if } \iota_G^-(t) \neq \iota_G(t)\,. \end{cases} \tag{7}$$

(ii) If there exists a copula $B \in \mathcal{C}_d$ such that $B_t^G = B$ for almost all t, then we use the notion $\circledast_{B,G} D^i := \circledast_{\mathbf{B},G} D^i$ and call it simplified $*$-product *of $D^1, \ldots, D^d$ w.r.t. B and G.*

(iii) If G is continuous, then the (simplified) $*$-product is abbreviated by $\circledast_{\mathbf{B}} D^i := \circledast_{\mathbf{B},G} D^i$ and $\circledast_B D^i := \circledast_{B,G} D^i$, respectively.

Note that the number d of bivariate copulas is typically clear from the context and therefore the simplified notation is used. We also sometimes use the notation $D^1 *_{\mathbf{B},G} \cdots *_{\mathbf{B},G} D^d := \circledast_{\mathbf{B},G} D^i$ for the $*$-product of d bivariate copulas $D^1 \ldots, D^d$ w.r.t. to $\mathbf{B}$ and G.

Note that for fixed $u_1, \ldots, u_d \in [0, 1]$ the integrand in (6) is well-defined as a consequence of Remark 2.1(b) because copulas are Lipschitz-continuous. The justification for the simplified notation in (iii) of the above definition is due to Proposition 2.14.

As usual, we denote by Π^d, M^d, and W^d, defined by

$$\Pi^d(u) := u_1 \cdots u_d\,, \qquad M^d(u) := \min_{1\leq i\leq d}\{u_i\}\,, \qquad W^d(u) := \max_{1\leq i\leq d}\left\{\sum_{i=1}^d u_i - d + 1\,, 0\right\}\,,$$

the *product copula*, the *upper Fréchet copula*, and the *lower Fréchet bound*, respectively, where W^d is a copula only for $d \leq 2$. As special $*$-products, we consider the following simplified products of bivariate copulas.

Definition 2.5 (Some specific simplified $*$-products)**.**

(i) The conditional independence product *is defined as* $\Pi_G D^i := \ast_{\Pi^d,G} D^i$.
(ii) The upper product *is defined as* $\bigvee_G D^i := D^1 \vee_G \cdots \vee_G D^d := \ast_{M^d,G} D^i$.
(iii) The lower product *is defined as* $\bigwedge_G D^i := D^1 \wedge_G \cdots \wedge_G D^d := \ast_{W^d,G} D^i$.

If G is continuous, we abbreviate the independence product by $\Pi D^i = \Pi_{i=1}^d D^i$ *, the upper product by* $\bigvee D^i = \bigvee_{i=1}^d D^i = D^1 \vee \cdots \vee D^d$ *, and the lower product by* $\bigwedge D^i = \bigwedge_{i=1}^d D^i = D^1 \wedge \cdots \wedge D^d$ *.*

Since W^d is a copula only if $d \leq 2$, we clarify that for $d \geq 3$, the lower product is defined in the sense of (6).

The following result shows that the $*$-product is a copula. It extends [2, Proposition 2.1] from continuous to general factor distribution functions.

Proposition 2.6. *For all measurable* $\mathbf{B} = (B_t)_{t\in[0,1]}$, $B_t \in \mathcal{C}_d$ *for all* t *, for all* $G \in \mathcal{F}^1$ *, and for all* $D^1, \dots, D^d \in \mathcal{C}_2$ *, the $*$-product* $\ast_{\mathbf{B},G} D^i$ *is a d-copula.*

Proof. Due to Theorem 2.2, the functions H_z^i , $1 \leq i \leq d$, defined by $H_z^i(u) := \lim_{v\searrow u} \partial_2^G D^i(v, G(z))$ for $u \in [0, 1)$ and $H_z^i(1) := \partial_2^G D^i(1, G(z)) = 1$ are univariate distribution functions for G-almost all $z \in \mathbb{R}$. Then, by Sklar's Theorem, F_z defined by

$$F_z(u_1, \dots, u_d) := B^G_{G(z)}(H_z^1(u_1), \dots, H_z^d(u_d)), \quad (u_1, \dots, u_d) \in [0, 1]^d ,$$

is a d-dimensional distribution function on $[0, 1]^d$, where (B_t^G) is defined by (7). It follows that

$$\begin{aligned}
\ast_{\mathbf{B},G} D^i\,(u_1, \dots, u_d) &= \int_0^1 B_t^G \left(\partial_2^G D^1(u_1, t), \dots, \partial_2^G D^d(u_d, t)\right) \mathrm{d}t \\
&= \int_0^1 B^G_{\iota_G(t)} \left(\partial_2^G D^1(u_1, \iota_G(t)), \dots, \partial_2^G D^d(u_d, \iota_G(t))\right) \mathrm{d}t \\
&= \int_{\mathbb{R}} B^G_{G(z)} \left(\partial_2^G D^1(u_1, G(z)), \dots, \partial_2^G D^d(u_d, G(z))\right) \mathrm{d}G(z) \\
&= \int_{\mathbb{R}} B^G_{G(z)} \left(H_z^1(u_1), \dots, H_z^d(u_d)\right) \mathrm{d}G(z) = \int_{\mathbb{R}} F_z(u_1, \dots, u_d)\, \mathrm{d}G(z) ,
\end{aligned} \tag{8}$$

where we apply (3) for the second equality and use that $B_t^G = B^G_{\iota_G(t)}$ which follows from Lemma A.1(v). The third equality follows from the transformation formula, see, e.g., [32, Theorem 2]. For the fourth equality, we use for fixed $(u_1, \dots, u_d) \in [0, 1]^d$ that $H_z^i(u_i) = \partial_2^G D^i(u_i, G(z))$, $1 \leq i \leq d$, for G-almost all z , see Theorem 2.2(i). Since the last integral is a mixture of distribution functions, the product $\ast_{\mathbf{B},G} D^i$ is a distribution function. The measurability of $F_z(u_1, \dots, u_d)$ in z is a consequence of the measurability of $\mathbf{B}$ and, by (4), of $t \mapsto \partial_2^G D^i(u_i, t)$ for all $u_i \in [0, 1]$, $1 \leq i \leq d$.

It remains to show that $\ast_{\mathbf{B},G} D^i$ has uniform marginals. For $i \in \{1, \dots, d\}$, let $v = (v_1, \dots, v_d)$ with $v_i \in [0, 1]$ and $v_j = 1$ for all $j \neq i$. Since $\partial_2^G D^j(v_j, t) = 1$ for all t and $j \neq i$, it follows that

$$\ast_{\mathbf{B},G} D^j\,(v_1, \dots, v_d) = \int_0^1 \partial_2^G D^i(v_i, t)\, \mathrm{d}t = \int_{\mathbb{R}} \partial_2^G D^i(v_i, G(z))\, \mathrm{d}G(z) = v_i ,$$

where the first equality holds due to the uniform marginals of the copula B_t^G , the second one is a consequence of the transformation formula and (3), and the last equality is given by Theorem 2.2 and the disintegration theorem. □

2.2 Sklar-type theorem for factor models

The following theorem describes the meaning of the notion of $*$-products. It is a version of Sklar's Theorem for completely specified factor models and states that the dependence structure of a random vector $(X_i)_{1\le i\le d}$ that follows a completely specified factor model, $X_i = f_i(Z, \varepsilon_i)$, is given by a $*$-product of the specifications $G = F_Z$, $C^i = C_{X_i,Z}$, and $B_t^G = C_{X_1,\dots,X_d|Z=G^{-1}(t)}$, $t \in [0,1]$.

Theorem 2.7 (Sklar's Theorem for completely specified factor models)**.**
Let $F_{1,\dots,d+1} \in \mathcal{F}^{d+1}$ be a $(d+1)$-dimensional distribution function with univariate marginal distribution functions $F_1,\dots,F_{d+1}$. Denote by $F_{i,d+1}$ the bivariate marginal distribution function of its $(i,d+1)$-marginal, by $F_{1,\dots,d}$ the distribution function of its first d components, and by $F_{1,\dots,d|F_{d+1}^{-1}(t)}$ the conditional distribution function of its first d components given that the $(d+1)$-st component equals $F_{d+1}^{-1}(t)$. Then, there exist bivariate copulas $C^1,\dots,C^d \in \mathcal{C}_2$ and a measurable family $\mathbf{B} = (B_t)_{t\in[0,1]}$ of d-copulas such that

$$F_{i,d+1}(x_i, x_{d+1}) = C^i(F_i(x_i), F_{d+1}(x_{d+1})) \qquad \text{for } i = 1,\dots,d, \tag{9}$$

$$F_{1,\dots,d|F_{d+1}^{-1}(t)}(x_1,\dots,x_d) = B_t^{F_{d+1}}\left(\left(\lim_{w_i\searrow x_i} \partial_2^{F_{d+1}} C^i(F_i(w_i), t)\right)_{1\le i\le d}\right) \qquad \text{for almost all } t \in [0,1], \tag{10}$$

$$F_{1,\dots,d}(x_1,\dots,x_d) = *_{\mathbf{B},F_{d+1}} C^i\,(F_1(x_1),\dots,F_d(x_d)) \tag{11}$$

for all $(x_1,\dots,x_{d+1}) \in \mathbb{R}^{d+1}$.
Conversely, for distribution functions $F_1,\dots,F_{d+1} \in \mathcal{F}^1$, bivariate copulas $C^1,\dots,C^d \in \mathcal{C}_2$ and a measurable family $\mathbf{B} = (B_t)_{t\in[0,1]}$ of d-copulas, the family $\left(F_{1,\dots,d|F_{d+1}^{-1}(t)}\right)_{t\in[0,1]}$ in (10) defines a $(d+1)$-dimensional distribution function $F_{1,\dots,d+1}$ with bivariate marginal distribution functions $F_{i,d+1}$ given by (9) and d-variate distribution function $F_{1,\dots,d}$ given by (11).

Further, for $1 \le i \le d$, the copula C^i is uniquely determined on $\mathrm{Ran}(F_i) \times \mathrm{Ran}(F_{d+1})$, and $B_t^{F_{d+1}}$ is uniquely determined on $\times_{i=1}^d \mathrm{Ran}\left(\lim_{w_i\searrow\cdot} \partial_2^{F_{d+1}} C^i(F_i(w_i), t)\right)$ for almost all $t \in [0,1]$.

Proof. Due to Sklar's Theorem in the bivariate case, there exist $C^1,\dots,C^d \in \mathcal{C}_2$ such that (9) holds for all $(x_1,\dots,x_{d+1}) \in \mathbb{R}^{d+1}$. The univariate marginal distribution functions of $F_{1,\dots,d|F_{d+1}^{-1}(t)}$ are given by

$$F_{i|F_{d+1}^{-1}(t)}(x) = \lim_{w\searrow x} \partial_2^{F_{d+1}} C^i(F_i(w), t), \quad \text{for all } x \in \mathbb{R} \text{ and for almost all } t \in [0,1], \tag{12}$$

$1 \le i \le d$, see Theorem 2.2(ii). Due to Sklar's Theorem in the d-variate case, $B_t \in \mathcal{C}_d$, $t \in [0,1]$, with

$$B_t(u) = F_{1\dots d|F_{d+1}^{-1}(t)}\left(F^{-1}_{1|F_{d+1}^{-1}(t)}(u_1),\dots,F^{-1}_{d|F_{d+1}^{-1}(t)}(u_d)\right), \quad u = (u_1,\dots,u_d) \in [0,1]^d,$$

for almost all t defines a family $\mathbf{B} = (B_t)_{t\in[0,1]}$ of d-copulas such that (10) holds true. Note that $\mathbf{B}$ is measurable because the mappings $[0,1]\times\mathbb{R}^d \ni (t,x) \mapsto F_{1\dots d|F_{d+1}^{-1}(t)}(x)$ and $[0,1]\times[0,1] \ni (t,u_i) \mapsto F^{-1}_{i|F_{d+1}^{-1}(t)}(u_i)$, $1 \le i \le d$, are measurable.

To show (11), we apply the disintegration theorem and obtain for all $x = (x_1,\dots,x_d) \in \mathbb{R}^d$ that

$$F_{1,\dots,d}(x) = \int_0^1 F_{1,\dots,d|F_{d+1}^{-1}(t)}(x)\,\mathrm{d}t = \int_0^1 B_t^{F_{d+1}}\left(\left(\partial_2^{F_{d+1}} C^i(F_i(x_i), t)\right)_{1\le i\le d}\right)\mathrm{d}t = *_{\mathbf{B},F_{d+1}} C^i\,(F_i(x_i)),$$

where for the second equality we use the representation in (10) and that $\lim_{w_i\searrow x_i} \partial_2^{F_{d+1}} C^i(F_i(w_i), t) = \partial_2^{F_{d+1}} C^i(F_i(x_i), t)$ for all t outside a Lebesgue-null set $N_x \subset [0,1]$, see Theorem 2.2.

For the converse direction, let $F_1,\dots,F_{d+1} \in \mathcal{F}_1$, $C^1,\dots,C^d \in \mathcal{C}_2$ and $\mathbf{B} = (B_t)_{t\in[0,1]}$ be measurable, $B_t \in \mathcal{C}_d$ for all t. Then, by Theorem 2.2 and Sklar's Theorem, $F_{1,\dots,d|F_{d+1}^{-1}(t)}$ given by (10) defines a d-variate distribution function for almost all $t \in [0,1]$. As a consequence of the measurability of $\mathbf{B}$, the mapping $t \mapsto F_{1,\dots,d|F_{d+1}^{-1}(t)}(x)$ is measurable for all $x \in \mathbb{R}^d$, compare (8). Hence, $F_{1,\dots,d+1}$ defined by

$$F_{1,\dots,d+1}(x,z) = \int_0^{F_{d+1}(z)} F_{1,\dots,d|F_{d+1}^{-1}(t)}(x_1,\dots,x_d)\,\mathrm{d}t,$$

$x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $z \in \mathbb{R}$, is a $(d+1)$-dimensional distribution function with marginal distribution function of the first d components given by

$$F_{1,\dots,d}(x) = F_{1,\dots,d+1}(x,\infty) = \int_0^1 B_t^{F_{d+1}}\left(\left(\partial_2^{F_{d+1}} C^i(F_i(x_i),t)\right)_{1\le i\le d}\right)\,\mathrm{d}t$$

$$= \int_0^1 B_t^{F_{d+1}}\left(\left(\lim_{w_i \searrow x_i} \partial_2^{F_{d+1}} C^i(F_i(w_i),t)\right)_{1\le i\le d}\right)\,\mathrm{d}t = \ast_{\mathbf{B},F_{d+1}} C^i\,(F_1(x_1),\dots,F_d(x_d))$$

and bivariate marginal distribution functions w.r.t. to the i-th and $(d+1)$-st component given by

$$F_{i,d+1}(x_i,z) = F_{1,\dots,d+1}(\infty,\dots,\infty,x_i,\infty,\dots,\infty,z) = \int_0^{F_{d+1}(z)} \partial_2^{F_{d+1}} C^i\,(F_i(x_i),t))\,\mathrm{d}t = C^i(F_i(x_i),F_{d+1}(z))\,.$$

The uniqueness properties follow directly from the uniqueness properties in Sklar's Theorem. □

Remark 2.8. *(a) For $F_1,\dots,F_d, G \in \mathcal{F}^1$, let $(X_1,\dots,X_d,Z)$ be a $(d+1)$-dimensional random vector with $X_i \sim F_i$, $1 \le i \le d$ and $Z \sim G$. Then, from Theorem 2.7 it follows that*

$$(X_1,\dots,X_d) \sim \ast_{\mathbf{B},G} D^i(F_1,\dots,F_d)\,,$$

for $D^i = C_{X_i,Z}$ and $\mathbf{B} = (B_t)_{t\in[0,1]}$ measurable such that $B_t^G = C_{X_1,\dots,X_d|Z=G^{-1}(t)}$ is the conditional copula of $(X_1,\dots,X_d)$ given $Z = G^{-1}(t)$.

(b) As a weakening of (10), there exists for all $x = (x_1,\dots,x_d) \in \mathbb{R}^d$ a Lebesgue-null set N_x such that

$$F_{1,\dots,d|F_{d+1}^{-1}(t)}(x) = B_t^{F_{d+1}}\left(\left(\partial_2^{F_{d+1}} C^i(F_i(x_i),t)\right)_{1\le i\le d}\right) \qquad \text{for all } t \in N_x^c\,,$$

compare Theorem 2.2.

As a consequence of Sklar's theorem 2.7 for factor models, the conditional independence product, the upper product, and the lower product is characterized by conditional independence, conditional comonotonicity, and conditional countermonotonicity, respectively.

Corollary 2.9. *For $1 \le i \le d$ and $F_i \in \mathcal{F}^1$, let $X_i \sim F_i$ be random variables on a non-atomic probability space. Then, for $G \in \mathcal{F}^1$ and $D^1,\dots,D^d \in \mathcal{C}_2$, the following statements hold true.*

(i) $(X_1,\dots,X_d) \sim \Pi_G D^i(F_1,\dots,F_d)$ if and only if there exists a random variable $Z \sim G$ such that $(X_1,\dots,X_d)|Z = z$ is independent for G-almost all z.

(ii) $(X_1,\dots,X_d) \sim \bigvee_G D^i(F_1,\dots,F_d)$ if and only if there exists a random variable $Z \sim G$ such that $(X_1,\dots,X_d)|Z = z$ is comonotonic for G-almost all z.

(iii) $(X_1,X_2) \sim D^1 \wedge_G D^2\,(F_1,F_2)$ if and only if there exists a random variable $Z \sim G$ such that $(X_1,X_2)|Z = z$ is countermonotonic for G-almost all z.

Throughout the following sections, the copula families **B** and **C** are assumed to be measurable.

2.3 Basic properties of $\ast$-products

For a d-copula C, denote by $\bar{C}$ the corresponding survival function and by $\widehat{C}$ its survival copula. Then, the survival function and the survival copula of the $\ast$-product are determined as follows.

Proposition 2.10 (Survival function and survival copula)**.**
The survival function $\overline{\ast_{\mathbf{B},G} D^i}$ *and the survival copula* $\widehat{\ast_{\mathbf{B},G} D^i}$ *of the* $\ast$*-product* $\ast_{\mathbf{B},G} D^i$ *are given by*

$$\overline{\ast_{\mathbf{B},G} D^i}(u) = \int_0^1 \widehat{B}_t^G \left(1 - \partial_2^G D^1(u_1, t), \dots, 1 - \partial_2^G D^d(u_i, t)\right) \,\mathrm{d}t\,, \tag{13}$$

$$\widehat{\ast_{\mathbf{B},G} D^i}(u) = \int_0^1 \widehat{B}_t^G \left(1 - \partial_2^G D^1(1 - u_1, t), \dots, 1 - \partial_2^G D^d(1 - u_i, t)\right) \,\mathrm{d}t\,,$$

for $u = (u_1, \dots, u_d) \in [0, 1]^d$ *, where* $\widehat{B}_t^G$ *is the survival copula of* B_t^G *.*

Proof. Let $(U_1, \dots, U_d, Z)$ be a random vector such that U_i is uniformly distributed on $(0, 1)$, $Z \sim G$, and

$$(U_1, \dots, U_d)|Z = G^{-1}(t) \sim B_t \left(\lim_{w_1 \searrow \cdot} \partial_2^G D^1(w_1, t), \dots, \lim_{w_d \searrow \cdot} \partial_2^G D^d(w_d, t)\right)$$

for almost all $t \in (0, 1)$ and $C_{U_i,Z} = D^i$ for all $1 \le i \le d$, compare Remark 2.8(a). Then, it holds by (11) that $\ast_{\mathbf{B},G} D^i(u) = P(U_i \le u_i\,,\ 1 \le i \le d)$. Further, we obtain

$$\begin{aligned}
\overline{\ast_{\mathbf{B},G} D^i}\,(u) = P(U_i > u_i\ \forall i) &= \int_0^1 P\left(U_i > u_i\ \forall i \mid Z = G^{-1}(t)\right) \mathrm{d}t \\
&= \int_0^1 \widehat{B}_t^G \left(1 - \lim_{w_1 \searrow u_1} \partial_2^G D^1(w_1, t), \dots, 1 - \lim_{w_d \searrow u_d} \partial_2^G D^d(w_d, t)\right) \mathrm{d}t \\
&= \int_0^1 \widehat{B}_t^G \left(1 - \partial_2^G D^1(u_1, t), \dots, 1 - \partial_2^G D^d(u_i, t)\right) \mathrm{d}t\,,
\end{aligned}$$

where the third equality follows by the application of Sklar's Theorem for survival functions to the conditional survival function in the integrand, see, e.g., Georges et al. [12, Theorems 1 and 2] using that the i-th conditional marginal survival function is given by $\bar{F}_{U_i|Z=G^{-1}(t)}(u_i) = 1 - F_{U_i|Z=G^{-1}(t)}(u_i) = 1 - \lim_{w_i \searrow u_i} \partial_2^G D^i(w_i, t)$. The fourth equality is a consequence of Theorem 2.2.
The second statement follows from the relationship $\widehat{C}(u_1, \dots, u_d) = \bar{C}(1 - u_1, \dots, 1 - u_d)$, $(u_1, \dots, u_d) \in [0, 1]^d$, between the survival copula $\widehat{C}$ and the survival function $\bar{C}$ of a copula $C \in \mathcal{C}_d$. □

For some particular specifications, the $\ast$-products simplify as follows.

Proposition 2.11 (Particular specifications)**.**
For all $u = (u_1, \dots, u_d) \in [0, 1]^d$ *, the following statements hold true.*

(i) If $D^j = M^2$ *for all* $j \neq i$*then* $\ast_{\mathbf{B}} D^k(u) = D^i(u_i, \min_{j \neq i}\{u_j\})$ *.*
(ii) If $D^j = W^2$ *for all* $j \neq i$*then* $\ast_{\mathbf{B}} D^k(u) = D_*^i(u_i, \min_{j \neq i}\{u_j\})$ *, where* $D_*(v_1, v_2) := v_1 - D(v_1, 1 - v_2)$ *.*
(iii) If $D^i = \Pi^2$ *for all* i *, then* $\ast_{\mathbf{B},G} D^i(u) = \int_0^1 B_t^G(u)\,\mathrm{d}t$ *.*
(iv) Marginalization property: For $J \subset \{1, \dots, d\}$ *, the* J*-marginal of* $\ast_{\mathbf{B},G} D^i$ *is given by* $\ast_{\mathbf{B}',G} D^j$ *with bivariate copulas* $(D^j)_{j \in J}$ *and conditional copulas* $\mathbf{B}' = (B_t')_t$ *where* $B_t' \in \mathcal{C}_{|J|}$ *is the* J*-marginal of* B_t *.*
(v) Identifiability property: If $D^j = M^2$ *, then the* (i, j)*-marginal of* $\ast_{\mathbf{B}} D^k$ *is given by* D^i *.*

Proof. To show statement (i), observe that $\partial_2 M^2(v,t) = \mathbb{1}_{\{t<v\}}$ for almost all t. This yields

$$\ast_{\mathbf{B}} D^k(u) = \int_0^1 B_t\left(\mathbb{1}_{\{t<u_1\}},\dots,\mathbb{1}_{\{t<u_{i-1}\}},\partial_2 D^i(u_i,t),\mathbb{1}_{\{t<u_{i+1}\}},\dots,\mathbb{1}_{\{t<u_d\}}\right)\mathrm{d}t$$
$$= \int_0^{\min_{j\neq i}\{u_j\}} \partial_2 D^i(u_i,t)\,\mathrm{d}t = D^i(u_i,\min_{j\neq i}\{u_j\})$$

for $u = (u_1,\dots,u_d) \in [0,1]^d$, where the second equality follows because all B_t have uniform univariate marginals and are grounded.
Statement (ii) follows similarly with $\partial_2 W^2(v,t) = \mathbb{1}_{\{t>1-v\}}$ for almost all t, and statement (iii) follows from $\partial_2^G \Pi^2(v,t) = v$.
(iv): For $u = (u_1,\dots,u_d)$ with $u_i = 1$ for $i \notin J$, it follows that $\ast_{\mathbf{B},G} D^i(u) = \ast_{\mathbf{B}',G} D^j(u_J)$, where $u_J = (u_j)_{j\in J}$.
Statement (v) is a consequence of (i) setting $u_k = 1$ for all $k \in \{1,\dots,d\}\setminus\{i,j\}$. □

Note that statements (i), (ii), and (v) in the above result are formulated w.r.t. continuous risk factor distribution functions and cannot be generalized to arbitrary $G \in \mathcal{F}^1$. A counterexample can be constructed from the following example.

Example 2.12. *Let $D^i = M^2$ for all i and $G = \mathbb{1}_{[0,\infty)}$ be the distribution function of the Dirac distribution in 0. Then, it holds that $\Pi_G D^i = \Pi^d \neq M^d$ using that $\iota_G(t) = 1$ and $\iota_G^-(t) = 0$ for all $t \in (0,1)$. In fact, for $Z \sim G$, it holds that $P(Z=0) = 1$, and, thus, the dependence specifications $C_{X_i,Z} = D^i = M^2$ do not yield any information on the X_i and cannot force comonotonicity of $(X_1,\dots,X_d)$.*

Next, we study the product $\ast_{\mathbf{B},G} D^i$ in the case where $D^i = M^2$ for all i. We make use of ordinal sums defined as follows.

Let $J \subset \mathbb{N}$ be a finite or countable subset of the natural numbers. Let $(a_k,b_k)_{k\in J}$ be a family of pairwise disjoint, open subinterval of $[0,1]$ and let $(C_k)_{k\in J}$ be a family of d-copulas. Then, the ordinal sum $(\langle a_k,b_k,C_k\rangle)_{k\in J}$ of $(C_k)_{k\in J}$ w.r.t. $(a_k,b_k)_{k\in J}$ is defined by

$$(\langle a_k,b_k,C_k\rangle)_{k\in J}(u) = \begin{cases} a_k + (b_k-a_k)C_k\left(\frac{\min\{u_1,b_k\}-a_k}{b_k-a_k},\dots,\frac{\min\{u_d,b_k\}-a_k}{b_k-a_k}\right), & \\ \quad \text{if } \min\{u_1,\dots,u_d\}\in(a_k,b_k) \text{ for some } k\in J \\ \min\{u_1,\dots,u_d\} & \text{else}, \end{cases}$$

where $u = (u_1,\dots,u_d) \in [0,1]^d$, see, e.g., Mesiar and Sempi [17].

The following proposition characterizes ordinal sums by $\ast$-products.

Proposition 2.13 (Ordinal sums).
For $G \in \mathcal{F}^1$, for a measurable family $\mathbf{B} = (B_t)_{t\in[0,1]}$ and a sequence $(C_k)_{k\in J}$ of d-copulas, and for pairwise disjoint open subintervals $(a_k,b_k)_{k\in J}$ of $(0,1)$, the following statements are equivalent:

(i) $\ast_{\mathbf{B},G} M^2 = (\langle a_k,b_k,C_k\rangle)_{k\in J}$
(ii) $(a_k,b_k)_{k\in J} = \{(\iota_G^-(t),\iota_G(t)) \mid \iota_G^-(t) \neq \iota_G(t),\, t\in(0,1)\}$ *and* $C_k = B_t^G$ *for* $t \in (a_k,b_k) = (\iota_G^-(t),\iota_G(t))$.

Proof. For $u = (u_1,\dots,u_d) \in [0,1]^d$, let $v := \min\{u_i\}$. Then, we have that

$$\ast_{\mathbf{B},G} M^2(u) = \int_0^1 B_t^G\left((\partial_2 \mathbb{1}_{\{t\le u_i\}})_{1\le i\le d}\right)\mathrm{d}t$$
$$= \begin{cases} v & \text{if } \iota_G^-(v) = \iota_G(v) \\ \iota_G^-(v) + (\iota_G(v)-\iota_G^-(v))\,B_v^G\left(\left(\frac{\min\{u_i,\iota_G(v)\}-\iota_G^-(v)}{\iota_G(v)-\iota_G^-(v)}\right)_{1\le i\le d}\right) & \text{if } \iota_G^-(v)\neq\iota_G(v), \end{cases}$$

which implies the assertion. Note that B_t^G is constant for $t \in (\iota_G^-(t), \iota_G(t))$. □

Denote by $\overline{A}$ the closure of a set $A \subset \mathbb{R}$. The following result justifies the simplified notation for the $*$-products where the argument G is omitted in the case that G is continuous, see Definition 2.4(iii). The proof is given in the Appendix.

Proposition 2.14.
Let $d \geq 2$. Then, the following statements are equivalent:

(i) $\underline{\ast_{\mathbf{B},G_1}} D^i = \underline{\ast_{\mathbf{B},G_2}} D^i$ *for all measurable families* $\mathbf{B} = (B_t)_{0 \leq t \leq 1}$ *of d-copulas and for all* $D^i \in \mathcal{C}_2$, $1 \leq i \leq d$,
(ii) $\overline{\mathrm{Ran}(G_1)} = \overline{\mathrm{Ran}(G_2)}$.
(iii) $\iota_{G_1}(t) = \iota_{G_2}(t)$ *for Lebesgue-almost all* $t \in [0, 1]$.

As a consequence of the above result, the $*$-product depends only on the closure of the range of the risk factor distribution G. Thus, the copula of a completely specified factor model is invariant under strictly increasing transformations of the factor variable.

The following result shows in which relevant cases the $*$-product attains the upper Fréchet copula.

Proposition 2.15 (Maximality)**.**
For the $*$-product, the following statements hold true.

(i) *If* $D^i = M^2$ *for all* i, *then* $\ast_{\mathbf{B}} D^i = M^d$.
(ii) *If* $D^i = W^2$ *for all* i, *then* $\ast_{\mathbf{B}} D^i = M^d$.
(iii) $\bigvee_G D^i = M^d$ *if and only if* $D^j = D^k$ *on* $[0, 1] \times \mathrm{Ran}(G)$ *for all* $j \neq k$.

Proof. Statements (i) and (ii) follow from Proposition 2.11(i) and (ii).
Statement (iii) is an extension of [2, Proposition 2.4(v)] to arbitrary $G \in \mathcal{F}^1$. We give the proof in the Appendix. □

The definition of the $*$-product also yields an invariance property under Lebesgue-measure preserving transformations.

Let λ be the Lebesgue measure on $\mathcal{B}([0, 1])$. Denote by $\mathcal{T}$ the set of measurable transformations $T\colon ([0, 1], \mathcal{B}([0, 1]), \lambda) \to ([0, 1], \mathcal{B}([0, 1]), \lambda)$ that are *measure preserving*, i.e. $\lambda^T = \lambda$, where $\lambda^T(A) := \lambda(T^{-1}(A))$ for all $A \in \mathcal{B}([0, 1])$ denotes the distribution of T w.r.t. λ. Let $\mathcal{T}_P$ be the set of all $T \in \mathcal{T}$ such that T is bijective. Then, elements of $\mathcal{T}_P$ are called *shuffles*, see [9].

The following statement shows that simplified $*$-products are invariant under joint shuffles of the factor variable Z assuming a continuous distribution function.

Proposition 2.16 (Invariance under shuffles)**.** *For all $T \in \mathcal{T}_P$ and $C \in \mathcal{C}_2$, the function $\mathcal{S}_T(C)\colon [0, 1]^2 \to [0, 1]$ given through*

$$\mathcal{S}_T(C)(u, v) := \int_0^v \partial_2 C(u, T(t))\, \mathrm{d}t$$

is a bivariate copula. Furthermore, for simplified $*$-products with continuous factor distribution function and $B \in \mathcal{C}_d$ holds

$$\ast_B C^i = \ast_B \mathcal{S}_T(C^i).$$

The proof is given in the Appendix.

2.4 Continuity results for $\ast$-products

In this section, we derive continuity properties of the $\ast$-product w.r.t. to all its specifications.

For the approximation of $\ast$-products w.r.t. the factor distribution, we need the following lemma. The proof is given in the Appendix.

Lemma 2.17. *For $G_n, G \in \mathcal{F}^1$, $n \in \mathbb{N}$, the following statements hold true.*

(i) ι_G determines ι_G^- uniquely by $\iota_G^-(t) = \inf\{s \mid \iota_G(s) \geq t\}$.
(ii) If $\iota_{G_n} \to \iota_G$, then $\iota_{G_n}^- \to \iota_G^-$, where each convergence is almost surely pointwise.

In the following example, we consider some typical approximations of distribution functions for which the corresponding transformations ι converge almost surely pointwise.

Example 2.18. *(a) Denote by $\mathcal{F}_0^1$ the set of distribution functions with compact support. For $G \in \mathcal{F}_0^1$, consider the discretization $(G_n)_n$ given by $G_n(x) := \frac{\lceil nG(x)\rceil}{n}$. Then, G_n is a distribution function for all n with $\mathrm{Ran}(G_n) \subseteq \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$. For $t \in (0,1)$ such that G^{-1} is continuous at t, it can be verified that $\iota_{G_n}(t) \to \iota_G(t)$. Thus, ι_{G_n} converges to ι_G almost surely pointwise.*
(b) For $G \in \mathcal{F}^1$, consider the discretization $(G_n)_n \in \mathcal{F}^1$ given by

$$G_n(x) := \begin{cases} \sup\{\frac{k}{n} \mid G(x) \geq \frac{k}{n}, k \in \mathbb{N}_0\}, & \text{if } G(x) < \frac{1}{2}, \\ \frac{1}{2} & \text{if } G(x) = \frac{1}{2}, \\ \inf\{\frac{k}{n} \mid G(x) \leq \frac{k}{n}, k \in \mathbb{N}_0\}, & \text{if } G(x) > \frac{1}{2}. \end{cases}$$

Similarly to the above example, it holds that $\mathrm{Ran}(G_n) \subseteq \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$ and $\iota_{G_n} \to \iota_G$ almost surely pointwise.

The following two counterexamples show that, in general, neither convergence in distribution (denoted by $\xrightarrow{\mathcal{D}}$) implies almost surely pointwise convergence of the corresponding transformations ι nor that the converse holds true.

Example 2.19 ($G_n \xrightarrow{\mathcal{D}} G \not\Rightarrow \iota_{G_n} \to \iota_G$)**.**
Let $G_n = F_{N(0,1/n)}$ be the distribution function of the normal distribution with mean 0 and variance $\frac{1}{n}$. Then, $G_n \xrightarrow{\mathcal{D}} G = \mathbb{1}_{[0,\infty)}$. But $\iota_{G_n} \not\to \iota_G$ almost surely pointwise because for all $t \in (0,1)$ it holds that $\iota_{G_n}(t) = t \neq \mathbb{1}_{(0,1)}(t) = \iota_G(t)$ for all $n \in \mathbb{N}$.

Example 2.20 ($\iota_{G_n} \to \iota_G \not\Rightarrow G_n \xrightarrow{\mathcal{D}} G$)**.**
Let $G, G' \in \mathcal{F}_c^1$ be continuous distribution functions with $G \neq G'$. Let $(G_n)_n$ be the approximation of G given by Example 2.18(b). Then, $\iota_{G_n} \to \iota_{G'}$ almost surely pointwise because $\iota_{G_n}(t) \to \iota_G(t) = t = \iota_{G'}(t)$ for all $t \in (0,1)$. But $G_n \not\xrightarrow{\mathcal{D}} G'$ because $G_n \xrightarrow{\mathcal{D}} G$ and $G' \neq G$.

For a continuity result of the $\ast$-product $\ast_{\mathbf{B},G} D^i$ w.r.t. the bivariate dependence specifications D^i, we consider as slightly generalized version of the ∂-convergence for bivariate copulas in Mikusiński and Taylor [18].

Definition 2.21 (∂_2-convergence)**.**
Let $D_n, D \in \mathcal{C}_2$ be bivariate copulas for all $n \in \mathbb{N}$. Then, the ∂_2-convergence $D_n \xrightarrow{\partial_2} D$ is defined by

$$\int_0^1 |\partial_2 D_n(x,t) - \partial_2 D(x,t)| \,\mathrm{d}t \xrightarrow{n\to\infty} 0 \quad \text{for all } x \in [0,1].$$

Remark 2.22. *a) Some typical approximations of copulas are the checkerboard, the checkmin and the Bernstein approximation, respectively. All these approximations are w.r.t. the ∂-convergence, see Mikusiński and Taylor [18], and, thus, also w.r.t. the ∂_2-convergence. In contrast, the ∂_2-convergence does not generally hold for the shuffle-of-min approximation, see Mikusiński and Taylor [18, Example 4].*

b) For a bivariate copula D, denote by D^T with $D^T(u,v) := D(v,u)$, $(u,v) \in [0,1]^2$, the transposed copula of D, and by K_D the associated Markov kernel defined by $K_D(t,[0,v]) := \lim_{u \searrow v} \partial_1 D(t,u)$ for all $u \in [0,1]$ and for Lebesgue-almost all $t \in [0,1]$. Then, the ∂_2-convergence is metrizable with a metric r given by $r(A,B) = \mathrm{D}_1(A^T, B^T)$, where D_1 denotes the metric defined by

$$\mathrm{D}_1(A,B) := \int_0^1 \int_0^1 \left| K_A(t,[0,v]) - K_B(t,[0,v]) \right| \, \mathrm{d}t \, \mathrm{d}v \tag{14}$$

for $A, B \in \mathcal{C}_2$, see [30]. Note that for all $v \in [0,1]$, there exists a Lebesgue-null set N_v such that $\partial_1 A(t,v) = K_A(t,[0,v])$ and $\partial_1 B(t,v) = K_B(t,[0,v])$ for all $t \in N_v^c$, see, e.g., Theorem 2.2(i).

As a main result, we give sufficient conditions for the continuity of the $*$-product w.r.t. all its arguments.

Theorem 2.23 (Continuity of $*$-products)**.**
Let $D_n^i, D^i \in \mathcal{C}_2$ be bivariate copulas, $1 \le i \le d$, $\mathbf{B^n} = (B_t^n)_{t \in [0,1]}, \mathbf{B} = (B_t)_{t\in[0,1]}$ be measurable families of d-copulas, and $G_n, G \in \mathcal{F}^1$ be distribution functions for all $n \in \mathbb{N}$. If

(i) $D_n^i \xrightarrow{\partial_2} D^i$ for all $1 \le i \le d$,
(ii) $B_t^n \xrightarrow{\mathcal{D}} B_t$ for Lebesgue-almost all $t \in [0,1]$, and
(iii) $\iota_{G_n}(t) \to \iota_G(t)$ for Lebesgue-almost all $t \in [0,1]$,

then it holds true that

$$*_{\mathbf{B^n}, G_n} D_n^i \;\to\; *_{\mathbf{B}, G} D^i \quad \textit{uniformly}\,.$$

Proof. We show for $u = (u_1, \dots, u_d) \in [0,1]^d$ that

$$*_{\mathbf{B^m}, G_k} D_n^i\,(u) \xrightarrow{n \to \infty} *_{\mathbf{B^m}, G_k} D^i\,(u) \quad \text{for all } k, m \in \mathbb{N}\,, \tag{15}$$

$$*_{\mathbf{B^m}, G_k} D_n^i\,(u) \xrightarrow{m \to \infty} *_{\mathbf{B}, G_k} D_n^i\,(u) \quad \text{for all } k, n \in \mathbb{N}\,, \tag{16}$$

$$*_{\mathbf{B^m}, G_k} D_n^i\,(u) \xrightarrow{k \to \infty} *_{\mathbf{B^m}, G} D_n^i\,(u) \quad \text{for all } n, m \in \mathbb{N}\,. \tag{17}$$

Due to the equicontinuity of copulas, the above $*$-products converge uniformly using Arzelà-Ascoli's theorem. Thus, the statement follows from the exchangeability of applying the limits and, again, from Arzelà-Ascoli's theorem.

First, we show (17). Assume w.l.g. that $D_n^i = D^i$ and $\mathbf{B^m} = \mathbf{B}$ for all $n, m \in \mathbb{N}$. From Lemma 2.17(ii) we obtain that $\iota_{G_k} \to \iota_G$ a.s. implies that $\iota_{G_k}^-(t) \to \iota_G^-(t)$ for all $t \in N_0^c \cap [0,1]$ outside a Lebesgue-null set N_0. Fix $t \in N_0^c \cap [0,1]$.

If $\iota_G^-(t) < \iota_G(t)$, then there exists $R \in \mathbb{N}$ such that for all $k \ge R$ it holds that $\iota_{G_k}^-(t) < \iota_{G_k}(t)$ and, thus,

$$B_t^{G_k}(u) = \frac{1}{\iota_{G_k}(t) - \iota_{G_k}^-(t)} \int_{\iota_{G_k}^-(t)}^{\iota_{G_k}(t)} B_s(u)\,\mathrm{d}s \xrightarrow{k \to \infty} \frac{1}{\iota_G(t) - \iota_G^-(t)} \int_{\iota_G^-(t)}^{\iota_{G}(t)} B_s(u)\,\mathrm{d}s = B_t^G(u)$$

and

$$\partial_2^{G_k} D^i(u_i, t) = \frac{D^i(u_i, \iota_{G_k}(t)) - D^i(u_i, \iota_{G_k}^-(t))}{\iota_{G_k}(t) - \iota_{G_k}^-(t)} \xrightarrow{k \to \infty} \frac{D^i(u_i, \iota_G(t)) - D^i(u_i, \iota_G^-(t))}{\iota_G(t) - \iota_G^-(t)} = \partial_2^G D^i(u_i, t)$$

for $i = 1, \dots, d$.

If $\iota_G^-(t) = \iota_G(t)$ and $\iota_{G_k}^-(t) = \iota_{G_k}(t)$ for all k, then it follows that

$$B_t^{G_k}(u) = B_t(u) = B_t^G(u) \quad \text{and} \quad \partial_2^{G_k} D^i(u_i, t) = \partial_2 D^i(u_i, t) = \partial_2^G D^i(u_i, t) \quad \text{for all } k\,.$$

If $\iota_G^-(t) = \iota_G(t)$ and $\iota_{G_{k_l}}^-(t) < \iota_{G_{k_l}}(t)$ for a subsequence $(k_l)_{l\in\mathbb{N}}$, then it follows from Lebesgue's differential theorem, see, e.g., [4, Theorem 8.4.6], that

$$B_t^{G_{k_l}} = \frac{1}{\iota_{G_{k_l}}(t) - \iota_{G_{k_l}}^-(t)} \int\limits_{\iota_{G_{k_l}}^-(t)}^{\iota_{G_{k_l}}(t)} B_s(u)\,\mathrm{d}s \xrightarrow{l\to\infty} B_t(u) = B_t^G(u)$$

and, since the partial derivative of a copula exists almost surely, that

$$\partial_2^{G_{k_l}} D^i(u_i, t) = \frac{D^i(u_i, \iota_{G_{k_l}}(t)) - D^i(u_i, \iota_{G_{k_l}}^-(t))}{\iota_{G_{k_l}}(t) - \iota_{G_{k_l}}^-(t)} \xrightarrow{l\to\infty} \partial_2 D^i(u_i, t) = \partial_2^G D^i(u_i, t)$$

if $t \in N_1^c \cap [0,1]$ is outside of a Lebesgue-null set $N_1 \supseteq N_0$.

Altogether, this yields

$$B_s^{G_k}\left(\left(\partial_2^{G_k} D^i(u_i, s)\right)_{1\le i\le d}\right) \xrightarrow{k\to\infty} B_s^G\left(\left(\partial_2^G D^i(u_i, s)\right)_{1\le i\le d}\right)$$

for all $s \in N_1^c \cap [0,1]$ using that $B_s^{G_k} \in \mathcal{C}_d$ is equicontinuous for all s. This implies

$$*_{\mathbf{B},G_k} D^i(u) = \int\limits_0^1 B_s^{G_k}\left(\left(\partial_2^{G_k} D^i(u_i, s)\right)_{1\le i\le d}\right) \mathrm{d}s \longrightarrow \int\limits_0^1 B_s^G\left(\left(\partial_2^G D^i(u_i, s)\right)_{1\le i\le d}\right) \mathrm{d}s = *_{\mathbf{B},G} D^i(u)$$

as $k \to \infty$, where we apply the dominated convergence theorem.

To show (15), let $j \in \{1, \dots, d\}$ and choose w.l.g. $G_k = G$, $\mathbf{B^m} = \mathbf{B}$, and $D_n^i = D_n$ for all $k, m, n \in \mathbb{N}$ and $i \neq j$. Let $(G^l)_{l\in\mathbb{N}}$ be the discrete approximation of G given in Example 2.18(b). Then, the family $(B_t^{G^l})_t$ is constant in t on the intervals $(\frac{\kappa-1}{l}, \frac{\kappa}{l})$, $1 \le \kappa \le l$, and each $B_t^{G^l}$ is Lipschitz continuous with Lipschitz constant 1.

Thus, for the Lebesgue measure λ on $[0,1]$, it holds that

$$\begin{aligned} &\lambda\left(\{t\colon |B_t^{G^l}((\partial_2^G D_n^i(u_i, t))_{1\le i\le d}) - B_t^{G^l}((\partial_2^{G^l} D_n^i(u_i, t))_{1\le i\le d})| > \varepsilon\} \cap (\tfrac{\kappa-1}{l}, \tfrac{\kappa}{l})\right) \\ &\le \lambda\left(\{t\colon |\partial_2^{G^l} D_n^j(u_j, t) - \partial_2^{G^l} D_n^j(u_j, t)| > \varepsilon\} \cap (\tfrac{\kappa-1}{l}, \tfrac{\kappa}{l})\right) \xrightarrow{n\to\infty} 0 \end{aligned}$$

for all $\varepsilon > 0$ and $1 \le \kappa \le l$, where the convergence follows from the assumption that $D_n^j \xrightarrow{\partial_2} D^j$. Then

$$\begin{aligned} &\int\limits_0^1 |B_t^{G^l}((\partial_2^{G^l} D_n^i(u_i, t))_{1\le i\le d}) - B_t^{G^l}(\partial_2^{G^l}(D^i(u_i, t))_{1\le i\le d})|\,\mathrm{d}t \\ &= \sum_{\kappa=1}^{l} \int\limits_{\frac{\kappa-1}{l}}^{\frac{\kappa}{l}} |B_t^{G^l}((\partial_2^{G^l} D_n^i(u_i, t))_i) - B_t^{G^l}((\partial_2^{G^l} D^i(u_i, t))_{1\le i\le d})|\,\mathrm{d}t \xrightarrow{n\to\infty} 0\,, \end{aligned}$$

which implies that $*_{\mathbf{B},G^l} D_n^i(u) \to *_{\mathbf{B},G^l} D^i(u)$ as $n \to \infty$ for all l. Thus, the statement follows from $*_{\mathbf{B},G^l} D_n^i \xrightarrow{l\to\infty} *_{\mathbf{B},G} D_n^i$ uniformly, see (17).

Statement (16) follows with the dominated convergence theorem. □

In the following remark, we note that a weak approximation of the bivariate dependence specifications or a weak approximation of the factor distribution does not guarantee the convergence of the corresponding $*$-products.

Remark 2.24. *(a) In general, the $*$-product $\ast_{\mathbf{B},G} D^i$ is not continuous in D^i w.r.t. weak convergence. A counterexample is given for the upper product and $G \in \mathcal{F}_c^1$ in [2, Example 2.7].*

(b) In general, the $*$-product is not continuous in the factor distribution w.r.t. weak convergence, i.e. $G_n \xrightarrow{\mathcal{D}} G \not\Rightarrow \ast_{\mathbf{B},G_n} D^i \xrightarrow{\mathcal{D}} \ast_{\mathbf{B},G} D^i$.

For a counterexample, let $(G_n)_n$ be the approximation of G given by Example 2.19. Then, $G_n \xrightarrow{\mathcal{D}} G = \mathbb{1}_{[0,\infty)}$. If the D^i do not coincide for all i, then the $*$-products do not necessarily converge because, e.g., for the upper products, it holds that

$$\bigvee_{G_n} D^i = \bigvee D^i \neq M^d = \bigvee_{G} D^i \,,$$

where the first equality holds due to the continuity of G_n for all n, and the inequality is true because of the maximality property of the upper product, see Proposition 2.15(iii)*. The last equality follows from*

$$\min\{u_1, \ldots, u_d\} = \int_0^1 \min\left\{\partial_2^G D^1(u_1, t), \ldots, \partial_2^G D^d(u_d, t)\right\} \mathrm{d}t$$

since

$$\partial_2^G D^i(u_i, t) = \frac{D^i\left(u_i, \iota_G(t)\right) - D^i\left(u_i, \iota_G^-(t)\right)}{\iota_G(t) - \iota_G^-(t)} = u_i$$

for all $u_i \in [0, 1]$ and $1 \le i \le d$ because $\iota_G(t) = 1$ and $\iota_G^-(t) = 0$ for all $t \in (0, 1)$.

2.5 The lower product of bivariate copulas

In the following proposition, we provide basic properties for the lower product of bivariate copulas which are parallel to some results in [2] for the upper product.

For a bivariate copula $D \in \mathcal{C}_2$, define the reflected copulas D_* and D^* by

$$D_*(u, v) := v - D(1 - u, v)\,, \quad \text{and} \quad D^*(u, v) := u - D(u, 1 - v)\,, \tag{18}$$

respectively, for $(u, v) \in [0, 1]^2$. Remember that the transposed copula D^T is defined by $D^T(u, v) := D(v, u)$, $(u, v) \in [0, 1]^2$.

Proposition 2.25. *For $D, E \in \mathcal{C}_2$ and for a random vector (U_1, U_2, U_3) the following statements hold true:*

(i) Minimality property: $D \wedge_G E = W^2$ if and only if $D = E_*$ on $[0, 1] \times \mathrm{Ran}(G)$.
(ii) $M^2 \wedge_G D \wedge_G E$ is a 3-copula if and only if $G \in \mathcal{F}_c^1$.
(iii) $(U_1, U_2, U_3) \sim M^2 \wedge D \wedge E \;\Leftrightarrow\; C_{U_1,U_2} = D^*$, $C_{U_1,U_3} = E^*$ and $(U_2, U_3)|U_1 = t$ is countermonotonic for almost all t.
(iv) $D \wedge_G M^2 = D$ on $[0, 1] \times \mathrm{Ran}(G)$ and $M^2 \wedge_G D = D^T$ on $\mathrm{Ran}(G) \times [0, 1]$.
(v) $D \wedge_G W^2 = D^*$ on $[0, 1] \times \mathrm{Ran}(G)$ and $W^2 \wedge_G D = (D^*)^T$ on $\mathrm{Ran}(G) \times [0, 1]$.
(vi) In general, the lower product is neither commutative nor associative.

The proof is given in the Appendix.

3 Ordering results for $*$-products

In this section, we establish lower and upper orthant ordering results for the $*$-product $\ast_{\mathbf{B},G} D^i$ w.r.t. the conditional copulas $\mathbf{B}$ and the bivariate specifications D^i. By the Sklar-representation theorem (Theorem

2.7) these results imply corresponding dependence ordering results for CSFM w.r.t. their specifications.

Definition 3.1 (Stochastic orderings).
Let ξ, ξ' be d-dimensional random vectors. Then, define the

(i) lower orthant ordering $\xi \le_{lo} \xi'$ *if for the corresponding distributions holds that* $F_\xi(x) \le F_{\xi'}(x)$ *for all* $x \in \mathbb{R}^d$,
(ii) upper orthant ordering $\xi \le_{uo} \xi'$ *if for the corresponding survival functions holds that* $\overline{F}_\xi(x) \le \overline{F}_{\xi'}(x)$ *for all* $x \in \mathbb{R}^d$,
(iii) concordance ordering $\xi \le_c \xi'$ *if it holds that* $\xi \le_{lo} \xi'$ *and* $\xi \le_{uo} \xi'$,

Note that all these orderings depend only on the distributions and, thus, are also defined for the corresponding distribution functions. A comparison w.r.t. the concordance ordering requires that the corresponding univariate marginal distributions are equal, i.e., $(\xi_1, \dots, \xi_d) \le_c (\xi_1, \dots, \xi_d)$ implies $\xi_i \overset{\mathrm{d}}{=} \xi_i'$ for all i. Further, if $d = 2$ and $\xi_i \overset{\mathrm{d}}{=} \xi_i'$, the orderings $\le_{lo}$, $\le_{uo}$, and $\le_c$ are equivalent and we denote them as the *standard bivariate dependence orderings*.

For an overview of stochastic orderings, see Müller and Stoyan [21], Shaked and Shanthikumar [28] and Rüschendorf [26].

In comparison to the ordering of $\ast_{\mathbf{B},G} D^i$ w.r.t. the specifications D^i, an ordering w.r.t. the copula family $\mathbf{B}$ is a simple task and given by the following proposition which extends Durante et al. [8, Proposition 3].

Proposition 3.2 (Ordering w.r.t. conditional copulas).
Let $\mathbf{B} = (B_t)_{0\le t\le 1}$, $\mathbf{C} = (C_t)_{0\le t\le 1}$ *be measurable families of d-copulas. If* $B_t \prec C_t$ *for almost all* t, *where* $\prec$ *is one of the orders* $\le_{lo}$, $\le_{uo}$, *and* $\le_c$, *respectively, then it holds true that*

$$\ast_{\mathbf{B},G} D^i \prec \ast_{\mathbf{C},G} D^i$$

for all $G \in \mathcal{F}^1$ *and for all copulas* $D^i \in \mathcal{C}_2$, $1 \le i \le d$.

Proof. The statement follows from the closure of these orders under mixtures (see Shaked and Shanthikumar [28, Theorems 6.G.3.(d)]). □

In the sequel, we are interested in ordering conditions for $\ast_{\mathbf{B},G} D^i$ w.r.t. the specifications D^i.

Intuitively, if the D^i increase in the standard bivariate dependence orderings, then the product $\ast_{\mathbf{B},G} D^i$ should increase due to the following reason: If all the D^i get closer to the upper Fréchet bound M^2, then each $X_i = f_i(Z, \varepsilon_i)$ depends more strongly positively on Z. Thus, the copula $C_{X_1,\dots,X_d}$ of $(X_1, \dots, X_d)$ should be closer to the upper Fréchet bound M^d. But it turns out that ordering criteria on D^i are more complicated. One can also couple each X_i more strongly negatively with Z which also leads to a stronger positive dependence among the X_i. Further, as we see in Theorem 3.7, general ordering conditions for $\ast_{\mathbf{B},G} D^i$ in D^i for fixed D^j, $j \ne i$, restrict the choice of the conditional copula family $\mathbf{B}$.

Another difficulty is that, for fixed $i \in \{1, \dots, d\}$, ordering results for $\ast_{j=1,\mathbf{B},G}^d D^j$ w.r.t. D^i always involve integral inequalities because

$$\ast_{j=1,\mathbf{B},G}^d D^j(u) = \int_0^1 B_t^G\left((\partial_2^G D^j(u_j, t))_{1\le j\le d}\right) \mathrm{d}t$$

depends on D^i through the (generalized) partial derivative $\partial_2^G D^i$ of D^i. More precisely, a pointwise ordering of the integrands w.r.t. D^i and E^i, i.e., $B_t^G\left((\partial_2^G D^j(u_j, t))_{1\le j\le d}\right) \le B_t^G\left((\partial_2^G E^j(u_j, t))_{1\le j\le d}\right)$ for all $(u_1, \dots, u_d) \in$

$[0,1]^d$ and $t \in (0,1)$, is not possible: If we set $u_j = 1$ for all $j \neq i$, then

$$\partial_2^G D^i(u_i,t) = B_t^G\left((\partial_2^G D^j(u_j,t))_{1\le j\le d}\right) \le B_t^G\left((\partial_2^G E^j(u_j,t))_{1\le j\le d}\right) = \partial_2^G E^i(u_i,t))$$

for all t implies $D^i = E^i$ on $[0,1] \times \mathrm{Ran}(G)$ and, thus, $\ast_{j=1,\mathbf{B},G}^d D^j = \ast_{j=1,\mathbf{B},G}^d E^j$.

In the remaining part of this section, we derive several lower and upper orthant ordering results for $\ast_{j=1,\mathbf{B},G}^d D^j$ w.r.t. the D^i verifying integral inequalities based on the Schur-ordering, the sign-change ordering, and the lower orthant ordering, respectively.

3.1 Ordering results for componentwise convex conditional copulas

Denote by $\prec_S$ the Schur-ordering for functions, i.e., for integrable functions $f, g\colon [0,1] \to \mathbb{R}$, the relation $f \prec_S g$ is defined by $\int_0^x f^*(t)\,\mathrm{d}t \le \int_0^x g^*(t)\,\mathrm{d}t$ for all $x \in (0,1)$ and $\int_0^1 f(t)\,\mathrm{d}t = \int_0^1 g(t)\,\mathrm{d}t$. Here h^* denotes the decreasing rearrangement of an integrable function h, i.e., the (essentially w.r.t. the Lebesgue measure λ) uniquely determined decreasing function h^* such that $\lambda(h^* \le t) = \lambda(h \le t)$ for all $t \in \mathbb{R}$.

We say that a family $(\Phi_t)_{t\in[0,1]}$ of functions $\Phi_t\colon \Theta \to \mathbb{R}$ is continuous, $\Theta = \mathbb{R}^d$ or $\Theta = [0,1]^d$, if the mapping $(t,x) \mapsto \Phi_t(x)$ is continuous for all $(t,x) \in [0,1] \times \Theta$. As a basic integral inequality result, we make use of the following theorem on rearrangements from Fan and Lorentz [11, Theorem 1].

Theorem 3.3 (Ky Fan–Lorentz Theorem).
Let $\Phi_t\colon \mathbb{R}^d \to \mathbb{R}$, $t \in [0,1]$, be a family of continuous functions. Then, the following statements are equivalent:

(i) For all bounded and decreasing functions f_i, g_i on $[0,1]$ with $f_i \prec_S g_i$, it holds that

$$\int_0^1 \Phi_t(f_1,\ldots,f_d)\,\mathrm{d}t \le \int_0^1 \Phi_t(g_1,\ldots,g_d)\,\mathrm{d}t\,. \tag{19}$$

(ii) Φ with $\Phi(t,\cdot) := \Phi_t(\cdot)$ satisfies the following conditions for all $0 \le t \le 1$, $0 \le a \le 1-2\delta$, $\delta > 0$, $u_k \ge 0$, $k = 1,\ldots,d$, $h \ge 0$ and $i \neq j$ where those arguments are omitted which are the same in each expression:

$$\Phi(u_i+h, u_j+h) - \Phi(u_i+h, u_j) - \Phi(u_i, u_j+h) + \Phi(u_i,u_j) \ge 0\,, \tag{20}$$

$$\Phi(u_i+h) - 2\Phi(u_i) + \Phi(u_i-h) \ge 0\,, \tag{21}$$

$$\int_0^\delta \left(\Phi_{a+\delta+s}(u_i) - \Phi_{a+\delta+s}(u_i+h) + \Phi_{a+s}(u_i+h) - \Phi_{a+s}(u_i)\right)\,\mathrm{d}s \ge 0\,. \tag{22}$$

For a function $f\colon \mathbb{R}^d \to \mathbb{R}$, let $\triangle_i^\varepsilon f(x) := f(x+\varepsilon e_i) - f(x)$ be the difference operator where $\varepsilon > 0$ and where e_i denotes the i-th unit vector w.r.t. the canonical base in $\mathbb{R}^d$. Then, f is said to be *supermodular*, respectively, *directionally convex* if $\triangle_i^{\varepsilon_i}\triangle_j^{\varepsilon_j} f(x) \ge 0$ for all $x \in \mathbb{R}^d$, for all $\varepsilon_i, \varepsilon_j > 0$, and for all $1 \le i < j \le d$, respectively, $1 \le i \le j \le d$. Note that in the literature, directionally convex functions are also called ultramodular or Wright convex.

Here, Condition (20) is supermodularity of Φ_t for all t, condition (21) is convexity of Φ_t in each component for all t. Functions that fulfill both conditions (20) and (21) are directionally convex. Motivated by Theorem 3.3, we consider the class $\mathcal{C}_d^{ccx}$ of componentwise convex d-copulas which is identical to the class of directionally convex copulas since copulas are supermodular.

Remark 3.4. *(a) As a consequence of the transformation formula, Theorem 3.3 also holds true if "decreasing" in* (i) *is substituted by "increasing" and the inequality in* (22) *is reversed, i.e.,*

$$\int_0^\delta \left(\Phi_{a+\delta+s}(u_i) - \Phi_{a+\delta+s}(u_i + h) + \Phi_{a+s}(u_i + h) - \Phi_{a+s}(u_i)\right) \mathrm{d}s \le 0\,. \tag{23}$$

for all $0 \le a \le 1 - 2\delta$, $\delta > 0$, $u_k \ge 0$, $k = 1, \dots, d$, $h \ge 0$.

(b) If Φ has continuous second partial derivatives w.r.t. all variables, then conditions (20), (21), (22), *and* (23), *respectively, are equivalent to*

$$\frac{\partial^2 \Phi}{\partial u_i \partial u_j} \ge 0 \quad \forall i \ne j\,, \qquad \frac{\partial^2 \Phi}{\partial u_i^2} \ge 0 \quad \forall i\,, \qquad \frac{\partial^2 \Phi}{\partial t \partial u_i} \le 0 \quad \forall i\,, \qquad \text{and} \quad \frac{\partial^2 \Phi}{\partial t \partial u_i} \ge 0 \quad \forall i\,,$$

respectively, see Lorentz [16].

In order to apply the Ky Fan–Lorentz Theorem to $*$-products, we consider an important class of bivariate copulas which are convex or concave in the second variable.

Definition 3.5 (CI/CIS/CDS copula). *A bivariate copula $D \in \mathcal{C}_2$ is said to be* conditionally increasing in sequence *(CIS), respectively,* conditionally decreasing in sequence *(CDS) if $\partial_2 D(u, t)$ is decreasing, respectively, increasing in t for all $u \in [0, 1]$.*
Further, D is conditionally increasing *(CI) if D and D^T are CIS, i.e., D is concave in both components.*

In the literature, the CIS property is often defined by the partial derivative w.r.t. the first component. However, we define it in this way because the $*$-product depends on the derivatives of the bivariate copulas w.r.t. the second component.

Note that for a random vector $(U_1, U_2) \sim D$ with $D \in \mathcal{C}_2$ CIS, the conditional distribution $U_1 \mid U_2 = v$ is stochastically increasing in v. This explains the denomination of conditional increasingness.

For the next theorem, we need the following lemma. The proof is given in the Appendix.

Lemma 3.6. *For $G \in \mathcal{F}^1$, conditions* (21), (22), *and* (23), *respectively, transfer from measurable* $\mathbf{B} = (B_t)_{t\in[0,1]}$, $B_t \in \mathcal{C}_d$, *to the mixtures* $\mathbf{B}^G = (B_t^G)_{t\in[0,1]}$.

As a consequence of the Ky Fan–Lorentz Theorem 3.3, we obtain that general $\le_{lo}$-ordering results for $*_{\mathbf{B},G} D^i$ w.r.t. D^i require convexity of B_t in each component for all t.

Theorem 3.7 ($\le_{\mathbf{lo}}$-ordering of componentwise convex $*$-products).
Assume that $\mathbf{B} = (B_t)_{t\in[0,1]}$ is a continuous family of d-copulas. Then, the following statements are equivalent:

(i) For all $G \in \mathcal{F}^1$ and for all CIS copulas $D^i, E^i \in \mathcal{C}_2$ with $D^i \le_{lo} E^i$, $1 \le i \le d$, it holds

$$*_{\mathbf{B},G} D^i \le_{lo} *_{\mathbf{B},G} E^i\,.$$

(ii) **B** *fulfills conditions* (21) *and* (22).

Proof. Assume (ii). Let $G \in \mathcal{F}^1$ and $D^i, E^i \in \mathcal{C}_2$ be CIS. For $(u_1, \dots, u_d) \in [0, 1]^d$, define $f_i(t) := \partial_2^G D^i(u_i, t)$ and $g_i(t) := \partial_2^G E^i(u_i, t)$, for almost all $t \in (0, 1)$. For $v \in (0, 1]$, we obtain from $D^i \le_{lo} E^i$ that

$$\begin{aligned}\int_0^v f_i(t)\,\mathrm{d}t &= D^i(u_i, \iota_G^-(v)) + (v - \iota_G^-(v))\,\partial_2^G D^i(u_i, v) \\ &\le E^i(u_i, \iota_G^-(v)) + (v - \iota_G^-(v))\,\partial_2^G E^i(u_i, v) = \int_0^v g_i(t)\,\mathrm{d}t\end{aligned}$$

with equality if $v = 1$. Since D^i and E^i are CIS, the functions f_i and g_i are decreasing; this yields $f_i \prec_S g_i$. Together with the boundedness of f_i and g_i it follows from the Ky Fan–Lorentz Theorem 3.3 that

$$\ast_{\mathbf{B},G} D^i(u) = \int_0^1 B_t^G\left(f_1(t),\dots,f_d(t)\right)\,\mathrm{d}t \le \int_0^1 B_t^G\left(g_1(t),\dots,g_d(t)\right)\,\mathrm{d}t = \ast_{\mathbf{B},G} E^i(u)\,,$$

because $(B_t^G)_t$ fulfills conditions (21) and (22), see Lemma 3.6. This proves (i).

The reverse direction follows in the same way as in the proof of the Ky Fan–Lorentz Theorem 3.3 (see Fan and Lorentz [11, Theorem 1]) because for all decreasing functions $f_i, g_i\colon [0,1] \to [0,1]$ with $f_i \prec_S g_i$, there exist $(u_1,\dots,u_d) \in [0,1]^d$ and copulas $D^i, E^i \in \mathcal{C}_2$ with $D^i \le_{lo} E^i$ such that $f_i(t) = \partial_2 D^i(u_i,t)$ and $g_i(t) = \partial_2 E^i(u_i,t)$ holds. □

A similar result holds true w.r.t. the upper orthant ordering as follows.

Theorem 3.8 ($\le_{\mathbf{uo}}$-ordering of componentwise convex $\ast$-products)**.**
Assume that $\mathbf{B} = (B_t)_{t\in[0,1]}$ is a continuous family of d-copulas. Then, the following statements are equivalent:

(i) For all $G \in \mathcal{F}^1$ and for all CIS copulas $D^i, E^i \in \mathcal{C}_2$ with $D^i \le_{lo} E^i$, $1 \le i \le d$, holds

$$\ast_{\mathbf{B},G} D^i \le_{uo} \ast_{\mathbf{B},G} E^i\,.$$

(ii) The survival copulas $\widehat{\mathbf{B}} = (\widehat{B}_t)_{t\in(0,1)}$ fulfill conditions (21) *and* (23).

Proof. The proof is similar to the proof of Theorem 3.7 applying the Ky Fan–Lorentz Theorem to the survival functions of the $\ast$-products given by (13). Since $1 - \partial_2^G D^i(u_i,t)$ is increasing in t for all u_i and i, the survival copulas $\widehat{\mathbf{B}}$ have to fulfill condition (23), see Remark 3.4(a). □

Remark 3.9. *(a) If $\mathbf{B}$ and the associated survival copulas $\widehat{\mathbf{B}}$ fulfill the convexity condition* (21) *as well as condition* (22) *and* (23)*, respectively, then it holds $\ast_{\mathbf{B},G} D^i \le_c \ast_{\mathbf{B},G} E^i$ for all CIS copulas D^i, E^i with $D^i \le_{lo} E^i$, $1 \le i \le d$. For simplified $\ast$-products, condition* (22) *and* (23) *are trivially fulfilled. If $d = 2$, then B_t is componentwise convex if and only if $\widehat{B}_t$ is componentwise convex.*

(b) The componentwise convexity condition (21) *for each B_t implies negative lower orthant dependence for all the bivariate marginals of B_t. To see this, let $i \ne j$ and $u = (u_1,\dots,u_d)$ with $u_k = 1$ for all $k \ne i,j$. Then, it holds true that*

$$B_t(u) = \int_0^{u_i} \partial_i B_t(u_1,\dots,u_{i-1},s,u_{i+1},\dots,u_d)\,\mathrm{d}s \le \int_0^{u_i} u_j\,\mathrm{d}t = \Pi^d(u)$$

using the uniform marginal condition $\int_0^1 \partial_i B_t(u)\,\mathrm{d}u_i = u_j$ and that $\partial_i B_t(u)$ is increasing in u_i. For a discussion of componentwise convex copulas, see, e.g., Klement et al. [14] and Klement et al. [13].
If B_t is not componentwise convex for some t outside a null set, then general lower orthant ordering results for $\ast_{\mathbf{B}} D^k$ w.r.t. D^i depend on D^j, $j \ne i$.
For example, the conditional copula M^d corresponding to the upper product $\bigvee = \ast_{M^d}$ is component-wise concave (and not convex). Due to the maximality property of the upper product, general ordering conditions for $\bigvee D^k$ w.r.t. D^i depend on D^j, see Proposition 2.15(iii).

(c) The ordering results for comonotonic random vectors in Rüschendorf [24, Corollary 3(b)] and for random vectors with common CI copula in Müller and Scarsini [19, Theorem 4.5], respectively, are based on the application of the Ky Fan–Lorentz Theorem 3.3 to (conditional) quantile functions. In contrast, Theorem 3.7 follows from the Ky Fan–Lorentz Theorem 3.3 comparing conditional distribution functions w.r.t. the conditioning variable.

We make use of another integral inequality due to Lorentz [16] as follows.

Theorem 3.10 (Lorentz). *Let $\Phi\colon [0,1]\times\mathbb{R}^d\to\mathbb{R}$ be continuous. The following statements are equivalent:*

(i) For all positive bounded measurable functions f_k on $[0,1]$, $1\le k\le d$, holds

$$\int_0^1 \Phi(t, f_1(t),\dots,f_d(t))\,\mathrm{d}t \le \int_0^1 \Phi(t, f_1^*(t),\dots,f_d^*(t))\,\mathrm{d}t\,.$$

(ii) Φ satisfies conditions (20) *and* (22).

Note that the above result also holds true if we replace the decreasing rearrangements f_i^* by the increasing rearrangements $f_{i\star}$ of f_i and condition (22) by (23).

As a consequence of the Lorentz Theorem 3.10, we obtain for continuous factor distribution functions $G\in\mathcal{F}_c^1$ the following result concerning shuffles.

Proposition 3.11. *Let $D^1,\dots,D^d\in\mathcal{C}_2$ be CIS copulas.*

(i) If $\mathbf{B}=(B_t)_{t\in[0,1]}$ is a continuous family of d-copulas that fulfills condition (22)*, then it holds true that*

$$\ast_{\mathbf{B}}\mathcal{S}_{T_i}(D^i)\le_{lo}\ast_{\mathbf{B}}D^i$$

for all shuffles $T_i\in\mathcal{T}_P$ of D^i.

(ii) For $B\in\mathcal{C}_d$, the simplified products satisfy

$$\ast_B\mathcal{S}_{T_i}(D^i)\le_{lo}\ast_B\mathcal{S}_T(D^i) \tag{24}$$

for all shuffles $T_i, T\in\mathcal{T}_P$.

Proof. For $u=(u_1,\dots,u_d)\in[0,1]^d$, define $g_{i,u_i}(t):=\partial_2\mathcal{S}_{T_i}(D^i)(u_i,t)$. Since D^i is conditionally increasing, the decreasing rearrangement is given by $g_{i,u_i}^*(t)=\partial_2 D^i(u_i,t)$ for almost all t. Hence, Theorem 3.10 implies

$$\ast_{\mathbf{B}}\mathcal{S}_{T_i}(D^i)(u)=\int_0^1 B_t\left((g_{i,u_i}(t))_{1\le i\le d}\right)\,\mathrm{d}t\le\int_0^1 B_t\left((g_{i,u_i}^*(t))_{1\le i\le d}\right)\,\mathrm{d}t=\ast_{\mathbf{B}}D^i(u)\,.$$

The second statement follows from the first one with Proposition 2.16. □

Remark 3.12. *(a) Note that the specifications on the right side of* (24) *are jointly shuffled.*

(b) A similar result to Proposition 3.11 holds true w.r.t. the upper orthant ordering. A generalization to arbitrary factor distribution functions $G\in\mathcal{F}^1$ is not possible because, in general, $g_{i,u_i}=\partial_2^G\mathcal{S}_{T_i}(D^i)(u_i,\cdot)\not\prec_S\partial_2^G D^i(u_i,\cdot)$ and, thus, $g_{i,u_i}^*\ne\partial_2^G D^i(u_i,\cdot)$, see also Example 3.15.

To apply Lorentz's Theorem 3.10 to the ordering of $\ast_{\mathbf{B},G}D^k$ w.r.t. D^i, we introduce and study the orderings $\le_{\partial_2 S,G}$ and $\le_{\partial_2 S}$ on the set $\mathcal{C}_2$ of bivariate copulas.

Definition 3.13 ($\le_{\partial_2 S}$, Schur order for copula derivatives)**.**
For $G\in\mathcal{F}^1$ and $D,E\in\mathcal{C}_2$, define the Schur order for the partial copula derivative *(w.r.t. the second variable) by*

$$D\le_{\partial_2 S,G}E\quad\text{if } \partial_2^G D(v,\cdot)\prec_S\partial_2^G E(v,\cdot)\quad\text{for all } v\in[0,1]\,.$$

For $G\in\mathcal{F}_c^1$, we abbreviate $\le_{\partial_2 S,G}$ by $\le_{\partial_2 S}$.

The least element in $\mathcal{C}_2$ w.r.t to the $\le_{\partial_2 S}$-order is given by the independence copula Π^2, i.e., it holds that $\Pi^2\le_{\partial_2 S}C$ for all $C\in\mathcal{C}_2$. In contrast, a greatest element does not exist. However, M^2 and W^2 as well as every shuffle of these copulas are maximal elements.

Let $\zeta_1 : \mathcal{C}_2 \to [0,1]$ be the dependence measure defined by

$$\zeta_1(A) := 3\,\mathrm{D}_1(A, \Pi^2)\,,$$

see [31]. By the following result, ζ_1 is increasing w.r.t. the $\leq_{\partial_2 S}$-ordering, compare Figure 4.

Proposition 3.14. *Let D and E be bivariate copulas. Then $D \leq_{\partial_2 S} E$ implies $\zeta_1(D^T) \leq \zeta_1(E^T)$.*

Proof. By definition of the D_1-metric in (14) and by the transpose of a copula, we have that

$$\begin{aligned}
\zeta_1(D^T) = \mathrm{D}_1(D^T, (\Pi^2)^T) &= \int_0^1 \int_0^1 \left| K_{D^T}(t,[0,v]) - K_{(\Pi^2)^T}(t,[0,v]) \right| \,\mathrm{d}t\,\mathrm{d}v \\
&= \int_0^1 \int_0^1 \left| \partial_1 D^T(t,v) - \partial_1 (\Pi^2)^T(t,v) \right| \,\mathrm{d}t\,\mathrm{d}v = \int_0^1 \int_0^1 \left| \partial_2 D(v,t) - v \right| \,\mathrm{d}t\,\mathrm{d}v \\
&\leq \int_0^1 \int_0^1 \left| \partial_2 E(v,t) - v \right| \,\mathrm{d}t\,\mathrm{d}v = \cdots = \zeta_1(E^T)\,,
\end{aligned}$$

where the inequality follows from the Hardy-Littlewood-Polya theorem which states that $f \leq_S g$ is equivalent to $\int_0^1 \varphi(f(t))\,\mathrm{d}t \leq \int_0^1 \varphi(g(t))\,\mathrm{d}t$ for all convex functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that the expectations exist, see, e.g., [26, Theorem 3.21]. □

In general, $D \leq_{\partial_2 S} E$ does not imply $D \leq_{\partial_2 S, G} E$ even if E is a CIS copula, which is shown by the following counterexample.

Example 3.15 ($\leq_{\partial_2 S} \not\Rightarrow \leq_{\partial_2 S,G}$).
Let $D = (\langle a_k, b_k, C_k \rangle)_{k \in \{1,2,3\}}$ be the ordinal sum of $C_1 = C_2 = \Pi^2$ and $C_3 = M^2$ w.r.t. the intervals $(a_1, b_1) = (0, \frac{1}{4})$, $(a_2, b_2) = (\frac{1}{4}, \frac{1}{2})$, and $a_3, b_3) = (\frac{1}{2}, 1)$. Consider the symmetric copula D^* of D defined by (18). *It can easily be seen that $D^* \leq_{\partial_2 S} M^2$ and $D^* \neq_{\partial_2 S} M^2$. Let $G \in \mathcal{F}^1$ be given by $G(x) = \frac{1}{2}(1 + (x \wedge 1))\mathbb{1}_{\{x \geq 0\}}$. Then, it holds that* $\mathrm{Ran}(G) = \{0\} \cup [\frac{1}{2}, 1]$, *$\iota_G(t) = \frac{1}{2}\mathbb{1}_{(0,1/2]}(t) + t \cdot \mathbb{1}_{(1/2,1]}(t)$, and $\iota_G^-(t) = t \cdot \mathbb{1}_{(1/2,1]}(t)$.*
For $u \leq \frac{1}{4}$, it holds that

$$\begin{aligned}
\partial_2^G M^2(u,t) &= \begin{cases} \frac{\min\{u, \iota_G(t)\} - \min\{u, \iota_G^-(t)\}}{\iota_G(t) - \iota_G^-(t)} = \frac{u-0}{\text{fk}\frac{1}{2}-0} = 2u\,, & \text{if } t \in (0, \frac{1}{2}]\,, \\ \lim_{s \nearrow t} \frac{\min\{u,t\} - \min\{u,s\}}{t-s} = 0\,, & \text{if } t \in (\frac{1}{2}, 1]\,, \end{cases} \\
&= 2u \cdot \mathbb{1}_{[0,1/2]}(t)\,, \quad \text{and} \\
\partial_2^G D^*(u,t) &= 4u \cdot \mathbb{1}_{(3/4,1]}(t)\,.
\end{aligned}$$

Hence, we obtain for $u \in (0, \frac{1}{4}]$ that $\partial_2^G D^*(u,\cdot) \succ_S \partial_2^G M^2(u,\cdot)$ and $\partial_2^G D^*(u,\cdot) \neq \partial_2^G M^2(u,\cdot)$. But this means that $D^* \not\leq_{\partial_2 S,G} M^2$.

However, if both D and E are CIS (or CDS), then it can easily be verified that $D \leq_{\partial_2 S} E$ yields $D \leq_{\partial_2 S,G} E$.

A relation of the $\leq_{\partial_2 S}$-ordering to the lower orthant ordering is given as follows. Note that we obtain from the definition of the reflected copula E^* of E in (18) that $E^* =_{\partial_2 S} E$, where, as usual, $=_{\partial_2 S}$ holds if $\leq_{\partial_2 S}$ and $\geq_{\partial_2 S}$ is fulfilled.

Lemma 3.16. *For $D, E \in \mathcal{C}_2$, the following statements hold true.*

(i) If E is CIS, then $D \leq_{\partial_2 S} E$ implies $E^* \leq_{lo} D \leq_{lo} E$.

(ii) If D and E are CIS, then $D \leq_{\partial_2 S} E$ and $D \leq_{lo} E$ are equivalent.

Proof. (i): Let $(u, v) \in [0, 1]^d$. For the decreasing rearrangement g_u^* of $\partial_2 D(u, \cdot)$, it follows that

$$D(u, v) = \int_0^v \partial_2 D(u, t)\,\mathrm{d}t \leq \int_0^v g_u^*(t)\,\mathrm{d}t \leq \int_0^v \partial_2 E(u, t)\,\mathrm{d}t = E(u, v)\,.$$

For the increasing rearrangement g_*^u of $\partial_2 D(u, t)$, it similarly holds that

$$E^*(u, v) = \int_0^v \partial_2 E^*(u, t)\,\mathrm{d}t = \int_0^v \partial_2 E(u, 1-t)\,\mathrm{d}t \leq \int_0^v g_*^u(t)\,\mathrm{d}t \leq \int_0^v \partial_2 D(u, t)\,\mathrm{d}t = D(u, v)\,.$$

(ii): If $D \leq_{lo} E$, then $D \leq_{\partial_2 S} E$ follows from

$$\int_0^v \partial_2 D(u, t)\,\mathrm{d}t = D(u, v) \leq E(u, v) = \int_0^v \partial_2 E(u, t)\,\mathrm{d}t$$

for all $u, v \in [0, 1]$, using that D and E are CIS. The reverse direction is given by (i). □

Consider the class

$$\mathcal{C}_2^E = \left\{D \in \mathcal{C}_2 \mid D \leq_{\partial_2 S} E\right\}\,,$$

of bivariate copulas that are closer than E to the independence copula or equal to E w.r.t. the $\leq_{\partial_2 S}$-ordering. Due to the following result, the class $\mathcal{C}_2^E$ has a least and a greatest element w.r.t. the lower orthant ordering given by a CDS and a CIS copula.

Proposition 3.17. *There exist a unique CDS copula $E_\downarrow \in \mathcal{C}_2^E$ and a unique CIS copula $E_\uparrow \in \mathcal{C}_2^E$ such that*

$$E_\downarrow =_{\partial_2 S} E =_{\partial_2 S} E_\uparrow\,. \tag{25}$$

It holds that $E_\downarrow = E_\uparrow^*$, where $E_\uparrow^*$ is defined by (18)*, and*

$$E_\downarrow \leq_{lo} D \leq_{lo} E_\uparrow \quad \text{for all } D \in \mathcal{C}_2^E\,. \tag{26}$$

Proof. To show (25), let $u \in [0, 1]$ and denote by $f_u\colon (0, 1) \to [0, 1]$ the essentially (w.r.t. the Lebesgue measure) unique decreasing rearrangement of $\partial_2 E(u, \cdot)$. For $(u, v) \in [0, 1]^2$, define

$$E_\uparrow(u, v) := \int_0^v f_u(t)\,\mathrm{d}t\,.$$

Then, $E_\uparrow$ is a bivariate copula, where the property of 2-increasingness follows for $(u_1, v_1) \leq (u_2, v_2)$ from

$$E_\uparrow(u_1, v_1) + E_\uparrow(u_2, v_2) - E_\uparrow(u_1, v_2) - E_\uparrow(u_2, v_1) = \int_{v_1}^{v_2} \underbrace{f_{u_2}(t) - f_{u_1}(t)}_{\geq 0}\,\mathrm{d}t \geq 0$$

because $\partial_2 E(u_2, t) \geq \partial_2 E(u_1, t)$ for all t .
Since $\partial_2 E_\uparrow(u, \cdot)$ is a rearrangement of $\partial_2 E(u, \cdot)$, it holds that $E =_{\partial_2 S} E_\uparrow$. Since $\partial_2 E_\uparrow(u, t) = f_u(t)$ for almost all t and f_u is the essentially uniquely determined decreasing rearrangement of $\partial_2 E(u, \cdot)$, it follows that $E_\uparrow$ is the uniquely determined CIS copula with $E =_{\partial_2 S} E_\uparrow$.

For the lower bound $E_\downarrow$, given by $E_\downarrow(u, v) := \int_{1-v}^1 f_u(t)\,\mathrm{d}t$, $(u, v) \in [0, 1]^2$, the statement follows similarly, so (25) is proved. Since $\int_0^1 f_u(t)\,\mathrm{d}t = u$ for all $u \in [0, 1]$, it follows that

$$E_\downarrow(u, v) = u - \int_0^{1-v} f_u(t)\,\mathrm{d}t = u - E_\uparrow(u, 1-v) = E_\uparrow^*(u, v)$$

for all $(u, v) \in [0, 1]^2$. Statement (26) follows with Lemma 3.16 (i). □

In the following, we give some examples of $\leq_{\partial_2 S}$-ordered copula families.

Example 3.18 (Elliptical copulas).
Let $(D^\rho)_{\rho\in[-1,1]}$ be a family of bivariate elliptical copulas with correlation parameter ρ . If $D^{|\rho_1|}$ and $D^{|\rho_2|}$ are CI, then

$$|\rho_1| \leq |\rho_2| \quad \Rightarrow \quad D^{\rho_1} \leq_{\partial_2 S} D^{\rho_2} . \tag{27}$$

A sufficient condition for $D^{|\rho_1|}$ to be CI is given by Abdous et al. [1, Proposition 1.2]. Then also $D^{|\rho_2|}$ is CI. Note that only in the Gaussian case, D^0 is CI, compare Rüschendorf [23, Theorem 2].

To show (27), let $0 \leq \rho_1 \leq \rho_2$. Since elliptical distributions are increasing w.r.t. the lower orthant ordering in the (generalized) correlation parameter, see Das Gupta et al. [7, Theorem 5.1], it follows that $D^{\rho_1} \leq_{lo} D^{\rho_2}$. Then, Lemma 3.16(ii) implies $D^{\rho_1} \leq_{\partial_2 S} D^{\rho_2}$ using that D^{ρ_1} and D^{ρ_2} are CI. The general case follows from the symmetry $(D^\rho)^* = D^{-\rho}$ of elliptical copulas.

Example 3.19 (Archimedean copulas).
Let C_ψ defined by $C_\psi(u,v) = \psi(\psi^{-1}(u) + \psi^{-1}(v))$ be the bivariate Archimedean copula with (strict) generator $\psi\colon \mathbb{R}_+ \to [0,1]$. The CI-property of C_ψ is characterized by the log-convexity of $-\psi'$, see Müller and Scarsini [20, Theorem 2.8]. Further, it holds that $C_{\psi_1} \leq_{lo} C_{\psi_2}$ if and only if $\psi_1^{-1} \circ \psi_2$ is subadditive, see Nelsen [22, Theorem 4.4.2]. Thus, we obtain from Lemma 3.16(ii) the following $\leq_{\partial_2 S}$-criterion for Archimedean copulas:
If $-\psi_i'$ is log-convex for $i = 1, 2$, then it holds that $C_{\psi_1} \leq_{\partial_2 S} C_{\psi_2}$ if and only if $\psi_1^{-1} \circ \psi_2$ is subadditive.

Sufficient conditions for the subadditivity are given in [22, Section 4.4]. We give some illustrating examples of $\leq_{\partial_2 S}$-increasing Archimedean copula families. The log-convexity of $-\psi'$ can be verified straightforwardly.

(a) The Clayton family $(C_{\psi_\theta})_{\theta\in[-1,\infty)}$ with (inverse) generator $\varphi_\theta(t) = \psi_\theta^{-1}(t) = \frac{1}{\theta}\left(t^{-\theta} - 1\right)$ for $\theta \neq 0$ and $\varphi_0(t) = \psi_0^{-1}(t) = -\ln(t)$ is $\leq_{lo}$-increasing, see Nelsen [22, Example 4.14]. Since $-\psi_\theta'$ is log-convex for $\theta \geq 0$, it follows that $(C_{\psi_\theta})_{\theta\geq 0}$ is $\leq_{\partial_2 S}$-increasing in θ .
(b) The Gumbel-Hougaard family $(C_{\psi_\theta})_{\theta\in[1,\infty)}$ with (inverse) generator $\varphi_\theta(t) = \psi_\theta^{-1}(t) = (-\ln(t))^\theta$ is $\leq_{lo}$-increasing, see Nelsen [22, Example 4.12]. Since $-\psi_\theta'$ is log-convex for all θ , it follows that $(C_{\psi_\theta})_{\theta\geq 1}$ is $\leq_{\partial_2 S}$-increasing in θ .
(c) The Frank family $(C_{\psi_\theta})_{\theta\in\mathbb{R}}$ with (inverse) generator $\varphi_\theta(t) = \psi_\theta^{-1}(t) = -\ln\left(\frac{\exp(-\theta t)-1}{\exp(-\theta)-1}\right)$ for $\theta \neq 0$ and $\varphi_0(t) = \psi_0^{-1}(t) = -\ln(t)$ is $\leq_{lo}$-increasing, see Nelsen [22, p. 150]. Since $-\psi_\theta'$ is log-convex for $\theta \geq 0$ and $C_{\psi_\theta}^* = C_{\psi_{-\theta}}$, see Nelsen [22, p. 133], it follows that $|\theta| \leq |\theta'|$ implies $C_{\psi_\theta} \leq_{\partial_2 S} C_{\psi_{\theta'}}$.

Combining the Ky Fan–Lorentz Theorem 3.3 and Lorentz's Theorem 3.10, we get the following main result.

Theorem 3.20 ($\leq_{\partial_2 S}$-ordering criterion).
Let $G \in \mathcal{F}^1$ and let $D^i, E^i \in \mathcal{C}_2$ be bivariate copulas with E^i CIS and $D^i \leq_{\partial_2 S, G} E^i$ for all $1 \leq i \leq d$. Assume that $\mathbf{B} = (B_t)_{t\in[0,1]}$ is continuous, $B_t \in \mathcal{C}_d$ for all t .

(i) If $\mathbf{B}$ fulfills condition (22) and if $B_t \in \mathcal{C}_d^{ccx}$ for all t , then

$$\ast_{\mathbf{B},G} D^i \leq_{lo} \ast_{\mathbf{B},G} E^i .$$

(ii) If $\widehat{\mathbf{B}} = (\widehat{B}_t)_{t\in[0,1]}$ fulfills condition (23) and if $\widehat{B}_t \in \mathcal{C}_d^{ccx}$ for all t , then

$$\ast_{\mathbf{B},G} D^i \leq_{uo} \ast_{\mathbf{B},G} E^i .$$

Proof. To show (i), define for $u = (u_1, \ldots, u_d) \in [0,1]^d$ the function $g_{i,u_i}(t) := \partial_2^G D^i(u_i, t)$ for almost all $t \in (0,1)$ and for $i = 1, \ldots, d$. Since E^i is CIS, it holds that $\partial_2^G E^i(u_i, \cdot)$ is decreasing. From the assumption that $D^i \leq_{\partial_2 S, G} E^i$, we obtain for the decreasing rearrangement g_{i,u_i}^* of g_{i,u_i} that $g_{i,u_i}^* \prec_S \partial_2^G E^i(u_i, \cdot)$. This yields the integral inequalities

$$\begin{aligned}
\ast_{\mathbf{B},G} D^i(u) &= \int_0^1 B_t^G\left(\partial_2^G D^1(u_1,t),\ldots,\partial_2^G D^d(u_d,t)\right)\,\mathrm{d}t \\
&\le \int_0^1 B_t^G\left(g^*_{1,u_1}(t),\ldots,g^*_{d,u_d}(t)\right)\,\mathrm{d}t \\
&\le \int_0^1 B_t^G\left(\partial_2^G E^1(u_1,t),\ldots,\partial_2^G E^d(u_d,t)\right)\,\mathrm{d}t = \ast_{\mathbf{B},G} E^i(u)
\end{aligned}$$

where we apply Theorems 3.10 and 3.3 using that also the copulas $(B_t^G)_t$ are componentwise convex and fulfill condition (22), see Lemma 3.6.

Statement (ii) follows similarly to (i) applying formula (13) for the survival function of the $\ast$-product. □

Since the independence copula coincides with its survival copula and is componentwise convex, we obtain the following result as a consequence of Theorem 3.20.

Corollary 3.21 (Ordering the conditional independence product)**.**
If $G \in \mathcal{F}^1$ and $D^i, E^i \in \mathcal{C}_2$ such that E^i is CIS and $D^i \le_{\partial_2 S,G} E^i$ for all $1 \le i \le d$, then

$$\Pi_G D^i \le_c \Pi_G E^i\,.$$

Remark 3.22. *(a) For simplified $\ast$-products, condition (22) of Proposition 3.11 and Theorem 3.20 are trivially fulfilled. In Proposition 3.11 there is no convexity condition w.r.t.* $\mathbf{B}$ *and B, respectively. The statement in Theorem 3.20 also holds true if the E^i are conditionally decreasing, i.e. if $\partial_2 E^i(u_i,\cdot)$ is increasing for all u_i.*

(b) Corollary 3.21 extends [15, Proposition 1] to arbitrary dimension and general factor distribution $G \in \mathcal{F}^1$ where the authors show that $\Pi_{i=1}^2 E^i(u,v) \ge \Pi^2(u,v) = uv$ for $u,v \in [0,1]$ and CIS copulas $E^1, E^2 \in \mathcal{C}_2$.

(c) The intuition why Theorem 3.20 is true can be seen in the following explanation. The condition $D^i \le_{\partial_2 S} E^i$ indicates that D^i is closer to the independence copula than E^i. Since additionally E^i is CIS, E^i is closer to the upper Fréchet bound than D^i, and $(E^i)^*$ is closer to the lower Fréchet bound than D^i. This yields a stronger positive dependence among the X_i in the factor model with specifications E^i. However, in general, such a statement cannot be obtained if the conditional copulas are not componentwise convex.

The following counterexample shows that the assumption $D^i \le_{\partial_2 S,G} E^i$ in Theorem 3.20 cannot be simplified to $D^i \le_{\partial_2 S} E^i$, and that, in general, $\ast_{\mathbf{B}} D^i \le_{lo} \ast_{\mathbf{B}} E^i$ does not imply $\ast_{\mathbf{B},G} D^i \le_{lo} \ast_{\mathbf{B},G} E^i$ for $G \in \mathcal{F}^1 \setminus \mathcal{F}^1_c$.

Example 3.23. *For $d \ge 2$, let $D^i = D^*$ and $E^i = M^2$ with D^* and $G \in \mathcal{F}^1$ given by Example 3.15, $i = 1,\ldots,d$. As shown there, it holds that $D^i \le_{\partial_2 S} E^i$ as well as $\partial_2^G D^i(u,\cdot) \succ_S \partial_2^G E^i(u,\cdot)$ for $u \le \frac{1}{4}$. This yields by Theorem 3.20 for all continuous $\mathbf{B} = (B_t)_{t\in[0,1]}$, $B_t \in \mathcal{C}_d^{ccx}$, satisfying condition (22) that $\ast_{\mathbf{B}} D^i \le_{lo} \ast_{\mathbf{B}} E^i$ and, as a consequence of Theorems 3.3 and 3.10, that $\ast_{\mathbf{B},G} D^i(u,\ldots,u) \ge \ast_{\mathbf{B},G} E^i(u,\ldots,u)$. So, for general $G \in \mathcal{F}^1 \setminus \mathcal{F}^1_c$ and for a general continuous family* $\mathbf{B}$ *of componentwise convex d-copulas which fulfills condition (22), we have the following diagram:*

$$\begin{array}{ccc}
D^i \le_{\partial_2 S} E^i & \not\Rightarrow & D^i \le_{\partial_2 S,G} E^i \\
\Downarrow & \not\searrow & \Downarrow \\
\ast_{\mathbf{B}} D^i \le_{lo} \ast_{\mathbf{B}} E^i & \not\Rightarrow & \ast_{\mathbf{B},G} D^i \le_{lo} \ast_{\mathbf{B},G} E^i\,.
\end{array}$$

3.2 Upper product ordering results

To derive ordering results for upper and lower products of bivariate copulas, consider on $\mathcal{C}_2$ the sign change ordering and the symmetric sign change ordering defined as follows.

For bivariate copulas $D, E \in \mathcal{C}_2$, define the function $f_{u,v}\colon [0,1] \to [-1,1]$ by

$$f_{u,v}(t) = \partial_2 E(u,t) - \partial_2 D(v,t)$$

for almost all $t \in (0,1)$ as the difference of the partial derivatives of E and D w.r.t. the second variable for fixed first components $u, v \in [0,1]$.

Definition 3.24 (Sign change orderings).
The sign change ordering $D \leq_{\partial\Delta} E$, *respectively, the* symmetric sign change ordering $D \leq_{s\partial\Delta} E$ *is defined via the property that for all* u, v, *respectively, for all* $u = v$, *the function* $f_{u,v}$ *has no* $(-,+)$*-sign change.*

The sign change orderings strengthen the standard bivariate dependence orderings. It holds true that

$$D \leq_{\partial\Delta} E \quad\Rightarrow\quad D \leq_{s\partial\Delta} E \quad\Rightarrow\quad D \leq_c E \quad\Leftrightarrow\quad D \leq_{lo} E\,,$$

see [2, Proposition 3.4]. Note that the lower and upper Fréchet copula are the least and greatest element, respectively, w.r.t. the $\leq_{\partial\Delta}$-ordering, i.e., it holds that $W^2 \leq_{\partial\Delta} D \leq_{\partial\Delta} M^2$ for all $D \in \mathcal{C}_2$. Examples of $\leq_{\partial\Delta}$-ordered copula families are elliptical copulas and some families of Archimedean copulas, see [2].

Each of both conditions

$$D^j = E^j \leq_{\partial\Delta} D^d, E^d\,, \quad D^d \leq_{s\partial\Delta} E^d\,, \quad \forall\, 1 \leq j \leq d-1\,, \tag{28}$$
$$\text{and} \quad D^j = E^j \geq_{\partial\Delta} D^d, E^d\,, \quad D^d \geq_{s\partial\Delta} E^d\,, \quad \forall\, 1 \leq j \leq d-1\,, \tag{29}$$

implies $\bigvee D^i \geq_c \bigvee E^i$, see [2, Proposition 3.6]. We generalize this result to arbitrary factor distributions as follows.

Theorem 3.25 (Sign-change ordering criterion for upper products).
Let $G \in \mathcal{F}^1$ *be a distribution function and let* $D^i, E^i \in \mathcal{C}_2$, $1 \leq i \leq d$, *be bivariate copulas. If either* (28) *or* (29) *holds, then it follows that*

$$\bigvee_G D^i \geq_c \bigvee_G E^i\,.$$

Proof. Assume (28). For $1 \leq i \leq d-1$ and $u_i, v \in [0,1]$, the functions $f_i, g_i, h\colon (0,1) \to [-1,1]$ given by

$$\begin{aligned} f_i(t) &= \partial_2 E^d(v,t) - \partial_2 D^i(u_i,t)\,,\\ g_i(t) &= \partial_2 D^d(v,t) - \partial_2 D^i(u_i,t)\,, \quad \text{and}\\ h(t) &= f_i(t) - g_i(t) = \partial_2 E^d(v,t) - \partial_2 D^d(v,t) \end{aligned}$$

have a.s. no $(-,+)$-sign change. Then, also the piecewise averaged functions f_i^G, g_i^G, $h^G\colon (0,1) \to [-1,1]$ given by

$$\begin{aligned} f_i^G(t) &= \partial_2^G E^d(v,t) - \partial_2^G D^i(u_i,t)\,,\\ g_i^G(t) &= \partial_2^G D^d(v,t) - \partial_2^G D^i(u_i,t)\,, \quad \text{and}\\ h^G(t) &= f_i^G(t) - g_i^G(t) = \partial_2^G E^d(v,t) - \partial_2^G D^d(v,t) \end{aligned}$$

have a.s. no $(-,+)$-sign change. Thus, the assertion follows in the same way as the proof of [2, Proposition 3.6].
Under the assumption of (29), the statement follows similarly with [2, Lemma 3.2], using that the functions f_i^G, g_i^G, and h_i^G have a.s. no $(+,-)$-sign change. □

Since we make use of it later on, we cite another concordance ordering criterion for upper products, based on the lower orthant ordering of the arguments.

Proposition 3.26 ($\leq_{lo}$-ordering criterion for upper products)**.**
For $D^2, \dots, D^d, E \in \mathcal{C}_2$, the following statements are equivalent:

(i) $D^i \leq_{lo} E$ *for all* $1 \leq i \leq d$,
(ii) $M^2 \vee D^2 \vee \cdots \vee D^d \leq_c M^2 \vee \underbrace{E \vee \cdots \vee E}_{(d-1)\text{-times}}$.

The result of Proposition 3.26 is given by [3, Theorem 1] even for the tighter supermodular ordering.

3.3 Lower product ordering results

An ordering criterion similar to the sign change criterion for upper products in Theorem 3.25 holds true for lower products. Remember that, in general, the lower products $M^2 \wedge_G D \wedge_G E$ and $W^2 \wedge_G D \wedge_G E$ are 3-copulas only for continuous G. The symmetric copula D_* associated with $D \in \mathcal{C}_2$ is defined in (18).

Theorem 3.27 (Sign-change ordering criterion for lower products)**.**
For bivariate copulas $D^1, D^2, D^3 \in \mathcal{C}_2$ and $G \in \mathcal{F}^1$, the following statements hold true:

(i) *If $D^1_* \leq_{\partial\Delta} D^2, D^3$ and $D^2 \leq_{s\partial\Delta} D^3$, then*

$$M^2 \wedge D^1 \wedge D^2 \leq_{lo} M^2 \wedge D^1 \wedge D^3 \quad \text{and} \quad D^1 \wedge_G D^2 \leq_{lo} D^1 \wedge_G D^3 .$$

(ii) *If $D^1_\star \geq_{\partial\Delta} D^2, D^3$ and $D^2 \geq_{s\partial\Delta} D^3$, then*

$$W^2 \wedge D^1 \wedge D^2 \leq_{lo} W^2 \wedge D^1 \wedge D^3 \quad \text{and} \quad D^1 \wedge_G D^2 \leq_{lo} D^1 \wedge_G D^3 .$$

Proof. To show the lower orthant ordering in (i), let $u = (u_1, u_2, u_3) \in [0,1]^3$. In the case that $G \in \mathcal{F}^1 \setminus \mathcal{F}^1_c$ is discontinuous, set $u_1 = 1$. Consider the functions $f, g, h\colon [0,1] \to [-1,1]$ defined by

$$\begin{aligned}
f(t) &:= \partial_2^G D^2(1-u_3, t) - \partial_2^G D^1_*(u_2, t),\\
g(t) &:= \partial_2^G D^3(1-u_3, t) - \partial_2^G D^1_*(u_2, t),\\
h(t) &:= \partial_2^G D^3(1-u_3, t) - \partial_2^G D^2(1-u_3, t) = g(t) - f(t).
\end{aligned}$$

Then f, g, h have no $(-,+)$-sign change and it holds that $\int f(t)\,\mathrm{d}t = \int g(t)\,\mathrm{d}t$. This yields the integral inequality

$$\int_0^{u_1} \max\{f(t), 0\}\,\mathrm{d}t \leq \int_0^{u_1} \max\{g(t), 0\}\,\mathrm{d}t,$$

compare [2, Lemma 3.2]. Thus, we obtain

$$\begin{aligned}
M^2 \wedge_G D^1 \wedge_G D^2(u) &= \int_0^{u_1} \max\left\{\partial_2^G D^1(u_2, t) + \partial_2^G D^2(1-u_3, t) - 1, 0\right\} \mathrm{d}t\\
&= \int_0^{u_1} \max\left\{\partial_2^G D^2(1-u_3, t) - \partial_2^G D^1_*(u_2, t), 0\right\} \mathrm{d}t\\
&= \int_0^{u_1} \max\left\{f(t), 0\right\} \mathrm{d}t\\
&\leq \int_0^{u_1} \max\left\{g(t), 0\right\} \mathrm{d}t = \cdots = M^2 \wedge_G D^1 \wedge_G D^3(u),
\end{aligned}$$

where the first equality follows from $\partial_2^G M^2(u_1, t) = \mathbb{1}_{\{u_1>t\}}$ for almost all t and for arbitrary $u_1 \in [0,1]$ in the case that G is continuous, respectively, for $u_1 = 1$ if G is discontinuous. This yields $M^2 \wedge D^1 \wedge D^2 \leq_{lo} M^2 \wedge D^1 \wedge D^3$ in the continuous case and $D^1 \wedge_G D^2 \leq_{lo} D^1 \wedge_G D^3$ for arbitrary G.
For the upper orthant ordering in (i), we obtain analogously that

$$\overline{M^2 \wedge_G D^1 \wedge_G D^2}(u) = \int_{u_1}^{1} \max\left\{1 - \partial_2^G D^1(u_2, t) - \partial_2^G D^2(1-u_3, t), 0\right\} \mathrm{d}t$$
$$\leq \int_{u_1}^{1} \max\left\{1 - \partial_2^G D^1(u_2, t) - \partial_2^G D^3(1-u_3, t), 0\right\} \mathrm{d}t = \overline{M^2 \wedge_G D^1 \wedge_G D^3}(u).$$

Statement (ii) follows analogously. □

Similarly to the $\leq_{lo}$-ordering criterion for the concordance ordering of upper products in Proposition 3.26, we obtain a concordance-ordering result for lower products based on a $\leq_{lo}$-ordering criterion for the bivariate dependence specifications.

Theorem 3.28 ($\leq_{lo}$-ordering criterion for lower products)**.**
Let $D, E^1, E^2 \in \mathcal{C}_2$ be bivariate copulas. Then, the following statements are equivalent:

(i) $D \leq_{lo} E^1$ and $D_* \leq_{lo} E^2$,
(ii) $M^2 \wedge D \wedge D_* \leq_c M^2 \wedge E^1 \wedge E^2$.

Proof. Assume (i). To show the lower orthant ordering, let $u = (u_1, u_2, u_3) \in [0,1]^3$. Then, it holds that

$$M^2 \wedge D \wedge D_*(u) = \int_0^{u_1} \max\left\{\partial_2 D(u_2, t) + \partial_2 D_*(u_3, t) - 1, 0\right\} \mathrm{d}t = \int_0^{u_1} \max\left\{\partial_2 D(u_2, t) - \partial_2 D(1-u_3, t), 0\right\} \mathrm{d}t$$
$$= \int_0^{u_1} \max\left\{\partial_2 D(u_2, t), \partial_2 D(1-u_3, t)\right\} \mathrm{d}t - D(1-u_3, u_1)$$
$$= \max\left\{D(u_2, u_1), D(1-u_3, u_1)\right\} - D(1-u_3, u_1)$$
$$\leq \max\left\{E^1(u_2, u_1), E^2_*(1-u_3, u_1)\right\} - E^2_*(1-u_3, u_1)$$
$$\leq \int_0^{u_1} \max\left\{\partial_2 E^1(u_2, t), \partial_2 E^2_*(1-u_3, t)\right\} \mathrm{d}t - E^2_*(1-u_3, u_1) = M^2 \wedge E^1 \wedge E^2(u),$$

where the first inequality follows from the assumptions using that $D_* \leq_{lo} E^2$ if and only if $D \geq_{lo} E^2_*$. The second inequality holds due to Jensen's inequality.
For the upper orthant ordering, we similarly obtain

$$\overline{M^2 \wedge D \wedge D_*}(u) = \int_{u_1}^{1} \max\left\{\partial_2 D(u_2, t) - \partial_2 D_*(u_3, t), 0\right\} \mathrm{d}t$$
$$\leq \int_{u_1}^{1} \max\left\{\partial_2 E^1(u_2, t) - \partial_2 E^2(u_3, t), 0\right\} \mathrm{d}t = \overline{M^2 \wedge E^1 \wedge E^2}(u).$$

Assume (ii). Then, (i) follows from the closure of the lower orthant ordering under marginalization and from the marginalization property of $*$-products, see Proposition 2.11(iv). □

3.4 Ordering results for convex combinations

In Section 3.1, we have established that general lower orthant ordering results for $\ast_{\mathbf{B},G} D^i$ in D^i for fixed D^j, $i \neq j$, are only possible if the conditional copulas $\mathbf{B} = (B_t)_t$ fulfill the convexity condition (21). Remember that this convexity condition implies negative dependence of the bivariate marginals of B_t.

Motivated by Theorem 3.20 for componentwise convex conditional copulas and by Proposition 3.26 concerning a $\leq_{lo}$-ordering criterion for the upper product, the question arises for which $\ast$-products ordering results of the form

$$D^i \prec_{\partial_2 S} E\,,\ \forall i\,, E \text{ CIS} \quad \Rightarrow \quad \ast_{\mathbf{B}} D^i \leq_{lo} \ast_{\mathbf{B}} E \tag{30}$$

hold true. Note that E is assumed to be a joint upper bound for the D^i.

To partly answer this question, we generalize the necessary integral ordering condition in the Ky Fan–Lorentz Theorem 3.3 under an additional ordering assumption on the upper bound.

Proposition 3.29. *If for all decreasing and bounded functions f_i, g_i on $[0,1]$ with $g_{i_1} \leq \ldots \leq g_{i_d}$, $i_1, \ldots, i_d \in \{1, \ldots, d\}$, such that $f_i \prec_S g_i$ the integral inequality* (19) *holds true, then Φ fulfills the milder convexity condition*

$$\Phi(u_i + h) - 2\phi(u_i) + \Phi(u_i - h) \geq 0 \quad \textit{for all } i\,, u_i\,, u_j \neq u_i\,, h \leq \min_{j \neq i} |u_i - u_j| \tag{31}$$

where those components are omitted which are the same in each expression.

Proof. We modify the proof of Fan and Lorentz [11, Theorem 1]: Let $0 \leq a < a+2\delta \leq 1$, and $u_1, \ldots, u_d \in [0,1]$. For some fixed i, let $h \in [0, \min_{j\neq i} |u_j - u_i|]$. Assume w.l.g. that $u_j \neq u_i$ for all $j \neq i$. Define

$$f_i(t) = \begin{cases} u_i + h\,, & 0 \leq t \leq a\,, \\ u_i\,, & a < t \leq a + 2\delta\,, \\ u_i - h\,, & a + 2\delta < t \leq 1\,, \end{cases} \qquad g_i(t) = \begin{cases} u_i + h\,, & 0 \leq t \leq a + \delta\,, \\ u_i - h\,, & a + \delta < t \leq 1\,, \end{cases}$$
$$f_j(t) = g_j(t) = u_j\,, \quad j \neq i\,.$$

Then, it holds true that $f_k \prec_S g_k$, $k = 1, \ldots, d$. Further, there exist $i_1, \ldots, i_d \in \{1, \ldots, d\}$ pairwise different such that $g_{i_1}(t) \leq \ldots \leq g_{i_d}(t)$ for all t. The inequality (19) reduces in this case to

$$\int_0^\delta \left\{\Phi(a+t, u_i + h) - \Phi(a+t, u_i) + \Phi(a+\delta+t, u_i - h_i) - \Phi(a+\delta+t, u_i)\right\} \mathrm{d}t \geq 0\,.$$

Dividing by δ and making $\delta \to 0$, yields (31). □

As a consequence, we obtain that lower orthant ordering results for $\ast$-products with a joint upper bound for all copulas also restrict the choice of conditional copulas.

Corollary 3.30. *If for all CIS copulas $D^i, E \in \mathcal{C}_2$ with $D^i \prec_{\partial_2 S} E$ the inequality*

$$\ast_{\mathbf{B}} D^i \leq_{lo} \ast_{\mathbf{B}} E \tag{32}$$

holds true, then **B** *fulfills the milder convexity condition* (31).

Proof. Let f_i, g_i be decreasing and bounded such that $f_i \prec_S g_i$ and $g_{i_1} \leq \ldots \leq g_{i_d}$, $i_1, \ldots, i_d \in \{1, \ldots, d\}$. Assume w.l.g. that $0 \leq f_i, g_i \leq 1$. Then, there exist $u_1, \ldots, u_d \in [0,1]$ and CIS copulas $D^i, E \in \mathcal{C}_2$ with $D^i \prec_{\partial_2 S} E$ such that $f_i(t) = \partial_2 D^i(u_i, t)$ and $g_i(t) = \partial_2 E(u_i, t)$. Thus, the statement follows from Proposition 3.29. □

Remark 3.31. *(a) Due to Corollary 3.30, ordering results of the form* (30) *can not be obtained for all continuous families* $\mathbf{B} = (B_t)_{t\in[0,1]}$ *of d-copulas.*

(b) The upper Fréchet copula M^d *fulfills the milder convexity condition* (31)*. In this case, inequality* (32) *is trivially fulfilled because* $\bigvee E^i = M^d$ *whenever* $E^i = E$ *for all* i *. Note that for the upper product the non-trivial generalized inequality*

$$M^2 \vee D^2 \vee \cdots \vee D^d \leq_c M^2 \vee \underbrace{E \vee \cdots \vee E}_{(d-1)\text{-times}} \tag{33}$$

holds true whenever $D^i \leq_{lo} E$ *(see Proposition 3.26).*

Denote by $co(M^d, \mathcal{C}_d^{ccx})$ the set of convex combinations of M^d with elements of $\mathcal{C}_d^{ccx}$. Then, we obtain the following result.

Theorem 3.32. *Let* $D^1 = E^1 = M^2$ *and* $D^i \in \mathcal{C}_2$ *such that for a CIS copula* $E \in \mathcal{C}_2$ *holds* $D^i \prec_{\partial_2 S} E = E^i$ *,* $2 \leq i \leq d$ *. Let* $B \in co(M^d, \mathcal{C}_d^{ccx})$ *. Then, for the simplified* $*$*-products, it holds true that*

$$*_B D^i \leq_{lo} *_B E^i .$$

Proof. The copula B is of the form $B = aM^d + (1-a)C$, for some $a \in [0,1]$, where $C \in \mathcal{C}_d^{ccx}$ fulfills the convexity condition (21). Thus, the statement follows from Theorem 3.20 and from (33) using that $D^i \prec_{\partial_2 S} E$ implies $D^i \leq_{lo} E$, see Lemma 3.16. □

Note that in the above result, $E^i = E$ for $i \in \{2,\dots,d\}$ is a joint upper bound for the copulas $D^2,\dots,D^d$.

4 Ordering results for completely specified factor models

In this section, we combine the ordering results on $*$-products in Section 3 with the ordering of the univariate marginal distributions. This leads to lower and upper orthant as well as concordance ordering results for CSFMs and, thus, to bounds w.r.t. these orderings in classes of CSFMs and PSFMs, respectively.

Suppose that $X = (X_1,\dots,X_d)$ with $X_i = f_i(Z,\varepsilon_i)$ and $Y = (Y_1,\dots,Y_d)$ with $Y_i = g_i(Z',\varepsilon_i')$ are d-dimensional random vectors that follow a completely specified factor model with factor distribution function $G = F_Z$ and $G' = F_{Z'}$, respectively, such that $\overline{\mathrm{Ran}(G)} = \overline{\mathrm{Ran}(G')}$. Then the corresponding copulas are given by the $*$-products

$$C_X = *_{\mathbf{B},G} D^i \quad \text{and} \quad C_Y = *_{\mathbf{C},G'} E^i ,$$

respectively, where $D^i = C_{X_i,Z}$, $E^i = C_{Y_i,Z'}$, $B_t^G = C_{X|Z=G^{-1}(t)}$, and $C_t^{G'} = C_{Y|Z'=G'^{-1}(t)}$, see Theorem 2.7. Further, by Sklar's Theorem, the corresponding distribution functions are given by

$$F_X = *_{\mathbf{B},G} D^i \,(F_{X_1},\dots,F_{X_d}) \quad \text{and} \quad F_Y = *_{\mathbf{C},G} E^i \,(F_{Y_1},\dots,F_{Y_d}) ,$$

using that $\overline{\mathrm{Ran}(G)} = \overline{\mathrm{Ran}(G')}$, see Proposition 2.14.

We establish conditions on the conditional copula families **B** and **C** assumed generally to be measurable, on the dependence specifications D^i and E^i , and on the distributions of the components X_i and Y_i to infer lower orthant, upper orthant and concordance comparison results for X and Y .

The following proposition compares CSFMs where the bivariate dependence specifications D^i and E^i coincide.

Proposition 4.1 (Ordering conditional copulas).
Assume that $D^i = E^i$ *for all* i *. Then, the following statements hold true.*

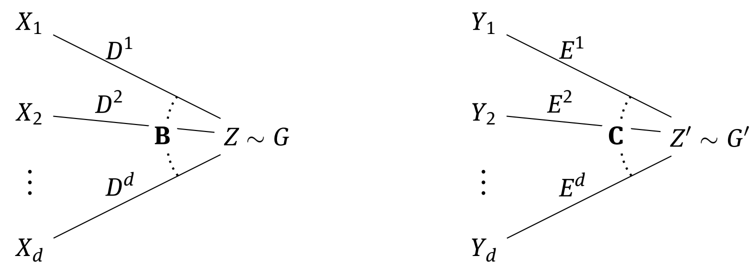

Figure 3 The setting in Section 4: completely specified factor models with dependence specifications D^i and $\mathbf{B} = (B_t)_t$ as well as E^i and $\mathbf{C} = (C_t)_t$, $1 \le i \le d$, and with factor distribution function G and G', respectively, such that $\overline{\mathrm{Ran}(G)} = \overline{\mathrm{Ran}(G')}$.

(i) If $\mathbf{B} \le_{lo} \mathbf{C}$ and $X_i \le_{lo} Y_i$ then $X \le_{lo} Y$.
(ii) If $\mathbf{B} \le_{uo} \mathbf{C}$ and $X_i \le_{uo} Y_i$ then $X \le_{uo} Y$.
(iii) If $\mathbf{B} \le_{c} \mathbf{C}$ and $X_i \overset{\mathrm{d}}{=} Y_i$ then $X \le_{c} Y$.

Proof. The statements follow from Proposition 3.2 for fixed marginal distributions together with Sklar's Theorem (respectively, Sklar's Theorem for survival functions) for fixed conditional copulas using that $X_i \le_{lo} Y_i$ (respectively, $X_i \le_{uo} Y_i$) implies $F_{X_i}(x) \le F_{Y_i}(x)$ (respectively, $F_{X_i}(x) \ge F_{Y_i}(x)$) for all $x \in \mathbb{R}$ and $1 \le i \le d$. □

In the remaining part of this section, we also establish ordering conditions w.r.t. the dependence specifications D^i and E^i.

For the following theorem, we need a family of componentwise convex conditional (survival) copulas that lies between $\mathbf{B}$ and $\mathbf{C}$. Then, we obtain a general ordering condition in dependence on the bivariate specifications, the conditional copulas and the marginal distributions.

Theorem 4.2. *Let $\mathbf{B}' = (B'_t)_{t\in[0,1]} \subset \mathcal{C}_d$ be continuous. Assume that all E^i are CIS.*

(i) If $\mathbf{B}'$ satisfies condition (22) and if $B'_t \in \mathcal{C}_d^{ccx}$ for all t, then

$$\mathbf{B} \le_{lo} \mathbf{B}' \le_{lo} \mathbf{C}\,,\; D^i \le_{\partial_2 S,G} E^i\,,\; \text{and } X_i \le_{lo} Y_i \text{ for all } i \;\Rightarrow\; X \le_{lo} Y\,.$$

(ii) If $\widehat{\mathbf{B}}'$ satisfies condition (23) and if $\widehat{B}_t \in \mathcal{C}_d^{ccx}$ for all t, then

$$\mathbf{B} \le_{uo} \mathbf{B}' \le_{uo} \mathbf{C}\,,\; D^i \le_{\partial_2 S,G} E^i\,,\; \text{and } X_i \le_{uo} Y_i \text{ for all } i \;\Rightarrow\; X \le_{uo} Y\,.$$

(iii) If $\mathbf{B}'$ and $\widehat{\mathbf{B}}'$ satisfy condition (22) and (23), respectively, and if $B_t, \widehat{B}_t \in \mathcal{C}_d^{ccx}$ for all t, then

$$\mathbf{B} \le_{c} \mathbf{B}' \le_{c} \mathbf{C}\,,\; D^i \le_{\partial_2 S,G} E^i\,,\; \text{and } X_i \overset{\mathrm{d}}{=} Y_i \text{ for all } i \;\Rightarrow\; X \le_{c} Y\,.$$

Proof. To show (i), we obtain from Proposition 3.2 and Theorem 3.20 that

$$\ast_{\mathbf{B},G} D^i \le_{lo} \ast_{\mathbf{B}',G} D^i \le_{lo} \ast_{\mathbf{B}',G} E^i \le_{lo} \ast_{\mathbf{C},G} E^i\,.$$

Then, the statement follows with Sklar's Theorem. Statements (ii) and (iii) follow analogously. □

Since the independence copula and its associated survival copula are componentwise convex, we obtain as a consequence of the above theorem ordering results for the standard factor model.

Corollary 4.3 (Ordering results for standard factor models)**.**
Assume that $\mathbf{B} = \mathbf{C} = \Pi^{\mathbf{d}} = (\Pi^d)$. If $D^i \le_{\partial_2 S,G} E^i$ and if E^i is CIS for all i, then

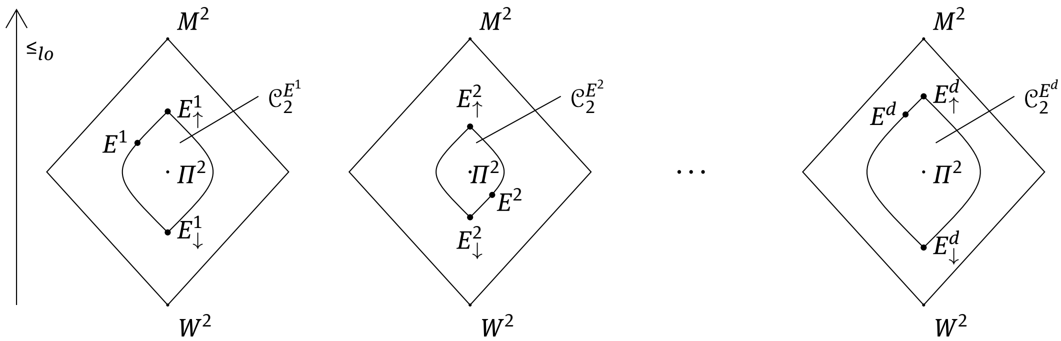

Figure 4 Classes $\mathcal{C}_2^{E^i} = \{C \in \mathcal{C}_2 \mid C \leq_{\partial_2 S} E^i\}$ of bivariate copulas generated by the copulas $E^i \in \mathcal{C}_2$, $i = 1, \ldots, d$, via the $\leq_{\partial_2 S}$-ordering. Note that M^2, Π^2, and W^2 denote the upper Fréchet copula, the independence copula, and the lower Fréchet copula, respectively. The copulas $E^i_\uparrow$ and $E^i_\downarrow$ are the uniquely determined copulas that are CIS and CDS, respectively, such that $E^i_\uparrow =_{\partial_2 S} E^i =_{\partial_2 S} E^i_\downarrow$, see Proposition 3.17. As a consequence of Proposition 3.14, it holds for all $D \in \mathcal{C}_2^{E^i}$ that $\zeta_1(D^T) \leq \zeta_1((E^i)^T)$.

(i) $X_i \leq_{lo} Y_i$ *for all i implies* $X \leq_{lo} Y$,
(ii) $X_i \leq_{uo} Y_i$ *for all i implies* $X \leq_{uo} Y$,
(iii) $X_i \overset{\mathrm{d}}{=} Y_i$ *for all i implies* $X \leq_c Y$.

In the following remark, we determine sharp bounds for some relevant classes of CSFMs including classes of standard factor models with bounded bivariate specification sets.

Remark 4.4. *Let* $F_i \in \mathcal{F}^1$ *for all* i. *Denote by* $\prec$ *one of the orderings* $\leq_{lo}$ *and* $\leq_{uo}$. *For* $E^i \in \mathcal{C}_2$, *denote by* $\mathcal{C}_2^{E^i} := \{C \in \mathcal{C}_2 \mid C \leq_{\partial_2 S} E^i\}$ *the class of bivariate copulas that is upper bounded w.r.t.* $\leq_{\partial_2 S}$ *by* E^i, $1 \leq i \leq d$. *For a risk factor* $Z \sim G$, $G \in \mathcal{F}_c^1$, *consider the class*

$$\mathcal{X}^f = \left\{\xi = (\xi_1, \ldots, \xi_d) \mid C_{\xi_i, Z} \in \mathcal{C}_2^{E^i}, \ F_{\xi_i} \prec F_i \text{ for all } i, \ C_{\xi | Z=z} \prec \Pi^d \text{ for all } z\right\}$$

of d-variate random vectors that are conditionally on $Z = z$ *negatively dependent w.r.t.* $\prec$, *have marginal distributions upper bounded by* F_i, *and have dependence specifications* $C_{\xi_i, Z} \in \mathcal{C}_2^{E^i}$, *see Figure 4. Then, for all* $\xi \in \mathcal{X}^f$, *it holds that*

$$F_\xi \prec \Pi_{i=1}^d E^i_\uparrow (F_1, \ldots, F_d), \tag{34}$$

where $E^i_\uparrow$ *is the uniquely determined CIS copula such that* $E^i_\uparrow =_{\partial_2 S} E^i$, *see Proposition 3.17. Further, a vector* $\xi \in \mathcal{X}^f$ *such that* $\xi \sim \Pi_{i=1}^d E^i_\uparrow (F_1, \ldots, F_d)$ *can explicitly be determined which implies that the bound in* (34) *is attained, compare Corollary 2.9.*

In PSFMs, the conditional copulas are not specified. For the comparison of upper bounds in classes of PSFMs, we note that the worst case distribution in a PSFM w.r.t. the orthant orders is obtained when the conditional copula specifications attain the upper Fréchet copula.

Theorem 4.5 (Upper bounds in classes of PSFMs).
Assume that $\mathbf{C} = \mathbf{M^d} = (M^d)$. *If* $D^j = E^j \leq_{\partial\Delta} D^d, E^d$ *for* $j = 1, \ldots, d-1$ *and* $D^d \leq_{s\partial\Delta} E^d$, *then*

(i) $X_i \leq_{lo} Y_i$ *for all i implies* $X \leq_{lo} Y$,
(ii) $X_i \leq_{uo} Y_i$ *for all i implies* $X \leq_{uo} Y$,
(iii) $X_i \overset{\mathrm{d}}{=} Y_i$ *for all i implies* $X \leq_c Y$.

Proof. From Proposition 3.2 and Theorem 3.25 we obtain that

$$\ast_{\mathbf{B},G} D^i \leq_{lo} \bigvee_G D^i \leq_{lo} \bigvee_G E^i = \ast_{\mathbf{C},G} E^i \,.$$

Then (i) follows with Sklar's Theorem. Statements (ii) and (iii) follow analogously. □

Similarly, we obtain for lower bounds in the two- and three-dimensional case the following result.

Theorem 4.6 (Lower bounds in classes of PSFMs, $d = 3$)**.**
Assume that $\mathbf{B} = \mathbf{W^3} = (W^3)$ *and* $D^1 = E^1 = M^2$ *. If* $D^2_* = E^2_* \leq_{\partial\Delta} D^3, E^3$ *and* $D^3 \leq_{s\partial\Delta} E^3$ *, Then*

(i) $X_i \leq_{lo} Y_i$ *implies* $(X_2, X_3) \leq_{lo} (Y_2, Y_3)$ *, and if* $G \in \mathcal{F}^1_c$ *then* $(X_1, X_2, X_3) \leq_{lo} (Y_1, Y_2, Y_3)$ *,*
(ii) $X_i \leq_{uo} Y_i$ *implies* $(X_2, X_3) \leq_{uo} (Y_2, Y_3)$ *, and if* $G \in \mathcal{F}^1_c$ *then* $(X_1, X_2, X_3) \leq_{uo} (Y_1, Y_2, Y_3)$ *,*
(iii) $X_i \stackrel{\mathrm{d}}{=} Y_i$ *implies* $(X_2, X_3) \leq_c (Y_2, Y_3)$ *, and if* $G \in \mathcal{F}^1_c$ *then* $(X_1, X_2, X_3) \leq_c (Y_1, Y_2, Y_3)$ *,*

Proof. For $G \in \mathcal{F}^1_c$, we obtain from Theorem 3.27 and Proposition 3.2 that

$$\ast_{\mathbf{B}} D^i = M^2 \wedge D^2 \wedge D^3 \leq_{lo} M^2 \wedge E^2 \wedge E^3 \leq_{lo} \ast_{\mathbf{C}} E^i \,.$$

Then, $(X_1, X_2, X_3) \leq_{lo} (Y_1, Y_2, Y_3)$ follows with Sklar's Theorem.
For general $G \in \mathcal{F}^1$, denote by $\mathbf{C^{23}} = (C^{23}_t)_{t\in[0,1]}$ the bivariate $(2, 3)$-marginal copulas of $\mathbf{C}$, i.e., $C^{23}_t(u, v) = C_t(1, u, v)$ for all $t, u, v \in [0, 1]$. Similarly, $\mathbf{B^{23}} = \mathbf{W^2} = (W^2)$. Then, we obtain that

$$D^2 \ast_{\mathbf{B^{23}},G} D^3 = D^2 \wedge_G D^3 \leq_{lo} E^2 \wedge_G E^3 \leq_{lo} E^2 \ast_{\mathbf{C^{23}},G} E^3 \,,$$

and, thus, $(X_2, X_3) \leq_{lo} (Y_2, Y_3)$. For the upper orthant and concordance ordering, the statements follow analogously. □

Note that the same results hold true if the inequality signs $\leq_{\partial\Delta}$ and $\leq_{s\partial\Delta}$ in Theorem 4.5 and Theorem 4.6 (with $D^1 = E^1 = W^2$) are reversed.

For classes of *partially specified internal factor models* (PSIFMs) where the first component of the risk vector in the PSFM coincides with (an increasing function of) the factor variable, see [3], we obtain the following results. Note that in this class, the first bivariate dependence specification is given by the upper Fréchet copula M^2 .

Theorem 4.7 (Upper bounds in classes of PSIFMs)**.**
Assume that $G \in \mathcal{F}^1_c$ *and* $\mathbf{C} = \mathbf{M^d} = (M^d)$ *. If* $D^1 = E^1 = M^2$ *and* $D^j \leq_{lo} E^j = E$ *for* $j = 2, \ldots, d$ *and* $E \in \mathcal{C}_2$ *, then*

(i) $X_i \leq_{lo} Y_i$ *for all i implies* $X \leq_{lo} Y$ *,*
(ii) $X_i \leq_{uo} Y_i$ *for all i implies* $X \leq_{uo} Y$ *,*
(iii) $X_i \stackrel{\mathrm{d}}{=} Y_i$ *for all i implies* $X \leq_c Y$ *.*

Proof. From Proposition 3.2 and Theorem 3.26, we obtain that

$$\ast_{\mathbf{B}} D^i \leq_{lo} \bigvee D^i \leq_{lo} \bigvee E^i = \ast_{\mathbf{C}} E^i \,.$$

Thus, the statement follows with Sklar's Theorem. Statements (ii) and (iii) follow analogously. □

For lower bounds in the three-dimensional case, we obtain the following result.

Theorem 4.8 (Lower bounds in classes of PSIFMs, $d = 3$)**.**
Assume that $G \in \mathcal{F}^1_c$ *and* $\mathbf{B} = \mathbf{W^3} = (W^3)$ *. If* $D^1 = E^1 = M^2$ *and* $D^2 = D^3_* \leq_{lo} E^j$ *for* $j = 2, 3$ *, then*

(i) $X_i \le_{lo} Y_i$ *for all i implies* $X \le_{lo} Y$,
(ii) $X_i \le_{uo} Y_i$ *for all i implies* $X \le_{uo} Y$,
(iii) $X_i \overset{\mathrm{d}}{=} Y_i$ *for all i implies* $X \le_c Y$.

Proof. From Theorem 3.28 and Proposition 3.2, we obtain

$$\ast_{\mathbf{B}} D^i = D^1 \wedge D^2 \wedge D^3 \le_{lo} E^1 \wedge E^2 \wedge E^3 \le_{lo} \ast_{\mathbf{C}} E^i .$$

Then, $(X_1, X_2, X_3) \le_{lo} (Y_1, Y_2, Y_3)$ follows with Sklar's Theorem. Statements (ii) and (iii) follow analogously. □

Conclusion

In this paper, we obtain some general ordering results for factor models w.r.t. the specifications of the joint distributions of the components with the risk factor variable. The results generalize the upper product ordering results in [2, 3] to general conditional dependence structures and are based essentially on a version of Sklar‘s theorem as well as on classical ordering results based on rearrangements. The results in this paper allow to determine worst case distributions w.r.t. the orthant orderings for classes of CSFMs as well as in subclasses of PSFMs for any $d \ge 2$ and, similarly, of best case distributions for $d = 2, 3$. Related ordering results w.r.t. the stronger supermodular and the directionally convex ordering need different techniques and are the subject of a subsequent study.

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A Appendix

Lemma A.1. *For* $G \in \mathcal{F}^1$ *, the following statements hold true.*

(i) ι_G *and* ι_G^- *are non-decreasing and Lebesgue-almost surely continuous.*
(ii) $G^{-1}(\iota_G(t)) = G^{-1}(t)$ *and* $\iota_G(G(x)) = G(x)$ *for all* $t \in [0, 1]$ *and* $x \in \mathbb{R}$.
(iii) *If* $G(x - \varepsilon) < G(x)$ *for all* $\varepsilon > 0$ *, then* $\iota_G^-(G(x)) = G^-(x)$.
(iv) $\iota_G^-(t) \le t \le \iota_G(t)\ \forall t \in (0, 1)$, $\iota_G^-(0) = 0 = \iota_G(0)$ *and* $\iota_G^-(1) \le 1 = \iota_G(1)$.
(v) $\iota_G \circ \iota_G = \iota_G$ *and* $\iota_G^- \circ \iota_G = \iota_G^-$.
(vi) ι_G^- *is left-continuous.*
(vii) *For all* $y \in \mathbb{R}$, ι_G *is left-continuous at* $G(y)$ *and* ι_G^- *is continuous at* $G^-(y)$.
(viii) *In general,* ι_G *is neither left-continuous nor right-continuous.*
(ix) $\iota_G^-(t) = t = \iota_G(t)$ *if and only if G is continuous at* $G^{-1}(t)$.
(x) $\iota_G^-(t) = t = \iota_G(t)$ *for all* $t \in [0, 1]$ *if and only if* $G \in \mathcal{F}_c^1$.
(xi) *If* $\iota_G(t) > t$ *, then* $\iota_G(t + \varepsilon) = \iota_G(t)$ *and* $\iota_G^-(t + \varepsilon) = \iota_G^-(t)$ *for all* $0 < \varepsilon \le \iota_G(t) - t$.
(xii) *If* $\iota_G^-(t) < t$ *, then* $\iota_G(t - \varepsilon) = \iota_G(t)$ *and* $\iota_G^-(t - \varepsilon) = \iota_G^-(t)$ *for all* $0 < \varepsilon < t - \iota_G^-(t)$.

Proof. (i): The non-decreasingness is clear. Since ι_G and ι_G^- can only have an at most countable number of jumps, the set of discontinuity points is a Lebesgue-null set.

(ii), (iii) and (iv) follow from the definition of G^{-1} and G^{-}, respectively, considering the cases where G is discontinuous and constant around x, respectively.
(v) is a consequence of (ii).
(vi): This follows from the left-continuity of G^{-} and G^{-1}.
(vii): To show the left-continuity of ι_G at $G(y)$, let $(t_n)_{n\in\mathbb{N}}$ be strictly increasing in $[0,1]$ with limit $G(y) > 0$. Then, we have

$$G(y) = \iota_G(G(y)) \geq \iota_G(t_n) \geq t_n \to G(y)$$

as $n \to \infty$ applying (ii), (i), and (iv). To show the right-continuity of ι_G^- at $G^-(y)$, let $(t_n)_{n\in\mathbb{N}}$ be strictly decreasing in $[0,1]$ with limit $G^-(y) < 1$. Then, we obtain similarly that

$$G^-(y) = \iota_G^-(G^-(y)) \leq \iota_G^-(t_n) \leq t_n \to G^-(y)\,.$$

(viii): Consider the distribution functions G and H defined by

$$G(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ \frac{2}{3}x & \text{if } x \in [0, \frac{1}{2})\,, \\ \frac{2}{3}x + \frac{1}{3} & \text{if } x \in [\frac{1}{2}, 1]\,, \\ 1 & \text{if } x > 1\,. \end{cases} \qquad H(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ x & \text{if } x \in [0, \frac{1}{2})\,, \\ \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1)\,, \\ 1 & \text{if } x \geq 1\,. \end{cases} \tag{35}$$

Then ι_G and ι_H are given by

$$\iota_G(t) = \begin{cases} t & \text{if } t \in [0, \frac{1}{3}) \cup (\frac{2}{3}, 1]\,, \\ \frac{2}{3} & \text{if } t \in [\frac{1}{3}, \frac{2}{3}]\,, \end{cases} \qquad \iota_H(t) = \begin{cases} t & \text{if } t \in [0, \frac{1}{2}]\,, \\ 1 & \text{if } t > \frac{1}{2}\,. \end{cases}$$

So, ι_G is not left-continuous at $t = \frac{1}{3}$ and ι_H is not right-continuous at $t = \frac{1}{2}$.
(ix): $\iota_G(t) = t = \iota_G^-(t)$ holds if and only if $G(G^{-1}(t)) = G^-(G^{-1}(t))$, which is equivalent to the continuity of G at $G^{-1}(t)$.
(x): If $G \in \mathcal{F}_c^1$, the statement follows from (ix). For the reverse direction, assume that $G \in \mathcal{F}^1 \setminus \mathcal{F}_c^1$ is discontinuous. Then there exists x such that $G(x) > G^-(x)$. For $t \in [G^-(x), G(x))$, it follows that $\iota_G(t) = G(G^{-1}(t)) = G(x) > t$.
(xi): Let $\varepsilon \in (0, \iota_G(t) - t]$. Then, the non-decreasingness of ι_G and (v) imply $\iota_G(t) \leq \iota_G(t+\varepsilon) \leq \iota_G(\iota_G(t)) = \iota_G(t)$ and $\iota_G^-(t) \leq \iota_G^-(t+\varepsilon) \leq \iota_G^-(\iota_G(t)) = \iota_G^-(t)$.
(xii): Let $x = G^{-1}(t)$. Then, the statement follows from $G^-(x) < t - \varepsilon \leq G(x)$. □

Proof of Theorem 2.2.
Consider the set $\mathcal{I}_c := \{(z_0, z_1) \mid z_0 < z_1\,,\ G \text{ is continuous on } (z_0, z_1)\}$ of open intervals on which G is continuous, and denote by $\mathcal{I}_s := \{\{z\} \mid z \in \mathbb{R}\}$ the set of one-point sets. Note that each element of $\mathcal{I}_c$ is the intersection of an open interval in $\mathbb{R}$ and the preimage of $(0,1)$ under G. We show that

$$\int_{(z_0,z_1)} F_{X|Z=z}(x)\,\mathrm{d}G(z) = \int_{(z_0,z_1)} \partial_2^G C(F(x), G(z))\,\mathrm{d}G(z)\,, \quad \text{and} \tag{36}$$

$$\int_{\{z\}} F_{X|Z=z}(x)\,\mathrm{d}G(z) = \int_{\{z\}} \partial_2^G C(F(x), G(z))\,\mathrm{d}G(z) \tag{37}$$

for all $(z_0, z_1) \in \mathcal{I}_c$ and $\{z\} \in \mathcal{I}_s$. Since G has at most countably many jump discontinuities, every open interval $(y_0, y_1) \subset \mathbb{R}$ can be written as a disjoint union of at most countably many elements of $\mathcal{I}_c$ and $\mathcal{I}_s$. Then, (36) and (37) imply

$$\int_{(y_0,y_1)} F_{X|Z=z}(x)\,\mathrm{d}G(z) = \int_{(y_0,y_1)} \partial_2^G C(F(x), G(z))\,\mathrm{d}G(z)$$

for all open intervals $(y_0, y_1) \subset \mathbb{R}$. Hence, the integrands coincide for G-almost all z, which yields (i).

To show (36), let $(z_0, z_1) \in \mathcal{I}_c$. Assume w.l.g. that $t_0 := G(z_0) < G^-(z_1) =: t_1$. Then we obtain from the disintegration theorem and Sklar's Theorem that

$$\begin{aligned}
\int_{(z_0,z_1)} F_{X|Z=z}(x)\,\mathrm{d}G(z) &= \lim_{z \nearrow z_1} F(x,z) - F(x,z_0) = C(F(x), G^-(z_1)) - C(F(x), G(z_0)) \\
&= \int_{(t_0,t_1)} \partial_2 C(F(x), s)\,\mathrm{d}s = \int_{(t_0,t_1)} \lim_{\varepsilon \searrow 0} \frac{C(F(x), s) - C(F(x), s-\varepsilon)}{\varepsilon}\,\mathrm{d}s \\
&= \int_{(t_0,t_1)} \lim_{\varepsilon \searrow 0} \frac{C(F(x), \iota_G(\iota_G(s))) - C(F(x), \iota_G^-(\iota_G(s)-\varepsilon))}{\iota_G(\iota_G(s)) - \iota_G^-(\iota_G(s)-\varepsilon)}\,\mathrm{d}s \\
&= \int_{(t_0,t_1)} \partial_2^G C(F(x), \iota_G(s))\,\mathrm{d}s = \int_{(z_0,z_1)} \partial_2^G C(F(x), G(z))\,\mathrm{d}G(z)\,,
\end{aligned} \tag{38}$$

where the third equality follows from the disintegration theorem applied on copulas. For the fourth equality, we use that the left-hand derivative and the derivative of the copula w.r.t. the second component coincide for Lebesgue-almost all s. The fifth equality follows from $\iota_G(s) = s = \iota_G^-(s)$ and $\iota_G^-(s-\varepsilon) = s - \varepsilon$ for all $s \in (t_0, t_1)$ and $\varepsilon \in (0, s - t_0)$ because G is continuous at $G^{-1}(s)$ and $G^{-1}(s-\varepsilon)$, respectively, see Lemma A.1(ix). The sixth equality holds by definition of the differential operator in (2), and the last equality is a consequence of the transformation formula.

To show (37), assume for $z \in \mathbb{R}$ w.l.g. that $G(z) > G^-(z)$. Then we obtain

$$\begin{aligned}
\int_{\{z\}} F_{X|Z=y}(x)\,\mathrm{d}G(y) &= F(x,z) - \lim_{w \nearrow z} F(x,w) = C(F(x), G(z)) - C(F(x), G^-(z)) \\
&= \frac{C(F(x), \iota_G(G(z))) - C(F(x), \iota_G^-(G(z)))}{\iota_G(G(z)) - \iota_G^-(G(z))} \cdot (G(z) - G^-(z)) \\
&= \lim_{\varepsilon \searrow 0} \frac{C(F(x), \iota_G(G(z))) - C(F(x), \iota_G^-(G(z)-\varepsilon))}{\iota_G(G(z)) - \iota_G^-(G(z)-\varepsilon)} \cdot (G(z) - G^-(z)) \\
&= \int_{\{z\}} \partial_2^G C(F(x), G(z))\,\mathrm{d}G(z)\,,
\end{aligned} \tag{39}$$

where we use $G(z) > G^-(z)$ and apply Lemma A.1(v) for the third equality. For the fourth equality, we use the left-continuity of ι_G^-, see Lemma A.1(vi). The last equality follows with the definition of the operator ∂_2^G in (2).

To show statement (ii) of Theorem 2.2, denote by $\mathbb{Q}$ the rational numbers. Due to part (i) it holds that

$$F_{X|Z=z}(x) = \partial_2^G C(F(x), G(z))$$

for all $x \in \mathbb{Q}$ and for all z outside the G-null set $N := \bigcup_{x \in \mathbb{Q}} N_x$. Then we obtain for $x \in \mathbb{R}$ that

$$F_{X|Z=z}(x) = \lim_{\substack{w \searrow x \\ w \in \mathbb{Q}}} F_{X|Z=z}(w) = \lim_{\substack{w \searrow x \\ w \in \mathbb{Q}}} \partial_2^G C(F(w), G(z)) =: H_z(x)$$

for all $z \in N^c$. For $z \in N^c$, the function H_z is by definition right-continuous. Since C is a 2-copula and thus 2-increasing, H_z is non-decreasing. Further, $H_z(-\infty) = 0$ and $H_z(\infty) = 1$. Hence, $H_z(x) = \lim_{w \searrow x} \partial_2^G C(F(w), G(z))$ coincides with $F_{X|Z=z}(x)$ for all $x \in \mathbb{R}$ and for all $z \in N^c$. This proves the assertion. □

Proof of Proposition 2.14.
(i) ⇒ (ii): Assume that $\overline{\mathrm{Ran}(G_1)} \neq \overline{\mathrm{Ran}(G_2)}$. As a consequence of Proposition 2.13, the d-variate products $\Pi_{G_1} M^2$ and $\Pi_{G_2} M^2$ do not coincide because for $G \in \mathcal{F}^1$, $\Pi_G M^2$ defines an ordinal sum with intervals $\{(\iota_G^-(t), \iota_G(t)) \mid \iota_G^-(t) \neq \iota_G(t)\,, t \in (0,1)\}$ which are different for $G = G_1$ and $G = G_2$ unless $\overline{\mathrm{Ran}(G_1)} = \overline{\mathrm{Ran}(G_2)}$.

(ii) ⇒ (iii): First assume that $\mathrm{Ran}(G_1) = \mathrm{Ran}(G_2)$. Then, for all $t \in (0,1)$, it holds that

$$\begin{aligned}\iota_{G_1}(t) &= G_1\left(\inf\{x \mid G_1(x) \geq t\}\right) = \inf\{u \in \mathrm{Ran}(G_1) | u \geq t\} \\ &= \inf\{u \in \mathrm{Ran}(G_2) | u \geq t\} = G_2\left(\inf\{x \mid G_2(x) \geq t\}\right) = \iota_{G_2}(t)\,.\end{aligned}$$

In the general case that $\overline{\mathrm{Ran}(G_1)} = \overline{\mathrm{Ran}(G_2)}$, it holds that $\mathrm{Ran}(G_1)$ and $\mathrm{Ran}(G_2)$ only differ in a Lebesgue-null set because distribution functions have at most countably many jump discontinuities. Hence, the first part implies that $\iota_{G_1}(t) = \iota_{G_2}(t)$ for Lebesgue-almost all t .
(iii) ⇒ (i): This follows from the definition of the $*$-product because $\ast_{\mathbf{B},G} D^i$ depends on G only through $\iota_G(t)$ for Lebesgue-almost all t . □

Proof of Proposition 2.15(iii). Assume that $D^i = D^j$ for all $i \neq j$. Then, for $u = (u_1, \ldots, u_d) \in [0,1]^d$, it holds true that

$$\bigvee_G D^i(u) = \int_0^1 \min_i \left\{\partial_2^G D^1(u_i, t)\right\} \,\mathrm{d}t = \int_0^1 \partial_2^G D^1(\min_i\{u_i\}, t)\,\mathrm{d}t = \int_0^1 \partial_2^G D^1(\min_i\{u_i\}, G(y))\,\mathrm{d}G(y) = \min_i\{u_i\}\,,$$

where the second equality holds because $\partial_2^G D^1(\cdot, t)$ is increasing for all t , the third equality follows from (3) and the transformation formula, see, e.g., [32, Theorem 2], and the last equality is a consequence of Theorem 2.2 and the disintegration theorem.

For the reverse direction, assume w.l.g. that $d = 2$ and $D^1(w_1, w_2) > D^2(w_1, w_2)$ for some $(w_1, w_2) \in [0,1] \times \mathrm{Ran}(G)$. Then, there exist $(u, v) \in (0,1) \times \mathrm{Ran}(G)$ and an ε-ball $B_\varepsilon(u,v) \subset (0,1)^2$ such that

$$\partial_2^G D^1(x,t) > \partial_2^G D^2(x,t) \quad \text{for almost all } (x,t) \in B_\varepsilon((u,v))\,, \tag{40}$$

because, otherwise, it would hold that

$$D^1(w_1, w_2) = \int_0^{w_2} \partial_2^G D^1(w_1, t)\,\mathrm{d}t \leq \int_0^{w_2} \partial_2^G D^2(w_1, t)\,\mathrm{d}t = D^2(w_1, w_2)\,.$$

which is a contradiction to $D^1(w_1, w_2) > D^2(w_1, w_2)$. As a consequence of (40), we obtain that

$$M^2(u,u) = u = \int_0^1 \partial_2^G D^1(u,t)\,\mathrm{d}t > \int_0^1 \min\left\{\partial_2^G D^1(u,t), \partial_2^G D^2(u,t)\right\} \,\mathrm{d}t = D^1 \vee_G D^2(u,u)\,.$$

This yields $D^1 \vee_G D^2 \neq M^2$. □

Proof of Proposition 2.16.
The first statement is a consequence of [31, Proposition 3].

Since $\mathcal{S}_T(C^i) \in \mathcal{C}_2$ for all i , the product $\ast_B \mathcal{S}_T(C^i)$ is well-defined. Hence, the second statement follows from

$$\begin{aligned}\ast_B \mathcal{S}_T(C^i)\,(u_1, \ldots, u_d) &= \int_0^1 B\left((\partial_2 \mathcal{S}_T(C^i)(u_i, t))_{1 \leq i \leq d}\right) \,\mathrm{d}t = \int_0^1 B\left((\partial_2 C^i(u_i, T(t)))_{1 \leq i \leq d}\right) \,\mathrm{d}t \\ &= \int_{[0,1]} B\left((\partial_2 C^i(u_i, s))_{1 \leq i \leq d}\right) \,\mathrm{d}\lambda^T(s) = \int_{[0,1]} B\left((\partial_2 C^i(u_i, s))_{1 \leq i \leq d}\right) \,\mathrm{d}\lambda(s) \\ &= \ast_B C^i\,(u_1, \ldots, u_d)\end{aligned}$$

for all $(u_1, \ldots, u_d) \in [0,1]^d$, using that $\partial_2 \mathcal{S}_T(C)(u,t) = \partial_2 C(u, T(t))$ for λ-almost all t . □

Proof of Lemma 2.17. (i): Let $t \in (0,1)$. Due to Lemma A.1, we consider three cases.
In the first case, assume that $\iota_G(t) = t$ and $\iota_G(t-\varepsilon) = t$ for some $\varepsilon > 0$. Define

$$t_0 := \inf\{s \mid \iota_G(s) = \iota_G(t)\}\,. \tag{41}$$

Then, Lemma A.1(xi) implies that ι_G^- in constant on $(t_0, t]$. We show that $\iota_G^-(t) = t_0$. Suppose that $\iota_G^-(t) > t_0$. Let $\eta = \iota_G^-(t) - t_0$. Then, $\iota_G^-(t_0+\delta) = t_0 + \eta$ for some $\delta \in (0,\eta)$. But this is a contradiction to Lemma A.1(iv). Suppose that $\iota_G^-(t) < t_0$. Then, Lemma A.1(xii) implies that ι_G is constant on $(\iota_G^-(t), t]$. But this is a contradiction to (41).
In the second case, assume that $\iota_G(t) = t$ and $\iota_G(t-\delta) = t-\delta$ for all $0<\delta<\varepsilon$ for some $\varepsilon > 0$. Then, Lemma A.1(xii) implies that $\iota_G^-(t) = t$.
In the third case, assume that $\iota_G(t) \neq t$. Then, Lemma A.1(iv) implies that $\iota_G(t) > t$. Lemma A.1(xi) implies that ι_G^- is constant on $(t_0, \iota_G(t)]$ for t_0 defined by (41). We show that $\iota_G^-(t_0+\delta) = t_0$ for all $0 < \delta < \iota_G(t) - t_0$ and, thus, $\iota_G^-(t) = t_0$. Suppose that $\iota_G^-(t_0+\delta) > t_0$ for some $\delta \in (0, \iota_G(t)-t_0)$. Then, there is a contradiction to Lemma A.1(iv). Suppose that $\iota_G^-(t_0+\delta) < t_0$ for some $\delta \in (0, \iota_G(t) - t_0)$. Then, Lemma A.1(xii) yields a contradiction to the minimality of t_0.
All of the three above considered cases imply that $\iota_G^-(t) = \inf\{s \mid \iota_G(s) \geq \iota_G(t)\}$. It remains to show that $\iota_G(s) \geq \iota_G(t) \Leftrightarrow \iota_G(s) \geq t$. From Lemma A.1(iv), we obtain that $\iota_G(t) \geq t$, which implies the direction from left to right. For the reverse direction, we obtain from Lemma A.1(v) and (i) that $\iota_G(s) = \iota_G(\iota_G(s)) \geq \iota_G(t)$.

(ii): Consider the functions $F_n, F\colon \mathbb{R} \to [0,1]$, $n \in \mathbb{N}$, defined by

$$F_n(t) = \begin{cases} 0 & \text{if } t<0\,,\\ \lim_{s\searrow t} \iota_{G_n}(s) & \text{if } t \in [0,1]\,,\\ 1 & \text{if } t>1\,,\end{cases} \qquad F(t) = \begin{cases} 0 & \text{if } t<0\,,\\ \lim_{s\searrow t} \iota_{G}(s) & \text{if } t \in [0,1]\,,\\ 1 & \text{if } t>1\,,\end{cases}$$

$$F_n^-(t) = \begin{cases} 0 & \text{if } t<0\,,\\ \lim_{s\nearrow t} \iota_{G_n}(s) & \text{if } t \in [0,1]\,,\\ 1 & \text{if } t>1\,,\end{cases} \qquad F^-(t) = \begin{cases} 0 & \text{if } t<0\,,\\ \lim_{s\nearrow t} \iota_{G}(s) & \text{if } t \in [0,1]\,,\\ 1 & \text{if } t>1\,.\end{cases}$$

Then, F_n and F are distribution functions with left-continuous version F_n^- and F^-, respectively. Since by assumption $\iota_{G_n} \to \iota_G$ almost surely pointwise, we obtain that $F_n(t) \to F(t)$ for all t at which F is continuous. This implies that the generalized inverse distribution functions converge almost surely, i.e.,

$$F_n^{-1}(t) \to F^{-1}(t) \quad \text{for almost all } t \in [0,1]\,. \tag{42}$$

Since $F^{-1}(t) = \inf\{s \mid F(s) \geq t\} = \inf\{s \mid F^-(s) \geq t\}$, it holds by construction of F and F^- and by (i) that $F^{-1}(t) = \inf\{s \mid \iota_G(s) \geq t\} = \iota_G^-(t)$. Similarly, we obtain that $F_n^{-1}(t) = \iota_{G_n}^-(t)$. Hence, (42) implies that $\iota_{G_n}^-(t) \to \iota_G^-(t)$ for almost all $t \in [0,1]$. □

Proof of Proposition 2.25. (i): Let $D = E_*$ on $[0,1] \times \mathrm{Ran}(G)$. Then, for $(u,v) \in [0,1]^2$, it holds that

$$\begin{aligned} D \wedge_G E(u,v) &= \int_0^1 \max\left\{\partial_2^G D(u,t) + \partial_2^G E(v,t) - 1, 0\right\} = \int_0^1 \max\left\{\partial_2^G D(u,t) - \partial_2^G E_*(1-v,t), 0\right\}\,\mathrm{d}t \\ &= \int_0^1 \max\left\{\partial_2^G D(u,t), \partial_2^G D(1-v,t)\right\}\,\mathrm{d}t - 1 + v = \int_0^1 \partial_2^G D(\max\{u, 1-v\}, t)\,\mathrm{d}t - 1 + v \\ &= \max\{u, 1-v\} - 1 + v = W^2(u,v)\,. \end{aligned}$$

For the reverse direction, assume w.l.g. that $D(w_1,w_2) < E_*(w_1,w_2)$ for some $(w_1,w_2) \in [0,1] \times \mathrm{Ran}(G)$. Then, there exist $(u,v) \in (0,1) \times \mathrm{Ran}(G)$ and an ε-ball $B_\varepsilon(u,v) \subset (0,1)^2$ such that

$$\partial_2^G D(x,t) < \partial_2^G E_*(x,t) \quad \text{for almost all } (x,t) \in B_\varepsilon((u,v))\,, \tag{43}$$

because, otherwise, it would hold that

$$D(w_1, w_2) = \int_0^{w_2} \partial_2^G D(w_1, t)\,\mathrm{d}t \geq \int_0^{w_2} \partial_2^G E_*(w_1, t)\,\mathrm{d}t = E_*(w_1, w_2)\,,$$

which is a contradiction to $D^1(w_1, w_2) < E_*(w_1, w_2)$. As a consequence of (43), we obtain that

$$\begin{aligned} W^2(u, 1-u) = 0 = \int_0^1 \partial_2^G D(u,t)\,\mathrm{d}t - u \\ &< \int_0^1 \max\left\{\partial_2^G D(u,t), \partial_2^G E_*(u,t)\right\}\,\mathrm{d}t - u = \int_0^1 \max\left\{\partial_2^G D(u,t), 1 - \partial_2^G E(1-u,t)\right\}\,\mathrm{d}t - u \\ &= \int_0^1 \max\left\{\partial_2^G D(u,t) + \partial_2^G E(1-u,t) - 1, 0\right\}\,\mathrm{d}t = D \wedge_G E\,(u, 1-u)\,. \end{aligned}$$

This yields $D \wedge_G E \neq W^2$.
(ii): If $G \in \mathcal{F}_c^1$ is continuous, then it holds that

$$\begin{aligned} M^2 \wedge_G D \wedge_G E(u) &= \int_0^1 \max\left\{\mathbb{1}_{\{u_1 \geq t\}} + \partial_2 D(u_2, t) + \partial_2 E(u_3, t) - 2\,, 0\right\}\,\mathrm{d}t \\ &= \int_0^{u_1} \max\left\{\partial_2 D(u_2, t) + \partial_2 E(u_3, t) - 1\,, 0\right\}\,\mathrm{d}t \end{aligned}$$

for $u = (u_1, u_2, u_3) \in [0,1]^3$. This defines a 3-copula, compare Durante et al. [8, Proposition 2].
For the reverse direction, assume that $G \in \mathcal{F}^1 \setminus \mathcal{F}_c^1$ is discontinuous and that $M^2 \wedge_G D \wedge_G E$ is a 3-copula. Then, Theorem 2.7 implies that there exists a random vector (U_1, U_2, U_3, Z) (under an extension of the probability space if necessary) such that $Z \sim G$, $C_{U_1,Z} = M^2$, $C_{U_2,Z} = D$, and $C_{U_3,Z} = E$ as well as

$$P\left((U_1, U_2, U_3) \leq u \,|\, Z = z\right) = W^3\left(\partial_2^G M^2(u_1, t), \partial_2^G D(u_2, t), \partial_2^G E(u_3, t)\right)$$

for all $z = G^{-1}(t)$, $t \in (0,1)$, and for all $u = (u_1, u_2, u_3) \in [0,1]^3$. Since G is discontinuous, there exists $t_0 \in (0,1)$ such that $\iota_G^-(t_0) < \iota_G(t_0)$. This implies that the conditional distribution functions $\partial_2^G M^2(\cdot, t_0)$, $\partial_2^G D(\cdot, t_0)$, and $\partial_2^G E(\cdot, t_0)$ are continuous. Now, choose $u = (u_1, u_2, u_3) \in [0,1]^3$ and $v = (v_1, v_2, v_3) = (1,1,1)$ such that

$$\begin{aligned} \partial_2^G D(u_1, t_0) = \partial_2^G E(u_2, t_0) = \partial_2^G M^2(u_3, t_0) = 0.5\,, \\ \partial_2^G D(v_1, t_0) = \partial_2^G E(v_2, t_0) = \partial_2^G M^2(v_3, t_0) = 1\,. \end{aligned}$$

Then, it follows that

$$P\left((U_1, U_2, U_3) \in [u, v] \,|\, Z = G^{-1}(t_0)\right) = V_{W^3}\left([\tfrac{1}{2}, 1]^3\right) = -0.5 < 0\,,$$

where $V_{W^3}([\frac{1}{2}, 1]^3)$ denotes the W^3-volume of the box $[\frac{1}{2}, 1]^3 \subset [0,1]^3$, see Nelsen [22, Exercise 2.36]. This yields a contradiction and, thus, $M^2 \wedge_G D \wedge_G E$ is not a copula.
(iii) is a consequence of Theorem 2.7 and Remark 2.8.
(iv) and (v): For $(u, v) \in [0,1] \times \mathrm{Ran}(G)$, it holds that

$$D \wedge_G M^2(u, v) = \int_0^1 \max\left\{\partial_2^G(u,t) + \partial_2^G M^2(v,t) - 1, 0\right\}\,\mathrm{d}t = \int_0^v \partial_2^G D(u,t)\,\mathrm{d}t = D(u,v)\,,$$

where the second equality holds true because $\partial_2^G M^2(v,t) = \mathbb{1}_{\{v>t\}}$ using that $v \in \mathrm{Ran}(G)$. The third equality follows from Theorem 2.2.

The other statements follow similarly.

(vi): As a consequence of (iv), $\bigwedge$ is not commutative if D is not symmetric. For a counterexample for associativity, let $D^i \in \mathcal{C}_2$ be a Gaussian copula with correlation $\rho_i \in (-1,1)$, $i = 1,2,3$. Then, $C^i \wedge C^j$ is a Gaussian copula with correlation $m(\rho_i,\rho_j) = \rho_i\rho_j - \sqrt{1-\rho_i^2}\sqrt{1-\rho_j^2}$. Obviously, in general, it holds that $m(\rho_1, m(\rho_2,\rho_3)) \neq m(m(\rho_1,\rho_2),\rho_3)$.

□

Proof of Lemma 3.6. For condition (21), the statement is trivial.

For condition (22), we need to show that

$$\int_0^\delta \left[B_{a+\delta+s}(u) - B_{a+\delta+s}(v) + B_{a+s}(v) - B_{a+s}(u)\right] \mathrm{d}s \geq 0\,, \quad \forall\, 0 \leq a \leq 1-2\delta\,,\ \forall \delta > 0\,, \tag{44}$$

implies

$$\int_0^\delta \left[B^G_{a+\delta+s}(u) - B^G_{a+\delta+s}(v) + B^G_{a+s}(v) - B^G_{a+s}(u)\right] \mathrm{d}s \geq 0\,, \quad \forall\, 0 \leq a \leq 1-2\delta\,,\ \forall \delta > 0\,, \tag{45}$$

where $u = (u_k), v = (v_k) \in [0,1]^d$ such that for some $i \in \{1,\dots,d\}$ and $u_i \leq v_i$, $u_j = v_j$ for all $j \neq i$.

Consider the function $f\colon [0,1] \to [-1,0]$ given by

$$f(t) = B_t(u) - B_t(v)\,.$$

Then, condition (44) is equivalent to

$$\int_0^\delta f(a+\delta+s)\,\mathrm{d}s \geq \int_0^\delta f(a+s)\,\mathrm{d}s \quad \text{for all } 0 \leq a \leq 1-2\delta \text{ and } \delta > 0\,,$$

which means that f is increasing. Thus, the smoothed function $f^G\colon [0,1] \to [-1,0]$ given by

$$\begin{aligned} f^G(t) &= \begin{cases} f(t)\,, & \text{if } \iota_G^-(t) = \iota_G(t)\,,\\ \frac{1}{\iota_G(t)-\iota_G^-(t)} \int_{\iota_G^-(t)}^{\iota_G(t)} f(s)\,\mathrm{d}s\,, & \text{if } \iota_G^-(t) \neq \iota_G(t) \end{cases}\\ &= B_t^G(u_i) - B_t^G(u_i+h) \end{aligned}$$

is also increasing. But this is equivalent to (45).

For condition (23), the statement follows analogously. □

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