Research Article Special Issue in memory of Abe Sklar

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New results on perturbation-based copulas

<https://doi.org/10.1515/demo-2021-0116>

Received May 30, 2021; accepted September 20, 2021

Abstract: A prominent example of a perturbation of the bivariate product copula (which characterizes stochastic independence) is the parametric family of Eyraud-Farlie-Gumbel-Morgenstern copulas which allows small dependencies to be modeled. We introduce and discuss several perturbations, some of them perturbing the product copula, while others perturb general copulas. A particularly interesting case is the perturbation of the product based on two functions in one variable where we highlight several special phenomena, e.g., extremal perturbed copulas. The constructions of the perturbations in this paper include three different types of ordinal sums as well as flippings and the survival copula. Some particular relationships to the Markov product and several dependence parameters for the perturbed copulas considered here are also given.

Keywords: copula, dependence parameter, Eyraud-Farlie-Gumbel-Morgenstern copula, ordinal sum, perturbation

MSC: 60E05, 62H05, 62H20

Dedicated to the memory of Professor Abe Sklar: we would like to pay homage to a great mathematician who made deep and innovative contributions to probabilistic metric spaces and who founded the theory of copulas.

1 Introduction

The earliest use of what is now called *perturbation theory* was to deal with otherwise unsolvable mathematical problems of celestial mechanics. When Kepler published his first law “*the orbit of every planet is an ellipse with the Sun at one of the two foci*,” in [66] and [67, book 5, part 1, III. De Figura Orbitæ]) at the beginning of the 17th century, he provided an analytical solution of a classical *two-body problem*, the two bodies being the planet under consideration and the Sun — in this scenario no perturbation occurred. Many decades later, when *three-body problems* were studied, e.g., the system Moon–Earth–Sun [59], one observed that the Moon (which has a much smaller mass than both Sun and Earth) does not move along a simple ellipse à la Kepler because of the competing gravitation of the Earth and the Sun, i.e., the constants describing the motion of a planet around the Sun are also influenced by the motion of other planets and may vary in time. In addition, in the second half of the 19th century the increasing accuracy of astronomical observations also required a

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higher accuracy of the solutions to Newton's gravitational equations [95] and motivated mathematicians such as Lagrange [80] and Laplace [81] to develop and study fundamental methods of perturbation theory (for a classical survey see, e.g., [9]).

In mathematics, physics, and chemistry, perturbation theory deals with mathematical methods for finding an approximate solution to a problem, by starting from the exact solution of a related, simpler problem. A critical feature of the technique is a middle step that breaks the problem into "solvable" and "perturbative" parts. Perturbation theory is widely used when the problem at hand does not have a known exact solution, but can be expressed as a "small" change to a known solvable problem.

In quantum mechanics (see [32] and [12]), perturbation theory can be used to describe a complicated unsolved system using a simple, solvable system. Starting with such a simple system whose mathematical solution is known, one may add a weak disturbance to the system (a so-called "Hamiltonian" [7]). If this disturbance is not too large, physical quantities related to the perturbed system (e.g., its energy levels) can be seen as "corrections" to those of the simple system. If these corrections are small compared to the size of the quantities themselves, approximate methods such as asymptotic series can help to calculate them.

Now let us turn to the concept of bivariate copulas (introduced in [119], see also [39, 64, 93]), i.e., to mathematical objects which capture the dependence structure among random variables and which are the topic of our current research. In this context, perturbation usually means that a (small) bivariate function (often called the perturbation factor) is added to a given copula, and one is interested to find out under which conditions the result is again a copula [36, 98]. A prominent example of such a perturbation is the family of Eyraud-Farlie-Gumbel-Morgenstern (or EFGM) copulas given in (2.7), where a parameterized family of perturbation factors is added to the product copula Π (for more details about EFGM copulas and for other families of copulas which can be considered as perturbations see [39, 68, 93]).

In this paper, we investigate two different types of perturbations of general copulas which will be introduced in Definitions 4.1 and 5.8 (where we can identify an interesting class of extremal elements), respectively.

In these constructions we make use of several techniques which are well-known in the theory of copulas, such as x - and y -flippings, the construction of the survival copula, and three variants of the concept of ordinal sums, to name just a few. We study in detail a number of mathematical properties of these perturbations and some interrelations between them. Some of the perturbations discussed here induce interesting extensions of the family of Eyraud-Farlie-Gumbel-Morgenstern copulas (compare [104]). Finally, some relationships to the Markov product of copulas are emphasized, and the values of four dependence parameters (Spearman's rho, Kendall's tau, Blomqvist's beta, and Gini's gamma) of our perturbations are given.

2 Preliminaries

Copulas are mathematical objects capturing the dependence structure among random variables. The name "copula" for functions linking an n -dimensional distribution and its one-dimensional marginals goes back to Sklar's paper [119] (compare also [120]), where he proved (for the case $n = 2$) a result which is now referred to as Sklar's Theorem. However, links between multivariate distributions and their one-dimensional marginals have been studied before, e.g., by Hoeffding [60, 61], Fréchet [50], Dall'Aglio [23–25], and Féron [48], and also later on without any reference to the concept of copulas (see, e.g., [108, 123]).

Formally, a (bivariate) copula [119] is a function $C: [0, 1]^2 \rightarrow [0, 1]$ which satisfies, for each $x \in [0, 1]$, the boundary conditions

$$C(x, 0) = C(0, x) = 0 \quad \text{and} \quad C(x, 1) = C(1, x) = x, \quad (\text{C1})$$

and which is 2-increasing, i.e., for all $(x, y), (x^*, y^*) \in [0, 1]^2$ with $(x, y) \leq (x^*, y^*)$

$$C(x, y) + C(x^*, y^*) - C(x, y^*) - C(x^*, y) \geq 0. \quad (\text{C2})$$

The set of all bivariate copulas will be denoted by $\mathcal{C}$.

There are infinitely many elements in $\mathcal{C}$: in the books [63, 93] and, more recently, [39] one finds plenty of examples of parametric families (usually with one or two parameters) of copulas, on the one hand, and

classes of copulas which can be constructed and characterized by functions in one variable (e.g., by additive and/or multiplicative generators [1, 83, 111, 112] in the case of Archimedean copulas), on the other hand.

Sklar's Theorem establishing bivariate copulas as the link between bivariate probability distributions and their marginals was given in [119] (see also [39, 64, 93]).

In brief, Sklar's Theorem states that, whenever (X, Y) is a random vector with its two marginal distributions $F_X, F_Y: \mathbb{R} \rightarrow [0, 1]$, then there exists a copula $C_{X,Y} \in \mathcal{C}$ (which is uniquely determined if and only if X and Y are continuous) such that the joint distribution $F_{X,Y}: \mathbb{R}^2 \rightarrow [0, 1]$ is given by

$$F_{X,Y}(u, v) = C_{X,Y}(F_X(u), F_Y(v)). \quad (2.1)$$

Conversely, for each copula $C \in \mathcal{C}$ the function $F: \mathbb{R}^2 \rightarrow [0, 1]$ given by

$$F(u, v) = C(F_X(u), F_Y(v))$$

is a two-dimensional probability distribution of the random vector (X, Y) such that $C_{X,Y} = C$.

Sklar's Theorem also holds for the general case of n -dimensional probability distributions (proofs for this general case and alternative proofs for the bivariate case can be found in [5, 33–35, 43, 46, 96, 100]).

In the language of Sklar's Theorem the following three basic copulas describe a pair of independent or comonotone dependent or countermonotone dependent random variables X and Y , respectively: the product copula $\Pi: [0, 1]^2 \rightarrow [0, 1]$ and the Fréchet-Hoeffding lower and upper bounds $W: [0, 1]^2 \rightarrow [0, 1]$ and $M: [0, 1]^2 \rightarrow [0, 1]$ given by, respectively, $\Pi(x, y) = xy$, $W(x, y) = \max(x + y - 1, 0)$, and $M(x, y) = \min(x, y)$.

If $C: [0, 1]^2 \rightarrow [0, 1]$ is an arbitrary copula then several related copulas can be considered, e.g., the x -flipping $C^{x\text{flip}}: [0, 1]^2 \rightarrow [0, 1]$ and the y -flipping $C^{y\text{flip}}: [0, 1]^2 \rightarrow [0, 1]$ of C , and the survival copula $C^{\text{surv}}: [0, 1]^2 \rightarrow [0, 1]$, which are defined (see [30, 93]) by, respectively,

$$C^{x\text{flip}}(x, y) = y - C(1 - x, y), \quad C^{y\text{flip}}(x, y) = x - C(x, 1 - y), \quad C^{\text{surv}}(x, y) = x + y - 1 + C(1 - x, 1 - y). \quad (2.2)$$

If X and Y are two continuous random variables and if $C_{X,Y} \in \mathcal{C}$ is the (unique) copula satisfying (2.1) in Sklar's Theorem then the following stochastic interpretation is an immediate consequence of [93, Theorem 2.4.4]:

$$(C_{X,Y})^{x\text{flip}} = C_{-X,Y}, \quad (C_{X,Y})^{y\text{flip}} = C_{X,-Y}, \quad (C_{X,Y})^{\text{surv}} = C_{-X,-Y}. \quad (2.3)$$

As an immediate consequence of (2.3), the x -flipping, the y -flipping and the construction of the survival copula (the latter being the composition of x -flipping and y -flipping) are involutive operations on $\mathcal{C}$, i.e., for each copula $C \in \mathcal{C}$ we have

$$C^{\text{surv}} = (C^{x\text{flip}})^{y\text{flip}} = (C^{y\text{flip}})^{x\text{flip}} \quad \text{and} \quad (C^{x\text{flip}})^{x\text{flip}} = (C^{y\text{flip}})^{y\text{flip}} = (C^{\text{surv}})^{\text{surv}} = C. \quad (2.4)$$

For the three basic copulas W , Π , and M the following relationships can be verified easily:

$$W^{x\text{flip}} = W^{y\text{flip}} = M, \quad \Pi^{x\text{flip}} = \Pi^{y\text{flip}} = \Pi, \quad M^{x\text{flip}} = M^{y\text{flip}} = W.$$

A copula $C \in \mathcal{C}$ which is invariant with respect to the construction of survival copulas, i.e., which satisfies $C^{\text{surv}} = C$, is also called radially symmetric [55] (see also [6, 8, 19]). The three basic copulas W , Π and M are trivial examples of radially symmetric copulas.

Given a copula $C \in \mathcal{C}$, we sometimes will work with a distinguished section of it, the so-called opposite diagonal section $\omega_C: [0, 1] \rightarrow \mathbb{R}$ defined by

$$\omega_C(x) = C(x, 1 - x). \quad (2.5)$$

There is an axiomatization for a function $\omega: [0, 1] \rightarrow [0, 1]$ to be an opposite diagonal section of copulas, and for each such ω there exists at least one $C \in \mathcal{C}$ such that $\omega_C = \omega$ (see, e.g., [31, 45, 52, 53]).

In Section 6, we shall be concerned with several dependence parameters of a copula $C \in \mathcal{C}$, in particular with Spearman's rho [121], Kendall's tau [65], Blomqvist's beta [11], and Gini's gamma [56] which can be

defined for each copula $C \in \mathcal{C}$ and assume their values in the interval $[-1, 1]$. The corresponding functions $\varrho, \tau, \beta, \gamma: \mathcal{C} \rightarrow [-1, 1]$ are given by (see, e.g., [93]), respectively:

$$\begin{aligned}\varrho(C) &= 12 \iint_{[0,1]^2} C(x, y) dx dy - 3, & \tau(C) &= 4 \iint_{[0,1]^2} C(x, y) dC(x, y) - 1, \\ \beta(C) &= 4 C(\tfrac{1}{2}, \tfrac{1}{2}) - 1, & \gamma(C) &= 4 \int_0^1 (C(x, x) + C(x, 1-x)) dx - 2.\end{aligned}\quad (2.6)$$

For the three basic copulas W, Π and M and for each function $\xi: \mathcal{C} \rightarrow [-1, 1]$ such that $\xi \in \{\varrho, \tau, \beta, \gamma\}$ we obtain the special values $\xi(W) = -1$, $\xi(\Pi) = 0$, and $\xi(M) = 1$.

From Theorems 5.1.1 and 5.1.9 in [93] (see also Definition 5.1.7 in this monograph) it follows that for each copula $C \in \mathcal{C}$ and for each function $\xi: \mathcal{C} \rightarrow [-1, 1]$ such that $\xi \in \{\varrho, \tau, \beta, \gamma\}$ we get for the x -flipping $C^{x\text{flip}}$ and the y -flipping $C^{y\text{flip}}$ of C and for the survival copula C^{surv} given by (2.2):

$$\xi(C^{x\text{flip}}) = \xi(C^{y\text{flip}}) = -\xi(C^{\text{surv}}) = -\xi(C).$$

A particularly interesting and important family of copulas, which is used quite often when the weak dependence of exchangeable random variables should be modeled, is $(C_\theta^{\text{EFGM}})_{\theta \in [-1, 1]}$, where each $C_\theta^{\text{EFGM}}: [0, 1]^2 \rightarrow [0, 1]$ is defined by

$$C_\theta^{\text{EFGM}}(x, y) = xy + \theta xy(1-x)(1-y). \quad (2.7)$$

This family was usually referred to as the family of *Farlie-Gumbel-Morgenstern copulas* [42, 58, 92, 93]. In [39] (see also [13, 14, 91]) it was pointed out that the corresponding distributions had already been studied in the earlier and, for many years, forgotten work by Eyraud [41]. In recognition of that we will consistently use the name *Eyraud-Farlie-Gumbel-Morgenstern copulas* (*EFGM copulas* for short) in this paper.

Recall the concept of an M -ordinal sum which was introduced first for triangular norms (see [109–111]) in [112] and then also for copulas [49]. This construction was based on earlier results for partially ordered sets [10, 40] and abstract semigroups [15–18] (compare also [3, 72, 74, 83, 113]).

In an analogous way, another type of ordinal sums of copulas based on the lower Fréchet-Hoeffding bound W was suggested in [90] (see also [39]).

A third type of ordinal sum of copulas, the so-called (vertical) Π -ordinal sums, will be used in Section 5. They were originally introduced in [76] as a generalization of some patchwork techniques [28, 37, 38] and of the gluing of copulas proposed in [118].

In each of the three cases we start with an arbitrary family $([a_k, b_k])_{k \in K}$ of non-empty, pairwise disjoint open subintervals of $[0, 1]$ and with an arbitrary family $(C_k)_{k \in K}$ of copulas. Then each of the three functions $C^{\text{Mos}}, C^{\text{Wos}}, C^{\Pi\text{os}}: [0, 1]^2 \rightarrow [0, 1]$ defined by, respectively,

$$C^{\text{Mos}}(x, y) = \begin{cases} a_k + (b_k - a_k)C_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}) & \text{if } (x, y) \in [a_k, b_k]^2, \\ M(x, y) & \text{otherwise,} \end{cases} \quad (2.8)$$

$$C^{\text{Wos}}(x, y) = \begin{cases} (b_k - a_k)C_k(\frac{x-a_k}{b_k-a_k}, \frac{y+b_k-1}{b_k-a_k}) & \text{if } (x, y) \in [a_k, b_k] \times [1-b_k, 1-a_k], \\ W(x, y) & \text{otherwise,} \end{cases} \quad (2.9)$$

$$C^{\Pi\text{os}}(x, y) = \begin{cases} a_k y + (b_k - a_k)C_k(\frac{x-a_k}{b_k-a_k}, y) & \text{if } x \in [a_k, b_k], \\ \Pi(x, y) & \text{otherwise,} \end{cases} \quad (2.10)$$

is a well-defined copula.

We call C^{Mos} the M -ordinal sum, C^{Wos} the W -ordinal sum, and $C^{\Pi\text{os}}$ the (vertical) Π -ordinal sum of the summands $([a_k, b_k], C_k)_{k \in K}$, and we shall write

$$C^{\text{Mos}} = M \cdot ((a_k, b_k, C_k))_{k \in K}, \quad C^{\text{Wos}} = W \cdot ((a_k, b_k, C_k))_{k \in K}, \quad C^{\Pi\text{os}} = \Pi \cdot ((a_k, b_k, C_k))_{k \in K}.$$

An ordinal sum $M\text{-}(\langle a_k, b_k, C_k \rangle)_{k \in K}$, $W\text{-}(\langle a_k, b_k, C_k \rangle)_{k \in K}$ or $\Pi\text{-}(\langle a_k, b_k, C_k \rangle)_{k \in K}$ is said to be non-trivial if the family of open intervals $(]a_k, b_k[)_{k \in K}$ does not consist of $]0, 1[$ only, i.e., if $\{]a_k, b_k[\mid k \in K\} \neq \{]0, 1[\}$.

Another extremal case which is covered by (2.8)–(2.10) is that of an empty index set: the empty ordinal sums $M\text{-}(\langle a_k, b_k, C_k \rangle)_{k \in \emptyset}$, $W\text{-}(\langle a_k, b_k, C_k \rangle)_{k \in \emptyset}$ and $\Pi\text{-}(\langle a_k, b_k, C_k \rangle)_{k \in \emptyset}$ coincide with M , W and Π , respectively.

Some deeper investigations of these and other types of ordinal sums can be found in [102] (see also, e.g., [38, 40, 69, 70, 74, 101]).

3 Some known results on perturbations of copulas

In a number of papers, various perturbations of copulas were introduced and studied from different points of view. In most cases, the authors fixed a copula $D \in \mathcal{C}$ and a suitable bivariate function $H: [0, 1]^2 \rightarrow \mathbb{R}$, and looked for conditions on H (and D) guaranteeing that also the function $C: [0, 1]^2 \rightarrow \mathbb{R}$ given by

$$C(x, y) = D(x, y) + H(x, y) \quad (3.1)$$

was a copula.

Remark 3.1. Several interesting variants and special cases of this general approach were investigated in detail. If not stated otherwise, $D: [0, 1]^2 \rightarrow [0, 1]$ always denotes a given copula.

(i) In [87] the following functional equation was studied:

$$C_\theta(x, y) = D(x, y) + \theta(x - D(x, y))(y - D(x, y)). \quad (3.2)$$

Observe that for $D = \Pi$ and $\theta \in [-1, 1]$ we obtain the Eyraud-Farlie-Gumbel-Morgenstern copula $C_\theta = C_\theta^{\text{EFGM}}$ as given in (2.7).

(ii) Put $D = M$ and $\theta \in [0, 1]$. Then it was mentioned in [87] that

$$C_\theta(x, y) = \max((1 - \theta)M(x, y) + \theta(x + y - 1), 0) \quad (3.3)$$

yields a copula.

(iii) As a variant of (ii), for a function $H: [0, 1]^2 \rightarrow \mathbb{R}$ also the functional equation

$$C(x, y) = \max(D(x, y) + H(x, y), 0) \quad (3.4)$$

was studied in [87] (compare also [77–79, 86]).

(iv) Let $f, g: [0, 1] \rightarrow \mathbb{R}$ be suitable functions and put

$$C(x, y) = D(x, y) + f(\max(x, y))g(\min(x, y)). \quad (3.5)$$

This situation was investigated in [36], while the special case $D = \Pi$ had already been discussed in [4].

(v) As a special case of (iv), a necessary and sufficient condition for the function

$$C(x, y) = xy + f(x)g(y), \quad (3.6)$$

to be a copula was given in [99, Theorem 2.3], and a simpler sufficient condition for being an absolutely continuous copula can be found in Theorem 2.5 in the same paper. Probably the simplest necessary and sufficient condition for C being a copula (see [39, Example 1.6.10]) is that the functions f and g are Lipschitz, vanish in 0 and 1 and satisfy $f'(x)g'(y) \geq -1$ for all $(x, y) \in [0, 1]^2$ for which the derivatives exist (compare also [39, Theorem 1.6.9]). In Section 5, we shall present some new results related to this type of perturbations.

(vi) As a generalization of (v), the parameterized function

$$C_\theta(x, y) = xy + \theta f(x)g(y), \quad (3.7)$$

was considered for the first time in the context of copulas in [105] and later in [82, 98, 117]. A generalized form of (3.7), where in the first summand on the right-hand side the product was replaced by an arbitrary copula, was discussed in [68]. For a survey and other generalizations of the cases (3.5)–(3.7) see [4].

(vii) For an interesting generalization of (vi), namely,

$$C(x, y) = \Pi(x, y) + \sum_{i=1}^n \theta_i f_i(x) g_i(y), \quad (3.8)$$

a necessary and sufficient condition for $\theta_i \in \mathbb{R}$ and $f_i, g_i: [0, 1] \rightarrow \mathbb{R}$ with $i = 1, 2, \dots, n$ was given in [88, Theorem 2.2] (also several sufficient conditions which are easier to check, together with a number of examples, were provided).

(viii) Let $N_1, N_2: [0, 1] \rightarrow [0, 1]$ be two convex strong negations on $[0, 1]$ (see, e.g., [72]). The problem, under which conditions on θ and N_1 and N_2 the function

$$C_\theta(x, y) = \max(xy + \theta N_1(x) N_2(y), 0) \quad (3.9)$$

is a copula was solved in [47] (see also [86]).

Remark 3.2. Other approaches to perturbations are based on weighted arithmetic and geometric means. For example, the weighted arithmetic mean of two arbitrary copulas is again a copula, and thus we can perturb a given copula D by means of some (arbitrary) copula C putting $E = (1 - \varepsilon)D + \varepsilon C$, where $\varepsilon \in]0, 1[$. In the case of the weighted geometric mean, it is known that the set of extreme-value copulas [57] is closed under this averaging operator. For some deeper study of this kind of problems see [20]. For some other types of perturbation, in particular for those connected with the diagonal expansion of a copula, we refer to [21]. More recently, other perturbations of copulas which are related to random noise were described and investigated in [89, 115, 116].

The family of EFGM copulas $(C_\theta^{\text{EFGM}})_{\theta \in [-1, 1]}$ has many nice properties (see, e.g., [104]), but also some drawbacks. One of them is that the values of the dependence functions of the four dependence parameters given in (2.6) are bounded by $-\frac{2}{9}$ and $\frac{2}{9}$ for each EFGM copula, so only weak dependencies can be modeled by these copulas.

A number of extensions of the family of EFGM copulas has been presented in the literature in order to overcome this constraint. A comprehensive extension (i.e., containing the three basic copulas W , Π and M) was introduced in [62], other approaches were based on quadratic constructions of copulas [75] or on some forms of convexity (such as ultramodularity [85] and Schur concavity [107]), see [103]. A very interesting and natural extension of EFGM copulas are polynomial copulas [122] (in [114] polynomial copulas of degree five are studied in detail) which can also be seen as special cases of the general perturbation (3.1). Also some of the perturbations investigated in [36, 68, 77, 78, 87, 98, 105] turn out to be extensions of the family of EFGM copulas.

4 Copula-based perturbations of copulas

We now consider perturbations of an arbitrary copula $C \in \mathcal{C}$ by means of the opposite diagonal sections given by (2.5) of two copulas $C_1, C_2 \in \mathcal{C}$. The study of these perturbations was motivated by the investigations in [36] or [86, 87]. Again, in some special cases we obtain extensions of the family of EFGM copulas.

Definition 4.1. Let C, C_1 and C_2 be three arbitrary copulas and $\theta \in \mathbb{R}$. Then we consider the function $[C, C_1, C_2]_\theta: [0, 1]^2 \rightarrow [0, 1]$ given by

$$[C, C_1, C_2]_\theta(x, y) = C(x, y) + \theta C_1(x, 1 - x) C_2(y, 1 - y). \quad (4.1)$$

Note that (4.1) is a variant of (3.7) in Remark 3.1(vi): in the first summand on the right-hand side, the product is replaced by an arbitrary copula C , and in the second summand the functions f and g equal the opposite diagonals (given by (2.5)) ω_{C_1} and ω_{C_2} , respectively, of two copulas C_1 and C_2 .

In order to find out under which conditions the function $[C, C_1, C_2]_\theta$ is a copula, we have to recall some measure theoretic facts about absolutely continuous and singular parts of copulas (compare [93, Section 2.4]).

If $\lambda: \mathcal{B}([0, 1]^2) \rightarrow [0, 1]$ denotes the Lebesgue measure on the σ -algebra $\mathcal{B}([0, 1]^2)$ of Borel subsets of $[0, 1]^2$ then, as a consequence of [93, Theorem 2.2.7], for each copula $C: [0, 1]^2 \rightarrow [0, 1]$ the mixed derivative $\frac{\partial^2 C(x, y)}{\partial x \partial y}$ exists almost everywhere on $[0, 1]^2$ (with respect to λ). Therefore, the function $\psi_{A_C}: [0, 1]^2 \rightarrow \mathbb{R}$ defined by

$$\psi_{A_C}(x, y) = \begin{cases} \frac{\partial^2 C(x, y)}{\partial x \partial y} & \text{if } \frac{\partial^2 C(x, y)}{\partial x \partial y} \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

is integrable over $[0, 1]^2$, and the function $A_C: [0, 1]^2 \rightarrow [0, 1]$ defined by

$$A_C(x, y) = \int_0^x \int_0^y \psi_{A_C}(u, v) dv du$$

is absolutely continuous. This function A_C is called the absolutely continuous part of the copula C , and the function $S_C: [0, 1]^2 \rightarrow [0, 1]$ given by $S_C = C - A_C$ the singular part of the copula C (see [93, (2.4.1)]). Finally, put

$$\alpha_C = \text{essinf}\{\psi_{A_C}(x, y) \mid (x, y) \in [0, 1]^2\} = \sup\{b \in \mathbb{R} \mid \lambda(\{(x, y) \in [0, 1]^2 \mid \psi_{A_C}(x, y) < b\}) = 0\}. \quad (4.2)$$

The following result provides a complete solution of the problem under which conditions the function $[C, C_1, C_2]_\theta$ is a copula.

Theorem 4.2. *Let $C: [0, 1]^2 \rightarrow [0, 1]$ be a copula, $A_C: [0, 1]^2 \rightarrow [0, 1]$ the absolutely continuous part of C , and α_C as given by (4.2). Then the following are equivalent:*

- (i) *for all copulas $C_1, C_2: [0, 1]^2 \rightarrow [0, 1]$ the function $[C, C_1, C_2]_\theta: [0, 1]^2 \rightarrow [0, 1]$ defined by (4.1) is a copula;*
- (ii) *$\theta \in [-\alpha_C, \alpha_C]$.*

Proof. Fix an arbitrary copula C and some $\theta \in \mathbb{R}$, and assume that condition (i) holds. Define the two functions $f, h: [0, 1] \rightarrow [0, 1]$ by $f(x) = \min(x, 1 - x)$ and $h(x) = 2x - \lfloor 2x \rfloor$, where $\lfloor u \rfloor$ denotes the floor of the real number u , and the sequence of functions $(g_n: [0, 1] \rightarrow [0, 1])_{n \in \mathbb{N}}$ given inductively by $g_1 = f$ and $g_{n+1} = \frac{1}{2}(g_n \circ h)$. Observe that, for each $n \in \mathbb{N}$, the functions f and g_n are non-negative and continuous, and that they vanish at the boundaries of $[0, 1]$, i.e., $f(0) = g_n(0) = f(1) = g_n(1) = 0$. Moreover, we have $\{f'(x), g'_n(y)\} \subseteq \{-1, 1\}$ for each point $(x, y) \in [0, 1]^2$ where these derivatives exist. Then each of these functions is the opposite diagonal section of some copula (see, e.g., [31, 45]), i.e., there exists a sequence of copulas $(C_n)_{n \in \mathbb{N}}$ such that $C_1(x, 1 - x) = f(x)$ and $C_n(x, 1 - x) = g_n(x)$ for each $x \in [0, 1]$ and all $n \in \mathbb{N} \setminus \{1\}$. For each $n \in \mathbb{N}$ consider the sets

$$A_n = \{(x, y) \in [0, 1]^2 \mid f'(x)g'_n(y) = 1\}, \quad B_n = \{(x, y) \in [0, 1]^2 \mid f'(x)g'_n(y) = -1\}$$

which may be written as

$$A_n = \bigcup_{i=0}^{2^{n-1}-1} \left(\left(\left[0, \frac{1}{2} \left[\times \right] \frac{2i}{2^n}, \frac{2i+1}{2^n} \right] \cup \left(\left[\frac{1}{2}, 1 \left[\times \right] \frac{2i+1}{2^n}, \frac{2i+2}{2^n} \right] \right) \right),$$

$$B_n = \bigcup_{i=0}^{2^{n-1}-1} \left(\left(\left[0, \frac{1}{2} \left[\times \right] \frac{2i+1}{2^n}, \frac{2i+2}{2^n} \right] \cup \left(\left[\frac{1}{2}, 1 \left[\times \right] \frac{2i}{2^n}, \frac{2i+1}{2^n} \right] \right) \right),$$

and observe that for each $n \in \mathbb{N}$

$$\lambda(A_n) = \lambda(B_n) = \frac{1}{2} \quad \text{and} \quad \lambda(A_n \cup B_n) = \lambda\left(\bigcup_{j \in \mathbb{N}} A_j\right) = \lambda\left(\bigcup_{j \in \mathbb{N}} B_j\right) = 1.$$

If, for some $\alpha > \alpha_C$ and for $H_{C,\alpha} = \{(x, y) \in [0, 1]^2 \mid \psi_{A_C}(x, y) < \alpha\}$ we have $\lambda(H_{C,\alpha}) > 0$, then there exist $m, n \in \mathbb{N}$ such that $\lambda(A_m \cap H_{C,\alpha}) > 0$ and $\lambda(B_n \cap H_{C,\alpha}) > 0$. Then, for any $(x, y) \in B_n \cap H_{C,\alpha}$, the mixed

derivative of C as well as the derivatives of f and g_n exist such that

$$\frac{\partial^2 [C, C_1, C_n]_\theta(x, y)}{\partial x \partial y} = \psi_{A_C}(x, y) + \theta f'(x) g'_n(y) < \alpha - \theta.$$

Clearly, if $\alpha - \theta < 0$ then $[C, C_1, C_n]_\theta$ has a negative density on a Borel subset of $[0, 1]^2$ with positive Lebesgue measure and, therefore, cannot be a copula. Thus, necessarily, $\theta \leq \alpha$. Similarly, $[C, C_1, C_m]_\theta$ cannot be a copula whenever $\alpha + \theta < 0$, and thus $\theta \geq -\alpha$. Since $\alpha > \alpha_C$ was chosen arbitrarily, these two inequalities imply $\theta \in [-\alpha_C, \alpha_C]$, showing that (i) implies (ii).

Conversely, fix a copula C and some $\theta \in [-\alpha_C, \alpha_C]$, and note that for all copulas $C_1, C_2 \in \mathcal{C}$ the two functions $f, g: [0, 1] \rightarrow [0, 1]$ given by $f(x) = C_1(x, 1-x)$ and $g(x) = C_2(x, 1-x)$ are 1-Lipschitz and that they satisfy the property $f(0) = f(1) = g(0) = g(1) = 0$. Also, the inequality

$$\frac{\partial^2 [C, C_1, C_2]_\theta(x, y)}{\partial x \partial y} = \psi_{A_C}(x, y) + \theta f'(x) g'(y) \geq \alpha_C - |\theta| \geq 0$$

holds on the set of all $(x, y) \in [0, 1]^2$ where $\frac{\partial^2 [C, C_1, C_2]_\theta(x, y)}{\partial x \partial y}$, $f'(x)$ and $g'(y)$ exist (this set has Lebesgue measure 1) and where the mixed derivative is bounded from below by α_C . Moreover, define the functions $f_0, g_0: [0, 1] \rightarrow \mathbb{R}$ by

$$f_0(x) = \begin{cases} f'(x) & \text{if } f'(x) \text{ exists,} \\ 0 & \text{otherwise,} \end{cases} \quad g_0(x) = \begin{cases} g'(x) & \text{if } g'(x) \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

and observe that the following sum of integrals

$$\iint_{[0,1]^2} \psi_{A_C}(x, y) dx dy + \iint_{[0,1]^2} f_0(x) g_0(y) dx dy = \iint_{[0,1]^2} \psi_{A_C}(x, y) dx dy + \int_0^1 f_0(x) dx \int_0^1 g_0(y) dy = \iint_{[0,1]^2} \psi_{A_C}(x, y) dx dy$$

equals the mass of the absolutely continuous part of the copula C , showing that the function $[C, C_1, C_2]_\theta$ is also a copula. $\square$

Theorem 4.2 clarifies, for a given copula C , the minimal range for the parameter θ ensuring that $[C, C_1, C_2]_\theta$ is a copula for an arbitrary choice of C_1 and C_2 . In case that C_1 and C_2 are fixed beforehand, too, also for values of θ outside of the interval $[-\alpha_C, \alpha_C]$ the function $[C, C_1, C_2]_\theta$ may be a copula (compare Remark 4.4(i) below as well as Remark 5.16(ii) for some particular examples).

Example 4.3. Consider the copula $C_{0.5}^{\text{CA}} = \sqrt{M \cdot \Pi}$ which is not absolutely continuous and which belongs to the family of Cuadras-Augé copulas first discussed in [22]. The formula (4.2) yields $\alpha_{C_{0.5}^{\text{CA}}} = 0.5$, so Theorem 4.2 tells us that, for all copulas $C_1, C_2 \in \mathcal{C}$ and for each $\theta \in [-0.5, 0.5]$, the triplet $[C_{0.5}^{\text{CA}}, C_1, C_2]_\theta$ given in (4.1) is a copula. The greatest perturbation of $C_{0.5}^{\text{CA}}$ of this type is obtained if we choose $\widehat{C}_1 = \widehat{C}_2 = M$ and $\theta = 0.5$, in which case we get for all copulas $C_1, C_2 \in \mathcal{C}$, for each $\theta \in [-0.5, 0.5]$ and for each $(x, y) \in [0, 1]^2$

$$\left| [C_{0.5}^{\text{CA}}, C_1, C_2]_\theta(x, y) - C_{0.5}^{\text{CA}}(x, y) \right| \leq [C_{0.5}^{\text{CA}}, M, M]_{0.5}(0.5, 0.5) - C_{0.5}^{\text{CA}}(0.5, 0.5) = 0.125.$$

Remark 4.4. Some properties of the three basic copulas W , M and Π follow directly from Theorem 4.2:

- (i) For all $C, C_1, C_2 \in \mathcal{C}$ we have $[C, C_1, C_2]_0 = C$ and, if $W \in \{C_1, C_2\}$, then we get $[C, C_1, C_2]_\theta = C$ for each $\theta \in \mathbb{R} \supseteq [-\alpha_C, \alpha_C]$.
- (ii) Observe that $\psi_{A_W}(x, y) = \psi_{A_M}(x, y) = 0$ for each $(x, y) \in [0, 1]^2$ and, therefore, $\alpha_W = \alpha_M = 0$. As a consequence, we get $[W, C_1, C_2]_0 = W$ and $[M, C_1, C_2]_0 = M$ for all $C_1, C_2 \in \mathcal{C}$, and neither $[W, C_1, C_2]_\theta$ nor $[M, C_1, C_2]_\theta$ is a copula if $\theta \neq 0$.
- (iii) On the other hand, we have $\psi_{A_\Pi}(x, y) = 1$ for each $(x, y) \in [0, 1]^2$ and, therefore, $\alpha_\Pi = 1$, i.e., $[\Pi, C_1, C_2]_\theta$ is a copula for all $C_1, C_2 \in \mathcal{C}$ and for all $\theta \in [-1, 1]$.

The following monotonicity properties are an immediate consequence of (4.1):

Corollary 4.5. Let C, C_1, C_2 and D_1, D_2 be copulas and $\eta, \theta \in [-\alpha_C, \alpha_C]$. Then we have:

- (i) if $\eta \leq \theta$ then $[C, C_1, C_2]_\eta \leq [C, C_1, C_2]_\theta$;
- (ii) if $C_1 \leq D_1$ and $C_2 \leq D_2$ then

$$\begin{aligned} [C, D_1, D_2]_\theta &\leq [C, C_1, C_2]_\theta && \text{whenever } \theta \in [-\alpha_C, 0[, \\ [C, C_1, C_2]_\theta &\leq [C, D_1, D_2]_\theta && \text{whenever } \theta \in [0, \alpha_C]; \end{aligned}$$

- (iii) $[C, M, M]_{-\alpha_C} \leq [C, C_1, C_2]_\theta \leq [C, M, M]_{\alpha_C}$.

Remark 4.6. Recall that $[\Pi, \Pi, \Pi]_\theta = C_\theta^{\text{EFGM}}$ for each $\theta \in [-1, 1]$. The family

$$([C, C_1, C_2]_\theta)_{(C, C_1, C_2, \theta) \in \mathcal{C}^3 \times [-\alpha_C, \alpha_C]}$$

can be considered as an extension of the family $(C_\theta^{\text{EFGM}})_{\theta \in [-1, 1]}$ of EFGM copulas, as well as several subfamilies thereof, e.g.,

$$([C, \Pi, \Pi]_\theta)_{(C, \theta) \in \mathcal{C} \times [-\alpha_C, \alpha_C]} \quad \text{and} \quad ([\Pi, C_1, C_2]_\theta)_{(C_1, C_2, \theta) \in \mathcal{C}^2 \times [-1, 1]}.$$

As a matter of fact, a copula C is not necessarily symmetric, i.e., we may have $C(x, y) \neq C(y, x)$ for some $(x, y) \in [0, 1]^2$. In the literature, there are several concepts to measure the asymmetry of a (quasi-)copula (see, for instance, [29] for a recent approach to study the asymmetry of a (quasi-)copula with respect to a curve).

Using the Chebyshev norm, in [71, 94] the degree of asymmetry of a copula $C \in \mathcal{C}$ was defined as

$$\text{asymm}(C) = \sup\{|C(x, y) - C(y, x)| \mid (x, y) \in [0, 1]^2\}. \quad (4.3)$$

Obviously, a copula C is symmetric if and only if $\text{asymm}(C) = 0$, and we always have $\text{asymm}(C) \leq \frac{1}{3}$, the maximal value $\frac{1}{3}$ being attained, e.g., by the copula $C^{[\frac{1}{3}]}$ given by $C^{[\frac{1}{3}]}(x, y) = \max(M(x, y - \frac{1}{3}), W(x, y))$.

Example 4.7. In special cases we can say something about the degree of asymmetry of copulas $[C, C_1, C_2]_\theta$ constructed as in (4.1), in particular about the relationship between $\text{asymm}(C)$ and $\text{asymm}([C, C_1, C_2]_\theta)$.

- (i) If C, C_1 and C_2 are arbitrary copulas such that the opposite diagonal sections of C_1 and C_2 coincide, i.e., $\omega_{C_1} = \omega_{C_2}$, then for each $\theta \in [-\alpha_C, \alpha_C]$ we get

$$\text{asymm}([C, C_1, C_2]_\theta) = \text{asymm}(C).$$

In other words, if $\omega_{C_1} = \omega_{C_2}$ and if $[C, C_1, C_2]_\theta$ is a copula then the construction (4.1) preserves the degree of asymmetry of the copula C .

- (ii) If C, C_1 and C_2 are symmetric copulas and $\theta \in [-\alpha_C, \alpha_C]$ then $[C, C_1, C_2]_\theta$ may be asymmetric: put $C = \Pi$, $C_1 = M$ and $C_2 = \Pi$ then we have

$$0 = \text{asymm}(C) < \text{asymm}([C, C_1, C_2]_1) = \frac{1}{32}.$$

Writing $D = [C, C_1, C_2]_1$ we see that $[D, M, \Pi]_{-1} = C$ and, therefore,

$$\frac{1}{32} = \text{asymm}(D) > \text{asymm}([D, C_1, C_2]_{-1}) = 0.$$

- (iii) Fix the two W -ordinal sums $\widehat{C}_1 = W(\langle \frac{1}{2}, 1, M \rangle)$ and $\widehat{C}_2 = W(\langle 0, \frac{1}{2}, M \rangle)$ which are asymmetric copulas ($\text{asymm}(\widehat{C}_1) = \text{asymm}(\widehat{C}_2) = \frac{1}{4}$) and whose opposite diagonal sections $\omega_{\widehat{C}_1}, \omega_{\widehat{C}_2} : [0, 1] \rightarrow [0, 1]$ are given by

$$\omega_{\widehat{C}_1}(x) = \max\left(\min\left(1 - x, x - \frac{1}{2}\right), 0\right) \quad \text{and} \quad \omega_{\widehat{C}_2}(x) = \omega_{\widehat{C}_1}(1 - x). \quad (4.4)$$

Then $\omega_{\widehat{C}_1}(x) \cdot \omega_{\widehat{C}_2}(y) \neq 0$ implies $(x, y) \in]\frac{1}{2}, 1] \times [0, \frac{1}{2}[$, in which case we have $\omega_{\widehat{C}_1}(y) \cdot \omega_{\widehat{C}_2}(x) = 0$. Thus the maximal absolute difference between the values $\omega_{\widehat{C}_1}(x) \cdot \omega_{\widehat{C}_2}(y)$ and $\omega_{\widehat{C}_1}(y) \cdot \omega_{\widehat{C}_2}(x)$ for $(x, y) \in [0, 1]^2$

equals $\frac{1}{16}$ and it is attained at the point $(\frac{3}{4}, \frac{1}{4})$, i.e., for each $\theta \in [-1, 1]$ we get for the degree of asymmetry of the copula $[\Pi, \widehat{C}_1, \widehat{C}_2]_\theta$:

$$\begin{aligned} \text{asymm}([\Pi, \widehat{C}_1, \widehat{C}_2]_\theta) &= \sup \left\{ \left| [\Pi, \widehat{C}_1, \widehat{C}_2]_\theta(x, y) - [\Pi, \widehat{C}_1, \widehat{C}_2]_\theta(y, x) \right| \mid (x, y) \in [0, 1]^2 \right\} \\ &= |\theta| \cdot \sup \left\{ \left| \omega_{\widehat{C}_1}(x) \cdot \omega_{\widehat{C}_2}(y) - \omega_{\widehat{C}_1}(y) \cdot \omega_{\widehat{C}_2}(x) \right| \mid (x, y) \in [0, 1]^2 \right\} \\ &= |\theta| \cdot \left| \omega_{\widehat{C}_1}(\tfrac{3}{4}) \cdot \omega_{\widehat{C}_2}(\tfrac{1}{4}) - \omega_{\widehat{C}_1}(\tfrac{1}{4}) \cdot \omega_{\widehat{C}_2}(\tfrac{3}{4}) \right| \\ &= \frac{1}{16} |\theta|. \end{aligned} \quad (4.5)$$

Here the relationship between the parameter θ and the degree of asymmetry of $[\Pi, \widehat{C}_1, \widehat{C}_2]_\theta$ is very simple, as it is given by a linear function of $|\theta|$. In Remarks 5.13 and 5.17 we present two families of copulas whose degrees of asymmetry depend on the respective parameters in a more complex way.

- (iv) If D is an arbitrary copula and if D_1 and D_2 are asymmetric copulas having the same opposite diagonal sections as the two copulas $\widehat{C}_1$ and $\widehat{C}_2$, respectively, which we considered in (iii), i.e., if ω_{D_1} and ω_{D_2} coincide with $\omega_{\widehat{C}_1}$ and $\omega_{\widehat{C}_2}$ as given in (4.4), respectively, then for each $\theta \in [-\alpha_D, \alpha_D]$ the degree of asymmetry of $[D, D_1, D_2]_\theta$ is bounded from above by the sum of the degrees of asymmetry of D and $[\Pi, \widehat{C}_1, \widehat{C}_2]_\theta$. In other words, for each $\theta \in [-\alpha_D, \alpha_D]$ we have

$$\text{asymm}([D, D_1, D_2]_\theta) \leq \text{asymm}(D) + \text{asymm}([\Pi, \widehat{C}_1, \widehat{C}_2]_\theta) = \text{asymm}(D) + \frac{1}{16} |\theta|. \quad (4.6)$$

If the copula D is symmetric then in (4.6) the equality holds.

5 Perturbations of the product copula and ordinal sums

For the copulas considered in this section, recall the constructions of M -ordinal sums and W -ordinal sums in (2.8) and (2.9), respectively, and the x -flipping, the y -flipping and the survival copula of a given ordinal sum defined in (2.2). The verification of the equalities in Lemma 5.1 is a matter of tedious computations.

Lemma 5.1. *The relationship between M -ordinal sums $M\langle (a_k, b_k, C_k) \rangle_{k \in K}$ and W -ordinal sums $W\langle (a_k, b_k, C_k) \rangle_{k \in K}$, on the one hand, and the x -flipping, the y -flipping and the survival copula of a given copula, on the other hand, can be formalized as follows:*

$$\begin{aligned} (M\langle (a_k, b_k, C_k) \rangle_{k \in K})^{\text{xflip}} &= W\langle (1 - b_k, 1 - a_k, (C_k)^{\text{xflip}}) \rangle_{k \in K}, \\ (W\langle (a_k, b_k, C_k) \rangle_{k \in K})^{\text{xflip}} &= M\langle (1 - b_k, 1 - a_k, (C_k)^{\text{xflip}}) \rangle_{k \in K}, \\ (M\langle (a_k, b_k, C_k) \rangle_{k \in K})^{\text{yflip}} &= W\langle (a_k, b_k, (C_k)^{\text{yflip}}) \rangle_{k \in K}, \\ (W\langle (a_k, b_k, C_k) \rangle_{k \in K})^{\text{yflip}} &= M\langle (a_k, b_k, (C_k)^{\text{yflip}}) \rangle_{k \in K}, \\ (M\langle (a_k, b_k, C_k) \rangle_{k \in K})^{\text{surv}} &= M\langle (1 - b_k, 1 - a_k, (C_k)^{\text{surv}}) \rangle_{k \in K}, \\ (W\langle (a_k, b_k, C_k) \rangle_{k \in K})^{\text{surv}} &= W\langle (1 - b_k, 1 - a_k, (C_k)^{\text{surv}}) \rangle_{k \in K}. \end{aligned} \quad (5.1)$$

Let us start with a special class of M -ordinal sums constructed by means of two copies of the product copula Π .

Definition 5.2. For each $r \in]0, \infty[$ consider the copula $\Pi^{(r)}: [0, 1]^2 \rightarrow [0, 1]$, defined as M -ordinal sum as follows:

$$\Pi^{(r)} = M\left(\left\langle 0, \frac{1}{r+1}, \Pi \right\rangle, \left\langle \frac{1}{r+1}, 1, \Pi \right\rangle\right). \quad (5.2)$$

If in Definition 5.2 we switch to the functional expression for M -ordinal sums given in (2.8), then for each $r \in]0, \infty[$ we obtain the following explicit formula for the copula $\Pi^{(r)} : [0, 1]^2 \rightarrow [0, 1]$:

$$\Pi^{(r)}(x, y) = \begin{cases} (r+1)xy & \text{if } (x, y) \in [0, \frac{1}{r+1}]^2, \\ \frac{(r+1)xy - x - y + 1}{r} & \text{if } (x, y) \in]\frac{1}{r+1}, 1]^2, \\ M(x, y) & \text{otherwise.} \end{cases}$$

If we take into account that $\Pi^{x\text{flip}} = \Pi^{y\text{flip}} = \Pi^{\text{surv}} = \Pi$, then the next three equalities follow in a straightforward way from (5.1):

Corollary 5.3. For each $r \in]0, \infty[$ we have

$$\begin{aligned} (\Pi^{(r)})^{x\text{flip}} &= W\left(\left\langle 0, \frac{r}{r+1}, \Pi \right\rangle, \left\langle \frac{r}{r+1}, 1, \Pi \right\rangle\right), \\ (\Pi^{(r)})^{y\text{flip}} &= W\left(\left\langle 0, \frac{1}{r+1}, \Pi \right\rangle, \left\langle \frac{1}{r+1}, 1, \Pi \right\rangle\right), \\ (\Pi^{(r)})^{\text{surv}} &= M\left(\left\langle 0, \frac{r}{r+1}, \Pi \right\rangle, \left\langle \frac{r}{r+1}, 1, \Pi \right\rangle\right). \end{aligned} \quad (5.3)$$

Comparing formulas (5.2) and (5.3), we see that the two copulas $\Pi^{(r)} : [0, 1]^2 \rightarrow [0, 1]$ and $(\Pi^{(r)})^{y\text{flip}}$ share the same pair of summands, namely, $(\langle 0, \frac{1}{r+1}, \Pi \rangle, \langle \frac{1}{r+1}, 1, \Pi \rangle)$, but they are ordinal sums of different types: the former is an M -ordinal sum, while the latter is a W -ordinal sum. For the sake of simplicity of the notations, let us fix a shortcut for the y -flipping of the copula $\Pi^{(r)}$:

Definition 5.4. For each $r \in]0, \infty[$, define the copula $\Pi^{(-r)} : [0, 1]^2 \rightarrow [0, 1]$ by

$$\Pi^{(-r)} = (\Pi^{(r)})^{y\text{flip}}. \quad (5.4)$$

Clearly, for each constant $r \in]-\infty, 0[\cup]0, \infty[$, we have $\frac{z}{z+1} = \frac{1}{r+1}$ if and only if $z = \frac{1}{r}$. This allows us to derive the notions $\Pi^{(1/r)}$ and $\Pi^{(-1/r)}$ directly, replacing the variable $r \in \mathbb{R} \setminus \{0\}$ in the formulas (5.2) and (5.4) by its reciprocal value $\frac{1}{r} \in \mathbb{R} \setminus \{0\}$.

Corollary 5.5. For each $r \in]-\infty, 0[\cup]0, \infty[$ we have

$$\Pi^{(-1/r)} = (\Pi^{(r)})^{x\text{flip}} \quad \text{and} \quad \Pi^{(1/r)} = (\Pi^{(r)})^{\text{surv}}. \quad (5.5)$$

Definition 5.6. Denote the set of all copulas $\Pi^{(r)}$ and $\Pi^{(-r)}$ given in (5.2) and (5.4), respectively, by $\mathcal{C}_\Pi^*$, i.e.,

$$\mathcal{C}_\Pi^* = \{\Pi^{(r)} \mid r \in]-\infty, 0[\cup]0, \infty[\}. \quad (5.6)$$

Six copulas $\Pi^{(r)} \in \mathcal{C}_\Pi^*$ as given in (5.2) and (5.4)–(5.5) are shown in Figure 1, as well as, in Figure 3, some contour plots of such copulas.

Remark 5.7. If we take into account (2.4), we immediately see that the set $\mathcal{C}_\Pi^*$ and the corresponding operations (x -flipping, y -flipping and the construction of the survival copula) induce a commutative diagram which is shown in the left-hand part of Figure 2 (compare also Figure 3).

Obviously, this commutative diagram is isomorphic to the commutative diagram in the right-hand part of Figure 2, where we “calculate” only with the parameters of the copulas $\Pi^{(r)}$, due to (5.4)–(5.5). A canonical isomorphism $\varphi : \mathbb{R} \setminus \{0\} \rightarrow \mathcal{C}_\Pi^*$ between these two commutative diagrams is given by $\varphi(r) = \Pi^{(r)}$.

However, the copulas $\Pi^{(r)} \in \mathcal{C}_\Pi^*$ have some additional properties. In particular, they are extremal elements (with respect to the usual partial order $\leq$ for functions from $[0, 1]^2$ to $[0, 1]$) of a special class of copulas which has been studied in a number of papers (see, e.g., [4, 36, 99, 105]).

From Example 1.6.10 in [39] we know that the function $C : [0, 1]^2 \rightarrow [0, 1]$ defined by $C(x, y) = xy + f(x)g(y)$ (see (3.6) in Remark 3.1(v)) is a copula if and only if the functions $f, g : [0, 1] \rightarrow \mathbb{R}$ are Lipschitz, vanish at the border of the unit interval $[0, 1]$ and satisfy $f'(x)g'(y) \geq -1$ for all $(x, y) \in [0, 1]^2$ where $f'(x)g'(y)$ exists.

Figure 1: 3D plots of three minimal elements given by (5.4) (top) and three maximal elements given by (5.2) (bottom) of the set $\mathcal{C}_\Pi^*$ (see Theorem 5.9).

Definition 5.8. Denote the set of Lipschitz functions on the unit interval vanishing at the boundaries of $[0, 1]$ by $\mathcal{L}_b$:

$$\mathcal{L}_b = \{f: [0, 1] \rightarrow \mathbb{R} \mid f \text{ is Lipschitz and } f(0) = f(1) = 0\}.$$

Furthermore, define the set $\mathcal{F}_\Pi$ by

$$\mathcal{F}_\Pi = \{(f, g) \in \mathcal{L}_b \times \mathcal{L}_b \mid f'(x)g'(y) \geq -1 \text{ whenever } f'(x)g'(y) \text{ exists}\}$$

and, whenever $(f, g) \in \mathcal{F}_\Pi$, the function $\Pi_{[f,g]}: [0, 1]^2 \rightarrow \mathbb{R}$ by

$$\Pi_{[f,g]}(x, y) = \Pi(x, y) + \Pi(f(x), g(y)).$$

Finally, since [39, Example 1.6.10] tells us that each function $\Pi_{[f,g]}$ with $(f, g) \in \mathcal{F}_\Pi$ is a copula, we can consider the partially ordered set $\mathcal{C}^{\text{pert}, \Pi}$ of copulas given by

$$\mathcal{C}^{\text{pert}, \Pi} = \{\Pi_{[f,g]} \mid (f, g) \in \mathcal{F}_\Pi\}. \quad (5.7)$$

With these notations in mind, we are ready to show that each of the copulas $\Pi^{(r)}$ with $r \in]-\infty, 0[\cup]0, \infty[$ is an extremal element of the partially ordered set of copulas $\mathcal{C}^{\text{pert}, \Pi}$.

Theorem 5.9. For each $r \in]0, \infty[$ we have:

- (i) the copula $\Pi^{(r)}$ is a maximal element of the set $\mathcal{C}^{\text{pert}, \Pi}$;
- (ii) the copula $\Pi^{(-r)}$ is a minimal element of the set $\mathcal{C}^{\text{pert}, \Pi}$.

Figure 2: Two isomorphic commutative diagrams discussed in Remark 5.7: the partially ordered sets $(\mathcal{C}_\Pi^*, \subseteq)$ and $(\mathbb{R} \setminus \{0\}, \leq)$ (compare also Figure 3).

Proof. Fix $a, b \in \mathbb{R}$ with $\min(a, b) > 0$ and a non-constant function $f_{\langle a, b \rangle} \in \mathcal{L}_b$ such that

$$\begin{aligned} a &= \sup\{-f'_{\langle a, b \rangle}(x) \mid x \in [0, 1] \text{ and } f'_{\langle a, b \rangle}(x) \text{ exists}\}, \\ b &= \sup\{f'_{\langle a, b \rangle}(x) \mid x \in [0, 1] \text{ and } f'_{\langle a, b \rangle}(x) \text{ exists}\}. \end{aligned}$$

It follows that $f_{\langle a, b \rangle}(x) \leq bx$ and $f_{\langle a, b \rangle}(x) \leq a(1-x)$ for all $x \in [0, 1]$, i.e., the function $f_{\langle a, b \rangle}^* : [0, 1] \rightarrow \mathbb{R}$ given by

$$f_{\langle a, b \rangle}^*(x) = \min(bx, a(1-x))$$

is the greatest function in $\mathcal{L}_b$ such that the range of its derivative is a subset of $[-a, b]$. Taking into account that $\Pi_{[f_{\langle a, b \rangle}, g]} \in \mathcal{C}^{\text{pert}, \Pi}$ implies $f'_{\langle a, b \rangle}(x)g'(y) \geq -1$ for all $(x, y) \in [0, 1]^2$ where the derivatives $f'_{\langle a, b \rangle}(x)$ and $g'(y)$ exist, we see that the function $g_{\langle 1/b, 1/a \rangle}^* : [0, 1] \rightarrow \mathbb{R}$ given by

$$g_{\langle 1/b, 1/a \rangle}^*(x) = \min\left(\frac{x}{a}, \frac{1-x}{b}\right)$$

is the greatest function in $\mathcal{L}_b$ such that the range of its derivative is a subset of $[-\frac{1}{b}, \frac{1}{a}]$. If we put $r = \frac{b}{a} > 0$, this implies that for each $g \in \mathcal{L}_b$ with $(f_{\langle a, b \rangle}, g) \in \mathcal{F}_\Pi$ we obtain

$$\Pi_{[f_{\langle a, b \rangle}, g]} \leq \Pi_{[f_{\langle a, b \rangle}^*, g]} \leq \Pi_{[f_{\langle a, b \rangle}^*, g_{\langle 1/b, 1/a \rangle}^*]} = \Pi^{(r)},$$

where $\Pi^{(r)} : [0, 1]^2 \rightarrow [0, 1]$ is given by (5.2). Since for any $r_1, r_2 > 0$ with $r_1 \neq r_2$ the copulas $\Pi^{(r_1)}$ and $\Pi^{(r_2)}$ are incomparable because of their M -ordinal sum structure, we see that $\Pi^{(r)}$ is a maximal element of $\mathcal{C}^{\text{pert}, \Pi}$ for each $r \in]0, \infty[$, showing that (i) holds.

In order to show (ii), choose an arbitrary $r \in]0, \infty[$. Note first that if $g \in \mathcal{L}_b$ then for the flipped function $g^- : [0, 1] \rightarrow [0, 1]$ given by $g^-(x) = -g(1-x)$ we also have $g^- \in \mathcal{L}_b$ and, thus, $(\Pi_{[f, g]})^{\text{flip}} = \Pi_{[f, g^-]}$ for each $\Pi_{[f, g]} \in \mathcal{C}^{\text{pert}, \Pi}$, i.e., the set $\mathcal{C}^{\text{pert}, \Pi}$ is closed under y -flipping. Since the y -flipping reverses the order and preserves the incomparability of copulas, a copula is maximal if and only if its y -flipping is minimal, showing that for each $r \in]0, \infty[$ the copula $\Pi^{(r)} = (\Pi^{(r)})^{\text{flip}}$ (compare also (5.2)) is a minimal element of $\mathcal{C}^{\text{pert}, \Pi}$. $\square$

Figure 3: Contour plots of some extremal copulas in $\mathcal{C}_\Pi^*$ (compare the two commutative diagrams in Figure 2): $\Pi^{(0.25)}$, $\Pi^{(-0.25)}$, $\Pi^{(4)}$, and $\Pi^{(-4)}$ (clockwise, from top right).

We can even show more: the copulas $\Pi^{(r)}$ with $r \in]0, \infty[$ are the only elements of $\mathcal{C}^{\text{pert}, \Pi}$ which have a non-trivial idempotent element. Recall that $u \in [0, 1]$ is called a non-trivial idempotent element of a copula $C: [0, 1]^2 \rightarrow [0, 1]$ if $0 < u < 1$ and $C(u, u) = u$.

Proposition 5.10. *Let $(f, g) \in \mathcal{F}_\Pi$ be a pair of functions such that the copula $\Pi_{[f, g]} \in \mathcal{C}^{\text{pert}, \Pi}$ has a non-trivial idempotent element. Then there exists an $r \in]0, \infty[$ such that $\Pi_{[f, g]} = \Pi^{(r)}$.*

Proof. Suppose that $\hat{u} \in]0, 1[$ is a non-trivial idempotent element of $\Pi_{[f, g]}$. Then there exist two copulas C_1 and C_2 such that $\Pi_{[f, g]}$ is an M -ordinal sum:

$$\Pi_{[f, g]} = M \cdot (\langle 0, \hat{u}, C_1 \rangle, \langle \hat{u}, 1, C_2 \rangle).$$

This means $\Pi_{[f, g]}(x, y) = x$ whenever $\hat{u} \in [x, y]$, and $\Pi_{[f, g]}(x, y) = y$ whenever $\hat{u} \in [y, x]$.

Therefore, if $\hat{u} \in [x, y]$ then $\Pi_{[f, g]}(x, y) = xy + f(x)g(y) = x$ and, as a consequence, $f(x)g(y) = x(1 - y)$, i.e., there is a constant $b > 0$ such that $f(x) = bx$ for all $x \in [0, \hat{u}]$ and $g(x) = \frac{1-x}{b}$ for all $x \in [\hat{u}, 1]$. Similarly, if $\hat{u} \in [y, x]$ then we get $f(x)g(y) = (1 - x)y$, implying that there is a constant $a > 0$ such that $f(x) = a(1 - x)$ whenever $x \in [\hat{u}, 1]$, and $g(x) = \frac{x}{a}$ whenever $x \in [0, \hat{u}]$. Moreover, we have $f(\hat{u}) = b\hat{u} = a(1 - \hat{u})$ and $g(\hat{u}) = \frac{1-\hat{u}}{b} = \frac{\hat{u}}{a}$, both implying $a = \frac{b\hat{u}}{1-\hat{u}}$. Putting $r = \frac{1-\hat{u}}{\hat{u}} > 0$ we have $a = \frac{b}{r}$.

Summarizing, we have identified a solution $(f_r, g_r) \in \mathcal{F}_\Pi$ of the functional equation $\Pi_{[f, g]}(x, y) = xy + f(x)g(y)$ which is unique up to pairs of positive multiplicative constants $(a, \frac{1}{a})$ and which is given by

$$f_r(x) = \min(rx, 1 - x) \quad \text{and} \quad g_r(x) = \min\left(x, \frac{1-x}{r}\right), \quad (5.8)$$

Now it is not difficult to check that $\Pi_{[f_r, g_r]} = \Pi^{(r)}$. □

Remark 5.11. Let $\Pi_{[f, g]}$ be a copula in $\mathcal{C}^{\text{pert}, \Pi}$ with $(f, g) \in \mathcal{F}_\Pi$.

Figure 4: Asymmetry in $\mathcal{C}_\Pi^*$: the degree of asymmetry $\text{asymm}(\Pi^{(r)})$ (see Remark 5.13) for $r \in [-5, 0[\cup]0, 2]$. Each copula $\Pi^{(r)}$ with $r > 0$ is symmetric, and so is $\Pi^{(-1)}$. The maximal degree of asymmetry in $\mathcal{C}_\Pi^*$, namely, $3 - 2\sqrt{2} \approx 0.171573$, is obtained for $r \in \{-1 - \sqrt{2}, 1 - \sqrt{2}\}$.

- (i) As follows from the proof of Proposition 5.10, $\Pi_{[f,g]}$ can have at most one non-trivial idempotent element $\hat{u} \in]0, 1[$, which uniquely determines the corresponding copula $\Pi^{((1-\hat{u})/\hat{u})} = \Pi_{[f,g]}$.
- (ii) Similarly, one can show that there can exist at most one element $\hat{u} \in]0, 1[$ such that for $\Pi_{[f,g]}(\hat{u}, 1 - \hat{u}) = 0$, which uniquely determines the corresponding copula $\Pi^{((\hat{u}-1)/\hat{u})} = \Pi_{[f,g]}$ (which, because of $\frac{\hat{u}-1}{\hat{u}} < 0$, necessarily is a minimal element of $\mathcal{C}^{\text{pert}, \Pi}$).

In Proposition 5.10 we have seen that, for $r \in]0, \infty[$, the functions f_r and g_r given by (5.8) generate the copula $\Pi^{(r)}$ in a canonical way: $\Pi^{(r)} = \Pi_{[f_r, g_r]}$. Also the copulas $\Pi^{(-r)}$, $\Pi^{(-1/r)}$ and $\Pi^{(1/r)}$ can be obtained in a similar way.

Remark 5.12. As a by-product of the second part of the proof of Theorem 5.9, it follows that the set of copulas $\mathcal{C}^{\text{pert}, \Pi}$ is closed under y-flipping. However, the set $\mathcal{C}^{\text{pert}, \Pi}$ is also closed under x-flipping and the construction of the survival copula: fix an arbitrary $r \in]0, \infty[$ and start with the pair of functions $(f_r, g_r) \in \mathcal{F}_\Pi$ given by (5.8). Define the functions $f_r^-, g_r^- : [0, 1] \rightarrow [0, 1]$ in the same way as in the proof of Theorem 5.9 by

$$f_r^-(x) = -f_r(1-x) = -\min(r(1-x), x) \quad \text{and} \quad g_r^-(x) = -g_r(1-x) = -\min\left(1-x, \frac{x}{r}\right),$$

and observe that $\{f_r^-, g_r^-\} \subseteq \mathcal{L}_b$. Then it is not difficult to check that

$$(\Pi^{(r)})^{x\text{-flip}} = \Pi_{[f_r^-, g_r^-]} = \Pi^{(-1/r)} \quad \text{and} \quad (\Pi^{(r)})^{\text{surv}} = \Pi_{[f_r^-, g_r^-]} = \Pi^{(1/r)}.$$

Finally, note that we also have the following limit properties:

$$\lim_{r \rightarrow -\infty} \Pi^{(r)} = \lim_{r \uparrow 0} \Pi^{(r)} = \lim_{r \downarrow 0} \Pi^{(r)} = \lim_{r \rightarrow \infty} \Pi^{(r)} = \Pi.$$

Remark 5.13. Figures 1 and 3 indicate clearly that the set $\mathcal{C}_\Pi^*$ contains symmetric and asymmetric copulas. More precisely, for each $\theta \in \mathbb{R} \setminus \{0\}$ the degree of asymmetry $\text{asymm}(\Pi^{(r)})$ of the copula $\Pi^{(r)}$ defined in (4.3) (see [71, 94]) is given by (the relationship between the parameter r and $\text{asymm}(\Pi^{(r)})$ is illustrated in Figure 4)

$$\text{asymm}(\Pi^{(r)}) = \begin{cases} \frac{1+r}{r(1-r)} & \text{if } r \in]-\infty, -1[, \\ \frac{r(r+1)}{r-1} & \text{if } r \in [-1, 0[, \\ 0 & \text{if } r \in]0, \infty[. \end{cases}$$

For the following family of copulas we shall make use of the so-called (vertical) Π -ordinal sums of copulas given in (2.10) (for more details see [102]).

Definition 5.14. For each $r \in]0, \infty[\setminus \{1\}$ define the copula $\Pi\text{-}(W, M)_{(r)} : [0, 1]^2 \rightarrow [0, 1]$ by

$$\Pi\text{-}(W, M)_{(r)} = \begin{cases} \Pi\text{-}(\langle 0, 1-r, W \rangle, \langle 1-r, 1, M \rangle) & \text{if } r \in]0, 1[, \\ \Pi\text{-}(\langle 0, \frac{1}{r}, M \rangle, \langle \frac{1}{r}, 1, W \rangle) & \text{if } r \in]1, \infty[. \end{cases} \quad (5.9)$$

Figure 5: 3D plots of several Π -ordinal sums $\Pi-(W, M)_{\langle r \rangle}$ with $r \in]0, \infty[\setminus \{1\}$, as given in Definition 5.14.

If in (5.9) we switch to the functional expression for Π -ordinal sums as given in (2.10), then we get the following explicit formulas for the copula $\Pi-(W, M)_{\langle r \rangle} : [0, 1]^2 \rightarrow [0, 1]$ in the two cases, $r \in]0, 1[$ and $r \in]1, \infty[$, respectively:

(i) in the case $r \in]0, 1[$ we have

$$\Pi-(W, M)_{\langle r \rangle}(x, y) = \begin{cases} 0 & \text{if } y \in [0, \frac{-x+1-r}{1-r}], \\ y & \text{if } y \in [\frac{rx-r}{1-r}, 1], \\ x + (y-1)(1-r) & \text{otherwise;} \end{cases}$$

(ii) in the case $r \in]1, \infty[$ we have

$$\Pi-(W, M)_{\langle r \rangle}(x, y) = \begin{cases} x & \text{if } y \in [rx, 1], \\ x + y - 1 & \text{if } y \in [\frac{rx-r}{1-r}, 1], \\ \frac{y}{r} & \text{otherwise.} \end{cases}$$

Remark 5.15. Recall the support of a copula $C \in \mathcal{C}$ which is defined as the complement of the union of all open subsets of $[0, 1]^2$ with C -measure zero, where the C -measure is the probability measure on $[0, 1]^2$ which is induced by the copula C (for more details see [93, Section 2.4]).

It is not difficult to see that, for $r \in]0, 1[$, the support of the singular copula $\Pi-(W, M)_{\langle r \rangle} \in \mathcal{C}$ consists of the line segments connecting the point $(1-r, 0)$ with $(0, 1)$ and $(1, 1)$ (blue in Figure 6), respectively. For $r \in]1, \infty[$, the support of $\Pi-(W, M)_{\langle r \rangle} \in \mathcal{C}$ is given by the line segments connecting the point $(\frac{1}{r}, 1)$ with $(0, 0)$ and $(1, 0)$ (red in Figure 6).

Figure 6: The singular copulas $\Pi\text{-}(W, M)_{\langle r \rangle}$ with $r \in]0, \infty[$ given in (5.9): Case 1, i.e., $r \in]0, 1[$ (left, support shown in blue) and Case 2, i.e., $r \in]1, \infty[$ (right, support shown in red).

Remark 5.16. The family of copulas $(\Pi\text{-}(W, M)_{\langle r \rangle})_{r \in]0, \infty[\setminus \{1\}}$ has a number of properties which can be derived directly from Definition 5.14.

(i) Because of

$$\lim_{r \uparrow 1} \Pi\text{-}(W, M)_{\langle r \rangle} = \lim_{r \downarrow 1} \Pi\text{-}(W, M)_{\langle r \rangle} = M \quad \text{and} \quad \lim_{r \rightarrow 0} \Pi\text{-}(W, M)_{\langle r \rangle} = \lim_{r \rightarrow \infty} \Pi\text{-}(W, M)_{\langle r \rangle} = W,$$

it makes sense to define $\Pi\text{-}(W, M)_{\langle 1 \rangle} = M$ and $\Pi\text{-}(W, M)_{\langle 0 \rangle} = \Pi\text{-}(W, M)_{\langle \infty \rangle} = W$. Then the family of copulas $(\Pi\text{-}(W, M)_{\langle r \rangle})_{r \in [0, \infty]}$ is continuous with respect to the parameter r , and non-decreasing on $[0, 1]$ and non-increasing on $[1, \infty]$ (also with respect to the parameter r). Six 3D plots of copulas of the form $\Pi\text{-}(W, M)_{\langle r \rangle}$ with $r \in]0, \infty[$ in Figure 7 illustrate this construction.

(ii) Consider the opposite diagonal section $\omega_{\Pi\text{-}(W, M)_{\langle r \rangle}} : [0, 1] \rightarrow [0, 1]$ of $\Pi\text{-}(W, M)_{\langle r \rangle}$ given by (2.5) and observe that we get $\omega_{\Pi\text{-}(W, M)_{\langle r \rangle}}(x) = \min(rx, 1 - x)$ whenever $r \in]0, 1[$, and $\omega_{\Pi\text{-}(W, M)_{\langle r \rangle}}(x) = \min(x, \frac{1-x}{r})$ whenever $r \in]0, \infty[$. Recalling the pair of functions (f_r, g_r) defined in (5.8), we see that

$$\omega_{\Pi\text{-}(W, M)_{\langle r \rangle}} = \begin{cases} f_r & \text{if } r \in]0, 1[, \\ g_r & \text{if } r \in]1, \infty[. \end{cases}$$

(iii) As a consequence, the copulas $\Pi\text{-}(W, M)_{\langle r \rangle}$ studied here provide an interesting bridge between the copulas $\Pi^{(r)}$ considered in (5.2) and the copulas $[C, C_1, C_2]_\theta$ defined in (4.1):

$$\Pi^{(r)} = \begin{cases} \left[\Pi, \Pi\text{-}(W, M)_{\langle r \rangle}, \Pi\text{-}(W, M)_{\langle r \rangle} \right]_{\frac{1}{r}} & \text{if } r \in]0, 1[, \\ \left[\Pi, \Pi\text{-}(W, M)_{\langle r \rangle}, \Pi\text{-}(W, M)_{\langle r \rangle} \right]_r & \text{if } r \in]1, \infty[. \end{cases} \quad (5.10)$$

Note that, with the exception of $r = 1$ where we have $\Pi^{(1)} = [\Pi, M, M]_1$, in all other cases the parameters $\frac{1}{r}$ (if $r < 1$) and r (if $r > 1$) of the expressions in formula (5.10) are not contained in the interval $[-1, 1] = [-\alpha_\Pi, \alpha_\Pi]$.

(iv) Defining, for an arbitrary copula $C \in \mathcal{C}$, the copula $\bar{C} : [0, 1]^2 \rightarrow [0, 1]$ by $\bar{C}(x, y) = C(y, x)$, we can extend the range of the parameter r of $\Pi^{(r)}$ from $[0, \infty]$ to $[-\infty, \infty]$ by introducing negative parameters:

$$\Pi^{(r)} = \begin{cases} \left[\Pi, \Pi\text{-}(W, M)_{\langle -r \rangle}, \overline{\Pi\text{-}(W, M)_{\langle -r \rangle}} \right]_{\frac{1}{r}} & \text{if } r \in]-1, 0[, \\ [\Pi, M, M]_{-1} & \text{if } r = -1, \\ \left[\Pi, \Pi\text{-}(W, M)_{\langle -r \rangle}, \overline{\Pi\text{-}(W, M)_{\langle -r \rangle}} \right]_r & \text{if } r \in]-\infty, -1[. \end{cases}$$

Observe that the definition $\Pi^{(-1)} = [\Pi, M, M]_{-1}$ makes sense because of $\lim_{r \uparrow -1} \Pi^{(r)} = [\Pi, M, M]_{-1} = \lim_{r \downarrow -1} \Pi^{(r)}$.

Remark 5.17. The 3D plots in Figure 5 suggest that the family $(\Pi(W, M)_{(r)})_{r \in]0, \infty[\setminus \{1\}}$ contains asymmetric copulas. Indeed for each $r \in]0, \infty[\setminus \{1\}$ the degree of asymmetry $\text{asymm}(\Pi(W, M)_{(r)})$ of the copula $\Pi(W, M)_{(r)}$ defined in (4.3) (see [71, 94]) is given by (for a visualization of the dependence of $\text{asymm}(\Pi(W, M)_{(r)})$ on the parameter r see Figure 7)

$$\text{asymm}(\Pi(W, M)_{(r)}) = \begin{cases} \frac{r(r-1)}{r^2-r-1} & \text{if } r \in]0, 1[, \\ \frac{r-1}{r^2+r-1} & \text{if } r \in]1, \infty[. \end{cases}$$

Figure 7: The degree of asymmetry $\text{asymm}(\Pi(W, M)_{(r)})$ (see Remark 5.13) for $r \in]0, 1[\cup]1, 6[$. No member of the family of copulas $(\Pi(W, M)_{(r)})_{r \in]0, \infty[\setminus \{1\}}$ is symmetric, and the maximal degree of asymmetry for this special set of Π -ordinal sums, namely, $\frac{1}{5}$, is obtained for $r \in \{\frac{1}{2}, 2\}$.

6 Markov product and dependence parameters

Several properties of copulas in the classes $\mathcal{C}_\Pi^*$ given by (5.6) and $\mathcal{C}^{\text{pert}, \Pi}$ given by (5.7) can be nicely related to some results on the Markov product of copulas. Finally, we shall discuss the dependence parameters of different perturbation-based copulas.

6.1 Perturbations which are idempotent with respect to the Markov product

Given two copulas $C_1, C_2 \in \mathcal{C}$, it is well-known that the partial derivatives $\frac{\partial C_1}{\partial y}$ and $\frac{\partial C_2}{\partial x}$ exist almost everywhere. Therefore the Markov product $*$: $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ (which was introduced as $*$ product in [26], see also [39, 73, 97] and [93, Section 6.4]) is well-defined, and the copula $C_1 * C_2$: $[0, 1]^2 \rightarrow [0, 1]$ is given by

$$(C_1 * C_2)(x, y) = \int_0^1 \frac{\partial C_1(x, t)}{\partial t} \cdot \frac{\partial C_2(t, y)}{\partial t} dt. \quad (6.1)$$

It is easy to see that the upper Fréchet-Hoeffding bound M and the product copula Π are the neutral element and the annihilator, respectively, of $\mathcal{C}$ with respect to the Markov product, i.e., for all $C \in \mathcal{C}$ we have

$$M * C = C * M = C \quad \text{and} \quad \Pi * C = C * \Pi = \Pi.$$

Moreover, for each $C \in \mathcal{C}$ the Markov product of C and the lower Fréchet-Hoeffding bound W are related to the x - and y -flipping and the survival copula of C as follows (see also [93]):

$$W * C = C^{\text{xflip}}, \quad C * W = C^{\text{yflip}}, \quad \text{and} \quad W * C = C^{\text{surv}} * W.$$

Since the Markov product is associative (see [26, Theorem 2.4]), the pair $(\mathcal{C}, *)$ is a monoid, i.e., a semigroup with neutral element:

Corollary 6.1. *The pair $(\mathcal{C}^{\text{pert}, \Pi}, *)$ is a sub-semigroup, and $(\mathcal{C}^{\text{pert}, \Pi} \cup \{M\}, *)$ is a sub-monoid of $(\mathcal{C}, *)$.*

Proof. We only have to show that the set $\mathcal{C}^{\text{pert}, \Pi}$ is closed under the Markov product $*$. Choose two arbitrary copulas $C_1, C_2 \in \mathcal{C}^{\text{pert}, \Pi}$ such that $C_1 = \Pi_{[f_1, g_1]}$ and $C_2 = \Pi_{[f_2, g_2]}$ for some pairs of functions $(f_1, g_1), (f_2, g_2) \in \mathcal{F}_\Pi$. If we put $c_{(g_1, f_2)} = \int_0^1 g'_1(t) f'_2(t) dt$ then straightforward calculations yield for each $(x, y) \in [0, 1]^2$

$$C_1 * C_2(x, y) = xy + f_1(x) \cdot g_2(y) \cdot \int_0^1 g'_1(t) \cdot f'_2(t) dt = \Pi_{[c_{(g_1, f_2)}, f_1, g_2]}(x, y).$$

Note that the constant $c_{(g_1, f_2)}$ may be split arbitrarily into two positive multiplicative factors, acting on f_1 and g_2 , respectively. $\square$

It has already been discussed in [93] that the family of Eyraud-Farlie-Gumbel-Morgenstern copula C_θ^{EFGM} is also closed under the Markov product, i.e., for all $\alpha, \beta \in [-1, 1]$

$$C_\alpha^{\text{EFGM}} * C_\beta^{\text{EFGM}} = C_{\alpha\beta/3}^{\text{EFGM}}.$$

Also the family of Fréchet copulas [51, 93] (and, therefore, the family of Mardia copulas [84, 93]) is closed under the Markov product $*$. Note that the family of Fréchet copulas $C_{\alpha, \beta}^{\text{Fréchet}}$ is defined, for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, and for all $(x, y) \in [0, 1]^2$ by

$$C_{\alpha, \beta}^{\text{Fréchet}}(x, y) = \alpha M(x, y) + (1 - \alpha - \beta) \Pi(x, y) + \beta W(x, y)$$

and can be seen as another type of perturbation of the product copula Π . For the Markov product the following holds (compare [93], but also [2]):

$$C_{\alpha_1, \beta_1}^{\text{Fréchet}} * C_{\alpha_2, \beta_2}^{\text{Fréchet}} = C_{\alpha_1\alpha_2 + \beta_1\beta_2, \alpha_1\beta_2 + \alpha_2\beta_1}^{\text{Fréchet}}.$$

Remark 6.2. In [2, Remark 3.2] (see also [27, 44]) copulas C which are idempotent with respect to the Markov product, i.e., which satisfy $C * C = C$, were studied. Several classes of idempotent copulas were identified, among them:

- (i) the class of M -ordinal sums $M(\langle a_k, b_k, C_k \rangle)_{k \in K}$ with $C_k = \Pi$ for each $k \in K$;
- (ii) the class of copulas of the form $\Pi_{[f, f]}(x, y) = xy + f(x)f(y)$, where the function $f: [0, 1] \rightarrow [0, 1]$ is Lipschitz with $f(0) = f(1) = 0$ and $f'(x) \in \{-1, 1\}$ wherever $f'(x)$ exists.

Remark 6.3. The set $\mathcal{C}^{\text{pert}, \Pi}$ given in (5.7) has also some nice properties related to the Markov product given in (6.1).

- (i) Based on Corollary 6.1, a copula $\Pi_{[f, g]} \neq \Pi$ is idempotent with respect to the Markov product if and only if $\Pi_{[f, g]} = \Pi_{[f, g]} * \Pi_{[f, g]} = \Pi_{[c_{(g, f)}, f, g]}$, which holds if and only if $c_{(g, f)} = 1$, i.e., if $\int_0^1 g'(t) f'(t) dt = 1$.
- (ii) From [2, Remark 3.2] (see also Remark 6.2) it follows that for each $r \in]0, \infty[$ the copula $\Pi^{(r)}$ is idempotent with respect to the Markov product given in (6.1), i.e., $\Pi^{(r)} * \Pi^{(r)} = \Pi^{(r)}$.

Because of (5.8) we have $\Pi^{(r)} = \Pi_{[f_r, g_r]} = \Pi_{[af_r, \frac{1}{a}g_r]}$ for each $a \in]0, \infty[$. Choosing $a = \frac{1}{\sqrt{r}}$ we obtain $\frac{1}{\sqrt{r}}f_r = g_r\sqrt{r}$. Defining the function $h: [0, 1] \rightarrow \mathbb{R}$ by $h_r = \frac{1}{\sqrt{r}}f_r = g_r\sqrt{r}$, i.e.,

$$h_r(x) = \min\left(x\sqrt{r}, \frac{1-x}{\sqrt{r}}\right),$$

then we have $(h_r, h_r) \in \mathcal{F}_\Pi$ and $\Pi^{(r)} = \Pi_{[h_r, h_r]}$.

If we want to check directly that $\Pi^{(r)}$ is idempotent with respect to $*$, we see that $\int_0^1 (h'_r(t))^2 dt = 1$ because of

$$h'_r(t) = \begin{cases} \sqrt{r} & \text{if } t \in]0, \frac{1}{r+1}[, \\ -\frac{1}{\sqrt{r}} & \text{if } t \in]\frac{1}{r+1}, 1[. \end{cases}$$

6.2 Dependence parameters

Finally, we shall look for the dependence parameters (Spearman's rho, Kendall's tau, Blomqvist's beta, and Gini's gamma) given in (2.6) of the perturbation-based copulas studied in this paper.

First, we will compute the dependence parameters of the perturbations discussed in Section 4 (for the exact formula for $[C, C_1, C_2]_\theta$ see Definition 4.1).

Proposition 6.4. *Let C, C_1 and C_2 be copulas and $\theta \in \mathbb{R}$ such that the function $[C, C_1, C_2]_\theta$ defined by (4.1) is a copula. For the dependence parameters given in (2.6) we obtain the following formulas:*

$$\varrho([C, C_1, C_2]_\theta) = \varrho(C) + 12 \cdot \theta \int_0^1 C_1(x, 1-x) dx \cdot \int_0^1 C_2(y, 1-y) dy,$$

$$\tau([C, C_1, C_2]_\theta) = \tau(C) + 8\theta \int_0^1 \int_0^1 C(x, y) \frac{d}{dx} C_1(x, 1-x) \frac{d}{dy} C_2(y, 1-y) dy dx$$

$$\beta([C, C_1, C_2]_\theta) = \beta(C) + 4 \cdot \theta \cdot C_1\left(\frac{1}{2}, \frac{1}{2}\right) \cdot C_2\left(\frac{1}{2}, \frac{1}{2}\right) = \beta(C) + \frac{1}{4}\theta(\beta(C_1) + 1)(\beta(C_2) + 1),$$

$$\gamma([C, C_1, C_2]_\theta) = \gamma(C) + 4\theta \int_0^1 C_1(x, 1-x)[C_2(x, 1-x) + C_2(1-x, x)] dx.$$

Proof. The formulas for $\varrho([C, C_1, C_2]_\theta)$, $\beta([C, C_1, C_2]_\theta)$ and $\gamma([C, C_1, C_2]_\theta)$ are straightforward. Concerning Kendall's tau, it follows from formula (5.1.12) and Theorem 5.1.5 in [93] that $\tau([C, C_1, C_2]_\theta)$ can be expressed in terms of the partial derivatives of $[C, C_1, C_2]_\theta$ with respect to x and y . Then partial integration together with the boundary conditions (C1) of C_1 and C_2 yields

Figure 8: Spearman's rho, Kendall's tau, Blomqvist's beta, and Gini's gamma for the extremal copulas $\Pi^{(r)} \in \mathcal{C}_\Pi^*$ with $r \in [-10, 10] \setminus \{0\}$.

$$\begin{aligned}
\tau([C, C_1, C_2]_\theta) &= 1 - 4 \iint_{[0,1]^2} \frac{\partial}{\partial x} [C, C_1, C_2]_\theta(x, y) \frac{\partial}{\partial y} [C, C_1, C_2]_\theta(x, y) dx dy \\
&= 1 - 4 \iint_{[0,1]^2} \frac{\partial}{\partial x} C(x, y) \cdot \frac{\partial}{\partial y} C(x, y) dx dy - 4\theta \iint_{[0,1]^2} C_1(x, 1-x) \cdot \frac{\partial}{\partial x} C(x, y) \cdot \frac{d}{dy} C_2(y, 1-y) dx dy \\
&\quad - 4\theta \iint_{[0,1]^2} C_2(y, 1-y) \cdot \frac{\partial}{\partial y} C(x, y) \cdot \frac{d}{dx} C_1(x, 1-x) dx dy \\
&\quad - 4\theta^2 \iint_{[0,1]^2} C_1(x, 1-x) \cdot \frac{d}{dx} C_1(x, 1-x) \cdot C_2(y, 1-y) \cdot \frac{d}{dy} C_2(y, 1-y) dx dy \\
&= \tau(C) - 4\theta \int_0^1 \frac{d}{dy} C_2(y, 1-y) \left(\int_0^1 C_1(x, 1-x) \cdot \frac{\partial}{\partial x} C(x, y) dx \right) dy \\
&\quad - 4\theta \int_0^1 \frac{d}{dx} C_1(x, 1-x) \left(\int_0^1 C_2(y, 1-y) \cdot \frac{\partial}{\partial y} C(x, y) dy \right) dx \\
&\quad - 4\theta^2 \int_0^1 C_1(x, 1-x) \cdot \frac{d}{dx} C_1(x, 1-x) dx \cdot \int_0^1 C_2(y, 1-y) \cdot \frac{d}{dy} C_2(y, 1-y) dy \\
&= \tau(C) + 8\theta \int_0^1 \int_0^1 C(x, y) \frac{d}{dx} C_1(x, 1-x) \frac{d}{dy} C_2(y, 1-y) dy dx,
\end{aligned}$$

which completes the proof. $\square$

Remark 6.5. There are several immediate consequences of Proposition 6.4 for special choices of either C or C_1 and C_2 :

- (i) For an arbitrary copula C and a parameter θ such that $[C, M, M]_\theta \in \mathcal{C}$ and $[C, \Pi, \Pi]_\theta \in \mathcal{C}$ we have the following relationships for the dependence parameters ϱ , β , and γ given in (2.6):

$$\begin{aligned}
\varrho([C, M, M]_\theta) &= \varrho(C) + \frac{3}{4}\theta, & \varrho([C, \Pi, \Pi]_\theta) &= \varrho(C) + \frac{1}{3}\theta, \\
\beta([C, M, M]_\theta) &= \beta(C) + \theta, & \beta([C, \Pi, \Pi]_\theta) &= \beta(C) + \frac{1}{4}\theta, \\
\gamma([C, M, M]_\theta) &= \gamma(C) + \frac{2}{3}\theta, & \gamma([C, \Pi, \Pi]_\theta) &= \gamma(C) + \frac{4}{15}\theta.
\end{aligned}$$

- (ii) In particular, for each $\theta \in [-1, 1]$ and for each of the dependence parameters given in (2.6) we obtain for the copula $[\Pi, M, M]_\theta: [0, 1]^2 \rightarrow [0, 1]$ given by (4.1):

$$\varrho([\Pi, M, M]_\theta) = \frac{3}{4}\theta, \quad \tau([\Pi, M, M]_\theta) = \frac{1}{2}\theta, \quad \beta([\Pi, M, M]_\theta) = \theta, \quad \gamma([\Pi, M, M]_\theta) = \frac{2}{3}\theta.$$

- (iii) If for some $C \in \mathcal{C}$ the equalities $\varrho(C) = \tau(C) = 0$ hold then we obtain for all copulas $C_1, C_2 \in \mathcal{C}$ satisfying $\varrho([C, C_1, C_2]_\theta) \cdot \tau([C, C_1, C_2]_\theta) \neq 0$

$$\varrho([C, C_1, C_2]_\theta) : \tau([C, C_1, C_2]_\theta) = 3 : 2.$$

We can also give some properties of the four dependence parameters related to perturbations discussed in Section 5, in particular to the copulas $\Pi^{(r)} \in \mathcal{C}_\Pi^*$ (see (5.2), (5.4) and Definition 5.6) and $\Pi_{[f,g]} \in \mathcal{C}^{\text{pert}, \Pi}$ (see Definition 5.8).

Remark 6.6. For the dependence parameters given in (2.6) we obtain the following formulas (for a visualization see Figure 8):

(i) For each $r \in \mathbb{R} \setminus \{0\}$ we get

$$\begin{aligned}\varrho(\Pi^{(r)}) &= \text{sign}(r) \frac{3|r|}{(|r|+1)^2}, & \tau(\Pi^{(r)}) &= \text{sign}(r) \frac{2|r|}{(|r|+1)^2}, \\ \beta(\Pi^{(r)}) &= \text{sign}(r) \min\left(|r|, \frac{1}{|r|}\right), & \gamma(\Pi^{(r)}) &= \begin{cases} \text{sign}(r) \frac{2|r|(3-|r|)}{3(|r|+1)} & \text{if } |r| \leq 1, \\ \text{sign}(r) \frac{2(3|r|-1)}{3|r|(|r|+1)} & \text{otherwise.} \end{cases}\end{aligned}$$

(ii) In each case, the extremal values are attained for $r = \pm 1$:

$$\varrho(\Pi^{(\pm 1)}) = \pm \frac{3}{4}, \quad \tau(\Pi^{(\pm 1)}) = \pm \frac{1}{2}, \quad \beta(\Pi^{(\pm 1)}) = \pm 1, \quad \gamma(\Pi^{(\pm 1)}) = \pm \frac{2}{3}.$$

(iii) Consider an open interval $\Theta \subseteq \mathbb{R}$ with $0 \in \Theta$ and a family of copulas $(C_\theta)_{\theta \in \Theta}$ which is continuous with respect to the parameter θ . In [54, Theorem 3.1] (compare also [106]), the authors have shown that, under mild regularity conditions, we have

$$\lim_{\theta \rightarrow 0} \frac{\varrho(C_\theta)}{\tau(C_\theta)} = \frac{3}{2}.$$

Putting $\Pi^{(0)} = \Pi$ we see that the family of copulas $(\Pi^{(r)})_{r \in \mathbb{R}}$ is continuous with respect to the parameter r , and it illustrates the result above because of

$$\varrho(\Pi^{(r)}) : \tau(\Pi^{(r)}) = 3 : 2$$

for each $r \neq 0$. This is the same situation as for the family of Eyraud-Farlie-Gumbel-Morgenstern copulas $(C_\theta^{\text{EFGM}})_{\theta \in [-1, 1]}$ since for each $\theta \neq 0$ we have

$$\varrho(C_\theta^{\text{EFGM}}) : \tau(C_\theta^{\text{EFGM}}) = 3 : 2.$$

Proposition 6.7. If $\xi : \mathcal{C} \rightarrow [-1, 1]$ is one of the dependence parameters ϱ, τ, β , and γ given in (2.6), then for each $\Pi_{[f, g]} \in \mathcal{C}^{\text{pert}, \Pi}$ we have

$$\xi([\Pi, M, M]_{-1}) \leq \xi(\Pi_{[f, g]}) \leq \xi([\Pi, M, M]_1).$$

Proof. Note that, whenever for some pairs of functions $(f_1, g_1), (f_2, g_2) \in \mathcal{F}_\Pi$ we have

$$\int_0^1 f_1(x) dx \int_0^1 g_1(x) dx \leq \int_0^1 f_2(x) dx \int_0^1 g_2(x) dx$$

then also $\xi(\Pi_{[f_1, g_1]}) \leq \xi(\Pi_{[f_2, g_2]})$ for each $\xi \in \{\varrho, \tau, \beta, \gamma\}$. Together with Corollary 4.5 and Remark 6.6 this shows that our claim holds. $\square$

Still keeping the notations of Proposition 6.7, we can identify the ranges of the dependence parameters for the copulas $\Pi_{[f, g]} \in \mathcal{C}^{\text{pert}, \Pi}$:

Corollary 6.8. For each $\Pi_{[f, g]} \in \mathcal{C}^{\text{pert}, \Pi}$ we obtain

$$\begin{aligned}|\varrho(\Pi_{[f, g]})| &\leq \frac{3}{4}, & |\tau(\Pi_{[f, g]})| &\leq \frac{1}{2}, \\ |\beta(\Pi_{[f, g]})| &\leq 1, & |\gamma(\Pi_{[f, g]})| &\leq \frac{2}{3}.\end{aligned}$$

7 Concluding remarks

Perturbation of functions has a long and multifaceted tradition — also for copulas starting from a copula and perturbing it either by some other function(s) or some other copulas or some derived copulas has been of interest to the scientific community for different reasons. Mostly, the question whether or not or under which conditions the newly built function is again a copula, has been in the focus of investigations.

Also in this contribution, we discussed classes of perturbed copulas — starting with some copula C being perturbed by the product of the opposite diagonal sections of two (possibly different) copulas C_1 and C_2 . We showed that the search for largest flexibility with respect to the choice of the copulas C_1, C_2 leads to a (possibly rather restricted) interval of possible values for the parameter. However, we also worked with different ordinal sums of copulas involving the three basic copulas, showing, e.g., that some particular ordinal sums with the independence copula as its only summand play an important role: they give rise to infinitely many, maximal and minimal elements of a set of perturbation copulas based on the independence copula and two one-place functions of a particular type.

Summarizing we may stress that an emphasis in this contribution has been the identification of new structural elements and properties of particular classes of perturbation copulas — revealing new properties, but also revealing new insights by connecting different types of perturbation copulas and by connecting our results to results known from different areas of copula theory.

For our further research on perturbation of copulas, we think, on the one hand, about the perturbation of copulas of higher dimensions. On the other hand, when considering random variables affected by some random noise, we see that the original copula, describing the stochastic structure of a random vector, is perturbed into a new copula, and our aim would be an analytical description of the perturbed copula. For some preliminary results in this direction see, e.g., [89].

Acknowledgements: Funded by the Johannes Kepler Open Access Publishing Fund. Anna Kolesárová is thankful for the support by the grant VEGA 1/0267/21 and, together with Radko Mesiar and Adam Šeliga, for the support by the grant APVV-18-0052. Radko Mesiar and Adam Šeliga also acknowledge the support by the grant APVV-17-0066.

The authors are also grateful to the two anonymous referees for their valuable comments and suggestions which eventually led to an improvement of an earlier version of this paper.

Conflict of interest statement: The authors declare that there is no conflict of interest.

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