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Dispersive order comparisons on extreme order statistics from homogeneous dependent random vectors

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Abstract: In this paper, we investigate sufficient conditions for preservation property of the dispersive order for the smallest and largest order statistics of homogeneous dependent random vectors. Moreover, we establish sufficient conditions for ordering with the dispersive order the largest order statistics from dependent homogeneous samples of different sizes.

Keywords: Dispersive order, copulas, Archimedean copulas, extreme order statistics

MSC: 60E15, 62G30, 62H05

1 Introduction

Order statistics play an important role in statistics, risk management, auction theory, reliability and many other theoretical and applied probability areas. They have received a lot of attention from many researchers. For comprehensive references one may refer to Balakrishnan and Rao ([1], [2]) and David and Nagaraja [3].

For a random vector $\mathbf{X} = (X_1, \dots, X_n)$, denote as $X_{i:n}$ the corresponding i th order statistic, $i = 1, \dots, n$. Most of the research on stochastic comparisons between order statistics has been dedicated to the case of independent and identically distributed (i.i.d.) random variables $X_1, \dots, X_n$. We can quote Boland et al. [4] who prove stochastic comparisons between order statistics with the hazard rate order and the likelihood ratio order, Raqab and Amin [5] who further prove comparison with the likelihood ratio order between order statistics from samples of different sizes, Kochar [6] who proves comparisons with the dispersive order between order statistics for decreasing failure rate (DFR) distributions or Khaledi and Kochar [7] who study comparisons with the dispersive order between order statistics from samples of different sizes for DFR distributions.

In practical situations, the observations are usually not i.i.d. During the last three decades, the case of independent but not necessarily identically distributed random variables has also got the attention of researchers. We refer the reader to Kochar [8] and Balakrishnan and Zhao [9] for comprehensive references. To mention a few with the dispersive order, Dykstra et al. [10] study comparisons of the largest order statistics with the dispersive order for independent exponential random variables. Khaledi and Kochar [11] extend the latter result from the exponential case to the proportional hazard PH) sample. More recently, Fang and Zhang [12] have obtained the dispersive order between maximums of one heterogeneous- and one homogeneous-independent samples for Weibull random variables sharing a common shape parameter.

In the last decade, some papers have been devoted to the study of the ordering properties of the order statistics from dependent samples. For example, Li and Fang [13] generalize the result in Fang and Zhang

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[12] to the dependent case where two proportional hazards samples have a common Archimedean copula and one has heterogeneous hazards and the other has the homogeneous arithmetic average hazards. For two Weibull samples having a common Archimedean survival copula, Li and Li [14] further prove dispersive order inequalities between minimums of one heterogeneous and one homogeneous samples. Fang et al. [15] derive the usual stochastic order, the dispersive order and the star order of order statistics from the PH sample with Archimedean survival copulas and the proportional reversed hazards (PRH) sample with Archimedean copulas. Li et al. [16] investigate order statistics from random variables following the scale model and obtain in the presence of the Archimedean copula or survival copula the usual stochastic order of the sample extremes and the second smallest order statistic, the dispersive order and the star order of the sample extremes.

In this paper, we consider homogeneous dependent samples and we establish sufficient conditions on the copulas or survival copulas and the marginal distribution functions in order to preserve the dispersive order for the smallest and largest order statistics. Moreover, we obtain sufficient conditions on the copulas and the marginal distribution functions for ordering with the dispersive order the largest order statistics from dependent homogeneous samples of different sizes.

The paper is structured as follows. In Section 2, we introduce the useful concepts that will be used in the rest of the paper. Section 3 is devoted to homogeneous dependent samples with different copulas and marginal distributions, and we obtain sufficient conditions for preserving the dispersive order for the smallest and largest order statistics. Finally, in Section 4, we derive sufficient conditions for the dispersive order of the sample extremes in the case of dependent homogeneous samples of different sizes with common copulas and marginal distributions.

2 Preliminaries

In this section, we recall the concepts that are important in the following.

Let X and Y be two random variables with their respective distribution functions F and G , survival functions $\bar{F}$ and $\bar{G}$ and right continuous inverses F^{-1} and G^{-1} .

Definition 2.1. X is said to be smaller than Y in the dispersive order, denoted $X \preceq_{disp} Y$, if

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha) \quad \text{for all } 0 < \alpha \leq \beta < 1,$$

or equivalently if $G^{-1}(F(x)) - x$ is increasing in x .

For more details on stochastic orders, we refer the reader to Shaked and Shanthikumar [17].

Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random vector with joint distribution function $\mathbf{F}$, joint survival function $\bar{\mathbf{F}}$, univariate marginal distribution functions $F_1, \dots, F_n$, survival functions $\bar{F}_1, \dots, \bar{F}_n$ and right continuous inverses $F_1^{-1}, \dots, F_n^{-1}$. If there exists $C : [0, 1]^n \rightarrow [0, 1]$ and $\hat{C} : [0, 1]^n \rightarrow [0, 1]$ such that

$$\begin{aligned} C(u_1, \dots, u_n) &= \mathbf{F}(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)) \\ \hat{C}(u_1, \dots, u_n) &= \bar{\mathbf{F}}(\bar{F}_1^{-1}(u_1), \dots, \bar{F}_n^{-1}(u_n)) \end{aligned}$$

for all $(u_1, \dots, u_n) \in [0, 1]^n$, then C and $\hat{C}$ are called the copula and the survival copula of $\mathbf{X}$, respectively. The functions $\delta_C(u) = C(u, \dots, u)$ and $\delta_{\hat{C}}(u) = \hat{C}(u, \dots, u)$ are known as the diagonal sections of C and $\hat{C}$.

The Archimedean copulas form an important class of copulas.

Definition 2.2. Let $\phi : [0, +\infty) \rightarrow [0, 1]$ with $\phi(0) = 1$ and $\phi(+\infty) = 0$. A n -dimensional copula C_ϕ is said to be an Archimedean copula with generator ϕ if, for all $(u_1, \dots, u_n) \in [0, 1]^n$,

$$C_\phi(u_1, \dots, u_n) = \phi(\psi(u_1) + \dots + \psi(u_n)),$$

where we denote $\psi = \phi^{-1}$ the right continuous inverse of ϕ for convenience.

For $(n-2)$ th differentiable ϕ , we know from McNeil and Nešlehová [18] that the function $C_\phi : [0, 1]^n \rightarrow [0, 1]$ is a n -dimensional copula if, and only if, the generator ϕ is a n -monotone function. Denoting $\phi^{(i)}(x)$ the i th derivative of ϕ , recall that ϕ is said to be n -monotone if $(-1)^i \phi^{(i)}(x) \geq 0$ for $i = 1, \dots, n-2$ and $(-1)^{n-2} \phi^{(n-2)}(x)$ is non-increasing and convex.

3 Comparison of the smallest and largest order statistics with the dispersive order

Let $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ be two homogeneous random vectors with copulas C_X and C_Y and univariate marginal distribution functions F and G , respectively. Under the conditions that both C_X and C_Y are the independence copula, Theorem 3.B.26 in Shaked and Shanthikumar [17] states that if $X \preceq_{disp} Y$, then $X_{i:n} \preceq_{disp} Y_{i:n}$ for $i = 1, \dots, n$, where $X_{i:n}$ and $Y_{i:n}$ are the i th order statistics of $\mathbf{X}$ and $\mathbf{Y}$, respectively. As mentioned in Shaked and Shanthikumar [17], this preservation property of the dispersive order is useful in reliability theory and in nonparametric statistics. Actually, the latter result can be extended for $\mathbf{X}$ and $\mathbf{Y}$ having a common copula, as shown next.

Proposition 3.1. *If $C_X = C_Y$ and $X \preceq_{disp} Y$, then $X_{i:n} \preceq_{disp} Y_{i:n}$ for $i = 1, \dots, n$.*

Proof. Since $\mathbf{X}$ and $\mathbf{Y}$ have a common copula, there exists a uniform random vector $(U_1, \dots, U_n)$ with distribution C_X such that $X_i = F^{-1}(U_i)$ and $Y_i = G^{-1}(U_i)$, so that we have $X_{i:n} = F^{-1}(U_{i:n})$ and $Y_{i:n} = G^{-1}(U_{i:n})$. This ensures that the distributions of $X_{i:n}$ and $Y_{i:n}$ are of the form $F_{i:n} = H_{i:n} \circ F$ and $G_{i:n} = H_{i:n} \circ G$, respectively, where $H_{i:n}$ denotes the distribution of $U_{i:n}$. The announced result then follows from $G_{i:n}^{-1}(F_{i:n}(x)) - x = G^{-1}(F(x)) - x$ for all x . $\square$

Proposition 3.1 considers samples $\mathbf{X}$ and $\mathbf{Y}$ with a common copula. The next result shows that the dispersive order can still be preserved for the smallest and largest order statistics when we consider different copulas for $\mathbf{X}$ and $\mathbf{Y}$ at the cost of requiring additional conditions on the marginal distribution G and the diagonal sections δ_{C_X} and δ_{C_Y} of the copulas C_X and C_Y for the largest order statistics, and on the survival function $\bar{G}$ and the diagonal sections $\delta_{\hat{C}_X}$ and $\delta_{\hat{C}_Y}$ of the survival copulas $\hat{C}_X$ and $\hat{C}_Y$ for the smallest order statistics.

Proposition 3.2. *For F and G twice differentiable,*

- (i) *if $\delta_{C_X}^{-1}(\alpha)/\delta_{C_Y}^{-1}(\alpha)$ is decreasing in $\alpha \in [0, 1]$, G is log-convex and $X \preceq_{disp} Y$, then $X_{n:n} \preceq_{disp} Y_{n:n}$;*
- (ii) *if $\delta_{\hat{C}_X}^{-1}(\alpha)/\delta_{\hat{C}_Y}^{-1}(\alpha)$ is decreasing in $\alpha \in [0, 1]$, $\bar{G}$ is log-convex and $X \preceq_{disp} Y$, then $X_{1:n} \preceq_{disp} Y_{1:n}$.*

Proof. (i) Obviously, the quantile functions of $X_{n:n}$ and $Y_{n:n}$ are $F^{-1}\{\delta_{C_X}^{-1}(\alpha)\}$ and $G^{-1}\{\delta_{C_Y}^{-1}(\alpha)\}$, $\alpha \in [0, 1]$, respectively. Thus, $X_{n:n} \preceq_{disp} Y_{n:n}$ if, and only if, $F^{-1}\{\delta_{C_X}^{-1}(\alpha)\} - G^{-1}\{\delta_{C_Y}^{-1}(\alpha)\}$ is decreasing in α . This latter difference can be rewritten as the sum of

$$A(\alpha) = F^{-1}\{\delta_{C_X}^{-1}(\alpha)\} - G^{-1}\{\delta_{C_X}^{-1}(\alpha)\}$$

and

$$B(\alpha) = G^{-1}\{\delta_{C_X}^{-1}(\alpha)\} - G^{-1}\{\delta_{C_Y}^{-1}(\alpha)\}.$$

Now, since $X \preceq_{disp} Y$ implies that $A(\alpha)$ is decreasing in α (since $\delta_{C_X}^{-1}(\alpha)$ is an increasing function), it suffices to prove that $B(\alpha)$ is decreasing in α as well. Simple calculations yield

$$\begin{aligned} B'(\alpha) &= (G^{-1})'\{\delta_{C_X}^{-1}(\alpha)\}(\delta_{C_X}^{-1})'(\alpha) - (G^{-1})'\{\delta_{C_Y}^{-1}(\alpha)\}(\delta_{C_Y}^{-1})'(\alpha) \\ &= \frac{\delta_{C_X}^{-1}(\alpha)}{G'(G^{-1}(\delta_{C_X}^{-1}(\alpha)))} \left(\ln\{\delta_{C_X}^{-1}(\alpha)\} \right)' - \frac{\delta_{C_Y}^{-1}(\alpha)}{G'(G^{-1}(\delta_{C_Y}^{-1}(\alpha)))} \left(\ln\{\delta_{C_Y}^{-1}(\alpha)\} \right)' \\ &= h\{G^{-1}(\delta_{C_X}^{-1}(\alpha))\} \left(\ln\{\delta_{C_X}^{-1}(\alpha)\} \right)' - h\{G^{-1}(\delta_{C_Y}^{-1}(\alpha))\} \left(\ln\{\delta_{C_Y}^{-1}(\alpha)\} \right)', \end{aligned} \quad (3.1)$$

where $h(t) = G(t)/G'(t)$. As $\delta_{C_X}^{-1}(\alpha)/\delta_{C_Y}^{-1}(\alpha)$ is decreasing in $\alpha \in [0, 1]$ and $\delta_{C_X}^{-1}(1)/\delta_{C_Y}^{-1}(1) = 1$, one has $0 \leq (\ln\{\delta_{C_X}^{-1}(\alpha)\})' \leq (\ln\{\delta_{C_Y}^{-1}(\alpha)\})'$ and $\delta_{C_X}^{-1}(\alpha) \geq \delta_{C_Y}^{-1}(\alpha)$. Hence, from (3.1), we have

$$B'(\alpha) \leq \left[h\{G^{-1}(\delta_{C_X}^{-1}(\alpha))\} - h\{G^{-1}(\delta_{C_Y}^{-1}(\alpha))\} \right] \left(\ln\{\delta_{C_X}^{-1}(\alpha)\} \right)'. \quad (3.2)$$

Finally, since G is log-convex, it is plain that h is a decreasing function so that the right-hand side in (3.2) is negative.

(ii) The quantile function of $X_{1:n}$ is clearly given by $F_{1:n}^{-1}(\alpha) = \bar{F}^{-1}(\delta_{C_X}^{-1}(1-\alpha))$, $\alpha \in [0, 1]$. So, $X_{1:n} \preceq_{disp} Y_{1:n}$ if, and only if, $\bar{F}^{-1}\{\delta_{C_X}^{-1}(\alpha)\} - \bar{G}^{-1}\{\delta_{C_Y}^{-1}(\alpha)\}$ is increasing in α , which is the case when $\hat{C}(\alpha) = \bar{F}^{-1}\{\delta_{C_Y}^{-1}(\alpha)\} - \bar{G}^{-1}\{\delta_{C_X}^{-1}(\alpha)\}$ is increasing in α since $X \preceq_{disp} Y$. Proceeding in a similar manner than in (3.1) and using the fact that $\delta_{C_X}^{-1}(\alpha)/\delta_{C_Y}^{-1}(\alpha)$ is decreasing in α and $\delta_{C_X}^{-1}(1)/\delta_{C_Y}^{-1}(1) = 1$, one has

$$C'(\alpha) \geq \left[\bar{h}\{\bar{G}^{-1}(\delta_{C_X}^{-1}(\alpha))\} - \bar{h}\{\bar{G}^{-1}(\delta_{C_Y}^{-1}(\alpha))\} \right] \left(\ln\{\delta_{C_X}^{-1}(\alpha)\} \right)', \quad (3.3)$$

where $\bar{h}(t) = \bar{G}(t)/\bar{G}'(t)$. Consequently, as $\bar{G}$ is log-convex, it is plain that $\bar{h}$ is a decreasing function so that the right-hand side in (3.3) is positive. $\square$

In particular, for Archimedean copulas, we directly get the following result.

Corollary 3.3. (i) For $C_X = C_{\phi_1}$ and $C_Y = C_{\phi_2}$, where C_{ϕ_1} and C_{ϕ_2} are two Archimedean copulas, if $\phi_1(\psi_1(\alpha)/n)/\phi_2(\psi_2(\alpha)/n)$ is decreasing in $\alpha \in [0, 1]$, G is log-convex and $X \preceq_{disp} Y$, then $X_{n:n} \preceq_{disp} Y_{n:n}$.
(ii) For $\hat{C}_X = \hat{C}_{\phi_1}$ and $\hat{C}_Y = \hat{C}_{\phi_2}$, where $\hat{C}_{\phi_1}$ and $\hat{C}_{\phi_2}$ are two Archimedean copulas, if $\phi_1(\psi_1(\alpha)/n)/\phi_2(\psi_2(\alpha)/n)$ is decreasing in $\alpha \in [0, 1]$, $\bar{G}$ is log-convex and $X \preceq_{disp} Y$, then $X_{1:n} \preceq_{disp} Y_{1:n}$.

Proof. It suffices to notice that the diagonal section of an Archimedean copula with generator ϕ is the form $\phi(\psi(\alpha)/n)$. Then, the result immediately follows from Proposition 3.2. $\square$

We illustrate Corollary 3.3 by an example where $\phi_1(\psi_1(\alpha)/n)/\phi_2(\psi_2(\alpha)/n)$ is decreasing in α .

Example 3.4. (i) Consider

$$\phi_1(t) = (\theta_1 t + 1)^{-1/\theta_1}$$

and

$$\phi_2(t) = (\theta_2 t + 1)^{-1/\theta_2}$$

with $0 < \theta_1 \leq \theta_2$, so that C_{ϕ_1} and C_{ϕ_2} are Clayton copulas. Simple calculations yield

$$\begin{aligned} k_1(\alpha) &= \phi_1(\psi_1(\alpha)/n)/\phi_2(\psi_2(\alpha)/n) \\ &= n^{1/\theta_1 - 1/\theta_2} \frac{(\alpha^{-\theta_1} - 1 + n)^{-1/\theta_1}}{(\alpha^{-\theta_2} - 1 + n)^{-1/\theta_2}}, \end{aligned}$$

so that $k_1(\alpha)$ is decreasing in α when $0 < \theta_1 \leq \theta_2$, as illustrated in Figure 3.1 for $n = 3$, $\theta_1 = 1$ and $\theta_2 = 2$. As a result, from Corollary 3.3 (i), we know that if G is log-convex and $X \preceq_{disp} Y$, then $X_{n:n} \preceq_{disp} Y_{n:n}$. A log-convex distribution function G whose support is $]-\infty, a]$ with $a \in \mathbb{R}$ can be expressed as

$$G(x) = (e^{h(x)} - 1)I[x < a] + 1, \quad x \in \mathbb{R}, \quad (3.4)$$

where h is an increasing and convex function such that $h(-\infty) = -\infty$ and $h(a) = 0$, and where $I[\cdot]$ denotes the indicator function, equal to 1 if the event appearing in the brackets is realized and to 0 otherwise. As an example, distribution functions of the form

$$G(x) = (e^{-|x|^a} - 1)I[x < 0] + 1, \quad x \in \mathbb{R},$$

with $\alpha \leq 1$ are log-convex. In particular, for two homogeneous samples with the same marginal distribution functions, that is when $F = G$, Corollary 3.3 (i) tells us that the variability (in terms of the dispersive order) of the largest statistics is increasing in the dependence parameter θ .

(ii) Likewise, if we consider two Clayton copulas $\hat{C}_{\phi_1}$ and $\hat{C}_{\phi_2}$ for the survival copulas with parameters $0 < \theta_1 \leq \theta_2$, respectively, then we know from Corollary 3.3 (ii) that if $\bar{G}$ is log-convex and $X \preceq_{disp} Y$, then $X_{1:n} \preceq_{disp} Y_{1:n}$. Well-known distributions have log-convex survival functions $\bar{G}$, as shown in [19]. Let us mention the Pareto distribution, the Gamma distribution with density function $f(x) = \frac{x^{c-1} \exp(-x)}{\Gamma(c)}$ and $0 < c < 1$ as well as the Weibull distribution with density function $f(x) = cx^{c-1} \exp(-x^c)$ and $0 < c < 1$.

The log-convexity condition on the marginal distribution function G in Proposition 3.2 and Corollary 3.3 is necessary to establish dispersive order inequalities among the largest order statistics, as illustrated in the following example.

Example 3.5. Consider $U = (U_1, \dots, U_n)$ and $V = (V_1, \dots, V_n)$ two homogeneous uniform random vectors distributed as C_{ϕ_1} and C_{ϕ_2} , respectively. Clearly, $U_{n:n} \preceq_{disp} V_{n:n}$ if, and only if $\phi_1(\psi_1(\alpha)/n) - \phi_2(\psi_2(\alpha)/n)$ is decreasing in α , which is actually not fulfilled for most of the Archimedean copulas. In particular, if we consider the Clayton copula, we get

$$\begin{aligned} k_2(\alpha) &= \phi_1(\psi_1(\alpha)/n) - \phi_2(\psi_2(\alpha)/n) \\ &= n^{1/\theta_1} (\alpha^{-\theta_1} - 1 + n)^{-1/\theta_1} - n^{1/\theta_2} (\alpha^{-\theta_2} - 1 + n)^{-1/\theta_2}. \end{aligned}$$

As a result, $k_2(\alpha)$ is not monotone, as illustrated in Figure 3.2 for $n = 3$, $\theta_1 = 1$ and $\theta_2 = 2$. Therefore, $U_{n:n}$ and $V_{n:n}$ cannot be compared with the dispersive order, the reason is that the uniform distribution is log-concave and not log-convex.

It is interesting to notice that $\delta_{C_X}^{-1}(\alpha)/\delta_{C_Y}^{-1}(\alpha)$ is decreasing in $\alpha \in [0, 1]$ if, and only if, $\ln \delta_{C_X}^{-1}(\alpha) - \ln \delta_{C_Y}^{-1}(\alpha)$ is decreasing in $\alpha \in [0, 1]$, which amounts to requiring $\ln U_{n:n} \preceq_{disp} \ln V_{n:n}$. Therefore the condition in Proposition 3.2 that $\delta_{C_X}^{-1}(\alpha)/\delta_{C_Y}^{-1}(\alpha)$ is decreasing in $\alpha \in [0, 1]$ can be rewritten as $\ln U_{n:n} \preceq_{disp} \ln V_{n:n}$.

Figure 3.1: $k_1(\alpha) = \phi_1(\psi_1(\alpha)/n)/\phi_2(\psi_2(\alpha)/n)$ for $\alpha \in [0, 1]$.

Figure 3.2: $k_2(\alpha) = \phi_1(\psi_1(\alpha)/n) - \phi_2(\psi_2(\alpha)/n)$ for $\alpha \in [0, 1]$.

4 Comparison of the largest order statistics with the dispersive order for different sample sizes

Let $\mathbf{X}_n = (X_1, \dots, X_n)$ be an homogeneous random vector with common distribution function F assumed to be twice differentiable and let $\mathbf{X}_{n-1} = (X_1, \dots, X_{n-1})$. In this section, we aim to compare the largest order statistics of samples $\mathbf{X}_n$ and $\mathbf{X}_{n-1}$ with the dispersive order. We denote by δ_n and δ_{n-1} the diagonal sections of the copulas of $\mathbf{X}_n$ and $\mathbf{X}_{n-1}$, respectively.

Proposition 4.1. *If F is log-convex and $\delta_n^{-1}(\alpha)/\delta_{n-1}^{-1}(\alpha)$ is decreasing in $\alpha \in [0, 1]$, then*

$$X_{n:n} \preceq_{disp} X_{n-1:n-1}.$$

Proof. First, we know that $F_{n:n}^{-1}(\alpha) = F^{-1}\{\delta_n^{-1}(\alpha)\}$ and $F_{n-1:n-1}^{-1}(\alpha) = F^{-1}\{\delta_{n-1}^{-1}(\alpha)\}$. Therefore, we clearly have that $X_{n:n} \preceq_{disp} X_{n-1:n-1}$ if, and only if, $D(\alpha) = F^{-1}\{\delta_n^{-1}(\alpha)\} - F^{-1}\{\delta_{n-1}^{-1}(\alpha)\}$ is a decreasing function. Similar calculations than in the proof of Proposition 3.2 yield

$$D'(\alpha) = k\{F^{-1}(\delta_n^{-1}(\alpha))\} \left(\ln\{\delta_n^{-1}(\alpha)\} \right)' - k\{F^{-1}(\delta_{n-1}^{-1}(\alpha))\} \left(\ln\{\delta_{n-1}^{-1}(\alpha)\} \right)'$$

where $k(t) = F(t)/F'(t)$. Since, $\delta_n^{-1}(\alpha)/\delta_{n-1}^{-1}(\alpha)$ is decreasing in $\alpha \in [0, 1]$, one has

$$D'(\alpha) \leq \left(k\{F^{-1}(\delta_n^{-1}(\alpha))\} - k\{F^{-1}(\delta_{n-1}^{-1}(\alpha))\} \right) \left(\ln\{\delta_n^{-1}(\alpha)\} \right)'.$$

Using the fact that $\delta_{n-1}^{-1}(\alpha) \leq \delta_n^{-1}(\alpha)$, it follows that $D'(\alpha) \leq 0$ if $k = F/F'$ is a decreasing function, that is, if F is log-convex. $\square$

For Archimedean copulas, Proposition 4.1 directly leads to the next result.

Corollary 4.2. *For $C_X = C_\phi$ where C_ϕ is an Archimedean copula with generator ϕ , if F is log-convex and $t\phi'(t)/\phi(t)$ is decreasing in $t > 0$, then*

$$X_{n:n} \preceq_{disp} X_{n-1:n-1}.$$

Proof. Since the copula C_ϕ is Archimedean with generator ϕ , we have $\delta_n^{-1}(\alpha) = \phi\left(\frac{\psi(\alpha)}{n}\right)$ and $\delta_{n-1}^{-1}(\alpha) = \phi\left(\frac{\psi(\alpha)}{n-1}\right)$. Consequently, $\delta_n^{-1}(\alpha)/\delta_{n-1}^{-1}(\alpha)$ is a decreasing function, if, and only if,

$$\frac{\phi'\left(\frac{\psi(\alpha)}{n}\right)}{\phi\left(\frac{\psi(\alpha)}{n}\right)} \frac{\psi(\alpha)}{n} \geq \frac{\phi'\left(\frac{\psi(\alpha)}{n-1}\right)}{\phi\left(\frac{\psi(\alpha)}{n-1}\right)} \frac{\psi(\alpha)}{n-1}$$

for all $\alpha \in (0, 1)$. The latter inequality is fulfilled for all $\alpha \in (0, 1)$ since $t\phi'(t)/\phi(t)$ is decreasing in $t > 0$. $\square$

The following example illustrates the condition on ϕ involved in Corollary 4.2. As mentioned in Li and Fang [20] and Mesfioui et al. [21], this condition is satisfied for most of the Archimedean copulas, such as Clayton, Frank and Gumbel copulas.

Example 4.3. Consider the generator $\phi(t) = (\theta t + 1)^{-1/\theta}$ with $\theta > 0$ of a Clayton copula. It verifies

$$\left(\frac{t\phi'(t)}{\phi(t)}\right)' = \frac{-1}{(\theta t + 1)^2} < 0,$$

so that for an homogeneous random vector $\mathbf{X} = (X_1, \dots, X_n)$ with such a dependence structure and a log-convex marginal distribution function F , we know from Corollary 4.2 that we have

$$X_{n:n} \preceq_{disp} X_{n-1:n-1} \preceq_{disp} \dots \preceq_{disp} X_{1:1}.$$

As already discussed in the previous section for Proposition 3.2 and Corollary 3.3, the log-convexity of F required in Proposition 4.1 and Corollary 4.2 is also crucial here to ensure the ordering of $X_{n:n}$ and $X_{n-1:n-1}$ with the dispersive order, as revealed by the next example.

Example 4.4. Consider the homogeneous uniform random vector $\mathbf{U} = (U_1, \dots, U_n)$ distributed as the independence copula C_ϕ . The generator of the independence copula, that is $\phi(t) = e^{-t}$, satisfies $t\phi'(t)/\phi(t) = -t$, so that it is well decreasing in $t > 0$, as required in Corollary 4.2. However, it is easy to see that the uniform distribution F is not log-convex but log-concave. In this case, the quantile functions of $X_{n-1:n-1}$ and $X_{n:n}$ are $F_{n-1:n-1}^{-1}(\alpha) = \alpha^{\frac{1}{n-1}}$ and $F_{n:n}^{-1}(\alpha) = \alpha^{\frac{1}{n}}$, $\alpha \in [0, 1]$, respectively, which implies that the function $g(\alpha) = F_{n:n}^{-1}(\alpha) - F_{n-1:n-1}^{-1}(\alpha)$ is not monotone, as illustrated in Figure 4.1 for $g(\alpha) = F_{3:3}^{-1}(\alpha) - F_{2:2}^{-1}(\alpha)$. As a result, $X_{n-1:n-1}$ and $X_{n:n}$ cannot be compared with the dispersive order.

Actually, for a uniform random vector $\mathbf{U} = (U_1, \dots, U_n)$ distributed as C_ϕ , a necessary and sufficient condition on the generator ϕ to get $U_{n:n} \preceq_{disp} U_{n-1:n-1}$ can be obtained, as shown next.

Proposition 4.5. For an homogeneous uniform random vector $\mathbf{U} = (U_1, \dots, U_n)$ distributed as C_ϕ , we have that

$$U_{n:n} \preceq_{disp} U_{n-1:n-1} \iff t\phi'(t) \text{ is a decreasing function in } t > 0.$$

Proof. Since the quantile function of $U_{n:n}$ is $F_{n:n}^{-1}(x) = \phi\left(\frac{\psi(u)}{n}\right)$, we get

$$\begin{aligned} U_{n:n} \preceq_{disp} U_{n-1:n-1} &\iff \phi\left(\frac{\psi(u)}{n}\right) - \phi\left(\frac{\psi(u)}{n-1}\right) \text{ is a decreasing function in } u \in [0, 1] \\ &\iff \phi\left(\frac{x}{n}\right) - \phi\left(\frac{x}{n-1}\right) \text{ is an increasing function in } x > 0 \\ &\iff \frac{x}{n-1} \phi'\left(\frac{x}{n-1}\right) \leq \frac{x}{n} \phi'\left(\frac{x}{n}\right) \text{ for all } x > 0 \\ &\iff t\phi'(t) \text{ is a decreasing function in } t > 0. \end{aligned}$$

$\square$

Figure 4.1: $g(\alpha) = F_{3:3}^{-1}(\alpha) - F_{2:2}^{-1}(\alpha)$ for $\alpha \in [0, 1]$.

Unfortunately, the function $t\phi'(t)$ is not monotone for most of the Archimedean copulas, including the independence, Clayton, Frank and Gumbel copulas. Note that this is the reason why an additional condition on F is needed for Proposition 4.1 to get the dispersive order for $X_{n-1:n-1}$ and $X_{n:n}$.

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