Research Article

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Maximal asymmetry of bivariate copulas and consequences to measures of dependence

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Abstract: In this article, we focus on copulas underlying maximal non-exchangeable pairs \((X, Y)\) of continuous random variables either in the sense of the uniform metric \(d_{\infty}\) or the conditioning-based metrics \(D_p\), and analyze their possible extent of dependence quantified by the recently introduced dependence measures \(\zeta_1\) and \(\xi_2\). Considering maximal \(d_{\infty}\)-asymmetry we obtain \(\zeta_1 \in \left[\frac{5}{6}, 1\right]\) and \(\xi_2 \in \left[\frac{3}{2}, 1\right]\), and in the case of maximal \(D_1\)-asymmetry we obtain \(\zeta_1 \in \left[\frac{3}{2}, 1\right]\) and \(\xi_2 \in \left(\frac{1}{2}, 1\right]\), implying that maximal asymmetry implies a very high degree of dependence in both cases. Furthermore, we study various topological properties of the family of copulas with maximal \(D_1\)-asymmetry and derive some surprising properties for maximal \(D_p\)-asymmetric copulas.

Keywords: asymmetry, copula, dependence measure, exchangeability, Markov kernel

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1 Introduction

Two random variables \(X\) and \(Y\) with joint distribution function \(H\) are called exchangeable if and only if the pairs \((X, Y)\) and \((Y, X)\) have the same distribution, or equivalently, if \(H(x, y) = H(y, x)\) holds for all \(x\) and \(y\). The study of exchangeable random variables has exhibited a lot of interest in statistics (see, for instance, [8] and references therein). In case \(X\) and \(Y\) are identically distributed and have distribution function \(F\), then \((X, Y)\) is exchangeable if and only if the underlying copula \(A\) coincides with its transpose \(A^t\) (defined as \(A^t(x, y) = A(y, x)\)). Hence, in what follows we consider continuous and identically distributed random variables \(X\) and \(Y\). While the class of continuous exchangeable random variables \(X\) and \(Y\) is uniquely characterized by the class of symmetric copulas, the exact opposite, i.e., maximal non-exchangeability of random variables, strongly depends on the choice of measure quantifying the degree of non-exchangeability. One natural measure of non-exchangeability was studied by Nelsen [21] as well as by Klement and Mesiar [15], who independently showed that

\[
d_{\infty}(A, A^t) = \sup_{x, y \in [0, 1]} |A(x, y) - A(y, x)| \leq \frac{1}{3}
\]

holds for every \(A \in C\) and introduced the \(d_{\infty}\)-based measure \(\delta: C \to [0, 1]\) via \(\delta(A) = 3d_{\infty}(A, A^t)\). Moreover, they characterized all copulas \(A \in C\) with maximal \(d_{\infty}\)-asymmetry and showed that these copulas always model slightly negatively correlated random variables \(X\) and \(Y\) in the sense of...
Spearman’s $\rho$. More precisely, $\delta(A) = 1$ implies $\rho(A) \in \left[ -\frac{5}{7}, -\frac{1}{3} \right]$. Similar results also hold for different measures of concordance (see [17]).

Considering other metrics on the space of copulas yields alternative measures of non-exchangeability ([13,25]): In [13] the stronger conditioning-based metric $D_t$ introduced in [27] was studied and the authors proved (among other things) that every copula $A \in C$ with maximal $D_t$-asymmetry (i.e., $D_t(A, A') = \frac{1}{2}$) is not maximal asymmetric with respect to $d_\infty$ and that no maximal $d_\infty$-asymmetric copula is maximal asymmetric with respect to $D_t$.

Building upon the results in [13] we here further investigate the family of copulas with maximal $D_1$-asymmetry, derive additional novel characterizations in terms of the Markov-product of copulas (see [3]), and study various topological properties; inter alia we prove that the family of mutually completely dependent copulas with maximal $D_1$-asymmetry is dense in the set of all copulas with maximal $D_1$-asymmetry. Furthermore, we extend the concept of maximal $D_1$-asymmetry to the general $D_p$-metrics ($p \in [1, \infty)$), defined by

$$D_p(A, B) := \left( \int \int_{[0,1] \times [0,1]} |K_p(x, [0, y]) - K_p(x, [0, y])|^p \, d\lambda(x) \, d\lambda(y) \right)^{\frac{1}{p}},$$

where $K_p(\cdot, \cdot)$ denote the Markov kernels (regular conditional distributions) of $A$, $B \in C$, respectively.

Although all $D_p$-metrics induce the same topology, we show the surprising result that maximal $D_1$-asymmetry is not equivalent to maximal $D_p$-asymmetry for $p \in (1, \infty)$. In fact, copulas with maximal $D_p$-asymmetry with $p \in (1, \infty)$ are always mutually completely dependent and maximal asymmetric w.r.t. $D_1$.

Moreover, we tackle the question on the degree of dependence of copulas exhibiting maximal asymmetry with respect to $d_\infty$ or $D_p$ for every $p \in [1, \infty]$. Since measures of concordance are generally not suitable for quantifying dependence (see, for instance, [11]) we consider the dependence measures $\zeta$ introduced in [27] and further studied in [10,11], as well as $\xi$, defined in [4] and reinvestigated in [2]. Both measures have recently attracted a lot of interest (see, e.g., [1,10,11,14,24,26]) since, in contrast to standard methods like Spearman’s $\rho$ or Kendall’s $\tau$, these measures are 1 if and only if $Y$ is a function of $X$ and 0 if and only if $X$ and $Y$ are independent; moreover, they can be estimated consistently without underlying smoothness assumptions. We prove that when considering maximal $d_\infty$-asymmetry $\zeta_1 \in \left[ \frac{5}{7}, 1 \right]$ and $\xi \in \left[ \frac{2}{3}, 1 \right]$ hold, and in the case of maximal $D_1$-asymmetry $\zeta_1 \in \left[ \frac{7}{9}, 1 \right]$ and $\xi \in \left[ \frac{1}{3}, 1 \right]$ follows. In other words, maximal non-exchangeable random variables (in the sense of $d_\infty$ or $D_p$) always imply a high degree of dependence w.r.t. $\zeta_1$ and $\xi$.

The rest of this article is organized as follows: Section 2 gathers preliminaries and notations that will be used throughout the article. In Section 3, we study possible values of $\zeta$ and $\xi$ for maximal $d_\infty$-asymmetric copulas and discuss an example illustrating differences of $\zeta$ and $\xi$ in the context of ordinal sums. In Section 4, we revisit copulas with maximal $D_1$-asymmetry and derive several topological properties. Extensions on maximal $D_p$-asymmetry for $p \in [1, \infty]$ and some interrelations are established in Section 5. Consequences on the dependence measures $\zeta$ and $\xi$ conclude the article (Section 6). Various examples and graphics illustrate both the obtained results and the ideas underlying the proofs.

## 2 Notation and preliminaries

For every metric space $(\Omega, d)$ the Borel $\sigma$-field in $\Omega$ will be denoted by $\mathcal{B}(\Omega)$, $\lambda$ will denote the Lebesgue measure on $\mathcal{B}(\mathbb{R})$, $\mathcal{T}$ will denote the class of all measurable $\lambda$-preserving transformations on $[0, 1]$, i.e.,

$$\mathcal{T} = \{ T : [0, 1] \to [0, 1] \text{ measurable with } \lambda(T^{-1}(E)) = \lambda(E) \, \forall E \in \mathcal{B}([0, 1]) \},$$

then $\mathcal{T}$ is a group under composition and $\lambda$ is $\mathcal{T}$-invariant.
and $\mathcal{T}_b$ the subclass of all bijective $T \in \mathcal{T}$. Throughout the article $C$ will denote the family of all two-dimensional copulas, $\mathcal{P}$ the family of all doubly-stochastic measures (for background on copulas and doubly stochastic measures we refer to [6,22] and references therein). Furthermore, $M$ denotes the upper Fréchet Hoeffding bound, $I$ the product copula, and $W$ the lower Fréchet Hoeffding bound. Additionally, the completely dependent copula induced by a measure-preserving transformation $h \in \mathcal{T}$ will be denoted by $C_h$ (see [27], Definition 9). The family of all completely dependent copulas will be denoted by $C_{cd}$ and the family of all mutually completely dependent copulas by $C_{mcd} = \{C_h \in C_{cd} : h \in \mathcal{T}_b\}$.

For every copula $C \in C$ the corresponding doubly stochastic measure will be denoted by $\mu_C$. As usual, $d_{\infty}$ denotes the uniform metric on $C$, i.e.,

$$d_{\infty}(A, B) = \max_{(x, y) \in [0, 1]^2} |A(x, y) - B(x, y)|$$

for every $A, B \in C$. It is well-known that $(C, d_{\infty})$ is a compact metric space (see [6]).

In what follows, Markov kernels will play an important role. A mapping $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ is called a Markov kernel from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ if the mapping $x \mapsto K(x, B)$ is measurable for every fixed $B \in \mathcal{B}(\mathbb{R})$ and the mapping $B \mapsto K(x, B)$ is a probability measure for every fixed $x \in \mathbb{R}$. A Markov kernel $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ is called regular conditional distribution of a (real-valued) random variable $Y$ given another random variable $X$ if for every $B \in \mathcal{B}(\mathbb{R})$

$$K(x(\omega), B) = \mathbb{E}(1_B \circ Y | X)(\omega)$$

holds $\mathbb{P}$-a.s. It is well-known that a regular conditional distribution of $Y$ given $X$ exists and is unique $\mathbb{P}^X$-almost sure (where $\mathbb{P}^X$ denotes the distribution of $X$, i.e., the push-forward of $\mathbb{P}$ via $X$). For every $A \in C$ (a version of) the corresponding regular conditional distribution (i.e., the regular conditional distribution of $Y$ given $X$ in the case that $(X, Y) \sim A$) will be denoted by $K_{A(\cdot, \cdot)}$. Note that for every $A \in C$ and Borel sets $E, F \in \mathcal{B}([0, 1])$ we have

$$\int_E K_A(x, F) d\lambda(x) = \mu_A(E \times F) \quad \text{and} \quad \int_{[0, 1]} K_A(x, F) d\lambda(x) = \lambda(F). \quad (2)$$

For more details and properties of conditional expectations and regular conditional distributions we refer to [12,16]. Expressing copulas in terms of their corresponding regular conditional distribution yields metrics stronger than $d_{\infty}$ (see [27]) and defined by

$$D_{\Phi}(A, B) := \left( \int_{[0, 1]} \int_{[0, 1]} |K_A(x, [0, y]) - K_B(x, [0, y])|^\Phi \ d\lambda(x) d\lambda(y) \right)^{1/\Phi}, \quad (3)$$

$$D_{\infty}(A, B) = \sup_{y \in [0, 1]} \int_{[0, 1]} |K_A(x, [0, y]) - K_B(x, [0, y])| \ d\lambda(x). \quad (4)$$

To simplify notation we will also write $\Phi_{A, B}(y) = \int_{[0, 1]} |K_A(x, [0, y]) - K_B(x, [0, y])| \ d\lambda(x)$. We will also work with $D_{\Phi}$, defined by

$$D_{\Phi}(A, B) = D_{\Phi}(A, B) + D_{\Phi}(A', B'),$$

whereby $A'$ denotes the transpose of $A \in C$. The metric $D_{\Phi}$ can be seen as metrization of the so-called $\Phi$-convergence, introduced and studied in [18,19]. In [27], it is shown that $(C, D_{1})$ is a complete and separable metric space with diameter $1/2$ and that the topology induced by $D_{1}$ is strictly finer than the one induced by $d_{\infty}$. For further background on $D_{1}$ and $D_{\Phi}$ as well as for possible extensions to the multivariate setting we refer to [6,7,10,27] and references therein.

The $D_{1}$-based dependence measure $\zeta_{1}$ (introduced in [27] and further investigated in [10,11]) is defined as

$$\zeta_{1}(X, Y) := \zeta_{1}(A) = 3D_{1}(A, I),$$
whereby \((X, Y)\) has copula \(A \in C\). In the sequel, we will also consider the dependence measure \(\xi\) (first introduced in [4] and reinvestigated in [2]) defined as \[
\xi(X, Y) = \frac{\int \text{Var}(E(1_{|Y_Z|}X))d\mu(t)}{\int \text{Var}(1_{|Y_Z|})d\mu(t)},
\]
where \(\mu\) is the law of \(Y\). In the copula setting, it is straightforward to verify that \(\xi\) can be expressed in terms of \(D_2\) and that \(\xi(X, Y) = \xi(A) = 6D_2^2(A, \Pi)\) holds. Both dependence measures attain values in \([0, 1]\) and are 0 if and only if \(A = \Pi\), and 1 if and only if \(A\) is completely dependent.

Letting \(S_0(A)\) denote the generalized shuffle of \(A\) w.r.t. the first coordinate, implicitly defined via the corresponding doubly stochastic measure \(\mu_A\) by \[
\mu_{S_0(A)}(E \times F) = \mu_A(h^{-1}(E) \times F),
\]
for all \(E, F \in \mathcal{B}([0, 1])\) (see, e.g., [5,9]), the following simple result holds:

**Lemma 2.1.** Let \(h \in \mathcal{T}_b\) be a \(\lambda\)-preserving bijection. Then \(\zeta(S_0(A)) = \zeta(A)\) and \(\xi(S_0(A)) = \xi(A)\) hold for every \(A \in C\).

**Proof.** According to Lemma 3.1 in [9] for \(h \in \mathcal{T}_b\) the Markov kernel of \(S_0(A)\) can be expressed as \(K_{S_0(A)}(x, [0, y]) = K_A(h^{-1}(x), [0, y])\) and for \(p \in [1, \infty)\) we obtain \[
D_p^p(S_0(A), \Pi) = \int \int |K_{S_0(A)}(x, [0, y]) - y|^p d\lambda(x) d\lambda(y)
\]
\[
= \int \int |K_A(h^{-1}(x), [0, y]) - y|^p d\lambda(x) d\lambda(y)
\]
\[
= \int \int |K_A(h^{-1}(x), [0, y]) - y|^p d\lambda(x) d\lambda(y)
\]
\[
= \int \int |K_A(h^{-1}(h(x)), [0, y]) - y|^p d\lambda(x) d\lambda(y) = D_p^p(A, \Pi),
\]
which proves the assertion. \(\square\)

In the sequel, we will also work with rearrangements [23] (see [26] for an elegant application of rearrangements in the copula context). We call \(f^* : [0, 1] \rightarrow \mathbb{R}\) the decreasing rearrangement of a Borel measurable function \(f : [0, 1] \rightarrow \mathbb{R}\) if it fulfills \(f^*(t) = \inf\{x \in \mathbb{R} : A([z \in [0, 1] : f(z) > x]) \leq t\}\). The stochastically increasing (SI)-rearrangement \(A^\downarrow\) of \(A\) is then defined as \[
A^\downarrow(x, y) = \int_{[0, x]} K_{\delta^y(t)}^\downarrow d\lambda(t),
\]
whereby the rearrangement is applied on the first coordinate of \(K_{\delta^y(t)}\), i.e., for every fixed \(y \in [0, 1]\) the rearranged Markov kernel is defined via \(K_{\delta^y(t)}(x, [0, y]) = \inf\{x \in [0, 1] : A([z \in [0, 1] : K_{\delta^y(z)}(z, [0, y]) > x]) \leq t\}\). In [26], it was shown that \(A^\downarrow\) is an SI copula and both dependence measures \(\zeta_i\) and \(\xi\) are invariant w.r.t. to the rearrangement, i.e., they fulfill \(\zeta(A^\downarrow) = \zeta(A)\) and \(\xi(A^\downarrow) = \xi(A)\), respectively. Recall that a copula \(A\) is called SI if there exists a Borel set \(\Lambda \subseteq [0, 1]\) with \(A(\Lambda) = 1\) such that for any \(y \in [0, 1]\) the mapping \(x \mapsto K_{\delta^y(t)}(x, [0, y])\) is non-increasing on \(\Lambda\). The family of all SI copulas will be denoted by \(C^\downarrow\). For further information we refer to [22] and references therein.

Given \(A, B \in C\) a new copula denoted by \(A * B\) can be constructed via the so-called star/Markov product \(A * B\) (see [3]) by \[
(A * B)(x, y) = \int_{[0,1]} \partial A(x, t) \partial B(t, y) d\lambda(t),
\]
(5)
where \( \partial_A x, y \) denotes the partial derivative of \( A \) with respect to the first coordinate. The star product \( A * B \) is always a copula, i.e., no smoothness assumptions on \( A, B \) are required. Translating to the Markov kernel setting the star product corresponds to the well-known composition of Markov kernels and the following lemma holds:

**Lemma 2.2.** [29] Suppose that \( A, B \in C \) and let \( K_A, K_B \) denote the Markov kernels of \( A \) and \( B \), respectively. Then the Markov kernel \( K_A \ast K_B \), defined by

\[
(K_A \ast K_B)(x, F) = \int_{[0,1]} K_B(y, F)K_A(x, dy),
\]

is a regular conditional distribution of \( A * B \).

### 3 Maximal \( d_\infty \)-asymmetric copulas and their extent of dependence with respect to \( \zeta_1 \) and \( \xi \)

Since ordinal sums will play an important role in what follows, we briefly recall their definition. We follow [6] and let \( I \subseteq \mathbb{N} \) be some finite index set, \( ((a_i, b_i))_{i \in I} \) be a family of non-overlapping intervals with \( 0 \leq a_i < b_i \leq 1 \) for each \( i \in I \) such that \( \bigcup_{i \in I} [a_i, b_i] = [0, 1] \) holds. Furthermore, \( (C_i)_{i \in I} \) denotes a family of bivariate copulas. Then the copula \( C \) defined by

\[
C(x, y) = \begin{cases} 
  a_i + (b_i - a_i)C_i \left( \frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i} \right), & \text{if } (x, y) \in (a_i, b_i)^2 \\
  \min\{x, y\} & \text{elsewhere}
\end{cases}
\]

is an ordinal sum, and we write \( C = (\langle a_i, b_i, C_i \rangle)_{i \in I} \). The following lemma gathers some useful formulas for \( D_1 \) and \( D_2 \), which will be used in the sequel.

**Lemma 3.1.** Let \( C = (\langle a_i, b_i, C_i \rangle)_{i \in I} \) be an ordinal sum with \( I = \{1, \ldots, n\} \) for some \( n \in \mathbb{N} \). Then

\[
D_2^2(C, \Pi) = \sum_{i=1}^{n} ((b_i - a_i)^2)D_2^2(C_i, \Pi) + f(a_1, \ldots, a_n, b_1, \ldots, b_n),
\]

\[
D_1(C, \Pi) = \sum_{i=1}^{n} (b_i - a_i)^2 \int_{[0,1]} \int_{[0,1]} |K_C(x, [0, y]) - (a_i + (b_i - a_i)y)|d\lambda(x)d\lambda(y) + g(a_1, \ldots, a_n, b_1, \ldots, b_n),
\]

whereby \( f \) and \( g \) are given by

\[
f(a_1, \ldots, a_n, b_1, \ldots, b_n) = \sum_{i=1}^{n} \left( \frac{(h_i - a_i)^2}{3} + (b_i - a_i)(1 - b_i) \right) = \frac{1}{3} \]

\[
g(a_1, \ldots, a_n, b_1, \ldots, b_n) = \sum_{i=1}^{n} b_i - a_i \left( \frac{1}{2} - b_i + \frac{b_i^2}{2} + \frac{a_i^2}{2} \right),
\]

respectively.

**Proof.** The definition of \( D_2^2 \) yields

\[
D_2^2(C, \Pi) = \int_{[0,1]} \int_{[0,1]} (K_C(x, [0, y]) - K_D(x, [0, y]))^2d\lambda(x)d\lambda(y)
\]

\[
= \int_{[0,1]} \int_{[0,1]} K_C(x, [0, y])^2d\lambda(x)d\lambda(y) - 2 \int_{[0,1]} y \int_{[0,1]} K_C(x, [0, y])d\lambda(x)d\lambda(y) + \int_{[0,1]} \int_{[0,1]} y^2d\lambda(x)d\lambda(y)
\]

\[
= \int_{[0,1]} \int_{[0,1]} K_C(x, [0, y])^2d\lambda(x)d\lambda(y) - \frac{1}{3}
\]
for every $C \in C$. Using the fact that (without loss of generality) the Markov kernel $K_C(x, [0, y])$ of $C$ is 0 below the squares $(a_i, b_i)^2$ and 1 above $(a_i, b_i)^2$, and applying change of coordinates yields

$$D_2^2(C, \Pi) + \frac{1}{3} = \sum_{i=1}^n \left( \int_{(a_i, b_i)} \int_{(a_i, b_i)} K_C(x, [0, y])^2 d\lambda(x) d\lambda(y) + \int_{(b_i, 1)} \int_{(a_i, b_i)} 1 d\lambda(x) d\lambda(y) \right)$$

$$= \sum_{i=1}^n \left( (b_i - a_i)^2 \int_{[0, 1]} \int_{[0, 1]} K_C(x, [0, y])^2 d\lambda(x) d\lambda(y) + (b_i - a_i)(1 - b_i) \right)$$

$$= \sum_{i=1}^n (b_i - a_i)^2 \left( D_2^2(C_i, \Pi) + \frac{1}{3} \right) + (b_i - a_i)(1 - b_i).$$

Analogously, we obtain

$$D_3(C, \Pi) = \sum_{i=1}^n \int_{[0, 1]} \int_{[0, 1]} |K_C(x, [0, y]) - y| d\lambda(x) d\lambda(y)$$

$$= \sum_{i=1}^n \int_{(a_i, b_i)} \int_{(a_i, b_i)} K_C \left( \frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i} \right) - y \right| d\lambda(x) d\lambda(y)$$

$$+ \sum_{i=1}^n \int_{(b_i, 1)} \int_{(a_i, b_i)} (1 - y) d\lambda(x) d\lambda(y) + \sum_{i=1}^n \int_{(0, a_i)} \int_{(a_i, b_i)} y d\lambda(x) d\lambda(y)$$

$$= \sum_{i=1}^n (b_i - a_i)^2 \left( \int_{[0, 1]} \int_{[0, 1]} |K_C(x, [0, y]) - (a_i + (b_i - a_i)y)| d\lambda(x) d\lambda(y) \right) + g(a_1, \ldots, a_n, b_1, \ldots, b_n),$$

with $g(a_1, \ldots, a_n, b_1, \ldots, b_n)$ as in the theorem.

As a direct consequence, the dependence measure $\xi$ of ordinal sums can easily be expressed in terms of $\xi(C_i)$:

**Corollary 3.2.** Let $C = (\langle a_i, b_i, C_i \rangle)_{i \in I}$ be an ordinal sum with $I = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$. Then

$$\xi(C) = \sum_{i=1}^n (b_i - a_i)^2 \xi(C_i) + 6f(a_1, \ldots, a_n, b_1, \ldots, b_n)$$

holds, where $f$ is defined according to Lemma 3.1 and only depends on the partition.

The following example shows that ordinal sums can be used to construct copulas attaining every possible dependence value w.r.t. to $\xi$ and $\zeta$.

**Example 3.3.** Consider $C_s = (\langle a_i, b_i, C_i \rangle)_{i \in \{1,2\}}$, whereby $a_1 = 0$, $a_2 = s$, $b_1 = s$, and $b_2 = 1$ for $s \in [0, 1]$ and set $C_1 = \Pi$ as well as $C_2 = M$. Figure 1 depicts the support of $\mu_{C_s}$ for different choices of $s \in [0, 1]$. Using Corollary 3.2 we have $\xi(C_s) = (1 - s)^2 + 6\left( \frac{s^2}{3} + s(1 - s) + \frac{(1 - s)^2}{3} - \frac{1}{3} \right) = 1 - s^2$. Therefore, the map $\varphi : [0, 1] \to [0, 1]$ defined by $s \mapsto \xi(C_s)$ is continuous and onto. The same holds for $\zeta(C_s) = 1 - s^3$.

Before deriving some first results concerning the range of the dependence measures $\xi(A)$ and $\zeta(A)$ for maximal $d_{\infty}$-asymmetric copulas $A$, we recall the characterizations of maximal $d_{\infty}$-asymmetry derived in [21,15]: $d_{\infty}(A, A')$ is maximal if and only if $A(\frac{2}{3}, \frac{1}{3}) = 0$ and $A(\frac{1}{3}, \frac{2}{3}) = \frac{1}{3}$ or $A'(\frac{2}{3}, \frac{1}{3}) = 0$ and $A'(\frac{1}{3}, \frac{2}{3}) = \frac{1}{3}$. Without loss of generality we may focus on the case $A(\frac{1}{3}, \frac{2}{3}) = \frac{1}{3}$ and $A(\frac{2}{3}, \frac{1}{3}) = 0$. Since $A$ is doubly stochastic
in this case we obviously have
\[
\begin{bmatrix}
\mu_1
\end{bmatrix} = \begin{bmatrix}
1/3
\end{bmatrix},
\begin{bmatrix}
\mu_2
\end{bmatrix} = \begin{bmatrix}
2/3
\end{bmatrix},
\begin{bmatrix}
\mu_3
\end{bmatrix} = \begin{bmatrix}
1/3
\end{bmatrix}.
\]
As a direct consequence, we can find copulas
\[A_s \in \mathcal{A}, 1 \leq s \leq 3\]
fulfilling
\[
\begin{bmatrix}
\xi(A)
\end{bmatrix} = \begin{bmatrix}
f_0
\end{bmatrix},
\]
whereby the functions
\[f_i : [0, 1]^2 \rightarrow \begin{bmatrix}
\frac{i-1}{3}, \frac{i}{3}
\end{bmatrix} \times \begin{bmatrix}
\frac{i-1}{3}, \frac{i}{3}
\end{bmatrix}
\]
is given by
\[f_i(x, y) = \left(\frac{x+iy-1}{3}, \frac{y+ix-1}{3}\right)\]
for each \((i, j) \in \{1, 2, 3\}^2\) (and \(\mu_{A_s}^{f_0}\) denotes the push-forward of \(\mu_A\) via \(f_i\)).

**Theorem 3.4.** If \(A \in \mathcal{C}\) has maximal \(d_{\gamma_s}\)-asymmetry, i.e., if \(\delta(A) = 3d_{\gamma_s}(A, A') = 1\) holds, then \(\xi(A) \in \left[\frac{2}{3}, 1\right]\). Moreover, for every \(s \in \left[\frac{2}{3}, 1\right]\) there exists a copula \(A\) with \(\delta(A) = 1\) fulfilling \(\xi(A) = s\).

**Proof.** We may assume that \(A\left(\begin{bmatrix}
1/3
\end{bmatrix}, \begin{bmatrix}
2/3
\end{bmatrix}\right) = 1/3\) and \(A\left(\begin{bmatrix}
2/3
\end{bmatrix}, \begin{bmatrix}
1/3
\end{bmatrix}\right) = 0\). Then there exist copulas \(A_1, A_2, A_3 \in \mathcal{C}\) such that
\[
\begin{bmatrix}
\mu_A
\end{bmatrix} = \begin{bmatrix}
\frac{1}{3}\mu_{A_1} + \frac{1}{3}\mu_{A_2} + \frac{1}{3}\mu_{A_3}
\end{bmatrix},
\]
holds. Defining \(h : [0, 1] \rightarrow [0, 1]\) by
\[
\begin{cases}
\frac{1}{3} + x & \text{if } x \in \left[0, \frac{2}{3}\right] \\
\frac{1}{3} - x & \text{if } x \in \left[\frac{2}{3}, 1\right],
\end{cases}
\]
we have \(h \in \mathcal{T}_b\) and \(S_b(A) = \left(\frac{1}{3}, h, A_i\right)\) \(\forall \{1, 2, 3\}\). Applying Lemmas 2.1 and 3.2 we therefore obtain
\[
\xi(A) = \xi(S_b(A)) = 6f(a_1, \ldots, a_n, b_1, \ldots, b_n) + \sum_{i=1}^{3} \frac{1}{9}\xi(A_i) = \frac{2}{3} + \sum_{i=1}^{3} \frac{1}{9}\xi(A_i) \geq \frac{2}{3},
\]
with equality if and only if \(A_i = \Pi\) for every \(i = 1, 2, 3\).

Defining \(A_s\) by
\[
\begin{bmatrix}
\mu_A
\end{bmatrix} = \begin{bmatrix}
\frac{1}{3}\mu_{A_1} + \frac{1}{3}\mu_{A_2} + \frac{1}{3}\mu_{A_3}
\end{bmatrix} \text{ with } C_s\]
as in Example 3.3 yields
\[
\xi(A_s) = \xi(S_b(A_s)) = \frac{1}{3}\xi(C_s) + \frac{2}{3}.
\]
Considering \(\xi(C_s) = 1 - s^2 \in [0, 1]\) for \(s \in [0, 1]\) and using the same arguments as in Example 3.3 it follows that for every \(s_0 \in \left[\frac{2}{3}, 1\right]\) we find a copula \(A \in \mathcal{C}\) with \(\xi(A) = s_0\) and \(3d_{\gamma_s}(A, A') = \delta(A) = 1\). \(\square\)
Since $\zeta_1$ and $\xi$ are similar by construction, one might expect the analogous statements for $\zeta_1$. Note, however, that a different proof is needed since according to Lemma 3.1 the formulas for $D$ are more involved.

**Theorem 3.5.** If $A \in C$ has maximal $d_{w-o}$-asymmetry, then $\zeta_1$ satisfies $\zeta_1(A) \in \left[\frac{5}{6}, 1\right]$. Furthermore, for every $s \in \left[\frac{5}{6}, 1\right]$ there exists a copula $A$ with $\delta(A) = 1$ fulfilling $\zeta_1(A) = s$.

**Proof.** Proceeding as in the proof of Theorem 3.4 we obtain $S_	heta(A) = \left(\left(\frac{i-1}{3}, \frac{i}{7}, A_i\right)\right)_{i \in \{1,2,3\}}$. Considering the (SI)-rearrangement $S_h(A)$ of $S_	heta(A)$ it is clear that $S_h(A)$ is an ordinal sum again and can be expressed as $S_h(A) = \left(\left(\frac{i-1}{3}, \frac{i}{7}, A_i\right)\right)_{i \in \{1,2,3\}}$. Since every $A_i$ is SI and hence fulfills $A_i(x,y) \geq \Pi_i(x,y) = \Pi(x,y)$ for every $(x,y) \in [0,1]^2$ and every $i \in \{1,2,3\}$ (see, e.g., [22][Section 5.2]), we obtain that

$$S_h(A) \geq C_{\Pi} = \left(\left(\frac{i-1}{3}, \frac{i}{7}, \Pi\right)\right)_{i \in \{1,2,3\}}$$

holds pointwise. Due to the fact that $\zeta_1$ is monotone w.r.t. the pointwise order on $C$ and $\zeta_1$ is invariant w.r.t. to (SI)-rearrangements (see [26]), we obtain

$$\zeta_1(A) = \zeta_1(S_h(A)) = \zeta_1(S_h(A)) \geq \zeta_1(C_{\Pi}) = \frac{5}{6},$$

where the last equality follows from Lemma 3.1 (the detailed calculations are deferred to Appendix A). To show the second assertion we can proceed analogously to the proof of Theorem 3.4 and use shrunk copies of the copula $C_{\Pi}$ defined in Example 3.3 (see Appendix A). \qed

While the minimum value of $\xi$ for a copula $A \in C$ with maximal $d_{w-o}$-asymmetry is attained if and only if $A_i = \Pi$ for every $i = 1, 2, 3$ in equation (7), $\zeta_1$ exhibits a different behavior as demonstrated in the following example:

**Example 3.6.** Let $A_i \in C$ be defined by

$$A_i(x,y) = xy + \frac{1}{2}x(1-x)y(1-y).$$

Then a version of the corresponding Markov kernel of $A_i$ is given by $K_A(x, [0,y]) = y + \frac{1}{2}(2x-1)y(y-1)$. Furthermore, we set $A_3 = A_1$ and $A_2 = \Pi$ and let $A$ denote the ordinal sum given by $A = \left(\left(\frac{i-1}{3}, \frac{i}{7}, A_i\right)\right)_{i \in \{1,2,3\}}$ and $C_{\Pi}$ be the ordinal sum given by $C_{\Pi} = \left(\left(\frac{i-1}{3}, \frac{i}{7}, \Pi\right)\right)_{i \in \{1,2,3\}}$ (Figure 2).

By construction we have $A \neq C_{\Pi}$, however, considering

$$\int_{[0,1]} \int_{[0,1]} \frac{1}{2}(2x - 1)y(y - 1)dA(x)dA(y) = 0$$

yields

$$\int_{[0,1]} \int_{[0,1]} K_A(x, [0,y]) - \frac{y}{3} dA(x)dA(y) = \int_{[0,1]} \int_{[0,1]} \frac{2y}{3} dA(x)dA(y)$$

$$= \int_{[0,1]} \int_{[0,1]} \left(\frac{2y}{3} + \frac{1}{2}(2x - 1)y(y - 1)\right)dA(x)dA(y)$$

$$= \int_{[0,1]} \int_{[0,1]} \left(\frac{2y}{3} + \frac{1}{2}(2x - 1)y(y - 1)\right)dA(x)dA(y)$$

$$= \int_{[0,1]} \int_{[0,1]} K_A(x, [0,y]) - \frac{y}{3} dA(x)dA(y)$$

Florian Griessenberger and Wolfgang Trutschnig
and analogously we obtain
\[ \int \int_{[0,1] \times [0,1]} K_{\Pi}(x, [0, y]) = \left( \frac{2}{3} + \frac{y}{3} \right) \, d\lambda(x)d\lambda(y) = \int \int_{[0,1] \times [0,1]} K_{A}(x, [0, y]) = \left( \frac{2}{3} + \frac{y}{3} \right) \, d\lambda(x)d\lambda(y). \]

Applying Lemma 3.1 we obtain \( \zeta(A) = \zeta(C_{0}) = \frac{5}{6} \).

**Remark 3.7.** Considering the monotonicity of \( \zeta \) with respect to the pointwise order on \( C^{1} \) as proved in [26], Example 3.6 shows that there exist copulas \( A, B \in C^{1} \) with \( A \leq B \) pointwise and \( A(x, y) < B(x, y) \) for some \( (x, y) \in [0, 1]^{2} \) fulfilling \( \zeta(A) = \zeta(B) \).

### 4 Maximal \( D_1 \)-asymmetry of copulas revisited

In this section, we complement characterizations of copulas with maximal \( D_1 \)-asymmetry going back to [13] and derive some topological properties of subclasses. To be consistent with the notation in [13], the family of copulas with maximal \( D_1 \)-asymmetry is denoted by

\[ C_{ex}^{1} := \{ A \in C : \kappa(A) = 2D_1(A, A') = 1 \} \subseteq C, \]

the subclass of mutually completely dependent copulas is denoted by \( C_{mcd}^{ex} \). We start with the family of mutually completely dependent copulas and show closedness w.r.t. the metric \( D_3 \).

**Proposition 4.1.** The set \( C_{mcd}^{ex} \) is closed in \( (C, D_3) \).

**Proof.** Let \( (A_n)_{n \in \mathbb{N}} \) be a sequence of mutually completely dependent copulas with \( D_3 \)-limit \( A \). Since according to [27] the family of completely dependent copulas is closed w.r.t. \( D_1 \), we obtain \( A \in C_{cd} \) and \( A' \in C_{cd} \). Using [27, Lemma 10] there exist \( \lambda \)-preserving transformations \( g, g' \in T \) such that a version of the Markov kernel \( K_{\lambda}(\cdot, \cdot) \) and \( K'_{\lambda}(\cdot, \cdot) \) is given by \( K_{\lambda}(x, E) = 1_{g}(g(x)) \) and \( K'_{\lambda}(x, E) = 1_{g'}(g'(x)) \), respectively. Furthermore, since a copula \( A \) is completely dependent if and only if it is left-invertible w.r.t. the \( \ast \)-product...
(see [27]) we have \( M = A' \ast A \). Applying Lemma 2.2 therefore yields that \( g \ast g'(x) = id(x) \) for \( \lambda \)-a.e. \( x \in [0, 1] \). Using the fact that \( g \) is surjective \( \lambda \)-almost everywhere, there exists a \( \lambda \)-preserving and bijective transformation \( \lambda \in \mathcal{T}_\lambda \) such that \( h = g \) holds \( \lambda \)-a.e., implying \( A = A_h \in C_{md} \). It remains to show that \( D_i(A_h, A_h') = \frac{1}{2} \), which can be done as follows. Using [13][Theorem 3.5] and the triangle inequality we obtain

\[
\frac{1}{2} = \int_{[0,1]} |h_n - h_{n}^{-1}| d\lambda(x) \leq \int_{[0,1]} |h_n - h| d\lambda(x) + \int_{[0,1]} |h - h_{n}^{-1}| d\lambda(x) + \int_{[0,1]} |h_{n}^{-1} - h_{n}| d\lambda(x)
\]

for every \( n \in \mathbb{N} \). Applying [27][Proposition 15 (ii)] yields

\[
D_i(A_h, A_h') = \int_{[0,1]} |h(x) - h^{-1}(x)| d\lambda(x) \geq \frac{1}{2} \left[ D_i(A_h, A_h) + D_i(A_h', A_h') \right] = \frac{1}{2} - D_i(A_h, A_h). \]

Together with the fact that the maximal distance cannot exceed \( \frac{1}{2} \) it follows that \( D_i(A_h, A_h') = \frac{1}{2} \), which completes the proof.

The following example shows that the set \( C_{md}^{\leq 1} \) is not closed w.r.t. the metric \( D_i \).

**Example 4.2.** Let \( A_h \in C_{md} \) be the mutually completely dependent copula induced by the bijective measure-preserving transformation \( h : [0, 1] \to [0, 1] \), given by

\[
h_n(x) = \begin{cases} 
x + \frac{j - 1}{n} & \text{if } x \in \left[ \frac{j - 1}{n}, \frac{j}{n} \right) \text{ and } j \in \left\{ 1, \ldots, \frac{n}{2} \right\} \\
x - 1 + \frac{j}{n} & \text{if } x \in \left( \frac{j - 1}{n}, \frac{j}{n} \right) \text{ and } j \in \left\{ \frac{n}{2} + 1, \ldots, n \right\} \\
1 & \text{if } x = 1 
\end{cases}
\]

for all \( n \in \mathbb{Z}^+ \) and let \( A_h \in C_{cd} \) be the completely dependent copula induced by the \( \lambda \)-preserving transformation \( h : [0, 1] \to [0, 1] \) given by \( h(x) = 2x(\text{mod}1) \) (see Figure 1 in [9]). Setting \( C_n = \left( \frac{j - 1}{n}, \frac{i}{n}, A_h \right)_{i,j \in \{1,2,3,4\}} \) and \( C = \left( \left\{ \frac{i - 1}{4}, \frac{i}{4}, A_h \right\}_{i \in \{1,2,3,4\}} \right) \), we have \( C_n \in C_{md} \) and \( C \in C_{cd} \) and according to [9, Example 3.3] it is straightforward to verify that \( \lim_{n \to \infty} D_i(C_n, C) = 0 \). As a next step, we reorder the shrunk copulas to obtain maximal \( D_i \)-asymmetry. Let \( f \) denote the \( \lambda \)-preserving interval exchange transformation \( f : [0, 1] \to [0, 1] \) defined by \( f(x) = (x - \frac{1}{4})h_{\{4,1\}}(x) + (x + \frac{3}{4})h_{\{0,4\}}(x) \) and, furthermore, let \( S_f(C_n) \in C_{md} \) and \( S_f(C) \in C_{cd} \) denote the respective shuffles (Figure 3). Due to the fact that the metric \( D_i \) is shuffle-invariant w.r.t. bijective transformations (using the same arguments as in the proof of Lemma 2.1) yields

\[
\lim_{n \to \infty} D_i(S_f(C_n), S_f(C)) = \lim_{n \to \infty} D_i(C_n, C) = 0.
\]

Setting \( U = \left[ 0, \frac{1}{4} \right] \cup \left[ \frac{3}{4}, 1 \right] \) and considering property (3) of Theorem 4.1 in [13] (see also Theorem 4.4(iii) in the sequel) we directly obtain that \( S_f(C_n) \) and \( S_f(C) \) are maximal asymmetric w.r.t. \( D_i \), which shows that \( C_{md}^{\leq 1} \) is not closed w.r.t. the metric \( D_i \).

Leaving the subclass of mutually completely dependent copulas we will now derive novel and handy characterizations of copulas with maximal \( D_i \)-asymmetry and then show some topological properties. The following lemma, showing that the \( \ast \)-product cannot increase the \( D_p \)-distance, will be useful in the sequel. The result has already been stated for \( D_i \) in a slightly different context in [28].

**Lemma 4.3.** For every \( A, B, C \in C \), the following inequality holds for every \( p \in [1, \infty) \):

\[
D_p^p(A \ast B, A \ast C) \leq D_p^p(B, C).
\]
Proof. Applying Lemma 2.2, Jensen’s inequality, disintegration and using the fact that \(\mu_A\) is doubly stochastic we obtain

\[
D^p_\infty(A \ast B, A \ast C) = \int_0^1 \int_0^1 \int_0^x K_B(t, [0, y])K_A(x, dt) - \int_0^1 K_C(t, [0, y])K_A(x, dt) \, d\lambda(x)d\lambda(y)
\]

\[
\leq \int_0^1 \int_0^1 \int_0^x |K_B(t, [0, y]) - K_C(t, [0, y])|^pK_A(x, dt) \, d\lambda(x)d\lambda(y)
\]

\[
= \int_0^1 \int_0^1 \int_0^x |K_B(t, [0, y]) - K_C(t, [0, y])|^p d\mu_A(x, t) \, d\lambda(y)
\]

\[
= \int_0^1 \int_0^1 |K_B(t, [0, y]) - K_C(t, [0, y])|^p \, d\lambda(t)d\lambda(y) = D^p_\infty(B, C),
\]

which completes the proof.

The next theorem gathers several equivalent characterizations of copulas having maximal \(D_1\)-asymmetry (see [13]), and the novel ones established here are (v) and (vi).

**Theorem 4.4.** For every \(A \in C\) the following statements are equivalent:

(i) \(\kappa(A) = 1\),

(ii) \(\Phi_{\lambda, \kappa}(\frac{1}{2}) = 1\) (or equivalently, \(A\) has maximal \(D_{\infty}\)-asymmetry),

(iii) there exists a Borel set \(U \in \mathcal{B}(0, 1)\) with the following properties:

\[
\lambda\left(U \cap \left[0, \frac{1}{2}\right]\right) = \lambda\left(U \cap \left[\frac{1}{2}, 1\right]\right) = \frac{1}{4}, \quad \mu_A\left(U \times \left[0, \frac{1}{2}\right]\right) = \frac{1}{2}, \quad \mu_A\left(\left[0, \frac{1}{2}\right] \times U\right) = 0,
\]

(iv) there exist sets \(U_1, U_2 \in \mathcal{B}(0, 1)\) with \(U_1 \subseteq \left[0, \frac{1}{2}\right], U_2 \subseteq \left[\frac{1}{2}, 1\right]\), \(\lambda(U_1) = \lambda(U_2) = \frac{1}{4}\) and \(V_1 = \left[0, \frac{1}{2}\right] \setminus U_1\) and \(V_2 = \left[\frac{1}{2}, 1\right] \setminus U_2\), and copulas \(C_1, C_2, C_3, C_4 \in C\) such that the following identity

\[
A(x, y) = \frac{1}{4}[C_1(F_1(x), G_1(y)) + C_2(G_1(x), G_2(y)) + C_3(F_2(x), F_1(y)) + C_4(G_2(x), F_2(y))]
\]
holds, whereby \( F_i(x) = 4\lambda(U_i \cap [0, x]) \) and \( G_i(x) = 4\lambda(V_i \cap [0, x]) \) for \( i = 1, 2 \).

(v) \((A \ast A) \left( \frac{1}{2}, \frac{1}{2} \right) = 0,\)

(vi) \( D_1(A \ast A, A \ast A') = \frac{1}{2}.\)

**Proof.** The equivalences of (i), (ii), (iii), and (iv) have already been proved in [13]. Note that the equivalence in property (ii) directly follows from the facts that \( \Phi_{A,A'} \) is Lipschitz continuous with Lipschitz constant 2 and the function \( \Phi_{A,A'} : [0, 1] \to [0, 1] \) fulfills \( \Phi_{A,A'}(y) \leq \min\{2y, 2(1 - y)\} \) for every \( y \in [0, 1] \) (see Lemma 5 in [27]). To show that (i) and (v) are equivalent we may proceed as follows: Suppose that \( \kappa(A) = 1 \). Then according to property (iii) there exists a Borel set \( U \in \mathcal{B}([0, 1]) \) with \( \lambda(U) = \frac{1}{2} \) such that \( K_{A}(x, \left[ 0, \frac{1}{2} \right]) = 1 \) for every \( x \in U \). Applying equation (2) and disintegration yields another Borel set \( V \subseteq U^c \) with \( \lambda(V) = \frac{1}{2} \) and \( K_{A}(x, \left[ 0, \frac{1}{2} \right]) = 0 \) for every \( x \in V \). Setting \( \tilde{V} = U^c \setminus V \), then obviously \( \lambda(\tilde{V}) = 0 \), holds, and applying Lemma 2.2 we obtain

\[
\mu_{A,A'} \left( \left\{ \left[ 0, \frac{1}{2} \right] \times \left[ 0, \frac{1}{2} \right] \right\} \right) = \int_{[0, \frac{1}{2}]} \int_{[0, \frac{1}{2}]} K_{A}(y, \left[ 0, \frac{1}{2} \right]) K_{A}(x, ds) d\lambda(x)
\]

\[
= \int_{[0, \frac{1}{2}]} \left( \int_{[0, \frac{1}{2}]} K_{A}(y, ds) + \int_{[0, \frac{1}{2}]} K_{A}(x, ds) + \int_{\tilde{V}} K_{A}(y, ds) \right) K_{A}(x, ds) d\lambda(x)
\]

\[
\leq \int_{[0, \frac{1}{2}]} K_{A}(x, U) d\lambda(x) + \int_{[0, \frac{1}{2}]} K_{A}(x, \tilde{V}) d\lambda(x)
\]

\[
\leq \mu_{A} \left( \left[ 0, \frac{1}{2} \right] \right) + \mu_{A} \left( [0, 1] \times \tilde{V} \right) = 0 + \lambda(\tilde{V}) = 0.
\]

Suppose now that \((A \ast A) \left( \frac{1}{2}, \frac{1}{2} \right) = 0\) holds. Then according to equation (5) we have

\[
\int_{[0, \frac{1}{2}]} K_{A}(x, \left[ 0, \frac{1}{2} \right]) K_{A}(x, \left[ 0, \frac{1}{2} \right]) d\lambda(x) = 0,
\]

so there exists a set \( \Lambda \in \mathcal{B}([0, 1]) \) with \( \lambda(\Lambda) = 1 \) such that \( K_{A}(x, \left[ 0, \frac{1}{2} \right]) K_{A}(x, \left[ 0, \frac{1}{2} \right]) = 0 \) holds for all \( x \in \Lambda \). Considering \( \min\{a, b\} = \frac{1}{2}(a + b - |a - b|) \) therefore yields

\[
\Phi_{A,A'} \left( \frac{1}{2} \right) = \int_{[0, \frac{1}{2}]} \left| K_{A}(x, \left[ 0, \frac{1}{2} \right]) \right| d\lambda(x)
\]

\[
= \int_{[0, \frac{1}{2}]} K_{A}(x, \left[ 0, \frac{1}{2} \right]) d\lambda(x) + \int_{[0, \frac{1}{2}]} K_{A}(x, \left[ 0, \frac{1}{2} \right]) d\lambda(x)
\]

\[
- 2 \int_{[0, \frac{1}{2}]} \min_{\Lambda} \left( K_{A}(x, \left[ 0, \frac{1}{2} \right]), K_{A}(x, \left[ 0, \frac{1}{2} \right]) \right) d\lambda(x)
\]

\[
= \frac{1}{2} + \frac{1}{2} - 2 \int_{\Lambda} \min \left( K_{A}(x, \left[ 0, \frac{1}{2} \right]), K_{A}(x, \left[ 0, \frac{1}{2} \right]) \right) d\lambda(x) = \frac{1}{2} + \frac{1}{2} - 2 \cdot 0 = 1.
\]

To show the equivalence of (i) and (vi) first assume that \( D_1(A \ast A, A \ast A') = \frac{1}{2} \). Then applying Lemma 4.3 directly yields \( D_1(A, A') \geq \frac{1}{2} \), hence \( \kappa(A) = 1 \). On the other hand, if \( \kappa(A) = 1 \) holds we may proceed as follows: There exists a Borel set \( U \in \mathcal{B}([0, 1]) \) with \( \lambda(U) = \frac{1}{2} \) and \( K_{A}(x, \left[ 0, \frac{1}{2} \right]) = 1 \) as well as \( K_{A}(x, \left[ 0, \frac{1}{2} \right]) = 0 \) for every \( x \in U \). Using disintegration and equation (2) there exists a Borel set \( V \subseteq U^c \)
with \( \lambda(V) = \frac{1}{2} \) and \( K_A(x, \left[ 0, \frac{1}{2} \right]) = 0 \) and \( K_{A'}(x, \left[ 0, \frac{1}{2} \right]) = 1 \) for every \( x \in V \). As before, set \( \tilde{V} = U^c \setminus V \).

Applying Lemma 2.2 yields

\[
A \ast A' \left( \frac{1}{2}, \frac{1}{2} \right) = \mu_{A \ast A'} \left( \left[ 0, \frac{1}{2} \right] \times \left[ 0, \frac{1}{2} \right] \right) = \int \int K_A(s, \left[ 0, \frac{1}{2} \right])K_A(x, ds)\,d\lambda(x)
\]

\[
\leq \int \int 1_{K_A(x, ds)}\,d\lambda(x) = \int K_A(x, U^c)\,d\lambda(x)
\]

\[
= \mu_A \left( \left[ 0, \frac{1}{2} \right] \times U^c \right) = \frac{1}{2} - \mu_A \left( 0, \frac{1}{2} \right) \times U = \frac{1}{2}
\]

as well as

\[
\mu_{A \ast A'} \left( \left[ 0, \frac{1}{2} \right] \times \left[ 0, \frac{1}{2} \right] \right) = \int \int 1_{K_A(x, ds)}\,d\lambda(x)
\]

\[
\leq \int K_A(x, V)\,d\lambda(x) = \mu_A \left( \left[ 0, \frac{1}{2} \right] \times V \right) = \frac{1}{2} - \mu_A \left( \left[ 0, \frac{1}{2} \right] \times V^c \right)
\]

\[
= \frac{1}{2} - \left( \mu_A \left( \left[ 0, \frac{1}{2} \right] \times U \right) + \mu_A \left( \left[ 0, \frac{1}{2} \right] \times V \right) \right) \geq \frac{1}{2}.
\]

Together with property (v) there exist Borel sets \( \Delta_1 \subseteq \left[ 0, \frac{1}{2} \right] \) and \( \Delta_2 \subseteq \left( \frac{1}{2}, 1 \right] \) with \( \lambda(\Delta_1) = \lambda(\Delta_2) = \frac{1}{2} \) such that

\[
K_{A \ast A'}(x_i, \left[ 0, \frac{1}{2} \right]) = 0, \quad K_{A \ast A'}(x_2, \left[ 0, \frac{1}{2} \right]) = 1 \quad \text{and} \quad K_{A \ast A'}(x_i, \left[ 0, \frac{1}{2} \right]) = 1, \quad K_{A \ast A'}(x_2, \left[ 0, \frac{1}{2} \right]) = 0
\]

for every \( x_i \in \Delta_1 \) and \( x_2 \in \Delta_2 \), which gives

\[
\Phi_{A \ast A' A \ast A'} \left( \frac{1}{2} \right) = \int K_{A \ast A'}(x, \left[ 0, \frac{1}{2} \right]) - K_{A \ast A'}(x, \left[ 0, \frac{1}{2} \right]) \,d\lambda(x) = \lambda(\Delta_1) + \lambda(\Delta_2) = 1.
\]

Since \( y \mapsto \Phi_{A,A'}(y) \) is Lipschitz continuous with Lipschitz constant 2 (see [27] [Lemma 5]), the property that \( D_h(A \ast A, A \ast A') = \frac{1}{2} \) follows immediately and the proof is complete.

\[\square\]

**Remark 4.5.** For mutually completely dependent copulas \( A_h \in C_{mcd} \), property (vi) of Theorem 4.4 simplifies to

\[
\kappa(A_h) = 1 \quad \text{if and only if} \quad D_h(A_h \ast A_h, M) = D_h(A_h', M) = h^2 - \|d\|_1 = \frac{1}{2},
\]

where the second equality of the right-hand side directly follows from [27] [Proposition 15 (ii)]. Furthermore, considering property (v) of Theorem 4.4 one might conjecture that \( (A \ast A')(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} \) also implies \( \kappa(A) = 1 \).

For the symmetric copula \( M \), however, it is clear that \( M(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} \) as well as \( M \ast M' = M \ast M = M \) holds.

Not surprisingly, the following result holds.

**Proposition 4.6.** The set \( C^{*} \) is closed in \( (C, D_h) \).

**Proof.** Let \( (A_n)_{n \in \mathbb{N}} \) be a sequence of maximal \( D_h \)-asymmetric copulas fulfilling \( \lim_{n \to \infty} D_h(A_n, A) = 0 \) for some \( A \in C \). Then applying Theorem 4.4, the triangle inequality and the fact that \( D_h \)-convergence implies both

\[
\lim_{n \to \infty} \Phi_{A_n} \left( \frac{1}{2} \right) = 0 \quad \text{and} \quad \lim_{n \to \infty} \Phi_{A_n'} \left( \frac{1}{2} \right) = 0
\]

we obtain
1 = \Phi_{A_n,A}(\frac{1}{2}) \leq \Phi_{A_n,A}(\frac{1}{2}) + \Phi_{A',A}(\frac{1}{2}) + \Phi_{A',A}(\frac{1}{2})

and hence

\Phi_{A',A}(\frac{1}{2}) \geq 1 - \lim_{n \to \infty} \Phi_{A_n,A}(\frac{1}{2}) - \lim_{n \to \infty} \Phi_{A',A}(\frac{1}{2}) = 1.

\□

Remark 4.7. Proposition 4.6 certainly is not surprising, however, the following result is. Key for proving the statement is property (v) of Theorem 4.4.

Theorem 4.8. The set \( C^{x=1} \) is closed in \((C, D_1)\).

Proof. Suppose that \( A, A_1, A_2, \ldots \) are copulas, that \( \kappa(A_n) = 1 \) for every \( n \in \mathbb{N} \), and that \( \lim_{n \to \infty} D_1(A_n, A) = 0 \).

Since the \(*\)-product is jointly continuous w.r.t. \( D_1 \) (see [29]) we have

\[ \lim_{n \to \infty} D_1(A_n * A_n, A * A) = 0. \]

Considering that \( D_1\)-convergence implies \( d_{\infty}\)-convergence, \( \lim_{n \to \infty} (A_n * A_n)\left(\frac{1}{2}, \frac{1}{2}\right) = A * A\left(\frac{1}{2}, \frac{1}{2}\right) \) follows, and applying Theorem 4.4 the proof is complete.

As analogous to the fact that shuffles are dense in \((C, d_{\infty})\) the set \( C^{x=1}_{\text{mod}} \) is dense in \( (C^{x=1}, d_{\infty}) \).

Theorem 4.9. The set \( C^{x=1}_{\text{mod}} \) is dense in \( (C^{x=1}, d_{\infty}) \).

Proof. Fix \( \varepsilon > 0 \) and let \( A \in C^{x=1} \) be a copula with maximal \( D_1 \)-asymmetry. According to property (iv) in Theorem 4.4 there exist sets \( U, U_1, V, V_1 \in \mathcal{B}[0, 1] \) and copulas \( C_i, C_{i_1}, C_{i_2}, C_i \in C \) such that

\[ A(x, y) = \frac{1}{4} [C_i(F_i(x), G_i(y)) + C_{i_1}(F_{i_1}(x), G_{i_1}(y)) + C_{i_2}(F_{i_2}(x), F_i(y)) + C_{i_2}(G_i(x), F_{i_2}(y))], \]

whereby \( F_i(x) = 4\lambda(U_i \cap [0, x]) = 4\int_{[0,x]} 1_{U_i}(s) \lambda(s) \) and \( G_i(x) = 4\lambda(V_i \cap [0, x]) = 4\int_{[0,x]} 1_{V_i}(s) d\lambda(s) \) for \( i \in \{1, 2\} \).

It is well-known that \( C_{\text{mod}} \) (in fact even the family of straight shuffles) is dense in \((C, d_{\infty})\) (see, e.g., [6] [Corollary 4.1.16]), hence, we can find mutually completely dependent copulas \( C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4} \in C_{\text{mod}} \) with \( d_{\infty}(C_i, C_{i_3}) < \varepsilon \) for every \( i \in \{1, 2, 3, 4\} \). Defining \( \tilde{A} \) by

\[ \tilde{A}(x, y) = \frac{1}{4} [C_{i_1}(F_{i_1}(x), G_{i_1}(y)) + C_{i_2}(G_{i_2}(x), G_{i_2}(y)) + C_{i_3}(F_{i_3}(x), F_{i_3}(y)) + C_{i_4}(G_{i_4}(x), F_{i_4}(y))], \]

and applying Theorem 4.4 we conclude that \( \tilde{A} \) has maximal \( D_1 \)-asymmetry too. Furthermore, using the triangle inequality we obtain

\[ \sup_{x, y \in [0, 1]} |A(x, y) - \tilde{A}(x, y)| \leq \frac{1}{4} \sup_{x, y \in [0, 1]} |C_i(F_i(x), G_i(y)) - C_{i_1}(F_{i_1}(x), G_{i_1}(y))| + \frac{1}{4} \sup_{x, y \in [0, 1]} |C_{i_2}(G_{i_2}(x), G_{i_2}(y)) - C_{i_4}(G_{i_4}(x), G_{i_4}(y))| + \frac{1}{4} \sup_{x, y \in [0, 1]} |C_{i_3}(F_{i_3}(x), F_{i_3}(y)) - C_{i_5}(F_{i_5}(x), F_{i_5}(y))| + \frac{1}{4} \sup_{x, y \in [0, 1]} |C_{i_4}(G_{i_4}(x), F_{i_4}(y)) - C_{i_6}(G_{i_6}(x), F_{i_6}(y))| \leq \frac{1}{4} \sum_{i=1}^{6} d_{\infty}(C_i, C_{i_3}) \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \]
As final step we have to show $\tilde{A} \in C_{mcd}$, which can be done as follows: Fix $y \in [0,1]$, then by applying Lemma 1 in [20] using the fact that $K_{\tilde{A}}(x, [0, y])$ is given by $K_{\tilde{A}}(x, [0, y]) = \mathds{1}_{[0,y]}(h(x))$ for $\lambda$-a.e. $x \in [0,1]$ and every $i \in \{1,\ldots,4\}$, the Markov kernel $K_{\tilde{A}}(x, [0, y])$ of $\tilde{A}$ can be expressed by

$$K_{\tilde{A}}(x, [0, y]) = \begin{cases} \mathds{1}_{[0,G_\lambda(y)]}(h_1 \circ F(x)) & \text{for } x \in U_1 \\ \mathds{1}_{[0,G_\lambda(y)]}(h_2 \circ G_\lambda(x)) & \text{for } x \in V_1 \\ \mathds{1}_{[0,F_\lambda(y)]}(h_3 \circ F_\lambda(x)) & \text{for } x \in U_2 \\ \mathds{1}_{[0,F_\lambda(y)]}(h_4 \circ G_\lambda(x)) & \text{for } x \in V_2 \end{cases}$$

for $\lambda$-a.e. $x \in [0,1]$. Since $\{U_1, U_2, V_1, V_2\}$ form a partition of $[0,1]$, for each $y \in [0,1]$ we have that $K_{\tilde{A}}(x, [0, y]) \in \{0,1\}$ for $\lambda$-a.e. $x \in [0,1]$, which is equivalent to $\tilde{A}$ being completely dependent (see [3]). Using the same arguments we also obtain for every $y \in [0,1]$ that $K_{\tilde{A}}(x, [0, y]) = (\partial_y \tilde{A}')(x, y) = (\partial_y \tilde{A})(y, x) \in [0,1]$ for $\lambda$-a.e. $x \in [0,1]$, i.e., $\tilde{A}'$ is completely dependent too. Altogether, we have shown that $\tilde{A} \in C_{mcd}$, which completes the proof. \square

### 5 Maximal $D_p$-asymmetry

Since the metrics $D_p$, $p \in [1,\infty]$ induce the same topology on $C$ one could conjecture that maximal $D_p$-asymmetry might be the same as maximal $D_1$-asymmetry. We will falsify this idea and start with three simple lemmata.

**Lemma 5.1.** [27] Suppose that $h_1$, $h_2$ are $\lambda$-preserving transformations on $[0,1]$ and let $A_{h_1}$, $A_{h_2}$ denote the corresponding completely dependent copulas. Then

$$D_p(A_{h_1}, A_{h_2}) = D_1(A_{h_1}, A_{h_2}) = \|h_1 - h_2\|_1$$

holds for every $p \in (1,\infty)$.

**Lemma 5.2.** The metric space $(C, D_p)$ has the following diameter:

1. $\text{diam}_{D_p}(C) = 2\frac{1}{2}$ for $p \in [1,\infty)$.
2. $\text{diam}_{D_p}(C) = 1$.

**Proof.** According to Lemma 5 in [27] we have

$$\text{diam}_{D_1}(C) = \int_{[0,1]} \min\{2y, 2(1-y)\} \mathrm{d}\lambda(y) = \frac{1}{2}.$$ 

Since $[K_{\tilde{A}}(x, [0, y]) - K_{\tilde{B}}(x, [0, y])] \in [0,1]$ it is straightforward to verify that

$$D_p^0(A,B) \leq D_1(A,B) \leq D_p(A,B)$$

holds for every $A, B \in C$ and $p \in [1,\infty)$. As a direct consequence, we obtain $D_p(A,B) \leq D_1(A,B) \leq 2^{1/p}$. On the other hand, there exist copulas $A, B \in C$ with $D_p(A,B) = 2^{1/p}$. Considering $A = M$ and $B = W$ and applying Lemma 5.1 yield

$$D_p^0(M,W) = D_1(M,W) = \frac{1}{2}.$$ 

The assertion for $p = \infty$ is a direct consequence of Lemma 5 in [27]. \square

Slightly adapting the notation of the previous section we will now focus on the family $C_{S_{p-1}}$ of all bivariate copulas with maximal $D_p$-asymmetry, i.e., $C_{S_{p-1}} = \{A \in C : K_p(A) = 2^{1/p}D_p(A, A') = 1\}$. Building
upon Lemma 5.1 and Theorem 3.5 in [13] there are mutually completely dependent copulas \( A \in C_{mcd} \) such that \( \kappa_p(A) = 1 \) is attained for every \( p \in [1, \infty) \). In fact, the copula \( A_h \) defined in Example 3.4 in [13] has maximal \( D_p \)-asymmetry for every \( p \in [1, \infty) \). The following lemma shows that a copula with maximal \( D_p \)-asymmetry for \( p \in (1, \infty) \) also has maximal \( D_1 \)-asymmetry.

**Lemma 5.3.** If \( A \in C \) satisfies \( \kappa_p(A) = 1 \) for some \( p \in (1, \infty) \), then \( \kappa(A) = 1 \) holds.

**Proof.** Using the inequality \( D_p(B, A) \leq D_1(B, A) \) as well as the fact that \( D_1(B, A) \leq \kappa(A) \) holds for all \( A, B \in C \), we obtain

\[
1 = 2 \delta D_p(A, A') \leq (2D_1(A, A'))^{1/2} \leq 1,
\]

which yields \( D_1(A, A') = \frac{1}{2} \).

The following example, however, shows that the reverse implication does not hold in general.

**Example 5.4.** Suppose that \( A \in C \) corresponds to the uniform distribution on the union of the four squares (Figure 4)

\[
\left( 0, \frac{1}{4} \right) \times \left( \frac{1}{4}, \frac{2}{4} \right), \left( \frac{1}{4}, \frac{2}{4} \right) \times \left( \frac{2}{4}, \frac{3}{4} \right), \left( \frac{2}{4}, \frac{3}{4} \right) \times \left( \frac{3}{4}, 1 \right), \left( \frac{3}{4}, 1 \right) \times \left( 0, \frac{1}{4} \right).
\]

Since \( A \) (and \( A' \)) is a checkerboard copula (see, for instance, [11,18]) a version of the Markov kernel of \( A \) is piecewise linear in \( y \) for fixed \( x \in [0, 1] \) and does not depend on the choice of the point \( x \in \left( \frac{i-1}{4}, \frac{i}{4} \right) \) for every \( i \in \{1, \ldots, 4\} \), (a version of) the corresponding Markov kernel is given by

\[
K_1(x, [0, y]) = \begin{cases} 
(4y - 1)I_{\left(\frac{1}{4}, \frac{1}{2}\right)}(y) + I_{\left(\frac{1}{2}, 1\right)}(y) & \text{for } x \in \left( 0, \frac{1}{4} \right), \\
(4y - 2)I_{\left(\frac{1}{2}, \frac{3}{4}\right)}(y) + I_{\left(\frac{3}{4}, 1\right)}(y) & \text{for } x \in \left( \frac{1}{4}, \frac{2}{4} \right), \\
(4y - 3)I_{\left(\frac{3}{4}, 1\right)}(y) & \text{for } x \in \left( \frac{2}{4}, \frac{3}{4} \right), \\
(4y)I_{\left[0, \frac{1}{4}\right]}(y) + I_{\left(\frac{1}{4}, 1\right)}(y) & \text{for } x \in \left( \frac{3}{4}, 1 \right).
\end{cases}
\]

**Figure 4:** Density of the copula \( A \) (left panel) and the copula \( A' \) (right panel) considered in Example 5.4. The copula \( A \) has maximal \( D_1 \)-asymmetry, i.e., \( \kappa(A) = 1 \), nevertheless \( \kappa_p(A) < 1 \) holds for \( p \in (1, \infty) \).
It is straightforward to verify $\kappa(A) = 1$ (e.g., by using property (iv) or property (v) in Theorem 4.4). On the other hand, simple calculations (see Appendix A) yield

$$D_p^p(A, A') = \frac{1}{4} + 2 \int_{[0,1]} (4x)^p d\lambda(x) + \frac{2}{4p + 4}$$

for every $p \in [1, \infty)$. As a direct consequence we obtain $D_p^p(A, A') < 2^{-1}$ for every $p \in (1, \infty)$, i.e., although $A$ has maximal $D_1$-asymmetry, it fails to have maximal $D_p$-asymmetry.

Contrary to $D_1$, the class $C^{\kappa_p-1}$, $p \in (1, \infty)$ only contains mutually completely dependent copulas.

**Theorem 5.5.** If $A \in C$ has maximal $D_p$-asymmetry for $p \in (1, \infty)$, then $A$ is a mutually completely dependent copula.

**Proof.** If $\kappa_p(A) = 1$ we have $\kappa(A) = 1$ and $D_p(A, A') = 2^{-1}$, which implies

$$\frac{1}{2} = \int \int \left| K_\lambda(x, [0, y]) - K_{A'}(x, [0, y]) \right|^p d\lambda(x) d\lambda(y)$$

$$\leq \int \int \left| K_\lambda(x, [0, y]) - K_{A'}(x, [0, y]) \right| d\lambda(x) d\lambda(y) = \frac{1}{2}.$$

Therefore, we obtain

$$\left| K_\lambda(x, [0, y]) - K_{A'}(x, [0, y]) \right|^p = \left| K_\lambda(x, [0, y]) - K_{A'}(x, [0, y]) \right|,$$

or equivalently, that

$$\left| K_\lambda(x, [0, y]) - K_{A'}(x, [0, y]) \right| \in [0, 1] \quad (9)$$

holds for $\lambda$-a.e. $(x, y) \in [0, 1]^2$. According to Lemma 5.3 and property (iii) in Theorem 4.4 there exist sets $U \in B([0, 1])$ and $V \in B([0, 1])$ such that $U \cap V = \emptyset$, $\lambda(U) = \lambda(V) = \frac{1}{2}$, and

$$K_\lambda(x, [0, y]) = \begin{cases} 1 & \text{for every } y \in \left[0, \frac{1}{2}\right] \\ 0 & \text{for every } y \in \left[\frac{1}{2}, 1\right] \end{cases}, \quad K_{A'}(x, [0, y]) = \begin{cases} 0 & \text{for every } y \in \left[0, \frac{1}{2}\right] \\ 1 & \text{for every } y \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

for every $x \in U$ as well as

$$K_\lambda(x, [0, y]) = \begin{cases} 0 & \text{for every } y \in \left[0, \frac{1}{2}\right] \\ \leq 1 & \text{for every } y \in \left[\frac{1}{2}, 1\right] \end{cases}, \quad K_{A'}(x, [0, y]) = \begin{cases} \leq 1 & \text{for every } y \in \left[0, \frac{1}{2}\right] \\ 1 & \text{for every } y \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

for every $x \in V$. Fix $x \in U$ such that equation (9) holds and suppose that $K_\lambda(x, [0, y]) = y_0 \in (0, 1)$ for some $y \in \left[0, \frac{1}{2}\right]$. Then due to equation (9) the Markov kernel of $A'$ must satisfy $K_{A'}(x, [0, y]) = y_0$, which is a contradiction to the fact that $K_{A'}(x, [0, y]) = 0$ for every $y \in \left[0, \frac{1}{2}\right]$. Hence, we obtain that $K_\lambda(x, [0, y]) \in [0, 1]$ for every $y \in [0, 1]$. In an analogous way, we obtain that $K_{A'}(x, [0, y]) \in [0, 1]$ holds for every $y \in [0, 1]$. Proceeding in the exactly same manner for $x \in V$ we obtain that for $\lambda$-a.e. $x \in [0, 1]$ and every $y \in [0, 1]$ the Markov kernels of $A$ and $A'$ satisfy $K_\lambda(x, [0, y]) \in [0, 1]$ and $K_{A'}(x, [0, y]) \in [0, 1]$. By Theorem 7.1 in [3] and Lemma 3.4 in [10], it follows that $A$ and $A'$ are completely dependent, implying that $A$ is mutually completely dependent. \hfill \Box
Altogether we have shown the following results:

**Corollary 5.6.** The following properties hold:

1. \( C^{\kappa_{se}=1} = C^{\kappa_{se}=1} \)
2. \( C^{\kappa_{se}=1} \supseteq C^{\kappa_{se}=1} \) for every \( p \in (1, \infty) \).
3. \( C^{\kappa_{se}=1} = C^{\kappa_{se}=1} \) for every \( p \in (1, \infty) \).

## 6 Maximal \( D_p \)-asymmetric copulas and their values for \( \zeta_1 \) and \( \xi_1 \)

In Section 3, we have shown that copulas \( A \in C \) with maximal \( d_{\infty} \)-asymmetry have very high dependence scores with respect to \( \zeta_1 \) and \( \xi_1 \). Here we now focus on the range of these dependence measures to maximal \( D_p \)-asymmetric copulas.

**Theorem 6.1.** If \( A \in C \) satisfies \( \kappa(A) = 1 \) for some \( p \in (1, \infty) \), then \( \zeta(A) = \zeta_1(A) = 1 \).

**Proof.** Since \( \zeta(A) \) and \( \xi(A) \) are 1 if and only if \( A \) is completely dependent, the assertion directly follows from Theorem 5.5.

For the case \( p = 1 \) different values for \( \xi_1 \) and \( \zeta_1 \) are possible.

**Theorem 6.2.** If \( A \in C \) satisfies \( \kappa(A) = 1 \), then \( \xi(A) \in \left( \frac{1}{2}, 1 \right] \) holds.

**Proof.** Proceeding analogously to the proof of Theorem 4.4 we obtain \( \xi(A) \ast \mu_{A,1} = \xi_1(A) \ast \mu_{A,1} \) if \( \xi(A) = \xi_1(A) = 1 \) (see Appendix A). Since \( A' \ast A \) is a copula, we find copulas \( A_1, A_2 \in C \) with \( (A' \ast A) = \left( \left( \frac{1}{2}, \frac{1}{2} \right], A_1 \right) \in [1,2] \).

Setting \( C_1 = \left( \left( \frac{1}{2}, \frac{1}{2} \right], \Pi \right) \), \( \mu_{A} \ast A \neq \mu_{A_1} \) follows. In fact, according to Theorem 4.4 there exists a set \( U \in \mathcal{B}([0, 1]) \) such that \( A \left( U \cap \left[ 0, \frac{1}{2} \right] \right) = A \left( U \cap \left[ \frac{1}{2}, 1 \right] \right) = \frac{1}{4} \) and \( 0 = \mu_1 \left( \left[ 0, \frac{1}{2} \right] \times \left[ 0, \frac{1}{2} \right] \right) = \int K_{\alpha}(x, y) d\lambda(x) \).

Hence, we can find Borel sets \( \Lambda_1 \subseteq \left[ 0, \frac{1}{2} \right] \), \( \Lambda_2 \subseteq \left( \frac{1}{2}, 1 \right] \) such that \( \lambda(\Lambda_1) = \lambda(\Lambda_2) = \frac{1}{2} \) and \( K_{\alpha}(x, U) = 0 \) for all \( x \in \Lambda_1 \) and \( K_{\alpha}(x, U) = 1 \) for all \( x \in \Lambda_2 \). The set \( \Lambda_3 \) defined by \( \left[ 0, 1 \right] \setminus (\Lambda_1 \cup \Lambda_2) \) obviously fulfills \( \lambda(\Lambda_3) = 0 \). Hence, we have

\[
\mu_{A} \ast A(U \times U) = \int_{U \in [0,1]} K_{\alpha}(s, U)K_{\alpha}(x, ds) d\lambda(x) = \int_{U \in \Lambda_1} K_{\alpha}(x, \Lambda_2) d\lambda(x) + \int_{U \in \Lambda_3} K_{\alpha}(s, U)K_{\alpha}(x, ds) d\lambda(x) \geq K_{\alpha}(x, \Lambda_2) d\lambda(x) = \mu_{A} \ast A(U \times \Lambda_2) = \mu_{A}(\Lambda_2 \times U) = \int_{\Lambda_2} K_{\alpha}(x, u) d\lambda(x) = \lambda_0(\left[ \frac{1}{2}, 1 \right]) \times U = \lambda(U) - \mu_1 \left( \left[ 0, \frac{1}{2} \right] \times \left[ 0, \frac{1}{2} \right] \right) = \frac{1}{2}.
\]

On the other hand,

\[
\mu_{A_1}(U \times U) = \int_{U \in [0,1]} K_{\alpha}(x, U) d\lambda(x) = \int_{U \in [0,\frac{1}{2}]} K_{\alpha}(x, U) d\lambda(x) + \int_{U \in [\frac{1}{2},1]} K_{\alpha}(x, U) d\lambda(x) = \int_{U \in [0,\frac{1}{2}]} 2\lambda(U \cap \left[ 0, \frac{1}{2} \right]) d\lambda(x) + \int_{U \in [\frac{1}{2},1]} 2\lambda(U \cap \left[ \frac{1}{2}, 1 \right]) d\lambda(x) = \frac{1}{4}.
\]
holds, implying \( A_i \neq \Pi \) for \( i = 1, 2 \), hence, considering that according to Lemma 4.3 we have 
\[
D_\lambda^2(A, \Pi) \geq D_\lambda^2(A' \ast A, A' \ast \Pi) = D_\lambda^2(A' \ast A, \Pi)
\]
and applying Corollary 3.2 finally yields
\[
1 \geq \zeta(A) \geq \zeta(\tilde{A}) = \frac{1}{4} \xi(A_0) + \frac{1}{4} \xi(A_\lambda) + \frac{1}{2} \geq \frac{1}{2}.
\]

**Theorem 6.3.** *If \( A \in C \) satisfies \( k_1(A) = 1 \), then \( \zeta(A) \in \left[ \frac{3}{4}, 1 \right] \) holds.*

**Proof.** Using the same arguments as in the proof of Theorem 6.2 we find copulas \( A_1, A_2 \in C \) such that
\[
(A' \ast A) = \left( \left( \frac{i - 1}{2}, \frac{i}{2}, A_1 \right) \right)_{i \in \{1,2\}} = \tilde{A} \text{ holds.}
\]
Since \( \tilde{A} \) is an ordinal sum it is clear that the (SI)-rearrangement \( \tilde{A}^! \) of \( \tilde{A} \) satisfies \( \tilde{A}^! = \left( \left( \frac{i - 1}{2}, \frac{i}{2}, A_1 \right) \right)_{i \in \{1,2\}} \). As SI copula, \( A_1^! \) fulfills \( A_1^!(x, y) \geq \Pi(x, y) \) for all \((x, y) \in [0, 1]^2 \) and \( i \in \{1, 2\} \), implying \( \tilde{A}^!(x, y) \geq C_\Pi(x, y) \) for every \((x, y) \in [0, 1]^2 \), whereby \( C_\Pi \) is defined as
\[
C_\Pi = \left( \left( \frac{i - 1}{2}, \frac{i}{2}, \Pi \right) \right)_{i \in \{1,2\}}.
\]
Hence, using Lemma 4.3 we obtain
\[
\frac{1}{3} \geq D_\lambda(A, \Pi) \geq D_\lambda(A' \ast A, A' \ast \Pi) = D_\lambda(A' \ast A, \Pi) = D_\lambda(\tilde{A}^!, \Pi) \geq D_\lambda(C_\Pi, \Pi),
\]

whereby we used the fact that \( D_\lambda(A, \Pi) \) is monotone with respect to the pointwise order in \( C^! \) (see [26]). Using Lemma 3.1 we obtain
\[
\zeta(C_\Pi) = 3D_\lambda(C_\Pi, \Pi) = \frac{3}{4} \int_{[0,1]} \int_{[0,1]} \left| y - \frac{y}{2} \right| \, d\lambda(x) \, d\lambda(y) + \frac{3}{4} \int_{[0,1]} \int_{[0,1]} \left| y - \frac{y}{2} \right| \, d\lambda(x) \, d\lambda(y) + \frac{3}{8}
\]
\[
= \frac{3}{16} + \frac{3}{8} - \frac{3}{16} + \frac{3}{8} = \frac{6}{8} = \frac{3}{4},
\]
which completes the proof. \( \square \)

The following example demonstrates that it is possible to find copulas \( A \in C^{p-1} \) such that \( \zeta(A) \) (or \( \xi(A) \), respectively) is arbitrarily close to the lower bound derived in Theorems 6.2 and 6.3.

**Example 6.4.** Let \( n \in \mathbb{N} \) be a natural number with \( n \geq 3 \), set \( N = 2^n \) and define the sets 
\( U^1_N, U^2_N, V^1_N, V^2_N \in \mathcal{B}([0, 1]) \) by 
\[
U^1_N = \bigcup_{j=1}^{N} \left( \frac{2j - 1}{N}, \frac{2j}{N} \right), \quad V^1_N = \bigcup_{j=1}^{N} \left( \frac{2j - 1}{N}, \frac{2j}{N} \right),
\]
\[
U^2_N = \bigcup_{j=1}^{N} \left[ \frac{1}{2}, \frac{2j - 2}{N} \right], \quad V^2_N = \bigcup_{j=1}^{N} \left[ \frac{1}{2}, \frac{2j - 2}{N} \right].
\]

Obviously, we have \( \lambda(U^1_N) = \lambda(U^2_N) = \lambda(V^1_N) = \lambda(V^2_N) = \frac{1}{4} \) and \( \lambda(U^1_N \cup U^2_N \cup V^1_N \cup V^2_N) = 1 \). Letting \( A_N \) denote the copula corresponding to the uniform distribution on the union of the four sets \( U^1_N \times V^1_N, U^1_N \times U^2_N, V^1_N \times U^2_N \) and \( V^2_N \times U^2_N \) (Figure 5), then by Theorem 4.4 \( A_N \) has maximal \( D_1 \)-asymmetry. As next step we calculate the dependence measure \( \zeta(A_N) \).

Applying Lemma 6.3 in [10] and using the fact that for every \( y \in [0, 1] \) the identity of \( K_{A_N}(x, [0, y]) = K_{A_N}(x, [0, y]) \) holds for \( \lambda \)-a.e. \( x, x_0 \in X \), whereby \( X \in \{U^1_N, U^2_N, V^1_N, V^2_N\} \), we obtain
\[
\frac{\zeta(A_N)}{3} = D_\lambda(A_N, \Pi) \leq \frac{2}{N} + \frac{1}{N} \sum_{j=1}^{N} \int_{[0,1]} K_{A_N}(x, \left[ 0, \frac{j}{N} \right]) \, d\lambda(x) + \frac{2}{N} \sum_{x \in U^1_N \cup U^2_N \cup V^1_N \cup V^2_N} \frac{1}{N} \sum_{j=1}^{N} \int_{X} K_{A_N}(x, \left[ 0, \frac{j}{N} \right]) \, d\lambda(x).
\]

where
\[
\zeta(A_N) = \frac{2}{N} \int_{X} \left( \frac{1}{N} \sum_{j=1}^{N} K_{A_N}(x, \left[ 0, \frac{j}{N} \right]) \right) \, d\lambda(x).
\]
Considering $X = U^j_N$ and $x \in U^j_N$, a version of the Markov kernel $K_{A^j}(x, \left[0, \frac{j}{N}\right])$ is given by

$$K_{A^j}(x, \left[0, \frac{j}{N}\right]) = \begin{cases} \frac{2j}{N} & \text{for } j \in \left\{2, 4, \ldots, \frac{N}{2}\right\} \\ \frac{2(j-1)}{N} & \text{for } j \in \left\{1, 3, \ldots, \frac{N}{2} - 1\right\} \\ 1 & \text{for } j > \frac{N}{2}, \end{cases}$$

which yields

$$m(U^j_N) = \frac{1}{4N} \sum_{j=1}^{N} \left| K_{A^j}(x, \left[0, \frac{j}{N}\right]) - \frac{j}{N} \right|$$

$$= \frac{1}{4N} \left( \sum_{j=2, 4, \ldots, N/2} \left| \frac{2j}{N} - \frac{j}{N} \right| + \sum_{j=1, 3, \ldots, N/2-1} \left| \frac{2(j-1)}{N} - \frac{j}{N} \right| + \sum_{j=1}^{N/2} \left( 1 - \frac{j}{N} \right) \right)$$

$$= \frac{1}{4N} \left( \sum_{j=1}^{N/2} \left| \frac{4j}{N} - \frac{2j}{N} \right| + \sum_{j=1}^{N/2-1} \left| \frac{4(j-1)}{N} - \frac{2j}{N} \right| + \sum_{j=1}^{N/2} \frac{1 - j}{N} \right)$$

$$= \frac{1}{4N} \left( \frac{N}{4} N^2 \right) \approx \frac{1}{16} + \frac{1}{2N^2}.$$
Remark 6.5. Slightly modifying the construction from Example 6.4 (which corresponds to copying shrunk versions of the product copula $\Pi$ in the small squares) we now construct the copula $B_N$ by copying shrunk versions of $M$ in every square of the “diagonal” of each of the four sets $U_0^1 \times V_N^1$, $U_0^2 \times V_N^2$, $V_0^1 \times U_N^1$, and $V_0^2 \times U_N^2$ as depicted in Figure 5 (magenta lines). The shuffle $B_N$ is obviously maximal $D_1$-asymmetric and, being completely dependent, fulfills $\zeta(B_N) = 1 = \zeta(B_N)$. Hence, setting $C_N^\alpha = aA_N + (1 - a)B_N$ for every $a \in [0, 1]$ (with $A_N$ according to Example 6.4) obviously yields a maximal $D_1$-asymmetric copula $C_N^\alpha$. Due to the fact that $\zeta(C_N^\alpha)$ and $\zeta(C_N^\beta)$ are continuous in $\alpha$ the intermediate value theorem implies that for every $s \in [\zeta(A_N), 1]$ we can find a copula $C_N^\alpha$ with $\zeta(C_N^\alpha) = s$ and the same result holds for $\zeta$ replaced by $\xi$. In other words, each point in the intervals mentioned in Theorems 6.2 and 6.3 is attained.

Remark 6.6. We have been able neither to find a copula $A \in C_{s=1}^\infty$ fulfilling $\zeta(A) = \frac{3}{4}$, nor to prove that such a copula does not exist.

Appendix

Calculations for the proof of Theorem 3.5:

To calculate $\zeta(C_{\tilde{\Pi}})$ we apply Lemma 3.1 and obtain

$$
\zeta(C_{\tilde{\Pi}}) = 3D_{\tilde{\Pi}}(C_{\tilde{\Pi}}, \Pi) \\
= \frac{1}{3} \int_0^1 \int_0^1 \left| y - \frac{Y}{3} \right| d\lambda(x)d\lambda(y) + \frac{1}{3} \int_0^1 \int_0^1 \left| y - \frac{1}{3} \right| d\lambda(x)d\lambda(y) + \frac{1}{3} \int_0^1 \int_0^1 \left| y - \frac{2}{3} \right| d\lambda(x)d\lambda(y) \\
+ \frac{10}{18} \\
= \frac{1}{3} \int_{[0,1]} 2\frac{y}{3} d\lambda(y) + \frac{1}{3} \int_{[0,1]} 2\frac{y}{3} - 1 d\lambda(y) + \frac{1}{3} \int_{[0,1]} 2\frac{y}{3} - 2 d\lambda(y) + \frac{10}{18} \\
= \frac{2}{18} + \frac{1}{18} + \frac{2}{18} + \frac{10}{18} = \frac{15}{18} = \frac{5}{6}.
$$

Let $A_s$ be the ordinal sum of $C_s$, i.e., $A_s = \left(\left(\frac{k-1}{3}, \frac{k}{3}, C_s\right)_{k \in \{1,2,3\}}\right)$, whereby $C_s$ is the ordinal sum considered in Example 3.3. Then for the integrals considered in Lemma 3.1 we obtain for $s \in \left[0, \frac{1}{3}\right]$:

$$
I_1 = \int_{[0,1]} \int_{[0,1]} |y - ys| dx dy = \frac{1 - s}{2}, \\
I_2 = \int_{[0,1]} \int_{[0,1]} \left| I_{0,y}(x) - \left( s + y\left(\frac{1}{3} - s\right) \right) \right| dx dy \\
= \int_{[0,1]} y(1 - \left( s + y\left(\frac{1}{3} - s\right) \right)) dy + \int_{[0,1]} (1 - y)\left( s + y\left(\frac{1}{3} - s\right) \right) dy \\
= \frac{1}{18}(7 - 3s) + \frac{1}{18}(6s + 1) = \frac{1}{18}(3s + 8), \\
I_3 = \int_{[0,1]} \int_{[0,1]} \left| y - \left(\frac{1}{3} + sy \right) \right| = \frac{9s^2 - 12s + 5}{18 - 18s}, \\
I_4 = \int_{[0,1]} \int_{[0,1]} \left| I_{0,y}(x) - \left(\frac{1}{3} + s + y\left(\frac{1}{3} - s\right) \right) \right| dx dy = \frac{1}{18}(4 - 3s) + \frac{1}{18}(6s + 4) = \frac{1}{18}(3s + 8), \\
I_5 = \int_{[0,1]} \int_{[0,1]} \left| y - \left(\frac{2}{3} + sy \right) \right| = \frac{9s^2 - 12s + 5}{18 - 18s}. 
$$
Applying Lemma 3.1 again we obtain
\[ D_1(A, \Pi) = s^2I_1 + \left( \frac{1}{3} - s \right)^2I_2 + s^2I_3 + \left( \frac{1}{3} - s \right)^2I_4 + s^2I_5 + \left( \frac{1}{3} - s \right)^2I_6 + \frac{5}{54}(2 + 9s - 27s^2) \]
for \( s \in \left[ 0, \frac{1}{3} \right]. \) Since \( f(s) = D_1(A, \Pi) \) is a continuous and decreasing function on \( s \in \left[ 0, \frac{1}{3} \right] \) and \( f(0) = \frac{1}{3} \) and \( f\left( \frac{1}{3} \right) = \frac{5}{18}, \) we have shown that \( \zeta(A, \lambda) \) attains every value in \( \left[ \frac{5}{6}, 1 \right]. \)

Calculations for Example 5.4:
Partitioning the integration area and using the fact that \( A \) is a continuous and decreasing function on \( s \in \left[ 0, \frac{1}{3} \right] \) and \( f(0) = \frac{1}{3} \) and \( f\left( \frac{1}{3} \right) = \frac{5}{18}, \) we have shown that \( \zeta(A, \lambda) \) attains every value in \( \left[ \frac{5}{6}, 1 \right]. \)

Calculations for Example 5.4:
Partitioning the integration area and using the fact that \( A \) is a continuous and decreasing function on \( s \in \left[ 0, \frac{1}{3} \right] \) and \( f(0) = \frac{1}{3} \) and \( f\left( \frac{1}{3} \right) = \frac{5}{18}, \) we have shown that \( \zeta(A, \lambda) \) attains every value in \( \left[ \frac{5}{6}, 1 \right]. \)

Calculations for the proof of Theorem 6.2:
To show that \( A^* \) follows from \( \kappa_1(A) = 1 \) we can proceed as follows: According to Theorem 4.4 property (iii) and using disintegration there exists a Borel set \( U \subseteq \mathcal{B}(\mathbb{R}) \) with \( \lambda(U) = \frac{1}{2} \) and
\[ K_A(x, \left[ 0, \frac{1}{2} \right] ) = 1 \]
for every \( x \in U. \) Using disintegration and equation (2) again yields the existence of a Borel set \( V \subseteq U^c \) with \( \lambda(V) = \frac{1}{2} \) and \( K_A(x, \left[ 0, \frac{1}{2} \right] ) = 0 \) and \( K_A(x, \left[ 0, \frac{1}{2} \right] ) = 1 \) for every \( x \in V. \) Set \( \tilde{V} = U^c \setminus \mathcal{V}, \) then applying Lemma 2.2 yields
\[ A^* = A \left( \frac{1}{2}, \frac{1}{2} \right) = \mu_A \left( \left[ 0, \frac{1}{2} \right] \times \left[ 0, \frac{1}{2} \right] \right) = \int \int K_A(x) \times \left[ 0, \frac{1}{2} \right] ) d\lambda(x) \]
\[ \leq \int \int 1K_A(x) d\lambda(x) + \int \int 1K_A(x) d\lambda(x) \]
\[ = \mu_A \left( \left[ 0, \frac{1}{2} \right] \times \left[ 0, \frac{1}{2} \right] \right) \leq \mu \left( \left[ 0, \frac{1}{2} \right] \times \left[ 0, \frac{1}{2} \right] \right) + \lambda(\tilde{V}) = \frac{1}{2} , \]
as well as
\[ \mu_{A'}(\left[ 0, \frac{1}{2} \right] \times \left[ 0, \frac{1}{2} \right]) = \int_{\left[ 0, \frac{1}{2} \right]} \int_{\left[ 0, \frac{1}{2} \right]} K_{A'}(x, ds) + \int_{\left[ 0, \frac{1}{2} \right]} 0K_{A'}(x, ds) + \int_{\left[ 0, \frac{1}{2} \right]} K_{A'}\left( s, \left[ 0, \frac{1}{2} \right] \right) K_{A'}(x, ds) \, d\lambda(x) \]
\[ \geq \int_{\left[ 0, \frac{1}{2} \right]} K_{A'}(x, U) d\lambda(x) = \mu_{A'}\left( \left[ 0, \frac{1}{2} \right] \times U \right) = \mu_{A'}(U \times \left[ 0, \frac{1}{2} \right]) = \frac{1}{2}. \]

Altogether we have shown \( A' + A\left( \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{2} \).

Additional calculations for Example 6.4:

First of all we derive a result similar to that of Lemma 6.3 in [10] for \( D^2_{\Pi}(A, \Pi) \). Using equation (2) we obtain

\[ \left| \int_{\left[ 0, \frac{1}{2} \right]} \int_{\left[ 0, \frac{1}{2} \right]} K_{A}(x, \left[ 0, y \right])^2 d\lambda(x) d\lambda(y) - \int_{\left[ 0, \frac{1}{2} \right]} \frac{1}{n} \sum_{i=1}^{n} K_{A}(x, \left[ 0, \frac{i}{n} \right])^2 d\lambda(x) \right| \]
\[ \leq \sum_{i=1}^{n} \int_{\left[ 0, \frac{1}{n} \right]} \int_{\left[ 0, \frac{1}{n} \right]} \left| K_{A}(x, \left[ 0, y \right])^2 - K_{A}(x, \left[ 0, \frac{i}{n} \right])^2 \right| d\lambda(y) d\lambda(x) \]
\[ \leq \frac{1}{n} \sum_{i=1}^{n} \int_{\left[ 0, \frac{1}{n} \right]} \left( K_{A}(x, \left[ 0, \frac{i}{n} \right])^2 - K_{A}(x, \left[ 0, \frac{i-1}{n} \right])^2 \right) d\lambda(x) \]
\[ = \frac{1}{n} \sum_{i=1}^{n} \int_{\left[ 0, \frac{1}{n} \right]} \left( K_{A}(x, \left[ 0, \frac{i}{n} \right]) - K_{A}(x, \left[ 0, \frac{i-1}{n} \right]) \right) \left( K_{A}(x, \left[ 0, \frac{i}{n} \right]) + K_{A}(x, \left[ 0, \frac{i-1}{n} \right]) \right) d\lambda(x) \]
\[ \leq \frac{2}{n} \sum_{i=1}^{n} \int_{\left[ 0, \frac{1}{n} \right]} \left( \left( \frac{i}{n} - 1 \right)^2 - \left( \frac{i}{n} \right)^2 \right) d\lambda(x) = \frac{2}{n} \sum_{i=1}^{n} \left( \frac{i}{n} - 1 \right)^2 = \frac{2}{n}. \]

Proceeding analogously to Example 6.4 and applying the previous inequality yield

\[ \frac{\xi(A_N) + 2}{6} = \frac{6D^2_{\Pi}(A_N, \Pi) + 2}{6} = D^2_{\Pi}(A_N, \Pi) + \frac{2}{3} = \int_{\left[ 0, \frac{1}{2} \right]} \int_{\left[ 0, \frac{1}{2} \right]} K_{A_0}(x, \left[ 0, y \right])^2 d\lambda(x) d\lambda(y) - \frac{1}{3} + \frac{1}{3} \]
\[ = \int_{\left[ 0, \frac{1}{2} \right]} \int_{\left[ 0, \frac{1}{2} \right]} K_{A_0}(x, \left[ 0, y \right])^2 d\lambda(x) d\lambda(y) \leq \frac{2}{N} + \frac{1}{N} \sum_{j=1}^{N} \int_{\left[ 0, \frac{1}{N} \right]} \left( K_{A_0}(x, \left[ 0, \frac{j}{N} \right])^2 \right) d\lambda(x) \]
\[ = \frac{2}{N} \sum_{X \in \{U_k, \bar{U}_k, V_k, \bar{V}_k\}} \frac{1}{N} \sum_{j=1}^{N} \int_{X} K_{A_0}(x, \left[ 0, \frac{j}{N} \right])^2 d\lambda(x) \]
\[ = m(X). \]

Considering \( X = U^N_k \) and \( x \in U^N_k \) we obtain

\[ m(U^N_k) = \frac{1}{4N} \sum_{j=1}^{N} K_{A_0}(x, \left[ 0, \frac{j}{N} \right])^2 = \frac{1}{4N} \left( \sum_{j=1}^{N} \left( \frac{N+4j}{N} \right)^2 \right) \]
\[ = \frac{1}{4N} \left( \frac{(N+2)(N+4)}{12N} + \frac{N^2 - 6N + 8}{12N} + \frac{N}{2} \right) \]
\[ = \frac{1}{6} + \frac{1}{3N^2}. \]

In an analogous manner, we obtain

\[ m(U^N_k) = \frac{1}{24} + \frac{1}{4N} + \frac{1}{3N^2}, \quad m(V^N_k) = \frac{1}{24} + \frac{1}{4N} + \frac{1}{3N^2}, \quad m(V^N_k) = \frac{1}{6} + \frac{1}{4N} + \frac{1}{3N^2}, \]

which altogether yields \( \frac{1}{2} < \xi(A_N) \leq \frac{1}{2} + \frac{15}{N} + \frac{8}{N^2}. \)
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