The evolution of mathematical concepts
An essay on analogy in mathematics

by André van Es

Can anything general be said about the evolution of mathematical ideas? And if so, can this be brought to bear on major problems in contemporary mathematics? The answer to both of these questions is yes. In this essay I will postulate the existence of a hidden evolutionary constant in the development of mathematical ideas. Formulating this requires an analysis of the notion of 'analogy' and of its role in mathematics.

Much of the evolution of mathematics and indeed of any science can be described (if not defined) as a process of abstraction and generalisation of its concepts. In turn, abstraction and generalisation in mathematics can be characterised as a process of distillation; it filters and isolates aspects of a specific mathematical object in which these are contained implicitly. Filtering these aspects, enumerating them in lists of axioms, thus making them explicit, makes it possible to study them in isolation, i.e., isolated from the original specific object and context. Usually these 'filtered' concepts open up a whole new range of applications. In this way, a generalisation can suddenly unify many different contexts and problems into one and the same mathematical framework. Without this kind of 'condensation', mathematics and science in general would be simply an accumulation of knowledge which would be bound to grow beyond the grasp of an individual mind. It is also this potential for condensation from which mathematicians derive a certain optimism; what presently seems to be a collection of different mathematical specialities of ever increasing complexity may at some point in the future be placed in a unifying framework.

The main thesis of this essay is that possibly this optimism can be only partially justified; the abstraction and condensation process described above may not be germane to the entire mathematical doctrine. There is a field of mathematics where something quite different seems to be going on. This 'something' is closely related to the occurence of 'analogy' in mathematics and more specifically in number theory.

Now, what do I mean by 'analogy'? Clearly a mathematical definition cannot be given and this may well be considered as its prime feature; it is a notion which eludes a precise mathematical definition. Yet, at the same time mathematicians agree that particular instances of analogy are somehow very real and (therefore) interesting. This is potentially very confusing; on the one hand, as a notion, 'analogy' lacks mathematical precision. On the other hand, particular examples of analogy are precise enough in a different sense; mathematicians 'know' about them and have a common understanding of them. It is this common understanding which enables them to exploit these analogies systematically. This is a very important criterion: In this essay, analogy does *not* refer to some subliminal creative thought process, some spark of intuition of an individual mathematician, 'genius' or whatever. It refers to its systematic exploitation by numerous mathematicians over numerous decades which makes its instances look like branches of mathematics in their own right.

The demarcation of 'analogy' from other phenomena is in itself not a mathematical enterprise. Therefore the demarcation line can not be expected to be precise in the absolute mathematical sense. However, given a certain mathematical knowledge and experience one should be able to understand which examples are more tyipcal than others. Indeed, it may be considered to be one of the aims of this essay to heighten a certain common metamathematical awareness of 'analogy' in mathematics. I hope to contribute to this by analysing a typical example of analogy, namely, the analogy between numbers and functions.

Are analogies equally distributed among the different fields of contemporary mathematics? It is my impression that this is not the case. I even suspect that they occur exclusively in the pursuit of problems which are arithmetical in nature. What seems to lie behind this postulated phenomenon is an observation about arithmetic and number theory. It is the simple fact that in number theory numbers are both tool and object of the description (this is the 'something' I was refering to earlier); like in any other science, numbers are used to measure and count. They are the basic tools of any science including mathematics. In number theory, numbers also become the *objects* of the science. Of course, this observation is metamathematical and plays no part in number theory itself. However, when one looks at the way in which analogies are exploited in number theory, this aspect somehow rises to the surface and seems to dominate many (metamathematical) phenomena which consistently accompany the exploitation of analogy there.

In this essay I hope to clarify many of the above statements and to make them plausible. Again, I hope to achieve this simultaneous goal by describing some elementary aspects of the analogy between numbers and functions.

It is this postulation of a sharp dichotomy between arithmetic and geometry which lies at the heart of this essay. This dichotomy is 'metamathematical' in nature. Of course, mathematically speaking, arithmetic and geometry may be heavily intertwined. Nonetheless, the spirit of this essay is that arithmetic has very different 'parameters of evolution'. These are not the ones described above as the isolation of aspects by abstraction and generalisation. One could describe them alternatively as the isolation of aspects by *analogy*. The (intuitive) explanation of this is also implicit in the description of the analogy between numbers and functions, which I will give.

In arithmetic, numbers are both tool and object of the description. How can this characteristic of number theory be visualised within number theory? Consider a positive real number x. Obviously, its prime use lies in its potential to quantify something. Any quantitive statement in mathematics (and in science in general) requires the (positive) real numbers. So primarily, when confronted with x, one is inclined to ask: 'What does x measure?'. In number theory a host of other questions also arise. When x is given as the value of some real valued function $x = f(z)$, one could ask: 'Is x transcendental or algebraic or even rational?'. In case x is rational, a number theorist typically examines if its numerator or denominator is divisible by some prime number p. The power of p occuring in it may be of great interest to him. The point is simply that in number theory numbers are interesting for more reasons than just their size. An example: consider the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} .$$

The size (or the modulus) of its values is crucial in studying distribution properties of the prime numbers, i.e., estimating the asymptotic behaviour of the number of prime numbers $\pi(x)$ below a certain bound x. On the other hand, the p-divisibility properties of its values at the negative integers are of great number theoretical importance as well. The moduli of these values have both a quantitative purpose and a qualitative, arithmetical meaning.

Another example is given by the partition function $p(n)$ which gives the number of distinct additive representations of a positive integer n. One may be merely interested in its rate of growth as n increases. A closer inspection of the first twenty five values or so led Ramanujan to conjecture that $p(n)$ also satisfies all kinds of strange congruences: it turns out that

$p(n)$ is congruent to 0 *mod* 5 whenever n is congruent to 4 *mod* 5. So the values of $p(n)$ are not merely quantities. They also have an arithmetical meaning.

Numbers clearly have an arithmetic aspect and a quantitative aspect. The interplay between these aspects can't be described explicitly in terms of a list of properties, rules, or axioms. It is somehow implicit in problems and objects of arithmetic. However, there is a perspective on number theory which shows how the realm of arithmetic 'would look like' if these two aspects existed in isolation from each other. This perspective is provided by the famous analogy between numbers and functions. I fervently hope to make clear what all of this means, at least in an intuitive manner.

Now let us analyse what precisely is meant 'the analogy between $\mathbb{Q}$ and $\mathbb{F}_q(t)$'. First of all these sets are fields and they both contain 'a ring of integers', i.e., $\mathbb{Z}$ lies in $\mathbb{Q}$ and $\mathbb{F}_q[t]$, the ring of polynomials lies in $\mathbb{F}_q(t)$. These rings share a lot of properties which can be enumerated in a list of axioms: They are both factorisation rings. They both have a division algorithm. Both of them have finite residue rings, i.e., the quotient ring by any ideal is a finite ring. All of this very much follows the classical abstraction process; in this case, these common properties can be formulated in the language and concepts of abstract commutative algebra. This enables one to study each property or aspect in an isolated (abstract) context. For instance, one could develop the theorems of factorisation rings which apply to $\mathbb{Z}$ and $\mathbb{F}_q[t]$ as special cases. Since there exists a formalism, i.e., the theory of factorisation rings, within which these theorems can be derived simultaneously for $\mathbb{Z}$ and $\mathbb{F}_q[t]$, describing these common theorems as 'analogous' would be an abuse of language.

Evidently, the term analogy should be reserved for something more subtle and mysterious. To show the subtlety of reasoning by analogy and to illustrate its heuristical nature I will now give an elaborate example, which will be important for a few other things to follow.

Consider the Kronecker-Weber theorem. This theorem states that every Galois extension with abelian Galois group of $\mathbb{Q}$ is contained in some cyclotomic field $\mathbb{Q}(\zeta_n)$, where ζ_n is some primitive nth root of unity. Of course, these roots of unity are values of the transcendental function $exp 2\pi iz$ at rational values z. For the polynomials very similar things exist.

The classical exponential function *exp* is no doubt the most fundamental analytical function utilised in number theory. Where could one find a similar function for the polynomials, satisfying (among other things) an analogous Kronecker-Weber theorem? For this it needs to have values which generate abelian extensions of $\mathbb{F}_q(t)$. Pursuing this kind of heuristic reasoning, one needs to have some notion of an 'analytical function' exp_C on some sort of space of polynomials Ω: $exp_C : \Omega \longrightarrow \Omega$. Now, what could be Ω?

Clearly, Ω should somehow be the analogue of the complex numbers. These are constructed out of the rationals by taking the algebraic closure of the completion of $\mathbb{Q}$. This completion is taken with respect to the euclidean metric on $\mathbb{Q}$. In constructing Ω one proceeds in an identical fashion although there is are some twists to it. First, one has to find an analogue of the euclidean norm on $\mathbb{F}_q(t)$ and take the completion with respect to this norm. Of course polynomials can't be used to count or to measure so there is no genuine euclidean metric or size. However, given a polynomial or rational function f one could simply assign (the negative of) its degree $-deg f$ to it. Indeed, this is a norm on $\mathbb{F}_q(t)$. The resulting completion is $\mathbb{F}_q\big((1/t)\big)$, which is the field of power series in $1/t$. This field could well be considered as the analogue of the reals. However, unlike the algebraic closure of $\mathbb{R}$, passing to the algebraic closure of $\mathbb{F}_q\big((1/t)\big)$ does not yield a complete space. So one has to extend the norm to this algebraic closure and completing once more with repect to this extended norm one finally obtains Ω. It has the same two properties which are essential for doing complex analysis: It is both complete and algebraically closed. One can see that the analogy is not completely straightforward. There are some distortions which should not bother the reader. He should only be aware that these are a common occurence in analogical reasoning.

On Ω one can define a function which is closely analogous to the exponential function $exp\ 2\pi iz$. This is the Carlitz exponential. It is defined by an infinite product:

$$exp_C(x) = x \prod_{0 \neq \alpha \in L} (1 - x/\alpha)$$

Here, L is the subset of Ω analogous to the subset $2\pi i\mathbb{Z}$ in $\mathbb{C}$.

This Carlitz exponential is analytical in the sense that it can be developed in a power series. In fact analysis in fields like Ω (i.e., non archimedean fields) is much less complicated than classical analysis. This is due to the fact that there are much easier criteria for the convergence of infinite sums. These converge whenever the leading terms of the partial sums converge, which is of course false in classical analysis.

Let us return to the Kronecker-Weber theorem. It turns out that evaluating exp_C at 'rational' values, i.e., rational functions, yields abelian extensions of $\mathbb{F}_q(t)$ which are very similar to $\mathbb{Q}(\zeta_n)$. In fact, every abelian extension of $\mathbb{F}_q(t)$ which does not involve an extension of the finite field $\mathbb{F}_q$ is contained in one of these extensions. They are therefore called cyclotomic function fields, although cyclotomy here has clearly lost its geometric aspect of 'circle division'.

The Kronecker–Weber theorem exhibits the quintessential aspect of analogy in mathematics; there is no mechanical way of translating it to the rational function field. Unlike the purely algebraic properties, there is no common frame work which describes the generation of cyclotomic fields for $\mathbb{Q}$ and $\mathbb{F}_q(t)$ simultaneously.

If there is no way of translating mechanically, would it be possible for one notion to have more than one analogue? It turns out that this is indeed a very common occurence. A very good example of this kind of ambiguity is provided by the notion of exponentiation: How would one proceed to translate the expression x^n, for positive integers x and n, to the polynomials? One alternative is the mechanical option: x^n refers to multiplication in the ring $\mathbb{Z}$. Its counterpart should therefore be f^n where f is a polynomial and n is a positive integer. The reasoning is simple: Multiplication in the ring $\mathbb{Z}$ corresponds to multiplication in the ring $\mathbb{F}_q[t]$. However, there is an alternative: Given the Carlitz exponential, one can easily define a Carlitz logarithm. This is the inverse power series of the Carlitz exponential. Now one can easily translate the classical expression *exp* (*nlog x*). Via the Carlitz exponential and logarithm one can give meaning to the expression f^n where n is no longer an integer but a polynomial.

Another instance of this sort of ambiguity can be seen in the role played by the real numbers. For the description of classical arithmetic and 'arithmetic for polynomials' one needs $\mathbb{R}$ for exactly the same thing; measurement purposes. On the other hand there, $\mathbb{R}$ is the completion of $\mathbb{Q}$ under the euclidean metric. As we have seen, this aspect of the reals translates to the field of power series $\mathbb{F}_q((1/t))$. So $\mathbb{R}$ seems to play a double role and each of these has a different counterpart.

Among all the facets of analogy treated so far, it is notably this potential for ambiguity which should enable the reader to understand the main theme of this essay: the relation between analogy and the metamathematical property of arithmetic: Numbers are tools and objects of the description.

This relation can be generalised succinctly in the following thesis: *There exists a natural fusion of aspects in the objects of arithmetic. Analogy serves to isolate these aspects in separate analogous contexts.*

How is this 'fusion of aspects' visible in our examples? As we have seen, the notion of exponentiation in $\mathbb{Z}$ can be formulated purely algebraically, but also by means of the analytical functions *exp* and *log*. These two different approaches can be carried over to the polynomials where they yield two distinct notions of exponentiation. So, exponentiation for numbers seems to carry two aspects at the same time. Each of these aspects can be translated to a different notion of polynomial exponentiation. This illustrates what is meant by isolation of aspects by means of analogy.

The other example exposes the fundamental reason behind this phenomenon. It shows the impossibility of translating the two fundamental roles played by the real numbers to one and the same polynomial analogue. This double role of $\mathbb{R}$ amounts to the observation that numbers are tool and object at the same time.

Can these arguments and conclusions be generalised to the wide and varied theme of analogy in mathematics? I believe that despite the variety of the theme, there is a clear pattern to be discerned in the way analogy manifests itself in mathematics. First of all, analogies seem to be disproportionately prominent in relation to number theory. Some of these analogies typically exist between an arithmetical object and some object which is a priori unrelated to arithmetic. An analogy of this type can comprise a very elaborate dictionary which serves as a heuristic device. Notions and theorems on the one side should have analogues on the other. Curiously, it turns out the number theoretical object is always the main beneficiary of this process. Somehow, the objects and theorems of number theory are very inaccessible in comparison to their non-arithmetical counterparts. It seems that analogy serves to reduce the complexity of an arithmetical object by studying another object which shares only a small portion of its multitude of aspects. This complexity seems to be the natural consequence of the fusion of aspects inherent to arithmetical objects.

André Weil is one of the few mathematicians who expressed himself on analogy. In his essay "De la métaphysique aux mathématiques" he writes:

> Rien n'est plus féconde, tous les mathématiciens le savent, que ces obscures analogies, ces troubles reflets d'une théorie à une autre, ces furtives caresses, ces brouilleries inexplicables ; rien aussi ne donne plus de plaisir au chercheur. Un jour vient ou l'illusion se dissipe ; le pressentiment se change en certitude ; les théories jumelles révèlent leur source commune avant de disparaitre ... La métaphysique est devenue mathématique, prete à former la matière d'un traité dont la froide beauté ne saurait plus nous émouvoir.

The view presented here could be concisely described as follows: Analogies in mathematics are hidden concepts. It is simply a matter of time before these concepts will be brought to the surface. The analogies will then reveal their common source. Interestingly, Weil gives no compelling reasons why this should always be the case. It is a view which he presents as a kind of common knowledge that all mathematicians will somehow recognise as the truth. He illustrates this common knowledge with the history of some mathematical concepts that now stand on solid foundations. He describes the intuitive notion of a group that Lagrange tried to come to grips with as a screen which he could only touch with the tip of his fingers. Then Galois brought the concept to light. Here the notion of analogy is understood in a very wide sense and includes all the various intuitive precursors of our modern concepts. Weil also gives a beautiful description of the analogy between number fields and function fields as a 'texte trilingue', although the Rosetta stone is still to be found.

It is surprising how deeply rooted this conviction is. Somehow mathematicians tend to attribute to their analogies the same kind of 'objectivity' as mathematical objects. The statement that 'X is the analogue of Y' is of course no rigorous mathematical statement but it can be so useful to mathematicians that it acquires quite often a degree of reliability that ressembles that of a mathematical theorem or concept. Such statements are considered to be the theorems and concepts of the future, couched in the metaphysics which is implicit in the word 'analogue'.

Can the discovery of analogies be considered as a mathematical discovery? Weil mentions the care taken by Hilbert to remove all traces of such discoveries from his collected works. He writes:

> Les lois non écrites de la mathématiques moderne interdisent en effet de publier des vues métaphysiques de cette espece.

Weil regretes that Hilbert's valuable ideas, notably those on the analogy between number fields and function fields, remained almost unpublished. Weil wrote his essay in the sixties. Probably it is fair to say that these 'unwritten laws of mathematics' have become out of fashion in modern mathematics. One can see this very clearly in subjects were the pursuit of an analogy has become to some extent an end in itself; Metaphysical discoveries of this kind are then published without any precise indication how these can be put to direct use. This can be seen in areas of mathematics where 'the mysteries of the infinite prime' are at the centre. No one would criticize these habits since this part of mathematics looks so promising (and has fullfilled some of these promises). Some people think it will lead to a proof of the Riemann hypothesis.

Ironically, Weil's own view on analogy as a hidden concept is in itself a very interesting metaphysical conviction. The reasoning behind it carries a subtle element of deceit. It is based on the observation that so many modern concepts of mathematics have 'grown out of analogies'. One could also say that these concepts were at some point in history 'less clear'. Reformulated in this way, the observation has almost become a tautological statement. Apart from the natural numbers, which Kronecker identified as a divine gift, all concepts have emerged over a certain period of time. Somehow this is implicit in the idea of 'concept'. As a tautological statement it can not be used as an argument. It certainly can't be used to demonstrate that concepts are hidden behind the analogies of today.

What could be an alternative for this belief? Perhaps a comparison with physics can be illuminating. Here, the mysterious relation between nature and mathematics may help to conceive o an alternative paradigm. It is the miracle described by Wigner as the unreasonable effectiveness of mathematics in physics. Yet, it is not considered to be a problem which physicists will ever adress as a problem of physics.

Analogy arising in number theory may well be considered to be a 'natural phenomenon'. Here, it appears to be the natural means of achieving the isolation of aspects which in other parts of mathematics is achieved by abstraction and generalisation. Only the future of mathematics can decide if this apparent difference is fundamental.

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Liebe MathematikerInnen,
ich bin in der 6. Klasse und wir haben gerade periodische Dezimalbrüche durchgenommen. Wir haben gelernt: $1/9 = 0.111\ldots$, $3/9 = 0.333$ usw.
Was aber ist dann $0.999\ldots$? Unsere Lehrerin hat gesagt, das wäre $9/9$. Das kann aber doch nicht sein. Das wäre doch 1 und $0.999\ldots$ ist doch ein Unendlichstel kleiner als 1. Gibt es $0.999\ldots$ überhaupt? Aber eine Zahl, die ich mir ausdenken kann, muss es doch geben. Wie kommt man an $0.999\ldots$?
Ich würde mich über eine Antwort freuen.

Lina