Research Article

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Student's Conceptions of Function Transformation

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Abstract: The purpose of this study is to assess object and schema conceptions of transformations of functions for undergraduate level students of Mongolian National University. The research participants were 37 undergraduate students who attended the Calculus course of the third author. To achieve our purpose two of the authors analyzed students' project work independently based on the pre-developed rubrics and further analyses were made. Students' project work included recognition of simple and complicated transformation of functions visually, expressing algebraic forms of such transformations and drawing a doll using transformations of a half circle of radius one. The research results show that students' object and schema conception of transformations of functions were poor. Finding the reason for these poor results is a subject for future research. Moreover, students who were able to recognize more complicated transformations visually could draw a doll using the half circle while the ones who could express transformations of both simple and complicated transformations in algebraic form were able to construct a doll using transformations of the half circle.

Keywords: object and schema conception; APOS theory.

1 Introduction

The study of function transformations has become important in pre-calculus courses at higher education institutions because it gives students new opportunities to use and reflect on the concept of function (Lage & Gaisman, 2006). Moreover, these concepts are fundamental for further study of mathematics and other subjects such as

physics. This topic is taught at different grades and levels of school depending on the countries (Zazkis, Liljedahl, & Gadowsky, 2003), where students find many of its components difficult to master and work with (Eisenberg & Dreyfus, 1994; Zazkis, Liljedahl, & Gadowsky, 2003).

Thus, the level of students' knowledge of these topics determines the future success of their learning of university-level mathematics and other related courses. Insufficient prior knowledge also causes difficulties in further study.

Researchers use various specific cognitive theories of learning mathematics to analyze students' conceptions of function transformations, including the cognitive theory called Action-process-object-schema (APOS), which was developed to analyze these conceptions (Baker, Hemenway, & Trigueros, 2001; Hollebrands, 2003; Lage & Gaisman, 2006). These researchers attributed students' difficulty with function transformation to their incomplete understanding of the function concept. Baker et al. (Baker, Hemenway, & Trigueros, 2001) pointed to the object conception of function as a condition for effective understanding of function transformations.

For the case of Mongolia, linear, power, exponential, logarithm and trigonometric functions transformations (horizontal and the vertical translations, stretch and compression) are usually taught at high school level pre-calculus courses. At university first grade level, these topics are only reviewed in calculus courses and used for further applications. Therefore, students' conception of functions transformation may be difficult for some university students with less fundamental prior knowledge.

In this research authors assumed that Mongolian high school students' action and process conceptions of transformations of basic elementary functions are well developed and set the goal to assess object and schema conception of transformation of functions. To achieve our goal, we analyzed students' project work conducted by one of the authors in her course taught in the fall semester of 2019/2020. In order to analyze, first we developed rubrics which will be discussed in detail in later sections and authors assessed students' works independently. Then we verified the consistency of assessments of authors and conducted further analysis.

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We posed the following research questions:

1. What are the characteristics of function transformation understanding and performance of first year university students? At what level are these characteristics?
2. What are the relationships between the characteristics of function transformation understanding of first level university students?

2 Theoretical framework and literature

The APOS theory is principally a model for describing how mathematical concepts can be learned. It is a framework used to explain how individuals mentally construct their understanding of mathematical concepts (Arnon, et al., 2014). From a cognitive perspective, a particular mathematical concept is framed in terms of its genetic decomposition, a description of how the concept may be constructed in an individual's mind. This differs from a mathematical formulation of the concept, which deals with how the concept is situated in the mathematical landscape – its role as a mathematical idea. Individuals make sense of mathematical concepts by building and using certain mental structures (or constructions) which are considered in the APOS theory to be stages in the learning of mathematical concepts (Piaget & Garcia, 1989). These structures arise through instances of reflective abstraction, which, in the APOS theory, consists of mental mechanisms such as interiorization, encapsulation, coordination, reversal, de-encapsulation, and thematization. Dubinsky (1991) discusses five types of reflective abstraction, or mental mechanisms (interiorization, coordination, reversal, encapsulation, and generalization) that lead to the construction of mental structures (Actions, Processes, Objects, and Schemas). Figure 1 illustrates the relationships between these structures and mechanisms.

The theoretical base of this research is built upon the APOS theory (Asiala, Brown, De Vries, & Dubinsky, 1996) for transformations of functions. Baker et al. (2001) developed a so-called genetic decomposition of the concept of transformation of functions and Lage and Gaisman (2006) refined it. Here we use the definition created by Lage and Gaisman for action, process, object and schema.

At an action level students are able to “perform operations on functions and variables step by step, and these operations can be applied either in the analytical or graphical representation context; rely on memorized

Figure 1: The APOS theory (based on Asiala et al., 1996).

facts or external signs”, “recognize differences between a function and its transformations only in terms of the syntax of the rule that defines the function, and recognize similarities between a function and its transformations, or between transformations, only in terms of some global property of the graph” (Lage & Gaisman, 2006, p. 2).

At the process level, students can “describe changes in the basic functions as a consequence of the application of the transformation without the need to perform each step of the transformation or move the graph of a function step by step. They are able to look at the graph of the transformed function and describe the changes that result from the transformation. These students are also able to reverse the process to identify the function on which a set of transformations was applied.” However, they show “difficulties in coordinating the information obtained from different representational contexts, and in flexibly translating information from one representational context to another” (Lage & Gaisman, 2006, p. 2).

Students who are at an object conception of transformation, can “apply actions on transformed functions and coordinate their properties in terms of possible changes in the original function, At this level, students are able to de-encapsulate any transformed function object into the process involved in its construction, and they are able to identify the basic function on which it is based and compare different transformed functions in terms of their properties in any representational context” (Lage & Gaisman, 2006, p. 2).

Schema conceptions are formed by the interconnection of several actions, processes, objects and other previously constructed schema, and the relationship between them. One possible example for a transformation of functions schema would include the schema for function, the transformation of functions object, and actions and processes on transformed functions to determine their properties or to classify them, for example into rigid or non-rigid transformations (Lage & Gaisman, 2006).

The concept of functions and their transformations are difficult not only for university level students but also for high school and secondary level students. Various research results on how modern technologies like GeoGebra assist in teaching these concepts can be found (Daher & Anabousi, 2015).

3 Research methods

The research methods were divided into research context and participants, data collecting tools, data analyzing tools and procedure and data analysis, as shown below.

3.1 Research context and participants

This research was done at the National University of Mongolia. The research sample included 37 students chosen from 66 students who studied the basic linear algebra and calculus course with course code Math102 during the fall semester of 2019/2020. The course was taught by the third author and the participants were students majoring in Science and Engineering. The lecturer used a Learning Management System based on GeoGebra, which she had developed. We call this system GeoGebra LMS for short. These 37 students are those who used the dynamic environment of the system and sent their project work through LMS. Since these students worked actively on the Geogebra LMS, we assume that their technological background was good enough to do the tasks. Some other students worked on a paper-and-pencil basis and sent their work separately. Those students weren't included in the sample.

Linear, power, exponential, logarithm and trigonometric functions transformations (horizontal and vertical translations, stretch and compression) were taught in high school. These topics are only reviewed in the course and used for application.

According to current high school math curricula of Mongolia, high school students study transformations of basic and elementary functions. At this level students relate change of equations of functions $y=f(x)$ with the graphs. They study how various simple changes of type $y=a f(bx+c)+d$ influence the graph of a function. The lecturer conducted an interactive project work on GeoGebra LMS with students on this topic as a part of course assessment.

The aim of the authors' research on this project work was to diagnose students' object and schema conceptions of function transformation. In addition,

Figure 2: Transformed dolls.

students' work with GeoGebra LMS enabled them to explore the function transformations dynamically, that is not only manipulating the algebraic representation of the function but its graphical representation too. GeoGebra LMS enables this graphical manipulation well because it allows the learner to drag the graph of the function. For example, to translate it and see the resulting change in the function's algebraic expression. Also, while performing the tasks, the lecturer worked as a facilitator of students' learning, directing them and requesting them to justify their answers. Providing immediate feedback was another regular job of the facilitator.

3.2 Data collecting tools

The APOS theory outlined above was used to diagnose students' understanding of the object and schema conceptions of functions transformations. So, the interactive project was based on the APOS theory and it consists of three tasks, listed below.

Task 1. Doll 1 shown in Figure 2 was transformed into shapes of dolls 2, 3, 4. Identify the transformation and related coefficient.

Task 2. Apply the transformations used for images 2, 3, 4 to a given function. The teacher assigned a function to students.

Task 3. Draw the doll using various transformations of half-circle.

Table 1: Rubrics criteria and descriptors for Task 1 and Task 2.

Criteria	Descriptors for Task 1			Descriptors for Task 2		
Transformation 1	Identify and interpret	1	Translation along the Ox -axis	Write the formula correctly	1	Translation along the Ox -axis
		2	Translation along the Oy -axis		2	Translation along the Oy -axis
		3	Oy -stretch		3	Oy -stretch
Transformation 2	Identify and interpret	1	Translation along the Ox -axis	Write the formula correctly	1	Translation along the Ox -axis
		2	Translation along the Oy -axis		2	Translation along the Oy -axis
		3	Ox -stretch		3	Ox -stretch
Transformation 3	Identify and interpret	1	Translation along the Ox -axis	Write the formula correctly	1	Translation along the Ox -axis
		2	Translation along the Oy -axis		2	Translation along the Oy -axis
		3	Ox -stretch		3	Ox -stretch
		4	Oy -stretch		4	Oy -stretch

3.3 Data analyzing tools

In order to answer the research questions, the authors developed three rubrics. The aim of these rubrics is to assess the characteristics of students' understanding of function transformations and application skills of the knowledge. In this study we considered the characteristics of students' understanding of function transformations such as translations and stretchings along axes Ox and Oy . Next, we give more detailed explanations about our rubrics. Rubrics for Task 1 and Task 2 each consist of 3 criteria with 3 ranges of performance and 10 descriptors as shown in Table 1. Rubrics criteria were defined as Transformation 1 (doll 1 to doll 2), Transformation 2 (doll 1 to doll 3) and Transformation 3 (doll 1 to doll 4). The rubric for Task 3 consists of 1 criterion with 6 descriptors and these are corresponding to drawing parts of the body (head, ears, paunch, arms, legs) and the "design".

The performance ranges of rubrics for Task 1 and Task 2 are defined as 2 – correct, 1 – incorrect and 0 – did nothing. For the "Design" descriptor of Task 3, ranges are defined as 2 – used more than two transformations of a semicircle, 1 – used up to two transformations of a semicircle and 0 – used only a semicircle. And, for the other descriptors of Task 3, ranges are defined as 2 – complete, 1 – incomplete and 0 – did not draw.

Here we use the code consisting of the letter D followed by 3 digits. The first digit stands for Task and the

second digit indicates the Criterion. The last digit shows the Description of the task. For example, code D123 means the Ox -stretch of Transformation 2 of Task 1.

3.4 Procedure and data analysis

Two of the authors used the rubrics and independently assessed all students' interactive project work, here they are referred to as evaluators. The interrater reliability of rubrics was tested by Cohens' Kappa coefficient. Additionally, the Pearson correlation coefficients were used to study the relationships between variables.

4 Research results

The research results were divided into reliability analysis, descriptive statistics, and correlational analysis, as shown below.

4.1 Results of the reliability analysis

For the nominal variable, the interrater reliability between the two evaluators was analyzed using the Cohen Kappa coefficient (k), and the statistical results are shown in the Table 2.

Table 4: Student performance on Task 2.

Range	D211	D212	D213	D221	D222	D223	D231	D232	D233	D234
2	18	1	17	0	16	12	0	0	0	11
1	6	2	7	5	5	12	14	14	19	9
0	13	34	13	32	16	13	23	23	18	17
Total	37	37	37	37	37	37	37	37	37	37

Table 5: Student performance on Task 3.

Range	D311	D312	D313	D314	D315	D316
2	21	19	19	15	15	11
1	0	2	2	2	1	6
0	16	16	16	20	21	20
Total	37	37	37	37	37	37

Table 6: Descriptive statistics of the performance for each criterion.

Criteria	M	SD	maximum
C11	0.78	0.44	1.33
C12	0.76	0.44	1.67
C13	0.70	0.38	1.25
C21	0.77	0.57	2.00
C22	0.72	0.66	2.00
C23	0.53	0.53	1.50
C31	0.95	0.90	2.00

Table 7: Results of correlational analysis.

	C11	C12	C13	C21	C22	C23	C3
C11	1.00	.90**	.70**	.20	.23	.19	.13
C12		1.00	.71**	.19	.20	.12	.18
C13			1.00	.11	.11	.21	.42*
C21				1.00	.87**	.74**	.54**
C22					1.00	.69**	.45**
C23						1.00	.65**
C3							1.00

**p<0.01, *p<0.05

drew arms and legs of the doll. Also, 29.72% (n=11) of the students used different transformations of a semi-circle to draw more elaborate designs.

4.3 Results of correlation analysis

Before answering the second research question in this study, we examined the descriptive statistics (mean and standard deviations, maximum) of the performance for each criterion. Here we use the code consisting of the letter C followed by 2 digits. The first digit represents the Task and the second digit indicates the Criterion. For example, code C13 means Criterion 3 (Transformation 3) of Task 1.

Table 6 shows that the mean score for all criteria is very poor. However, our goal is not to study factors that affect the performance of students. The purpose of this research is to find the relationships among the characteristics of the understanding of function transformation. To answer the second research question, we examined the correlation between all criteria. Statistical results are shown in Table 7.

The results of the analysis show that there are statistically significant relationships between criteria C11 and C12 (.90, p<0.01), C11 and C13 (.70, p<0.01), C12 and C13 (.71, p<0.01). We also found that the three criteria for Task 2 were statistically significantly related with one another. Specifically, correlation coefficients for C21 and C22, C21 and C23, C22 and C23 are higher, that is 0.87 (p<0.01), 0.74 (p<0.01) and 0.69 (p<0.01), respectively.

Those results indicate that Task 1 and Task 2 detect different types of students' understanding of functions transformations. In other words, criteria C11, C12, C13 indicate the same group of characteristics, while criteria C21, C22 and C23 indicate a different characteristic of understanding of function transformation. We call the first group of criteria the **ability to express function transformation in words**, the second group of criteria are called the **ability to express function transformation in a formula or algebraically**.

The hardest criterion for Task 1 is C13 and it has the highest correlation with C3 (0.42, p<0.05) than any other criterion in the task. In other words, students who can perform harder transformation of functions can interpret or use their knowledge in more advanced situations.

In addition, all criteria for Task 2 are highly correlated with C3 (0.54, 0.45, 0.65 and $p < 0.01$ for all values). That means, students who are able to write formulas for functions can successfully design and draw a doll completely.

5 Discussion and conclusion

From the results of analysis, we can see that student performances are not satisfactory. Particularly students' object and schema conceptions of function transformation are poor. On the one hand, the reason for the poor performance might be influenced by students' poor acquisition of action and process conceptions. Therefore, we need to focus on activities such as student project work that improves students' action and process conceptions of function transformations. The problem of how Learning Management Systems (LMS's) which are equipped with dynamic environments assist the development of student's action and process conceptions is interesting from this point of view. On the other hand, the main area of the students' poor performance was the calculation of translation and stretching coefficients. According to the statistical results of our research, students were more capable of working on transformations with integer coefficients than on those with rational coefficients. Therefore, when teaching this topic, teachers should focus on the choice of examples and problems. That is, teachers should use various transformation examples and problems both with integer and rational coefficients.

For future research we need to diagnose students' action and process conceptions of function transformations and analyze the relationships between high school and university level math calculus curricula.

At this time the authors weren't concerned with the reasons for poor performance and low achievement. We hypothesize the following factors to be the main reasons for students' poor performance:

- insufficient proficiency with technology;
- poor achievement in high school level pre-calculus content;
- poor action and process conception of transformations of functions;
- poor wording or formulation of tasks.

These areas are interesting directions for a future study.

Moreover, based on the results of correlational analysis, we can conclude that students who are able to recognize more complicated transformations can also

visually draw a doll using a half circle, while those who can express transformations of both simple and complicated transformations in algebraic form are able to construct a doll using transformations of the half circle. In this research we studied the discrimination of characteristics by correlational analysis. We need to improve this result and develop a model for students' understanding of function transformation by doing higher level analysis.

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