Research Article

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Teaching Linear Algebra in Engineering Courses Using Critical Thinking

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Abstract: Nowadays it is important to improve in higher education students the critical thinking skills and dispositions in the different curricular units that are part of their schooling. Since “Linear Algebra” is a curricular unit part of any degree in engineering, we show that it is possible to integrate activities in their classes that enable the development of skills and critical thinking dispositions by students. We present examples of such activities for one item/topic of the syllabus of Linear Algebra as a proposal for future implementation in the teaching process. We will apply principles of instructional design planning and evaluation of those activities that develop critical thinking skills and dispositions in students. We will also apply active learning principles and cooperative learning strategies, including the learning based on problem solving, using questioning.

Keywords: Critical thinking, higher education, mathematics, linear algebra.

1 Introduction

According to Nascimento, Catarino, Morais, and Vasco (2019), “(...) the current knowledge society requires citizens prepared to the world around them and aware of the decisions they have to make in their daily lives. In this sense, several studies report that critical thinking is essential in the job market, and it is recommended that this competence be integrated in the training of higher education students (...)” (p. 191).

After the participation of the first author (teacher of the Curricular Unit “Linear Algebra” in a Portuguese

public higher education institution) in a teacher training course in Education for Critical Thinking (held at the University Degli Studi Roma Tre, Italy, in February 2018, inserted in the Erasmus + project entitled CRITHINKEDU), it began to make sense, for this first author, to change the way of teaching this Curricular Unit to engineering degree students. The second author is also a teacher of the same Curricular Unit and participated in a course “Promover o pensamento crítico na minha Unidade Curricular¹” held at a Portuguese public higher education institution (HEI), in July 2018. The last-mentioned course was a replication of the Rome course and so many participants of that HEI and others attended. Some questions guided our participation in these training courses, namely: Why are we interested in promoting students’ critical thinking in our course? What specific critical thinking aspects can and/or need to be promoted in our course(s)? How can these critical thinking aspects contribute to the expected course outcomes?, among others. Questions related to our daily lives and our future professionals also occurred to us, such as: What is critical thinking?, What is reflective critical thinking?, What is critical thinking for?, How does critical thinking work?, What are the advantages of critical thinking?, What are the characteristics of critical thinking?, What are the skills related to critical thinking?, How to stimulate critical thinking?, How to become a more critical person?.

Critical thinking is the analysis of facts to form a judgment (Glaser, 2017). The subject is complex, and several different definitions exist, which generally include words such as the rational, skeptical, unbiased analysis, evaluation of factual evidence. Critical thinking is self-directed, self-disciplined, self-monitored, and self-corrective thinking (Clarke, 2019).

As teachers of Linear Algebra we started to think about these questions and soon we were faced with a major question: what is the definition of critical thinking? In the literature there exist some definitions about

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¹ Promote critical thinking in my curricular unit – translation made by the authors.

critical thinking, but there is no unanimity regarding the definition of critical thinking. In this work we decided to adopt the definition of Facione (1990), which consists of “(...) critical thinking is a purposeful and self-regulating judgment that results in interpretation, analysis, evaluation and inference, as well as in the explanation of the conceptual, methodological, criteriological or contextual considerations on which this judgment is based (...)” (p. 2).

According to Dominguez (2018), the critical thinking skills (interpretation, analysis, inference, evaluation, explanation, self-regulation (Figure 1) and the critical thinking dispositions (search for truth, open mind, analytical mind, systematicity, self-confidence, intellectual curiosity, maturity) contribute to the students' success and facilitate the transition to the labor market in all professional areas, hence the need to address them, in particular in the field of engineering.

Also, critical thinking skills and disposition about critical thinking are other indicators of achievement, giving information on students' abilities as future learners and problem solvers (Ernest, & Monroe, 2004, p. 508).

Now we present a brief description with the main ideas of each critical thinking skill and each disposition, namely, interpretation, analysis, inference, evaluation, explanation and self-regulation for skills, and search for the truth, open mind, analytical mind, systematicity, self-confidence, intellectual curiosity and maturity for dispositions. Starting with critical thinking skills, include in *Interpretation* we consider: to understand and to express the meaning or meaning of a wide variety of experiences, situations, events, judgments, conventions, beliefs, rules, procedures, or criteria; for *Analysis*, the idea is: to identify intentional relationships and actual inferential relationships between statements, questions, concepts, descriptions, or other forms of representation designed to express beliefs, judgments, experiences, fundamentals, information, or opinions; for *Inference*: to identify and ensure the elements necessary to draw reasonable conclusions, to develop conjectures and hypotheses; to consider the relevant information and to infer the consequences arising from data, statements, principles, evidence, judgments, beliefs, opinions, concepts, descriptions, questions, or other forms of representation; for *Evaluation*: to assess the credibility of statements or other representations that constitute accounts or descriptions of one's perception, experience, situation, judgment, belief, or opinion; and to evaluate the logical robustness of actual or intentional inferential relationships between statements, descriptions, questions, or other forms of representation; for *Explanation*: to present the

Figure 1: The critical thinking skills (Facione, 2013, p. 5).

results of a reasoning; to justify this reasoning in terms of evidence considerations, concepts, methodologies, criteria, and contexts on which the results were based and, finally, for *Self-Regulation*: to check, consciously, their cognitive activities and the elements used in these activities, particularly by applying skills of analysis and evaluation of own inferential judgments with a view to question, to confirm, to validate or to correct the reasoning itself and / or reasoning of another person.

Now about the dispositions of critical thinking, we consider for *Search for the truth*: to search for the best knowledge in each context, to be brave in asking questions, honest and objective in the search for an answer, even if the results are not in accordance with your own interests or preconceived opinions; for *Open mind*: to tolerate divergent points of view and to be open to the possibility of your own bias. Valuing tolerance and understanding of other people's beliefs and lifestyles; for *Analytical mind*: to focus on the use of reasoning and evidence to solve problems, anticipating potential conceptual or practical difficulties, and consciously being aware of the need to intercede; for *Systematicity*: to be organized, focused and careful in your questioning and in looking for approaches to problem solving and decision making; for *Self-confidence*: confidence in the strength of your own reasoned judgments and the ability to lead others in the process of rational problem solving. To believe in your own reasoning process; for *Intellectual curiosity*: to have intellectual curiosity and the desire to learn, even when the application of knowledge is not easily understood; and finally, for *Maturity*: to address problems, research and decision making with the position that some of these problems may be poorly structured, accepting that actions

/ situations admit more than one reasonable option, and, often, in the context in which judgments should be made, based on patterns, contexts and evidence that do not make it possible to find consensual certainty.

According to Huang, Ricci and Mnatsakanian (2016), in the perspective of the teaching of this curricular unit in mathematics nature, students can improve and develop their critical thinking when they learn mathematics to solve math problems, identifying possible solutions and evaluating and justifying their reasons for the results and thus gaining confidence in the way they think. In higher education institutions critical thinking promoting strategies should be integrated in the teaching of mathematics. In that way students will be able to find the necessary information, evaluate this information, to solve problems (Jacob, 2012) and to improve their communication. They will also be able to make appropriate decisions to achieve solving problems independently, systematic, and logical (Huang, Ricci, & Mnatsakanian, 2016; Jacob, 2012). In this sense, we present the approaches used in one of the first items in the syllabus of the curricular unit “Linear Algebra”, outlining the critical thinking in the teaching practices and the changes made at the level of planning, strategies adopted and assessment of learning. We emphasize the development of the skills and dispositions of critical thinking.

2 Critical thinking in linear algebra

In this section we show that it is possible to integrate these kinds of activities in the classes. We present it for the first item/topic of the syllabus of Linear Algebra. We will apply different principles of instructional design that support the design, planning, and evaluation of activities that develop these skills and dispositions in students. We will also apply active learning principles and cooperative learning strategies, including the learning based on problem solving, and questioning.

In the first subsection we introduce this curricular unit, describing its weekly workload and the type of classes this curricular unit has. In the second section we state a proposal to be experimented with which is related to the promotion of Critical Thinking in this Curricular Unit.

2.1 The Curricular Unit Linear Algebra

This curricular unit is part of the 1st year/1st semester of any program in engineering education and has a weekly

teaching load distributed between theoretical and theoretical-practical classes.

Any curricular unit planned for students in the 1st year/1st semester of the courses of the first cycle of studies (Post-Bologna era), as in this case, brought some challenges to the teacher of HEI.

For the students it is their first year in a HEI so they have several curricular units that demand a lot of work, a lot of effort, a lot of persistence, conditions that they were not used to until then.

Another challenge for a teacher in HEI is the need to be updated in all the aspects of a world in constant change. As well as the pedagogical approach of HEI, the teacher must develop the critical, inventive, and creative spirit of the students, supporting and stimulating their cultural, scientific, professional, and human background.

The authors of this work are both teachers of this curricular unit, the first author being the teacher of the unique theoretical class and the teacher of one practical class, and the second author being the teacher of two practical classes. We think that the way of teaching using only a classroom exposure of the syllabus is outdated. In our opinion it is essential to change this teaching methodology. It should be noted that the theoretical class is the first contact that students have related to this curricular unit. In practical classes (after the theoretical class) exercises/tasks are proposed related to the topics/items covered in the theoretical class taught that week.

Starting to think about the introduction of critical thinking in this curricular unit, we should change the teaching approach. Now students can have an active role in the classroom, they can therefore be involved in their own learning. Next, we will try to summarize the changes that should be adopted from now on. We show what we should do in one setting (setting 1, figure 2) of the planning of the syllabus of this curricular unit.

The syllabus contents presented are those usually included in this type of curricular unit, and the sequence may often not be the one adopted here.

It is usual for the 1st Engineering Cycles (Post-Bologna era) in Portugal (Pinto, 2004) and outside Portugal (Perceval, 2017), that the syllabus starts with the topic related to matrices, followed by the determinant topics and systems of linear equations, and only, subsequently, the topics vector spaces and linear applications and still other sequential options (Cristovão & Spiller, 2009).

It is also worth mentioning that there are already studies that point to the inclusion of some of these syllabuses in Secondary Education (Pinho, 2010). Day and Kalman (1999) also present a list of concepts and

Figure 2: The planning using the model 4C/ID (Van Merriënboer, Clark & Croock, 2002).

procedures that it is possible to address before students enter Higher Education.

This curricular unit addresses basic knowledge about matrices, determinants, vector spaces and linear applications that will be used in other curricular units throughout the respective study cycle.

2.2 The proposal

The planning we use is related to the model 4C/ID described in Van Merriënboer, Clark and Croock (2002). This model consists of a design theory with four basic steps and is an integrated teaching design method: design overall learning tasks and learning sub-tasks for complete tasks; develop evaluation tools and sort learning tasks; decompose generative layer and regenerative layer; design special exercises.

The curricular unit of Linear Algebra is a semester length curricular unit with 30 hours of theoretical classes and 30 hours of practical classes presented within fifteen weeks. The syllabus was divided into settings having in mind a general objective (outcome) as follows.

The Figure 2 shows the general planning using this model, having previously defined a general objective

(outcome) to be achieved by students at the end of the semester. Subsequently, a partition was designed in six stages/settings and six evaluation moments.

We propose the use of tasks/activities that develop the students' critical thinking skills and dispositions. We will also apply active learning principles and cooperative learning strategies, including the learning based on problem solving and questioning.

Each setting is divided into several tasks and in each task the use of the amount of information provided decreases and, in all tasks, we try to develop critical thinking skills and dispositions. Figure 3 shows the number of the tasks into which we intend to divide setting 1 and we can also see that each learning task includes: 1. Specific objective; 2. Supportive information (provided beforehand); 3. Just-in-time information (provided during performance); 4. Assessment.

Notice that from learning task 1 to learning task 4, the amount of information provided decreases. With the execution of each task, the student evolves and need less and less information.

For this setting we define a specific objective to find an appropriate matrix definition. The information given beforehand consists of images of matrices and non-matrices of different types and some basic elements of

Figure 3: The planning of setting 1.

the matrix as the entries and the dimension of a matrix. We propose specific questions that should be presented to the students in order to guide them towards the specific purpose of the task. For example, “What do you mean by matrix?”, “Which elements do you find in the images that can be useful to the definition of matrix?”, “Which different type of matrices can you identify?” The just-in-time information (provided during performance) consists in the presentation of specific examples/problems with all the step-by-step information until they find the solution.

The assessment we propose is that the students use a linear algebra book to compare the definition found in the book with the one they write, and evaluate its veracity.

Sometimes, the application of active learning principles and cooperative learning strategies (Lopes & Silva, 2009), including problem-based learning and questioning, is essential for the promotion of thinking both critical and creative in our students.

In the following example we present the elements of learning task 1 that we propose for setting 1. For this task we should state the four elements mentioned before.

We start learning task 1 by giving the students the specific objective, which in this case is to find one appropriate definition of a matrix. Recall that the students have never had any contact with this type of content before. In the supportive information we can, for instance, present images with matrices of different type and images with elements which are not matrices (Figure 4). Also, some elements of the Elementary Matrix Theory

can be given to the students as a supportive information (provided beforehand).

The students look the images and try to record which are the important characteristics related to this specific objective. During this performance, we can help the students with some examples with all the step-by-step information until the students find the solution for their problem. All this help also can be done using appropriate questioning such as:

We can use the strategy “think @ pair @ share” technique (Lyman, (1981), that the following Figure 5 illustrates.

Basically, the technique aims to help us to think for ourselves about a specific problem/topic/situation and to improve our thinking based on feedback from colleagues. Sharing it with the group using this technique, we contribute to the development of some critical thinking skills (interpretation, analysis, evaluation, explanation).

For learning task 1, we can use specific questions that are presented to students to guide them towards the specific purpose of the task as shown in Figure 6.

Different types of questions can be used. For instance, as questions of clarification we suggest the following examples: What do you mean by matrix? Which different type of matrices can you identify? Which elements do you find in the images that can be useful to the definition of a matrix? For questions that probe assumptions we suggest the following examples: Are there images that are not matrices? Please use the book and compare your

Figure 4: Images that can be used as an example of learning task 1 of setting 1.

Figure 5: Think-pair-share (Lyman, 1981).

definition of a matrix with the definition that you find in the book and verify the accuracy of its definition. Could you explain the reasons you used to adopt this matrix definition? Are the reasons adequate?

Another example we can use is to find out why the students need to study matrices in this engineering course. This is presented in Figure 7.

Next, we present a summary of the plan that we propose to adopt for learning task 1 in setting 1:

- During the class, first the students work individually, and then work in pairs to compare the definitions of a matrix.
- When each pair of students has a common definition, they can share their definition with the rest of the class.
- There will be a discussion about it and in the end, there will be a new definition – the definition of the class.
- Using a book of Linear Algebra, the students can compare their definition with the book's definition.

We think that with these strategies we are trying to develop some critical skills and some dispositions such as those shown in Figure 6. Also, the communication between the students can develop the way of presenting a reasoned decision and also shows the necessity of listening and debate with others.

Figure 6: The questioning that we propose for learning task 1.

Figure 7: Images that we propose to use for another situation in the setting 1.

3 Final Remarks

We leave here some remarks:

- It is a curricular unit that requires commitment of the teacher and of the students and requires scientific rigor from both.
- As it is the first formal contact that students have with mathematical abstraction, the subjects to be developed require the presentation of different mathematical examples that the students should master, as well as generalizations. So they are presented in a way that facilitates the introduction of the concepts to be addressed.
- We think that it is easy to motivate students in this curricular unit since students will use all their contents in subsequent curricular units.
- The different type of classes should be a time for sharing knowledge, where teaching should be centred on the students, giving them the opportunity to be “the classroom researcher”.
- The dynamics of working in pairs/groups should be promoted, with feedback as a powerful tool for student learning.

We have intended to show how these strategies can be used in the teaching of Linear Algebra in engineering degrees. We have presented an example that may allow other teachers to rethink their curricular unit to contribute to the promotion of critical thinking for our future engineers.

We invite all teachers of Linear Algebra in the engineering degrees to reformulate their curricular unit cooperatively working with their colleagues to exchange ideas and debate the outcomes.

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