Research Article

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Diverse Representations to The Sum of Powers of Two as a Means For Enhancing Interest and Creative Thinking in Mathematics Study

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Abstract: In this paper, we present several different approaches to formula for the sum of integer powers of two in accordance with different representations of this sum or different algebraic methods for its computation. Our long-term experience shows the effectiveness of discussion on this theme for enhancing interest and creative thinking of the students about solutions of various problems, not only in mathematics but also in others fields of knowledge.

Keywords: creative thinking; different representations; diversity of approaches; deeper understanding.

1 Introduction

The purpose of this article is to draw the attention of high school, college and university teachers to didactic possibilities of different representations of one well-known problem that in most tutorials is presented (and accordingly taught) in the same key. As is well known, from the mathematical point of view, one proof of the theorem or formula is enough; however, this is not the case from didactic point of view. Different approaches to one task, especially to such a basic, as the sum $1 + 2 + 2^2 + \dots + 2^n$ let the student see a lot of mathematical ideas in simple terms, improving understanding and getting interested in learning mathematics.

2 Theoretical background

Mathematics teachers are always looking for new challenges for their students to attract interest in learning mathematics and provoke creative thinking. Such trials may be connected to different mathematical problems, but they may also integrate different themes by applying different methods to solving a single problem. The latter is in the spirit of the Poincaré's saying: "Mathematics is the art of giving the same name to different things." (Henry Poincaré Quotes, n.d.)

By thinking about different approaches to a single problem, students are likely to gain a deeper understanding of the problem. Moreover, by analyzing different kinds of mathematical ideas, students are provided with a range of effective patterns for their own creative mathematical thinking and we take as a motto the well-known G. Polya statement: "It is better to solve one problem five different ways, than to solve five problems one way." (George Polya Quotes n.d.)

In this paper, we will show several different approaches to discover and prove the formula for the sum of non-negative integer powers of two:

$$\sum_{k=0}^n 2^k = 1 + 2 + 2^2 + \dots + 2^n$$

These approaches are based on simple arithmetic and algebraic operations, or on finding interesting connections (analogies) of this sum with some real or abstract processes.

We believe that such a diversity of approaches to the single problem stimulates creative thinking and increases interest in the study of mathematics. There are many researches on the creative thinking of the gifted students (Krutetskii, 1976); (Leikin, & Pitta-Pantazi, 2013); (Tabach & Friedlander, 2017). This direction of education research takes special significance nowadays not only for special

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schools or classes for the gifted students, but for the ordinary schools and for colleges and universities too. It happens because algorithmic skills, requiring a lot of study time and effort from teachers and students, become less and less relevant for human activity with fast development of computer technology. In the modern world, performers of standard procedures are not needed, but creative minded people are essential. This is noted by many researchers, for example in the article (Mann, 2006, p. 236) says: “The goal of mathematical education should be to think creatively, not simply to arrive at the right answer”. Therefore, the main problem of education nowadays is to develop interest in the matter, and foster creative thinking of the “ordinary” students. First, you need to be interested because there is no creativity without interest. The question: “What creative thinking is?” has been discussed many times by many researchers. We only quote one of them: “Creativity is a process of becoming sensitive to problems, deficiencies, gaps in knowledge, missing elements, disharmonies, and so on; identifying the difficult; searching for solutions, making guesses or formulating hypotheses about the deficiencies; testing and re-testing these hypotheses and possibly modifying and re-testing them; and finally communicating the results.” (Torrance, 1966, p. 8).

3 Preliminary motivation examples

3.1 Ancient chess-game legend

According to one ancient legend (Mikes Dr. Math games for kids), the Emperor of China offered the inventor of the chess-game anything he wished, as a reward for this beautiful entertainment. The request of the inventor was: “give me only one grain of rice for the first square of a chess board, two grains for the second square, four for the third, and so on, until the last square”. “What a foolishly modest request” - thought the Emperor, but when the required amount of grains $1 + 2 + 2^2 + \dots + 2^{63}$ was calculated, it became clear that this quantity of rice was a lot more than there was in the whole world. Actually, to understand this enormous number of the grains of rice, it is enough to think about the last and the biggest term in this sum and approximate it in the following simple way: $2^{63} = 2^3 \times (2^{10})^6 > 8 \times (10^3)^6 = 8 \times 10^{18}$. Assuming 10 grains of rice weigh 1 gram, the weight of 8×10^{18} grains will be more than 8×10^{17} grams or 8×10^{14} kg = 8×10^{11} ton of rice, and that is just for the last (64) square of the chess board. It is clear that $2^{64} = 2^4 \times 2^{60} = 16 \times (2^{10})^6 > 16 \times (10^3)^6 = 16 \times 10^{18}$.

Therefore, the promised reward, assuming that 10 grains of rice weigh 1 gram, will be more than 16×10^{17} gram = 16×10^{14} kg = 16×10^{11} ton = 16 000 billions ton of rice that is far more rice than has ever existed on the earth. Further, we will present several different approaches to the formula for computation of the total amount of the needed grains of rice that is the sum $1 + 2 + 2^2 + \dots + 2^{63}$.

3.2 Family tree analyses and pedigree collapse

One student started thinking about his school home assignment “pedigree research” and began to build his family tree:

He quickly came to the conclusion, that for each generation the number of ancestors doubled. He gave the name “ancestors of order n ” to the people of n generation back in this tree. Therefore, the parents are the first order ancestors, grandparents are the second order ancestors, and so on. According to this model, everyone had 2^n ancestors of order n . Assuming generation changes every 20 years, he figured out that 200 years ago everyone had $2^{10} = 1024$ ancestors of order 10. Thus, 1000 years ago according to this model, everyone had $2^{50} = (2^{10})^5 > (10^3)^5 = 10^{15} = 10^6 \times 10^9$ ancestors of order 50, which is more than million billion people. This is impossible because it is much more than the number of people who have ever lived on the earth, and number of ancestors of anyone in all 50 generations should be the sum $2 + 2^2 + \dots + 2^{50}$, which is even more. What is the reason of this incompatibility? What was wrong with the seemingly logical inferences?

We constantly refer to these family tree examples in calculus courses and discuss this paradox, named “pedigree collapse”, with students. It is always of great interest, and except in rare cases, we soon get the correct answer. It’s worthy of discussion and reading the appropriate literature, for example, “The Mountain of Names: a history of the human family” (Shoumatoff, A., 1995).

4 Different approaches to the sum of powers of 2

4.1 Multiplication by 2

$$\begin{aligned} S_n &= 1 + 2 + 2^2 + 2^3 + \dots + 2^n \quad (n \in N) \\ 2S_n &= 2 + 2^2 + 2^3 + \dots + 2^n + 2^{n+1} \Rightarrow 2S_n = \\ &= S_n + 2^{n+1} - 1 \Leftrightarrow S_n = 2^{n+1} - 1. \end{aligned}$$

It is the most widespread school approach to this sum because in a similar way the computation formula for the sum $S_n(q) = 1 + q + q^2 + q^3 + \dots + q^n$ may easily found (with the help of multiplication by q). However, this is a formal and not very impressive method. We have often checked our first-year students competence about this sum and found that only about a quarter of them were able to quickly give the precise formula for its computation and no more than 5% were able to prove this formula. This observation concerns the little attention the high school teachers give to this sum and give it only as a formal mathematical equation without clarifying its intriguing philosophical essence, as you can see in some of the next approaches.

4.2 Decay Integer Model (Whole Decomposition approach)

$$\begin{aligned} 2^{n+1} &= 2^n + 2^n \\ 2^{n+1} &= 2^n + 2^{n-1} + 2^{n-1} \\ 2^{n+1} &= 2^n + 2^{n-1} + 2^{n-2} + 2^{n-2} \\ 2^{n+1} &= 2^n + 2^{n-1} + 2^{n-2} + 2^{n-3} + 2^{n-3} \\ &\dots\dots\dots \\ 2^{n+1} &= 2^n + 2^{n-1} + 2^{n-2} + 2^{n-3} + \dots + 2 + 1 + 1 \end{aligned}$$

This solution can be interpreted as a numerical presentation to the “nuclear decay” model in nuclear physics. Let us think about some material that consists initially from 2^{n+1} atoms and which begins to decay with the half-life period of one day (that is, at the end of each day there will be half the number of atoms that were there at the beginning of the day). At the end of the first day half of the 2^{n+1} atoms, i.e. 2^n , atoms were annihilated and other half was still present. At the end of the second day half of the 2^n remained atoms, i.e. 2^{n-1} atoms, annihilated and other half i.e. 2^{n-1} was still present. Here we designate by help of the frame and green color, the number of atoms still present at the proper stage of decay, and by red color

the numbers of annihilated atoms. In the first line of the schema 1 we see the results of the first day of decay, in the second line, the results of the second day and so on, till the last final day number $n+1$, that only one atom remained and the process will stop. From this last line we get instantly the required result: $2^n + 2^{n-1} + 2^{n-2} + 2^{n-3} + \dots + 2 + 1 = 2^{n+1} - 1; (n \in N)$.

4.3 Candy eating strategy

Of course, the idea of the previous proof (2) may be realized, without the use of nuclear physics knowledge and without using the sad connotation word “decay” and may be clarified, for the secondary, and even for the primary school students by the next example. Let one child receive on his birthday $128 = 2^7$ candies and decide to eat each day half of candies available at the beginning of the day. So, on the first day he eats $64 = 2^6$ candies, on the second day, $32 = 2^5$ and so on, till the eighth day, when only one last candy remains. Therefore, we get all the eighth day candy eating process in the form of the sum:

$$128 = 64 + 32 + 16 + 8 + 4 + 2 + 1 + 1$$

Otherwise, in powers of two notation:

$$2^7 = 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 1 + 1.$$

4.4 Compound interest (Financial Model)

Let someone invest 1\$ in a bank at 100% per annum. If so, the initial sum will double every year. For the first year the investor’s income will be $2\$ - 1\$ = 1\$$, for the second year his income will be $(2^2 - 2)\$ = 2\$$, for the third year $(2^3 - 2^2)\$ = 2^2\$$, and so on. In this way his total income for n years will be: $(2 - 1) + (2^2 - 2) + (2^3 - 2^2) + \dots + (2^n - 2^{n-1})$ dollars that equals to $(2^n - 1)$ dollars, as a differences between the final amount $2^n\$$ and initial invested amount of 1\$

$$(2-1) + (2^2-2) + (2^3-2^2) + \dots + (2^n-2^{n-1}) = 2^n - 1 \quad \text{or} \quad \left(\begin{aligned} 1 + 2 + 2^2 + \dots + 2^{n-1} &= 2^n - 1 \end{aligned} \right), \quad (n \in N)$$

4.5 Tournament Analogy Model

K players play hand-to-hand games in a tournament and the loser leaves the field. How many games will there be to the end of this tournament?

Logical Solution

After one game, one player leaves the tournament. At the end of this tournament $K - 1$ of K players leave and so $K - 1$ games were played in the tournament and this is the total number of games.

Special case $K = 2^{n+1}$ arithmetic solution

If there were $K = 2^{n+1}$ players in the beginning, as in the first round 2^n games will be played; in the 2nd round 2^{n-1} games and so on, till the last final round, where only 1 game will be played. Therefore, the total number of games is $2^n + 2^{n-1} + \dots + 1$.

Comparing the results obtained by two different methods of solution (logical and arithmetical), we have:

$$1 + 2 + 2^2 + \dots + 2^n = K - 1 = 2^{n+1} - 1$$

4.6 Addition of 1 (Absorption Method)

$$S_n = 1 + 2 + 2^2 + 2^3 + \dots + 2^n$$

$$\begin{aligned} 1 + S_n &= \underline{1+1} + 2 + 2^2 + 2^3 + \dots + 2^n = \underline{2+2} + 2^2 + 2^3 + \dots + 2^n = \\ &= \underline{2^2+2^2} + 2^3 + \dots + 2^n = \dots = \underline{2^3+2^3} + 2^4 + \dots + 2^n = \dots = 2^n + 2^n = \\ &= 2^{n+1} \Rightarrow S_n = 2^{n+1} - 1 \end{aligned}$$

We gave it the name “Absorption Method” since the previous number absorbs the next as a result of the use of the simple identity: $2^k + 2^k = 2 \times 2^k = 2^{k+1}$, $k = 0, 1, 2, \dots$. So, the total sum $S_n + 1 = 1 + 1 + 2 + 2^2 + \dots + 2^n$ turns into the one resulting number 2^{n+1} .

4.7 Method of Differences (Telescoping Model)

This method based on the simple property of powers of two: $2^n = 2^{n+1} - 2^n$ and may be used in financial model 4 too.

$$S_n = 1 + 2 + 2^2 + 2^3 + \dots + 2^n = (2 - 1) + (2^2 - 2) + (2^3 - 2^2) + \dots + (2^{n+1} - 2^n) = 2^{n+1} - 1$$

This method can be generalized to the sum of integer powers of q ($q \neq 0 \wedge q \neq 1$)

$$1 + q + q^2 + q^3 + \dots + q^n \quad (n \in \mathbb{N})$$

By using of the identity:

$$q^{n+1} - q^n = q^n(q - 1) \Leftrightarrow q^n = \frac{q^{n+1}}{q - 1} - \frac{q^n}{q - 1} \quad (q \neq 0 \wedge q \neq 1)$$

Figure 1: Double Increase Jumps on x-axis.

$$1 + q + q^2 + q^3 + \dots + q^n = \frac{1}{q - 1} (q - 1 + q^2 - q + q^3 - q^2 + \dots + q^{n+1} - q^n) = \frac{q^{n+1} - 1}{q - 1}$$

4.8 Sum of Distances (Double Increases Jumps Model)

Imagine a Jumper, starting from the point 1 on the figure 1 with a jump of 1 m and then doubling the distance of each of his next jumps. After the first jump, he will be at the point 2, after the second jump – at the point 4, after the third jump – at the point 8 and after n jumps, he will be at the point 2^n on the figure 1. From one point of view, the distance the Jumper travels

during the whole process will be the sum of jumps, that is $1 + 2 + 2^2 + \dots + 2^{n-1}$.

From another point of view, the distance the Jumper travels is equal to the distance between the start point 1 and final point 2^n on the x-axes, that is $2^n - 1$. Therefore, we will get $1 + 2 + 2^2 + \dots + 2^{n-1} = 2^n - 1$.

4.9 Binary Arithmetic Method

In the binary notation we have $1 = 1$; $2 = 10$; $2^2 = 100$; $2^3 = 1000$ and so on.

By this way the sum $S_n = 1 + 2 + 2^2 + 2^3 + \dots + 2^n$ in the binary notation will be

$$\begin{aligned} S_n &= 1 + 10 + 100 + 1000 + \dots + 100\dots0 = 111\dots1 \text{ (n + 1 times "1")} \\ S_n + 1 &= \underset{n+1 \text{ times } 1}{111\dots1} + 1 = \underset{n+1 \text{ times } 0}{1000\dots0} = 2^{n+1}. \text{ Therefore, } S_n = 2^{n+1} - 1. \end{aligned}$$

4.10 Combinatorial Analogy Model

The number of all binary sequences of the length $n+1$ is, as known, 2^{n+1} .

Let us divide all these sequences into $n+1$ subsets $A_0, A_1, A_2, \dots, A_{n+1}$ in accordance with the place (from the left) of the first digit 1 in a sequence. A_0 includes only 1 sequence without the digit 1 at all; A_1 includes 2^n sequences, A_2 includes 2^{n-1} sequences and generally A_k includes 2^{n+1-k} sequences ($k=0, 1, 2, \dots, n+1$). Thus, we obtain:

$$2^{n+1} = 1 + 1 + 2 + 2^2 + 2^3 + \dots + 2^n \Rightarrow 1 + 2 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 1.$$

4.11 Sheet Bisection Model

It is very attractive and easy to show in the classical “physical” proof process of the formula

$$\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n} \quad (n \in \mathbb{N})$$

A teacher breaks a piece of paper to two equal parts. He puts one-half on the table and the other half holds in his hand. Then he repeats this process with the half of the sheet that was in his hand. The teacher stresses each time that the last part of the paper placed on the table is the same as the piece remaining in his hand. Therefore, if the piece, remaining in his hand, added to the parts that are on the table to this moment, we find all the paper on the table:

$$\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} + \frac{1}{2^n} = 1 \Rightarrow \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n} \quad (n \in \mathbb{N}).$$

This demonstration is always appreciated by students and gives the beautiful example of solving an abstract mathematical problem by means of analogy with a simple real and visual process.

4.12 Fraction decay model (Unit Decomposition approach)

Let $S_n = 1 + 2 + 2^2 + 2^3 + \dots + 2^n$ and $F_n = 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}$. It is clear that $2^n F_n = 2^n + 2^{n-1} + 2^{n-2} + \dots + 2 + 1 = S_n$.

For computation of F_n , let us think about the one unit decompositions

$$1 = \frac{1}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16}$$

From this unit decomposition process, it is clear that for any natural n it is true

$$1 = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} + \frac{1}{2^n}$$

and so

$$F_n = 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} = 2 - \frac{1}{2^n} \\ \Rightarrow S_n = 1 + 2 + 2^2 + 2^3 + \dots + 2^n = 2^n F_n = 2^n \left(2 - \frac{1}{2^n} \right) = 2^{n+1} - 1.$$

4.13 Probability Analogy Model

Let us think about the question: “What is the probability of random event A that in a family with n children at least one is a daughter [assume that $p_{\text{male}} = p_{\text{female}} = 0.5$]?”

It is clear that the probability of complementary to A event $\bar{A}$, that all n children in a family with n children are boys, equals $\frac{1}{2} \times \frac{1}{2} \times \dots \times \frac{1}{2} = \frac{1}{2^n}$. Divide all the elementary events in the births ordering possible process, for a family with n children, into n subsets A_k ($k = 1, 2, 3, \dots, n$), according to the number of the birth, when the youngest girl was born.

It is clear, in the assumption $p_{\text{male}} = p_{\text{female}} = 0.5$, that

$$p(A_k) = \frac{1}{2} \times \frac{1}{2} \times \dots \times \frac{1}{2} = \frac{1}{2^k}$$

k times

and

$$p(A) = \sum_{k=1}^n p(A_k) = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}, \quad P(\bar{A}) = \frac{1}{2^n} \Rightarrow p(A) = 1 - p(\bar{A}) = 1 - \frac{1}{2^n} \\ \Rightarrow \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

5 Conclusions

Some of the above approaches were presented at different educational conferences (4-6) and seminars for lecturers, high school teachers and used many times in the calculus courses in the Shamoon College of Engineering, always arousing great interest in the audience. These representations demonstrate the variety of ways of thinking about the problem and show the strength and beauty of mathematical thinking. In this article we have given a brief description of the above approaches, but each of them may be used as a base for interesting and useful discussion with the students.

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