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The variational iteration method for solving the Volterra integro-differential forms of the Lane-Emden and the Emden-Fowler problems with initial and boundary value conditions

Abstract: In this paper, the variational iteration method (VIM) is used to examine the Volterra integro-differential forms of the singular Lane–Emden and the Emden–Fowler initial value problems and boundary value problems arising in physics, astrophysics and stellar structures. The Volterra integro-differential forms of the Lane–Emden and the Emden–Fowler equations overcome the singularity behavior at the origin $x = 0$. The Lagrange multiplier, needed for the VIM, is $\lambda = -1$ for the various cases of the specified equations having distinct shape factors. We illustrate our work by analyzing few initial value problems and boundary value problems to emphasize the convergence of the acquired results.

Keywords: Lane-Emden problem; Emden-Fowler problem; variational iteration method

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1 Introduction

Many problems in physics, astrophysics and stellar structures, which occur on semi-infinite interval, are related to the Emden–Fowler equation [1–10]

$$y_{xx} + \frac{k}{x} y_x + g(x)f(y) = 0, \ x > 0, \tag{1}$$

subject to the conditions

$$y(0) = 1, \ y'(0) = 0, \tag{2}$$

where $k > 0$ is a constant, called shape factor. For $g(x) = 1$, Equation (1) is the standard Lane–Emden equation

$$y_{xx} + \frac{k}{x} y_x + f(y) = 0, \ x > 0, \tag{3}$$

subject to the conditions

$$y(0) = 1, \ y'(0) = 0, \tag{4}$$

that has been used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure and the thermal behavior of a spherical cloud of gas [11–14]. The Emden-Fowler Equation (1) is sometimes called the generalized Lane–Emden equation. The Lane–Emden Equation (3) was first studied by the astrophysicists Jonathan Homer Lane and Robert Emden, where they considered the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [11]. For other special forms of $f(y)$, the well-known Lane–Emden equation was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and theory of thermionic currents [11–14]. A substantial amount of work has been done on this type of problems for various structures of $f(y)$ in [1–18].

The Emden–Fowler Equations (1) are singular differential equations which describe the equilibrium density distribution in self-gravitating sphere of polytropic isothermal gas [7, 8] and has a regular singularity at the origin $x = 0$. The polytropic theory of stars essentially follows out of thermodynamic considerations that deals with the issue of energy transport, through the transfer of materials between different levels of the star and modeling of clusters of galaxies [7]. The Emden–Fowler equation arises in the study of fluid mechanics, relativistic mechanics, and in the study of chemically reacting systems

The singularity behavior that occurs at $x = 0$ is the main difficulty of Equations (1) and (3). In [1–5], the Lane–Emden–Fowler Equations (1)–(2) were handled by using the Adomian decomposition method and the modified decomposition method. In [6, 7], the same equation was handled by using the homotopy perturbation method and the variational iteration method. In [8, 9], the variational iteration method was applied to study this problem. In [10–27],

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a variety of methods were used to investigate this problem for various cases of $g(y)$.

A considerable amount of research work has been invested to study the Lane–Emden–Fowler equations, where a variety of powerful methods were employed [1–10]. The Adomian decomposition method (ADM) was mostly used to examine this equation analytically and numerically. The variational iteration method was also applied in a parallel manner to the ADM for numerical and analytical purposes. The Haar wavelet approach was used in [8] which has influenced additional progress in this field. It is normal in all aforementioned approaches to try to seek alternative effective schemes that surmount the singularity behavior of this equation.

Recently, a novel effective strategy was established in [1] to convert the Lane–Emden and the Emden–Fowler equations to equivalent Volterra integral equations or to equivalent Volterra integro-differential equations of any order. The new Volterra integral forms were used in [1], combined with the ADM, to address the singularity issue. This strategy was proven to be reliable and efficient as confirmed in [1].

A great deal of attention has been devoted to the study of the VIM [22] aimed at investigating various models, singular and nonsingular, linear and nonlinear, as well as ODEs and PDEs. The VIM accurately computes the solution in a series solution, that often converges to the exact solution, if such a solution exists.

Our approach in this work will run in two streams. First, we plan to convert the Lane–Emden and the Emden–Fowler Equations (1) and (3) to its equivalent Volterra integro-differential equations following the strategy used in [1]. The second aim is to apply the VIM for the transformed Volterra integro-differential equations. The convergence concept will be emphasized by the obtained approximations that lead to the exact closed form solutions.

In particular, we will examine the Lane–Emden and the Emden–Fowler equations given by

$$y_{xx} + \frac{k}{x} y_x + f(y) = 0, \quad x > 0, \; k > 0, \tag{5}$$

and

$$y_{xx} + \frac{k}{x} y_x + f(x, y) = 0, \; x > 0, \; k > 0, \tag{6}$$

respectively, subject to the initial conditions

$$y(0) = a, \quad y'(0) = 0, \tag{7}$$

and the boundary conditions

$$y(1) = b, \quad y'(0) = 0. \tag{8}$$

As stated before, we will manipulate the useful transformation used in [1] to convert each equation to an equivalent Volterra integro-differential equation. In what follows, we only summarize the necessary steps for the conversion process, more details can be found in [1].

1.1 The Lane-Emden equation of shape factor of $k > 1$

The generalized Lane-Emden equation of shape factor of $k > 1$ reads

$$y'' + \frac{k}{x} y' + f(y) = 0, \quad y(0) = \alpha, \quad y'(0) = 0, \quad k > 1, \tag{9}$$

where $f(y)$ can take one of several linear or nonlinear forms.

To convert (9) to an integral form, we first set

$$y(x) = \alpha - \frac{1}{k-1} \int_0^x t \left(1 - \frac{t^{k-1}}{x^{k-1}}\right) f(y(t))\, dt. \tag{10}$$

Differentiating (10) twice, using the Leibniz rule, gives

$$\begin{aligned} y'(x) &= -\int_0^x \left(\frac{t^k}{x^k}\right) f(y(t))\, dt, \\ y''(x) &= -f(y(x)) + \int_0^x k \left(\frac{t^k}{x^{k+1}}\right) f(y(t))\, dt. \end{aligned} \tag{11}$$

In [1], the Volterra integral forms (10)–(11) were proved to be equivalent to the generalized Lane-Emden Equation (9).

However, for $k = 1$, the integral form is

$$y(x) = \alpha + \int_0^x t \ln\left(\frac{t}{x}\right) f(y(t))\, dt, \tag{12}$$

which can be obtained by letting $k \to 1$ in Equation (10).

Based on the latter results, we set the Volterra integral forms for the Lane–Emden equations as:

$$y(x) = \begin{cases} \alpha + \int_0^x t \ln\left(\frac{t}{x}\right) f(y(t))\, dt, & \text{for } k = 1, \\ \alpha - \frac{1}{k-1} \int_0^x t \left(1 - \frac{t^{k-1}}{x^{k-1}}\right) f(y(t))\, dt, & \text{for } k > 1. \end{cases} \tag{13}$$

1.2 The Emden-Fowler equation of shape factor of $k > 1$

The generalized Emden-Fowler equation of shape factor of $k > 1$ reads

$$y'' + \frac{k}{x} y' + f(x, y) = 0, \quad y(0) = \alpha, \quad y'(0) = 0, \quad k > 1, \tag{14}$$

where $f(x,y)$ can take one of several linear or nonlinear forms.

To convert (14) to an integral form, we first set

$$y(x) = \alpha - \frac{1}{k-1}\int_0^x t\left(1-\frac{t^{k-1}}{x^{k-1}}\right) f(t,y(t))\,dt. \quad (15)$$

Differentiating (15) twice, using the Leibniz rule, gives

$$\begin{aligned} y'(x) &= -\int_0^x \left(\frac{t^k}{x^k}\right) f(t,y(t))\,dt, \\ y''(x) &= -f(y(x)) + \int_0^x k\left(\frac{t^k}{x^{k+1}}\right) f(t,y(t))\,dt. \end{aligned} \quad (16)$$

However, for $k = 1$ the integral form is

$$y(x) = \alpha + \int_0^x t\ln\left(\frac{t}{x}\right) f(t,y(t))\,dt, \quad (17)$$

which can be obtained by setting $k \to 1$ in Equation (15).

Based on the last results, we set the Volterra integral forms for the Lane–Emden equations as

$$y(x) = \begin{cases} \alpha + \int_0^x t\ln\left(\frac{t}{x}\right) f(t,y(t))\,dt, & \text{for } k=1, \\ \alpha - \frac{1}{k-1}\int_0^x t\left(1-\frac{t^{k-1}}{x^{k-1}}\right) f(t,y(t))\,dt, & \\ & \text{for } k>1. \end{cases} \quad (18)$$

1.3 The variational iteration method

Consider the differential equation

$$Lu + Nu = g(x), \quad (19)$$

where L and N are linear and nonlinear operators respectively, and $g(x)$ is the inhomogeneous source term. The variational iteration method admits the use of a correction functional for Equation (19) in the form

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\left(Lu_n(t) + N\tilde{u}_n(t) - g(t)\right)\,dt, \quad (20)$$

where λ is a general Lagrange's multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. The Lagrange multiplier λ is crucial and critical in the method, and it can be a constant or a function. Having λ determined, an iteration formula should be used for the determination of the successive approximations $u_{n+1}(x)$, $n \geq 0$ of the solution $u(x)$. The initial approximation u_0 can be any selective function. However, the initial values $u(0)$, $u'(0)$, and $u''(0)$ are preferably used for the selective zeroth approximation u_0 as will be seen later. Consequently, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x). \quad (21)$$

It is interesting to point out that the distinct forms of the Lagrange multipliers λ were formally derived in [1], hence we skip the details. It suffices to present a brief summary of such results:

For first order ODE of the form

$$u' + p(x)u = q(x), \quad u(0) = \alpha, \quad (22)$$

it was found that $\lambda = -1$, and the correction functional gives the iteration formula

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) + p(t)u_n(t) - q(t)\right)\,dt. \quad (23)$$

Generally, for the nth-order ODE

$$\begin{aligned} &u^{(n)} + f(u,u',u'',\cdots,u^{(n-1)}) = g(x), \ u(0) = \alpha_0, \\ &u'(0) = \alpha_1, \ \cdots, \ u^{(n-1)}(0) = \alpha_{n-1}, \end{aligned} \quad (24)$$

we found that $\lambda = \frac{(-1)^n}{(n-1)!}(t-x)^{n-1}$, and the iteration formula takes the form

$$\begin{aligned} u_{n+1}(x) &= u_n(x) + \frac{(-1)^n}{(n-1)!}\int_0^x (t-x)^{n-1}\Big(u^{(n)} + \\ &+ f(u,u',u'',\cdots,u^{(n-1)}) - g(t)\Big)\,dt. \end{aligned} \quad (25)$$

The successive approximations u_{n+1}, $n \geq 0$ of the solution $u(x)$ will be readily obtained upon using any selective function $u_0(x)$. Consequently, the solution is expressed as

$$u(x) = \lim_{n\to\infty} u_n(x). \quad (26)$$

In other words, the correction functional (27) will give several approximations, and therefore the exact solution is obtained via taking the limit of the resulting successive approximations.

1.4 The VIM and the Lagrange multipliers

It is interesting to point out that the VIM can be used directly to solve the Lane–Emden and the Emden–Fowler equations as studied in [1–5]. In this case we summarize the steps to find the Lagrange multipliers λ. In this section, we will describe the essential steps for using the variational iteration method and the determination of the Lagrange multipliers for various values of α.

The correction functional for the Lane-Emden-Fowler equation reads

$$\begin{aligned} y_{n+1}(x) &= y_n(x) + \int_0^x \lambda(\xi)\Bigg((y_n(\xi))_{\xi\xi} \\ &+ \frac{k}{\xi}(y_n(\xi))_\xi + f(\xi)\tilde{g}(y_n(\xi))\Bigg)\, d\xi, \end{aligned} \tag{27}$$

where $\delta(\tilde{g}(y_n(\xi))) = 0$. In [1–5], three important cases were derived:
(i) For the cylindrical problems, we have $k = 1$. In this case, λ is given by

$$\lambda(\xi) = \xi \ln\left(\frac{\xi}{x}\right). \tag{28}$$

(ii) For the spherical problems, we have $k = 2$. The Lagrange multiplier λ is given by

$$\lambda(\xi) = \frac{\xi(\xi - x)}{x}. \tag{29}$$

(iii) For the general case with $k > 1$, the Lagrange multiplier λ is given by

$$\lambda(\xi) = \frac{\xi(\xi^{\alpha-1} - x^{\alpha-1})}{(\alpha-1)x^{\alpha-1}}. \tag{30}$$

2 Lane–Emden initial value problems

Next, we will study three Lane–Emden initial value models having singular behavior at $x = 0$, for the special cases $k = 1$ and 2.

Example 1

We first study the homogeneous nonlinear Lane–Emden type with $k = 1$:

$$y'' + \frac{1}{x}y' - 4e^{1-2y} = 0, \tag{31}$$

with initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{32}$$

The first order Volterra integro-differential form for (31) reads

$$y'(x) = 4\int_0^x \frac{t}{x} e^{1-2y(t)}\, dt. \tag{33}$$

To use the variational iteration method for (33), the correction functional is set by

$$\begin{aligned} y_{n+1}(x) = \ & y_n(x) - \int_0^x \Bigg(y_n'(t) \\ & -4\left(\int_0^t \frac{r}{t} e^{1-2y_n(r)}\, dr\right) dt\Bigg), \quad n \ge 0, \end{aligned} \tag{34}$$

where the Lagrange multiplier $\lambda = -1$. Considering the given initial values, we can choose $y_0(x) = 1$. Using this selection into (34), we get the following successive approximations

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= 1 + \frac{1}{e}x^2, \\ y_2(x) &= 1 + \frac{1}{e}x^2 - \frac{1}{2e^2}x^4 + \frac{2}{9e^3}x^6 - \cdots, \\ y_3(x) &= 1 + \frac{1}{e}x^2 - \frac{1}{2e^2}x^4 + \frac{1}{3e^3}x^6 - \frac{17}{72e^4}x^8 + \cdots, \\ y_4(x) &= 1 + \frac{1}{e}x^2 - \frac{1}{2e^2}x^4 + \frac{1}{3e^3}x^6 - \frac{1}{4e^4}x^8 \\ &\quad + \frac{179}{900e^5}x^{10} - \cdots, \\ &\vdots \ . \end{aligned} \tag{35}$$

It is worth noting that y_1 was computed exactly, however the minor setback of the VIM is that when iterating more than once, the integral cannot be evaluated symbolically. Hence, we evaluated the integral numerically by replacing the integrand, namely $\frac{r}{t}e^{1-2y_n(r)}$, by its Taylor polynomial $P_N(r)$ about $r = 0$. This resulted in noise terms, for instance, when estimating $y_2(x)$, we got $\frac{2}{9e^3}x^6$ as a noise term which eventually disappeared as we computed higher approximations. The previous computations clearly yield the exact solution of (31), which is given by

$$y(x) = \ln(x^2 + e), \quad e = \exp(1). \tag{36}$$

Example 2

First, we consider the homogeneous nonlinear Lane–Emden type with $k = 1$:

$$y'' + \frac{2}{x}y' - 2y(3 + 2\ln y) = 0, \tag{37}$$

with initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{38}$$

The first order Volterra integro-differential form for (37) reads

$$y'(x) = 2\int_0^x \frac{t^2}{x^2}\, y(t)\left(3 + 2\ln y(t)\right) dt. \tag{39}$$

To use the variational iteration method for (39), the correction functional for (39) is set by

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n'(t) \right. \\ \left. -2 \left(\int_0^t \frac{r^2}{t^2} \, y_n(r)(3 + 2\ln y_n(r))\, dr \right) dt \right), \quad n \geq 0, \tag{40}$$

where the Lagrange multiplier $\lambda = -1$. Considering the given initial values, we can select $y_0(x) = 1$. Using this selection into (40) we obtain the following successive approximations

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= 1 + x^2, \\ y_2(x) &= \frac{-15x + 80x^3 + 27x^5}{225x}, \\ &+ \frac{225x \ln\left(1 + x^2\right) + 150x^3 \ln\left(1 + x^2\right)}{225x} \\ &+ \frac{45x^5 \ln\left(1 + x^2\right) + 240 \arctan x}{225x} \\ y_3(x) &= 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{19}{756}x^8 + \cdots, \\ y_4(x) &= 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 \\ &+ \frac{71}{10395}x^{10} + \cdots, \\ &\vdots \;. \end{aligned} \tag{41}$$

Note that $y_1(x)$ and $y_2(x)$ were evaluated exactly. However, as we move to higher iterated the integral cannot be evaluated symbolically. Therefore, analogous to the previous example 1, we have estimated the integral via replacing the integrand with its Taylor's polynomial expansion about $r = 0$. This resulted in noise terms (such as $\frac{19}{756}x^8$ in $y_3(x)$) which eventually disappear as we evaluate higher approximations. Disregarding these apparent noise terms, it is obvious that the iterates appear to converge towards the exact solution of (31), which is

$$y(x) = e^{x^2}. \tag{42}$$

3 Emden–Fowler initial value problems

In this section, we will consider two Emden–Fowler initial value models with singular behavior at $x = 0$. The two problems are selected for the cases $k = 2$ and $k = 3$.

Example 1

Consider the homogeneous nonlinear Emden–Fowler type with $k = 2$:

$$y'' + \frac{2}{x}\, y' + (3 - x^2)y = 0, \tag{43}$$

with initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{44}$$

As we stated before, the Volterra integro-differential form can be extended to the Emden–Fowler Equation where $f = f(x, y)$. The first order Volterra integro-differential form for (43) reads

$$y'(x) = -\int_0^x \frac{t^2}{x^2}\,(3 - t^2)y(t)\, dt. \tag{45}$$

To use the variational iteration method for (45), the correction functional is set by

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n'(t) \right. \\ \left. + \left(\int_0^t \frac{r^2}{t^2}(3 - r^2)y_n(r)\, dr \right) dt \right), \quad n \geq 0, \tag{46}$$

where the Lagrange multiplier $\lambda = -1$. Considering the specified initial values, we can select $y_0(x) = 1$. Using this selection into (46), results in the following successive approximations

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{20}x^4, \\ y_2(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{13}{840}x^6 + \frac{1}{1440}x^8, \\ y_3(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \frac{1}{420}x^8 + \cdots, \\ y_4(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \frac{1}{384}x^8 + \cdots, \\ &\vdots \;. \end{aligned} \tag{47}$$

This gives the exact solution of (31), namely,

$$y(x) = e^{-\frac{1}{2}x^2}. \tag{48}$$

Example 2

Consider the homogeneous nonlinear Lane–Emden type with $k = 3$:

$$y'' + \frac{3}{x}\, y' + (15x - 9x^4)y = 0, \tag{49}$$

with initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{50}$$

As we stated before, the Volterra integro-differential form can be extended to the Emden–Fowler equation where $f = f(x, y)$. The first order Volterra integro-differential form for (49) reads

$$y'(x) = -\int_0^x \frac{t^3}{x^3}\,(15t - 9t^4)y(t)\, dt. \tag{51}$$

To use the variational iteration method for (51), the correction functional is set by

$$\begin{aligned} y_{n+1}(x) &= y_n(x) - \int_0^x \left(y_n'(t) + \left(\int_0^t \frac{r^3}{t^3} (15r \right.\right. \\ &\left.\left. -9r^4) y_n(r)\, dr \right) dt \right), \qquad n \geq 0, \end{aligned} \tag{52}$$

where the Lagrange multiplier $\lambda = -1$. Considering the given initial values, we can select $y_0(x) = 1$. Using this selection into (52), we obtain the following successive approximations

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= 1 - x^3 + \frac{3}{16}x^6, \\ y_2(x) &= 1 - x^3 + \frac{1}{2!}x^6 - \frac{21}{176}x^9 + \frac{9}{896}x^{12}, \\ y_3(x) &= 1 - x^3 + \frac{1}{2!}x^6 - \frac{1}{3!}x^9 + \frac{369}{9856}x^{12} + \cdots, \\ y_4(x) &= 1 - x^3 + \frac{1}{2!}x^6 - \frac{1}{3!}x^9 + \frac{1}{4!}x^{12} + \cdots, \\ &\vdots \;. \end{aligned} \tag{53}$$

This results in the following exact solution:

$$y(x) = e^{-x^3}. \tag{54}$$

4 Lane–Emden boundary value problems

In this section, we will study a Lane–Emden boundary value models with singular behavior at $x = 0$. The boundary value problem we selected is for the case $k = 1$.

Example 1

We consider the homogeneous nonlinear Lane–Emden boundary value problem:

$$y'' + \frac{1}{x}y' + \left(3y^5 - y^3\right) = 0, \tag{55}$$

with boundary conditions

$$y(1) = \frac{1}{\sqrt{2}}, \quad y'(0) = 0. \tag{56}$$

The first order Volterra integro-differential form for (55) reads as

$$y'(x) = \int_0^x \frac{t}{x}\left(3y^5 - y^3\right)\, dt. \tag{57}$$

To use the variational iteration method, we set the correction functional for (57) by

$$\begin{aligned} y_{n+1}(x) &= y_n(x) - \int_0^x \left(y_n'(t) + \left(\int_0^t \frac{r}{t} (3y_n^5(r) \right.\right. \\ &\left.\left. -y_n^3(r))\, dr \right) dt \right), \qquad n \geq 0, \end{aligned} \tag{58}$$

where the Lagrange multiplier $\lambda = -1$. Considering the given initial values, we can select $y_0(x) = a$, where a is unknown to be determined. Using this selection into (58), we obtain the following successive approximations:

$$\begin{aligned} y_0(x) &= a, \\ y_1(x) &= a + \frac{1}{4}a^3x^2, \\ y_2(x) &= a + \frac{1}{4}a^3x^2 + \frac{3}{4}a^5\left(1 - 8a^2 + 15a^4\right)x^4 \\ &\quad + \frac{1}{192}a^7\left(1 - 16a^2 + 69a^4 - 90a^6\right)x^6 + \cdots, \\ y_3(x) &= a + \frac{1}{4}a^3x^2 + \frac{3}{4}a^5\left(1 - 8a^2 + 15a^4\right)x^4 \\ &\quad + \frac{7}{768}a^7\left(1 - 103a^2 + 63a^4 - \frac{585}{7}a^6\right)x^6 + \cdots, \\ &\vdots\;, \end{aligned} \tag{59}$$

where $y_4(x)$ and $y_5(x)$ are computed but not listed. Substituting the boundary condition $u(1) = \frac{1}{\sqrt{2}}$ into the approximations $y_j(x), j \geq 2$, and solving the resulting equations for a we obtain a sequence of approximations for a given by

$$\begin{aligned} &0.7947533704, 0.9303561845, \\ &0.9456536689, 0.9989160879, \cdots. \end{aligned} \tag{60}$$

It is clear that this sequence converges to $a = 1$. Substituting $a = 1$ into $y_5(x)$ gives the approximation

$$y(x) = 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \frac{35}{128}x^8 + \cdots, \tag{61}$$

that converges to the exact solution

$$y(x) = \frac{1}{\sqrt{1 + x^2}}. \tag{62}$$

5 Emden–Fowler boundary value problems

In this section, we will study Emden–Fowler boundary value models with singular behavior at $x = 0$. The boundary value problems we selected are for the case $k = 3$.

Example 1

First, we consider the homogeneous nonlinear Emden–

Fowler boundary value problem:

$$y'' + \frac{3}{x}y' + (8-4x^2)y = 0, \quad (63)$$

with boundary conditions

$$y(1) = \frac{1}{e}, \quad y'(0) = 0, \quad e = \exp(1). \quad (64)$$

For comparison reasons, we will apply the variational method to the Emden–Fowler problem without using the Volterra integro-differential form. To use the variational iteration method for (63), we set the correction functional by

$$\begin{aligned} y_{n+1}(x) = \quad & y_n(x) + \int_0^x \left(\frac{t(t^2-x^2)}{2x^2} \left(y_n''(t) + \frac{3}{t}y_n'(t) \right.\right. \\ & \left.\left. +(8-4t^2)y_n(t) \right) dt \right), \quad n \ge 0, \end{aligned} \quad (65)$$

where the Lagrange multiplier $\lambda = \frac{t(t^2-x^2)}{2x^2}$. Considering the given initial values, we can select $y_0(x) = a$, where a is unknown that will be determined. Substituting this selection into (65), we obtain the following successive approximations

$$\begin{aligned} y_0(x) &= a, \\ y_1(x) &= a\left(1 - x^2 + \frac{1}{6}x^4\right), \\ y_2(x) &= a\left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{9}x^6 + \frac{1}{120}x^8\right), \\ y_3(x) &= a\left(1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{13}{360}x^8 + \cdots\right), \\ y_4(x) &= a\left(1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{13}{4!}x^8 + \cdots\right), \\ &\vdots\ , \end{aligned} \quad (66)$$

where $y_5(x)$ and $y_6(x)$ are computed but not listed. Using the boundary condition $u(1) = e^{-1}$ into the approximations $y_j(x), j \ge 2$, and solving the resulting equations for a we obtain a sequence of approximations for a given by

$$\begin{aligned} &0.9261300616, 1.006830748, 0.9996200812, \\ &1.000015115, \cdots . \end{aligned} \quad (67)$$

It is clear that this sequence converges to $a = 1$. Substituting $a = 1$ into $y_5(x)$ gives the approximation

$$y(x) = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \cdots, \quad (68)$$

that converges to the exact solution

$$y(x) = e^{-x^2}. \quad (69)$$

Example 2

We now consider the inhomogeneous Emden–Fowler type with $\alpha = 2$:

$$y'' + \frac{2}{x}y' - (6+4x^2)y = (6-6x^2-4x^4), \quad (70)$$

with initial conditions

$$y(0) = 1, \quad y'(0) = 0. \quad (71)$$

The correction functional for (70) reads as

$$\begin{aligned} y_{n+1}(x) = \quad & y_n(x) + \int_0^x \left(\frac{\xi(\xi-x)}{x}\right) \left(y_n''(\xi) + \frac{2}{\xi}y_n'(\xi) \right. \\ & \left. - \left(6+4\xi^2\right) y_n - \left(6-6\xi^2-4\xi^4\right) \right) d\xi, \end{aligned} \quad (72)$$

where we used $\lambda(\xi) = \frac{\xi(\xi-x)}{x}$ as given previously in (29).

Considering the given initial values, we can select $y_0(x) = 1$. Using this selection into (72) we obtain the following successive approximations:

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= 1 + 2x^2 - \frac{1}{10}x^4 - \frac{2}{21}x^6, \\ y_2(x) &= 1 + 2x^2 + \frac{1}{2}x^4 + \frac{17}{210}x^6 - \frac{17}{1260}x^8 + \cdots, \\ y_3(x) &= 1 + 2x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{29}{840}x^8 + \cdots, \\ y_4(x) &= 1 + 2x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \cdots, \\ &\vdots\ , \\ y_n(x) &= x^2 + \sum_{m=0}^{kn} \frac{1}{m!}x^{2m}, \end{aligned} \quad (73)$$

for some positive integer k. Taking the limit of the latter equation and recalling that

$$y(x) = \lim_{n\to\infty} y_n(x), \quad (74)$$

this in turn gives the exact solution

$$y(x) = x^2 + e^{x^2}. \quad (75)$$

Example 3

We next consider the inhomogeneous Emden–Fowler type with $\alpha > 2$:

$$y'' + \frac{4}{x}y' - \left(18x + 9x^4\right)y = \left(20 - 36x^3 - 18x^6\right), \quad (76)$$

with initial conditions

$$y(0) = 1, \quad y'(0) = 0. \quad (77)$$

The correction functional for (76) is given by

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(\frac{\xi(\xi^3 - x^3)}{3x^3}\right)\left(y_n''(\xi) + \frac{4}{\xi}y_n'(\xi)\right.$$
$$\left. -\left(18\xi + 9\xi^4\right)y_n(\xi) - \left(20 - 36\xi^3 - 18\xi^6\right)\right)d\xi, \quad (78)$$

where we used $\lambda(\xi) = \dfrac{\xi(\xi^3 - x^3)}{3x^3}$ as was given previously in (30).

Considering the given initial values, we can select $y_0(x) = 1$. Substituting this selection into (78), we obtain the following successive approximations

$$\begin{aligned}
y_0(x) &= 1,\\
y_1(x) &= 1 - 2x^2 + x^3 + \frac{9}{10}x^5 + \frac{1}{6}x^6 + \frac{9}{44}x^8,\\
y_2(x) &= 1 - 2x^2 + x^3 + \frac{1}{2}x^6 + \frac{1}{9}x^9 + \frac{1}{120}x^{12} + \cdots,\\
y_3(x) &= 1 - 2x^2 + x^3 + \frac{1}{2}x^6 + \frac{1}{6}x^9 + \frac{13}{3600}x^{12} + \cdots,\\
y_4(x) &= 1 - 2x^2 + x^3 + \frac{1}{2}x^6 + \frac{1}{6}x^9 + \frac{1}{24}x^{12} + \cdots,\\
&\vdots\ ,\\
y_n(x) &= -2x^2 + \sum_{m=0}^{kn}\frac{1}{m!}x^{3m},
\end{aligned} \quad (79)$$

for some positive integer k. Taking the limit of the latter equation and recalling that

$$y(x) = \lim_{n\to\infty} y_n(x), \quad (80)$$

this yields the exact solution

$$y(x) = -2x^2 + e^{x^3}. \quad (81)$$

6 Emden–Fowler boundary value problems

In this section, we will study three Emden–Fowler boundary value models with singular behavior at $x = 0$. The three problems are selected for $\alpha = 1, 2$ and > 2, where the three distinct Lagrange multipliers in (28)–(30) will be used, respectively.

Example 4

We first consider the inhomogeneous Emden–Fowler type with $\alpha = 1$:

$$y'' + \frac{1}{x}y' - 9y = -9 - 9x^3 + \frac{1}{x}, \quad (82)$$

with boundary conditions

$$y(0) = 1, \quad y(1) = 3. \quad (83)$$

The correction functional for (82) reads as

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(\xi \ln\left(\frac{\xi}{x}\right)\right)\left(y_n''(\xi) + \frac{1}{\xi}y_n'(\xi)\right.$$
$$\left. -9(y_n)(\xi) + \left(9 + 9\xi^3 - \frac{1}{\xi}\right)\right)d\xi, \quad (84)$$

where we used $\lambda(\xi) = \xi \ln\left(\dfrac{\xi}{x}\right)$, as given previously in (28).

Considering the given initial values, we can select $y_0(x) = 1 + ax$ where $y'(0) = a$. Substituting this selection into (84), we obtain the following successive approximations:

$$\begin{aligned}
y_0(x) &= 1 + ax,\\
y_1(x) &= 1 + x + ax^3 - \frac{9}{25}x^5,\\
y_2(x) &= 1 + x + x^3 + \frac{9}{25}(a-1)x^5 - \frac{81}{1225}x^7,\\
y_3(x) &= 1 + x + x^3 + \frac{81}{1225}(a-1)x^7 - \frac{9}{1225}x^9,\\
y_4(x) &= 1 + x + x^3 + \frac{9}{1225}(a-1)x^9 - \frac{81}{148225}x^{11},\\
y_5(x) &= 1 + x + x^3 + \frac{81}{148225}(a-1)x^{11}\\
&\quad -\frac{729}{25050025}x^{13},\\
&\vdots\ .
\end{aligned} \quad (85)$$

Imposing the boundary condition $y(1) = 3$ in all approximations $y_0, y_1, \ldots$, and solving each result for a we obtain the following sequence of values for a:

$$2, 1.36, 1.183673469, 1.111111111,$$
$$1.074380165, 1.053254438, \cdots. \quad (86)$$

It is obvious that this sequence converges to $a = 1$. Substituting $a = 1$ in (85) gives the approximations

$$\begin{aligned}
y_0(x) &= 1 + x,\\
y_1(x) &= 1 + x + x^3 - \frac{9}{25}x^5,\\
y_2(x) &= 1 + x + x^3 - \frac{81}{1225}x^7,\\
y_3(x) &= 1 + x + x^3 - \frac{9}{1225}x^9,\\
y_4(x) &= 1 + x + x^3 - \frac{81}{148225}x^{11},\\
y_5(x) &= 1 + x + x^3 - \frac{729}{25050025}x^{13},\\
&\vdots\ ,\\
y_n(x) &= 1 + x + x^3 + \cdots.
\end{aligned} \quad (87)$$

Note that there exists a noise term which obviously will vanish in the limit, hence the resulting exact solution of

(82) is given by

$$y(x) = 1 + x + x^3. \tag{88}$$

Example 5

We next consider the homogeneous Emden–Fowler type with $\alpha = 2$:

$$y'' + \frac{2}{x}y' - (4x^2 - 6)y = 0, \tag{89}$$

with boundary conditions

$$y(0) = 1, \quad y(1) = e^{-1}. \tag{90}$$

The correction functional for (89) is

$$\begin{aligned} y_{n+1}(x) = \; & y_n(x) + \int_0^x \left(\frac{\xi(\xi - x)}{x}\right)\left(y_n''(\xi) + \frac{2}{\xi}y_n'(\xi)\right. \\ & \left. - \left(4\xi^2 - 6\right) y_n\right) d\xi, \end{aligned} \tag{91}$$

where we used $\lambda(\xi) = \dfrac{\xi(\xi - x)}{x}$ as given in (29).

Considering the given initial value, we can select $y_0(x) = 1 + bx$, where $y'(0) = b$. Substituting this selection into (91), we obtain the following successive approximations

$$\begin{aligned} y_0(x) &= 1 + bx, \\ y_1(x) &= 1 - x^2 - \frac{1}{2}bx^3 + \frac{1}{5}x^4 + \frac{2}{15}bx^5, \\ y_2(x) &= 1 - x^2 + \frac{1}{2}x^4 + \frac{2}{10}bx^5 - \frac{13}{105}x^6 - \frac{1}{20}bx^7 \\ &\quad + \frac{1}{90}x^8 + \frac{4}{675}bx^9, \\ y_3(x) &= 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 - \frac{3}{280}bx^7 + \frac{4}{105}x^8 \\ &\quad + \frac{7}{900}bx^9 - \frac{59}{11550}x^{10} - \frac{53}{2970}bx^{11} + \cdots, \\ y_4(x) &= 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 + \frac{1}{1400}bx^9 \\ &\quad - \frac{47}{5775}x^{10} - \frac{47}{69300}bx^{11} + \frac{151}{128700}x^{12} + \cdots, \\ y_5(x) &= 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{120}x^{10} \\ &\quad - \frac{1}{30800}bx^{11} + \frac{2489}{1801800}x^{12} \\ &\quad + \frac{4}{105105}bx^{13} + \cdots, \\ &\vdots \; . \end{aligned} \tag{92}$$

Imposing the boundary condition $y(1) = e^{-1}$ in all approximations $y_0, y_1, \ldots$ and solving each result for b, we obtain the following sequence for values of b:

$$\begin{aligned} &-0.6321, -0.4579, -0.3473, -0.2776, \\ &-0.2305, -0.1967, \cdots . \end{aligned} \tag{93}$$

It is obvious that this sequence converges to $b = 0$. Substituting $b = 0$ in the approximations (92) gives

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= 1 - x^2 + \frac{1}{5}x^4, \\ y_2(x) &= 1 - x^2 + \frac{1}{2}x^4 - \frac{13}{105}x^6 + \frac{1}{90}x^8, \\ y_3(x) &= 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{4}{105}x^8 \\ &\quad - \frac{59}{11550}x^{10} + \cdots, \\ y_4(x) &= 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 \\ &\quad - \frac{47}{5775}x^{10} + \frac{151}{128700}x^{12} + \cdots, \\ y_5(x) &= 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 \\ &\quad - \frac{1}{120}x^{10} + \frac{2489}{1801800}x^{12} + \cdots, \\ &\vdots \; , \\ y_n(x) &= \sum_{m=0}^{kn} \frac{(-1)^m}{m!}x^{2m}, \end{aligned} \tag{94}$$

for some positive integer k. Upon taking the limit of the last equation, the exact solution will be given by

$$y(x) = e^{-x^2}. \tag{95}$$

Example 6

We close this work by studying the inhomogeneous Emden–Fowler type with $\alpha > 2$:

$$y'' + \frac{6}{x}y' - \left(14 + 4x^2\right)y = \left(24x - 14x^3 - 4x^5\right), \tag{96}$$

with boundary conditions

$$y(0) = 1, \quad y(1) = 1 + e. \tag{97}$$

The correction functional for (96) reads

$$\begin{aligned} y_{n+1}(x) = y_n(x) &+ \int_0^x \left(\frac{\xi\left(\xi^5 - x^5\right)}{5x^5}\right)\left(y_n'(\xi) + \frac{6}{\xi}y_n'(\xi)\right. \\ & \left. - \left(14 + 4\xi^2\right) y_n(\xi) - \left(24\xi - 14\xi^3 - 4\xi^5\right)\right) d\xi, \end{aligned} \tag{98}$$

where we used $\lambda(\xi) = \dfrac{\xi(\xi^5 - x^5)}{5x^5}$ as given above in (30).

Considering the given boundary values, we can select $y_0(x) = 1 + cx$, where $y'(0) = c$. Using this selection into (98) we obtain the following successive approximations

$$\begin{aligned} y_0(x) &= 1 + cx, \\ y_1(x) &= 1 + x^2 + \frac{7c + 12}{12}x^3 + \frac{1}{9}x^4 + \frac{2c - 7}{25}x^5 - \frac{1}{21}x^7, \end{aligned}$$

$$
\begin{aligned}
y_2(x) &= 1+x^2+x^3+\frac{1}{2}x^4+\frac{49}{300}cx^5+\frac{25}{297}x^6 \\
&+\frac{37c-42}{900}x^7+\frac{1}{234}x^8+\frac{12c-67}{4725}x^9-\frac{1}{924}x^{11}, \\
y_3(x) &= 1+x^2+x^3+\frac{1}{2}x^4+\frac{1}{6}x^6+\frac{49}{1800}cx^7+\frac{118}{3861}x^8 \\
&+\frac{79c-42}{8100}x^9+\frac{1531}{579150}x^{10} \\
&+\left(\frac{c}{880}-\frac{13}{5940}\right)x^{11}+\cdots, \\
y_4(x) &= 1+x^2+x^3+\frac{1}{2}x^4+\frac{1}{6}x^6+\frac{1}{24}x^8+\frac{49c}{16200}x^9 \\
&+\frac{2113}{289575}x^{10} \\
&+\left(\frac{497c}{356400}-\frac{49}{118800}\right)x^{11}+\cdots, \\
y_5(x) &= 1+x^2+x^3+\frac{1}{2}x^4+\frac{1}{6}x^6+\frac{1}{24}x^8+\frac{1}{120}x^{10} \\
&+\frac{343c}{1425600}x^{11}+\cdots, \\
y_6(x) &= 1+x^2+x^3+\frac{1}{2}x^4+\frac{1}{6}x^6+\frac{1}{24}x^8+\frac{1}{120}x^{10} \\
&+\frac{1}{720}x^{12}+\frac{2401}{166795200}cx^{13}+\cdots, \\
&\vdots\ .
\end{aligned} \tag{99}
$$

Imposing the boundary condition $y(1) = 1+e$ in all approximations $y_0, y_1, \ldots$, and solving each result for c we obtain the following sequence for values of c:

$$
\begin{aligned}
&2.718281828, 1.409230799, 0.92645776, \\
&0.68198291, 0.5362808, 0.440314, 0.3726, \cdots .
\end{aligned} \tag{100}
$$

It is obvious that this sequence converges to $c = 0$. Substituting $c = 0$ in (99) we find the approximations

$$
\begin{aligned}
y_0(x) &= 1, \\
y_1(x) &= 1+x^2+x^3+\frac{1}{9}x^4-\frac{7}{25}x^5-\frac{1}{21}x^7, \\
y_2(x) &= 1+x^2+x^3+\frac{1}{2}x^4+\frac{25}{297}x^6-\frac{7}{150}x^7+\frac{1}{234}x^8 \\
&-\frac{67}{4725}x^9-\frac{1}{924}x^{11}, \\
y_3(x) &= 1+x^2+x^3+\frac{1}{2}x^4+\frac{1}{6}x^6+\frac{118}{3861}x^8-\frac{7}{300}x^9 \\
&+\frac{1531}{579150}x^{10}-\frac{13}{5940}x^{11}+\cdots, \\
y_4(x) &= 1+x^2+x^3+\frac{1}{2}x^4+\frac{1}{6}x^6+\frac{1}{24}x^8+\frac{2113}{289575}x^{10} \\
&-\frac{49}{118800}x^{11}+\cdots, \\
y_5(x) &= 1+x^2+x^3+\frac{1}{2}x^4+\frac{1}{6}x^6+\frac{1}{24}x^8+\frac{1}{120}x^{10} \\
&+\cdots, \\
y_6(x) &= 1+x^2+x^3+\frac{1}{2}x^4+\frac{1}{6}x^6+\frac{1}{24}x^8 \\
&+\frac{1}{120}x^{10}+\frac{1}{720}x^{12}+\cdots, \\
&\vdots\ , \\
y_n(x) &= x^3+\sum_{m=0}^{kn}\frac{1}{m!}x^{2m}.
\end{aligned} \tag{101}
$$

Noting that the noise terms vanish in the limit, the exact solution is given by

$$
y(x) = x^3 + e^{x^2}. \tag{102}
$$

7 Conclusion

The present work exhibits the reliability of combining the Volterra integro-differential equations of the Lane–Emden and the Emden–Fowler equations and variational iteration method for the solution of these singular equations. In this work, we demonstrate that this strategy can be well suited to attain numerical and analytical solutions. The challenge arising due to the existence of a singularity at $x = 0$, can be easily addressed.

The Lagrange multiplier used, for all cases of the parameter k, is $\lambda = -1$. To support our work, we examined six distinct examples, initial value problems and boundary value problems. Some of the examples are linear and others are nonlinear. This is a significant feature of the variational iteration method in that it handles all problems in a straightforward manner without imposing any restriction that may change the physical nature of the solution. The results demonstrate the reliability and effectiveness of the presented analysis.

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