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A reliable algorithm for positive solutions of nonlinear boundary value problems by the multistage Adomian decomposition method

Abstract: In this paper, we present a reliable algorithm to calculate positive solutions of homogeneous nonlinear boundary value problems (BVPs). The algorithm converts the nonlinear BVP to an equivalent nonlinear Fredholm–Volterra integral equation. We employ the multistage Adomian decomposition method for BVPs on two or more subintervals of the domain of validity, and then solve the matching equation for the flux at the interior point, or interior points, to determine the solution. Several numerical examples are used to highlight the effectiveness of the proposed scheme to interpolate the interior values of the solution between boundary points. Furthermore we demonstrate two novel techniques to accelerate the rate of convergence of our decomposition series solutions by increasing the number of subintervals and adjusting the lengths of subintervals in the multistage Adomian decomposition method for BVPs.

Keywords: boundary value problem; positive solution; Adomian decomposition method; multistage decomposition; nonlinear differential equation

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1 Introduction

In this paper, by using the multistage Adomian decomposition method (ADM) for homogeneous boundary value problems (BVPs) with Dirichlet boundary conditions, we consider positive solutions of a class of homogenous nonlinear BVPs

$$u''(x) + \alpha_1 u'(x) + \alpha_0 u(x) + f(u(x), u'(x)) = 0, \quad a < x < b, \tag{1}$$

$$u(a) = 0, \; u(b) = 0, \tag{2}$$

where α_0 and α_1 are constants, and f is an analytic nonlinear function and satisfies $f(0, 0) = 0$.

Notice that the formula of Equation (1) subsumes the cases of the various anharmonic oscillator equations, including the pendulum equation with a sinusoidal nonlinearity, the Duffing oscillator and the van der Pol oscillator equations, and so forth.

Considerable research has been invested in studying nonlinear BVPs. The existence of a positive solution for problem (1) and (2) under suitable conditions of $f(u(x), u'(x))$ has been studied by Agarwal et al. [1]. Several techniques, such as the shooting method, finite difference method, and Green functions have been used to handle this type of problems. Wazwaz [2] used the method of undetermined coefficients in the ADM combined with the Padé approximants technique to obtain the positive solutions of homogeneous nonlinear BVPs.

We remark that the nonlinear BVP (1) and (2) has a trivial solution, i.e. the zero solution. It has been previously demonstrated that the native command 'NDSolve' for numerically solving differential equations in MATHEMATICA cannot generate the positive solution, but yields the zero solution.

For this type of nonlinear BVPs, we find by the single-stage ADM for BVPs proposed in [3], that we can only obtain the zero solution. In order to obtain the positive solution, we instead apply the multistage ADM for BVPs, and partition the original domain into at least two subdomains such that $[a, b] = [a, \xi] \cup [\xi, b]$, etc. Thus we have changed our original problem of a second-order homogeneous nonlinear BVP into two contiguous second-order inhomogeneous nonlinear BVPs, where one of the boundary conditions is $u(\xi) > 0$ by our assumption of a positive solution.

We can expound the present approach, which is a novel technique of solution interpolation in sequential stages using the ADM between the boundary values of the solution at the boundary points, by analogy to the tech-

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nique of solution extrapolation in sequential stages using the ADM beginning from the initial values of the solution at the initial point, which is founded upon the concept of analytic continuation. The multistage ADM for BVPs is a technique of solution interpolation, where the combined solution on the entire domain is indeed a spline assembled from the sequence of analytic approximate sub-solutions, in other words, the sequence of interpolants on their respective subintervals, using the well-known boxcar function. We denote this combined function as the decomposition spline in analogy to interpolation by polynomials, rational functions, trigonometric functions and so on.

The proposed approach converts the original nonlinear BVP into an equivalent nonlinear Fredholm-Volterra integral equation for the positive solution in each subinterval. A separate series in each subinterval is computed by using the modified recursion scheme of Duan and Rach [3] in the ADM for nonlinear BVPs. The two or more sub-solutions, i.e. interpolants, are combined by applying the conditions of continuity at the interior boundary point to form the decomposition spline in analogy to the multistage ADM for IVPs [4–12]. We then solve the matching equations for the flux at the interior point, or interior points, thus obtaining a rapidly convergent sequence of analytic approximations for the positive solution.

The ADM is a well-known systematic method for practical solution of linear or nonlinear and deterministic or stochastic operator equations, including ordinary differential equations, partial differential equations, integral equations, integro-differential equations, etc. [13–21]. A substantial amount of research has been devoted to the study of the ADM in a wide area of scientific and engineering applications. The ADM permits us to solve both nonlinear IVPs and BVPs [2, 22–39], including cases for nonlinear fractional differential equations [32, 40–42], without unphysical restrictive assumptions such as required by linearization, perturbation, ad hoc assumptions, guessing the initial term or a set of basis functions, and so forth. Furthermore the ADM does not require the use of Green's functions which are not easily determined in most cases. A key notion is the Adomian polynomials, which are tailored to the particular nonlinearity to easily and systematically solve nonlinear differential equations.

In the ADM, the solution $u(x)$ is represented by the decomposition series and the nonlinearity $Nu(x)$ is represented by the series of the Adomian polynomials

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \text{ and } Nu(x) = \sum_{n=0}^{\infty} A_n(x), \tag{3}$$

respectively, where the Adomian polynomials $A_n(x)$ are dependent upon the solution components from $u_0(x)$ through $u_n(x)$, inclusively, i.e. $A_n(x) = A_n(u_0(x), \ldots, u_n(x))$, thus facilitating computation of the solution components by recursion.

Adomian and Rach [13] first published the definitional formula for the Adomian polynomials for the one-variable simple nonlinearity $Nu = f(u(x))$, where f is assumed to be analytic, as

$$A_n(x) = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n} f\left(u(x;\lambda)\right)\Big|_{\lambda=0}, \quad n \geq 0, \tag{4}$$

where

$$u(x;\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n(x), \tag{5}$$

and

$$f(u(x;\lambda)) = \sum_{n=0}^{\infty} \lambda^n A_n(x), \tag{6}$$

where λ is a grouping parameter of convenience. In computational practice, we truncate the decomposition series after $n = M$ for some finite M, since the higher order solution components $u_n(x)$ for $n > M$ do not contribute to the calculation of the $A_n(x)$ for $n \leq M$ in practice, thus

$$A_n(x) = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n} f\left(\sum_{n=0}^{M} \lambda^n u_n(x)\right)\Bigg|_{\lambda=0}, \quad 0 \leq n \leq M, \tag{7}$$

from which we can calculate the first $M+1$ Adomian polynomials from $A_0(x)$ through $A_M(x)$, inclusively, such that $A_n(x) = A_n(u_0(x), u_1(x), \ldots, u_n(x))$ for $n = 0, 1, \ldots, M$.

We list the formulas of the first several Adomian polynomials for the one-variable simple analytic nonlinearity $Nu = f(u(x))$ from A_0 through A_5, inclusively, for convenient reference as

$$\begin{aligned}
A_0 &= f(u_0),\\
A_1 &= u_1 f'(u_0),\\
A_2 &= u_2 f'(u_0) + \frac{1}{2!}u_1^2 f''(u_0),\\
A_3 &= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{1}{3!}u_1^3 f^{(3)}(u_0),\\
A_4 &= u_4 f'(u_0) + \left(u_1 u_3 + \frac{1}{2!}u_2^2\right) f''(u_0)\\
&\quad + \frac{1}{2!}u_1^2 u_2 f^{(3)}(u_0) + \frac{1}{4!}u_1^4 f^{(4)}(u_0),\\
A_5 &= u_5 f'(u_0) + (u_2 u_3 + u_1 u_4) f''(u_0)\\
&\quad + \left(\frac{1}{2!}u_1 u_2^2 + \frac{1}{2!}u_1^2 u_3\right) f^{(3)}(u_0)\\
&\quad + \frac{1}{3!}u_1^3 u_2 f^{(4)}(u_0) + \frac{1}{5!}u_1^5 f^{(5)}(u_0).
\end{aligned}$$

We recognize that

$$A_0 = f(u_0), \; A_n = \sum_{k=1}^{n} C_n^k f^{(k)}(u_0), \text{ for } n \geq 1, \tag{8}$$

where the C_n^k are the sums of all possible products of k components from $u_1, u_2, \cdots, u_{n-k+1}$, whose subscripts sum to n, divided by the factorial of the number of repeated subscripts [43], which is called Rach's Rule (P. 16 [16], P. 51 [17]).

Other algorithms for the Adomian polynomials have been developed by Adomian and Rach [44, 45], Wazwaz [46], Abdelwahid [47], Rach [48] and several others [49–52]. Recently Duan [53–55] has developed several new, more efficient algorithms for fast generation of the one-variable and multi-variable Adomian polynomials. For the case of the one-variable Adomian polynomials, Duan's Corollary 3 algorithm [55] does not involve the differentiation operator, but only requires the operations of addition and multiplication, which is eminently convenient for computer algebra systems such as MATHEMATICA, MAPLE or MATLAB,

$$\begin{aligned} &C_n^1 = u_n, \ n \geq 1, \\ &C_n^k = \frac{1}{n}\sum_{j=0}^{n-k}(j+1)u_{j+1}C_{n-1-j}^{k-1}, \ 2 \leq k \leq n, \end{aligned} \tag{9}$$

from which we can quickly and easily calculate the first $M+1$ Adomian polynomials. We list the corresponding MATHEMATICA code in Appendix A.

For convenient reference for the multi-variable Adomian polynomials, we present the MATHEMATICA code for Duan's Corollary 1 algorithm [54] in Appendix B, and also list the first six two-variable Adomian polynomials for the differential nonlinearity $Nu = f(u(x), u'(x))$ in Appendix C.

In Section 2, we present the multistage ADM for BVPs for computation of positive solutions. In Section 3, we investigate four homogeneous nonlinear examples including a nonlinear BVP with a quadratic nonlinearity in the first-order derivative, a nonlinear BVP with a quadratic nonlinearity in the solution with a singular variable system coefficient, a nonlinear BVP for the unforced pendulum equation and a nonlinear BVP for the unforced van der Pol equation, for which we consider partition of the original domain into two, three and four subintervals to demonstrate an accelerated rate of convergence by increasing the number of subintervals as well as by adjusting the lengths of subintervals in the multistage ADM for BVPs. Section 4 summarizes our findings.

2 Description of the proposed approach

We shall assume that the homogeneous nonlinear BVP (1) and (2) has a positive solution, i.e. the solution $u(x)$ satisfies $u(x) > 0$, for $a < x < b$. For necessary conditions for the existence and uniqueness of a positive solution of nonlinear BVPs, see [1].

Using Adomian's operator-theoretic notation, we rewrite Equation (1) as

$$Lu(x) + Ru(x) + Nu(x) = 0, \ a < x < b, \tag{10}$$

where $Lu(x) = u''(x)$, $Ru(x) = \alpha_1 u'(x) + \alpha_0 u(x)$, and $Nu(x) = f(u(x), u'(x))$.

In order to obtain the positive solution, we apply the multistage ADM for BVPs [56], and partition the original domain into at least two subdomains such that $[a, b] = [a, \xi] \cup [\xi, b]$, etc. Thus we have changed our original problem of a second-order homogeneous nonlinear BVP into two contiguous second-order inhomogeneous nonlinear BVPs, where one of the boundary conditions is $u(\xi) > 0$ by our assumption of a positive solution.

We next derive the modified recursion scheme in the ADM for nonlinear BVPs used in each subdomain. Then we present the multistage decomposition scheme for positive solutions.

2.1 The modified recursion scheme in the ADM for nonlinear BVPs

In this subsection, we review the modified recursion scheme by Duan and Rach [3] in the ADM for the solution of the inhomogenous nonlinear BVP used in each subdomain in the form of

$$Lu(x) + Ru(x) + Nu(x) = 0, \ a < x < b, \tag{11}$$

$$u(a) = \alpha, \ u(b) = \beta, \tag{12}$$

where not all of α and β are 0.

We take the inverse linear operator as

$$L_{a,a}^{-1}(\cdot) = \int_a^x \int_a^x (\cdot) dx dx.$$

Applying the operator $L_{a,a}^{-1}(\cdot)$ to both sides of Equation (11) yields

$$u(x) = \alpha + (x-a)u'(a) - L_{a,a}^{-1}(Ru + Nu), \tag{13}$$

where the boundary value at $x = a$ is used. Using Equation (13), we evaluate $u(x)$ at $x = b$ to obtain

$$u(b) = \alpha + (b-a)u'(a) - \left[L_{a,a}^{-1}(Ru + Nu)\right]_{x=b}, \tag{14}$$

where

$$\left[L_{a,a}^{-1}(\cdot)\right]_{x=b} = \int_a^b \int_a^x (\cdot) dx dx.$$

Substituting the boundary value at $x = b$ from Equation (12) into Equation (14) and after appropriate algebraic manipulations, we obtain

$$u'(a) = \frac{\beta - \alpha}{b - a} + \frac{1}{b - a}\left[L_{a,a}^{-1}(Ru + Nu)\right]_{x=b}. \quad (15)$$

Substituting Equation (15) into Equation (13) yields

$$u(x) = \frac{\beta(x-a) + \alpha(b-x)}{b-a} + \frac{x-a}{b-a}\left[L_{a,a}^{-1}(Ru + Nu)\right]_{x=b} - L_{a,a}^{-1}(Ru + Nu), \quad (16)$$

which is an equivalent nonlinear Fredholm-Volterra integral equation for the nonlinear BVP (11) and (12).

Substituting the Adomian decomposition series for the solution $u(x)$ and the series of Adomian polynomials in Equation (3) into Equation (16), we have

$$\sum_{n=0}^{\infty} u_n = \frac{\beta(x-a) + \alpha(b-x)}{b-a} + \frac{x-a}{b-a}\left[L_{a,a}^{-1}\left(R\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n\right)\right]_{x=b} - L_{a,a}^{-1}\left(R\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n\right).$$

The solution components $u_n(x)$ may then be determined by the classic Adomian recursion scheme

$$u_0(x) = \frac{\beta(x-a) + \alpha(b-x)}{b-a}, \quad (17)$$

$$u_{n+1}(x) = \frac{x-a}{b-a}\left[L_{a,a}^{-1}(Ru_n + A_n)\right]_{x=b} - L_{a,a}^{-1}(Ru_n + A_n), \quad n \geq 0. \quad (18)$$

Other recursion schemes, such as the modified recursion scheme of Wazwaz [57, 58], or the parametrized recursion scheme of Duan [3, 53, 59] can be used instead to facilitate computation or increase the rate of convergence. The nth-stage approximation obtained by the ADM is the truncated decomposition series

$$\varphi_n(x) = \sum_{k=0}^{n-1} u_k(x). \quad (19)$$

We note that if we use the alternate inverse linear operator

$$L_{b,b}^{-1}(\cdot) = \int_b^x \int_b^x (\cdot)dxdx,$$

the equivalent nonlinear Fredholm-Volterra integral equation for the nonlinear BVP (11) and (12) is

$$u(x) = \frac{\beta(x-a) + \alpha(b-x)}{b-a} + \frac{b-x}{b-a}\left[L_{b,b}^{-1}(Ru + Nu)\right]_{x=a} - L_{b,b}^{-1}(Ru + Nu). \quad (20)$$

2.2 The multistage decomposition scheme for positive solutions

We return to the homogenous nonlinear BVP (1) and (2). For this type of nonlinear BVPs, we find by the standard recursion scheme (17) and (18), in other words the single-stage ADM for BVPs, that we can only obtain the trivial solution, i.e. the zero solution, because the original problem is not in the canonical form of Adomian. In order to obtain the positive solution, we instead apply the multistage ADM for BVPs.

We first partition the original interval $[a, b]$ into two subintervals $[a, b] = [a, \xi] \cup [\xi, b]$. Suppose the value of the solution at the interior point is $u(\xi) = \eta$, which represents a positive undetermined constant.

Subinterval 1:

We solve the inhomogeneous nonlinear BVP

$$Lu(x) + Ru(x) + Nu(x) = 0, \ a < x < \xi, \quad (21)$$

$$u(a) = 0, \ u(\xi) = \eta. \quad (22)$$

From Equation (16), we obtain the equivalent nonlinear Fredholm-Volterra integral equation for the solution

$$u(x) = \frac{\eta(x-a)}{\xi-a} + \frac{x-a}{\xi-a}\left[L_{a,a}^{-1}(Ru + Nu)\right]_{x=\xi} - L_{a,a}^{-1}(Ru + Nu), \quad (23)$$

which leads to the modified recursion scheme

$$u_0 = \frac{\eta(x-a)}{\xi-a}, \quad (24)$$

$$u_{n+1} = \frac{x-a}{\xi-a}\left[L_{a,a}^{-1}(Ru_n + A_n)\right]_{x=\xi} - L_{a,a}^{-1}(Ru_n + A_n), \quad n \geq 0, \quad (25)$$

where we denote the mth-stage approximation in the first subinterval as

$$\varphi_m^{(1)}(x) = \varphi_m^{(1)}(x;\eta) = \sum_{k=0}^{m-1} u_k(x). \quad (26)$$

Subinterval 2:

We solve the inhomogeneous nonlinear BVP

$$Lu(x) + Ru(x) + Nu(x) = 0, \ \xi < x < b, \quad (27)$$

$$u(\xi) = \eta, \ u(b) = 0. \quad (28)$$

From Equation (16), we obtain the equivalent nonlinear Fredholm-Volterra integral equation for the solution

$$u(x) = \frac{\eta(b-x)}{b-\xi} + \frac{x-\xi}{b-\xi}\left[L_{\xi,\xi}^{-1}(Ru + Nu)\right]_{x=b} - L_{\xi,\xi}^{-1}(Ru + Nu), \quad (29)$$

which leads to the modified recursion scheme

$$u_0 = \frac{\eta(b-x)}{b-\xi}, \quad (30)$$

$$u_{n+1} = \frac{x-\xi}{b-\xi}\left[L_{\xi,\xi}^{-1}(Ru_n + A_n)\right]_{x=b} - L_{\xi,\xi}^{-1}(Ru_n + A_n), \qquad n \geq 0, \quad (31)$$

where we denote the mth-stage approximation in the second subinterval as

$$\varphi_m^{(2)}(x) = \varphi_m^{(2)}(x;\eta) = \sum_{k=0}^{m-1} u_k(x). \quad (32)$$

Next we match the two approximations $\varphi_m^{(1)}(x)$ and $\varphi_m^{(2)}(x)$ at the interior point ξ by applying the continuity condition for the flux,

$$\left.\frac{d\varphi_m^{(1)}(x;\eta)}{dx}\right|_{x=\xi} = \left.\frac{d\varphi_m^{(2)}(x;\eta)}{dx}\right|_{x=\xi}, \quad (33)$$

which determines the positive value of the constant η.

We denote the $\eta[\varphi_m]$ as the positive approximate value of η obtained by matching $\varphi_m^{(1)}(x)$ and $\varphi_m^{(2)}(x)$ according to Equation (33).

Combining the interpolants $\varphi_m^{(1)}(x;\eta[\varphi_m])$ and $\varphi_m^{(2)}(x;\eta[\varphi_m])$ by using the boxcar function, we obtain the matched mth-stage approximation of the positive solution on the entire interval $[a, b]$ known as the decomposition spline

$$\varphi_m(x) = \varphi_m^{(1)}(x;\eta[\varphi_m])\Pi(x;a,\xi) + \varphi_m^{(2)}(x;\eta[\varphi_m])\bar{\Pi}(x;\xi,b), \quad (34)$$

where the boxcar functions are defined as

$$\Pi(x;c,d) = \begin{cases} 1 & c \leq x < d, \\ 0 & \text{otherwise,} \end{cases} \quad (35)$$

and

$$\bar{\Pi}(x;c,d) = \begin{cases} 1 & c \leq x \leq d, \\ 0 & \text{otherwise.} \end{cases} \quad (36)$$

We note that the matching equation for $m = 1$ is $\frac{\eta}{\xi-a} = \frac{-\eta}{b-\xi}$ from Equations (24) and (30), which implies that $\eta = 0$. So we consider the sequence of the $\eta[\varphi_m]$ beginning with $m \geq 2$.

3 Numerical examples

We consider four homogenous nonlinear BVPs in the form of (1) and (2).

We remark that the command 'NDSolve', i.e. the native command for numerically solving differential equations in MATHEMATICA, cannot give the positive solution, but can only give the zero solution for Examples 1, 3 and 4, and cannot give any solution whatsoever for Example 2 due to the presence of singularities.

Example 1. Consider the homogeneous nonlinear BVP with a quadratic nonlinearity in the first-order derivative

$$u''(x) + 8u(x) + 2(u'(x))^2 = 0, \; 0 \leq x \leq 1, \quad (37)$$

$$u(0) = 0, \; u(1) = 0. \quad (38)$$

The positive solution of this BVP is $u^*(x) = x - x^2$.

We partition the interval $[0, 1]$ into two subintervals $[0, 0.5]$ and $[0.5, 1]$, and let $u(0.5) = \eta$, which is a positive undetermined constant.

By the decomposition of the solution $u(x) = \sum_{n=0}^{\infty} u_n(x)$, the Adomian polynomials for the differential nonlinearity $Nu = (u'(x))^2$ are

$$A_0 = (u_0')^2, \; A_n = \sum_{k=0}^{n} u_{n-k}' u_k', \; n \geq 1.$$

In the subinterval $[0, 0.5]$, using Equations (23)–(25), we calculate the solution components

$$u_1 = -\frac{8x^3\eta}{3} - 4x^2\eta^2 + x\left(\frac{2\eta}{3} + 2\eta^2\right),$$

$$u_2 = x\left(\frac{7\eta}{45} + \eta^2 + \frac{4\eta^3}{3}\right) + x^2\left(-\frac{8\eta^2}{3} - 8\eta^3\right) + x^3\left(-\frac{8\eta}{9} - \frac{8\eta^2}{3} + \frac{32\eta^3}{3}\right) + 8x^4\eta^2 + \frac{16x^5\eta}{15}, \; \ldots,$$

where the nth-stage approximation in the first subinterval is $\varphi_n^{(1)}(x;\eta) = \sum_{k=0}^{n-1} u_k(x)$.

In the subinterval $[0.5, 1]$, using Equations (29)–(31), we calculate the solution components

$$u_1 = -2\eta + \frac{8x^3\eta}{3} - 2\eta^2 + x^2\left(-8\eta - 4\eta^2\right) + x\left(\frac{22\eta}{3} + 6\eta^2\right),$$

$$u_2 = \frac{\eta}{3} + \frac{11\eta^2}{3} + 4\eta^3 + x\left(-\frac{127\eta}{45} - \frac{59\eta^2}{3} - \frac{52\eta^3}{3}\right) + x^2\left(8\eta + \frac{112\eta^2}{3} + 24\eta^3\right) + x^3\left(-\frac{88\eta}{9} - \frac{88\eta^2}{3} - \frac{32\eta^3}{3}\right) + x^4\left(\frac{16\eta}{3} + 8\eta^2\right) - \frac{16x^5\eta}{15}, \; \ldots,$$

where we denote the nth-stage approximate solution in the second subinterval as $\varphi_n^{(2)}(x;\eta) = \sum_{k=0}^{n-1} u_k(x)$.

By solving the matching equation for the flux at the interior point $x = 0.5$

$$\left.\frac{d\varphi_n^{(1)}(x;\eta)}{dx}\right|_{x=0.5} = \left.\frac{d\varphi_n^{(2)}(x;\eta)}{dx}\right|_{x=0.5}, \qquad (39)$$

we can compute the sequence of approximate values for the undetermined constant η.

The matching equation for $n = 2$ reads $3\eta^2 = \eta$, which has a positive root $\eta[\varphi_2] = 1/3$. The matching equation for $n = 3$ reads $\eta\left(22 - 75\eta + 60\eta^2\right) = 0$. In order to avoid the complexity of spurious multiple roots, we select the nearest positive root from the first computed real positive root $\eta[\varphi_2]$. The MATHEMATICA command 'FindRoot' with a user-defined initial value can realize this aim. Thus we obtain $\eta[\varphi_3] = 0.470215$.

In a similar manner, we obtain the sequence of values of $\eta[\varphi_n]$ for $n = 4, 5, \ldots$. In Table 1, we list the values of $\eta[\varphi_n]$ for $n = 2, 3, \ldots, 10$. We note that the true value of η is 0.25.

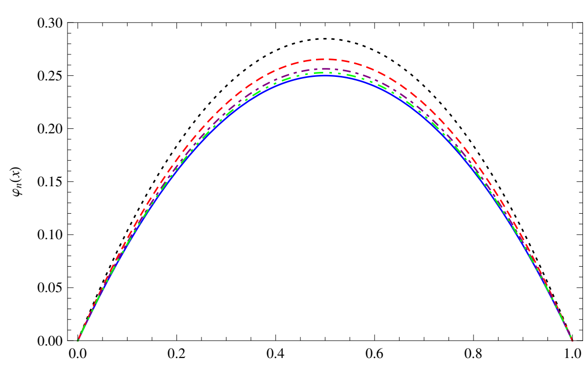

Figure 1: The exact solution $u^*(x)$ (solid line) and the matched nth-stage approximations $\varphi_n(x)$ for $n = 4$ (dot line), $n = 5$ (dashed line), $n = 6$ (dot-side line), $n = 7$ (dot-dot-side line).

Thus we express the matched nth-stage approximant as

$$\varphi_n(x) = \varphi_n^{(1)}(x;\eta[\varphi_n])\Pi(x;0,0.5) + \varphi_n^{(2)}(x;\eta[\varphi_n])\bar{\Pi}(x;0.5,1).$$

The curves of the exact solution $u^*(x)$ and the matched nth-stage approximations $\varphi_n(x)$ for $n = 4, 5, 6, 7$ are shown in Figure 1.

Furthermore, we have computed the error function

$$E_n(x) = \varphi_n(x) - u^*(x), \qquad (40)$$

and the maximal error parameters

$$ME_n = \max_{0\le x\le 1} |\varphi_n(x) - u^*(x)|, \qquad (41)$$

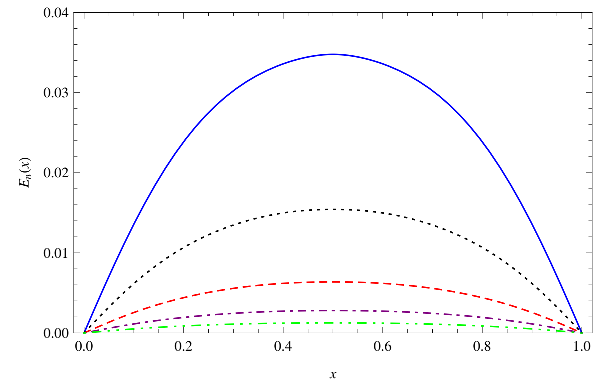

Figure 2: Curves of the error functions $E_n(x)$ for $n = 4$ (solid line), $n = 5$ (dot line), $n = 6$ (dashed line), $n = 7$ (dot-side line), and $n = 8$ (dot-dot-side line).

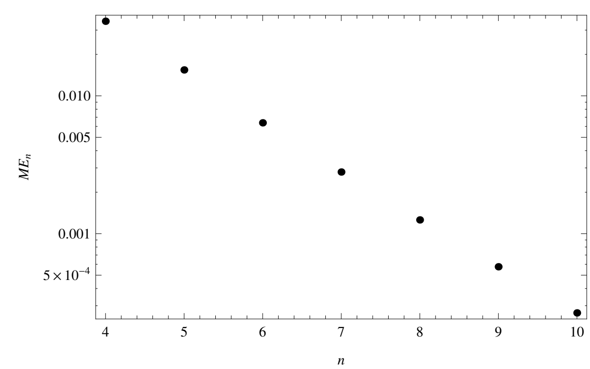

Figure 3: Logarithmic plots of the ME_n for $n = 4$ through 10.

for the nth-stage approximation $\varphi_n(x)$. The curves of the error functions $E_n(x)$ for $n = 4$ through 8 are plotted in Figure 2, which displays a rapid rate of convergence on the domain of validity. The MATHEMATICA command 'NMaximize' can be conveniently used to compute the maximal error parameters ME_n. We list the values of the ME_n for $n = 2, 3, \ldots, 10$ in Table 2. The logarithmic plots of the values of ME_4 through ME_{10} are displayed in Figure 3, which demonstrates an approximately exponential rate of convergence for the obtained decomposition series.

Example 2. Consider the homogeneous nonlinear BVP with a quadratic nonlinearity and a singular variable system coefficient

$$u''(x) + \pi^3 \frac{u^2(x)}{\sin(\pi x)} = 0, \quad 0 < x < 1, \qquad (42)$$

$$u(0) = 0, \quad u(1) = 0. \qquad (43)$$

The positive solution of this BVP is $u^*(x) = \frac{1}{\pi}\sin(\pi x)$.

We partition the interval $[0, 1]$ into two subintervals $[0, 0.5]$ and $[0.5, 1]$, and let $u(0.5) = \eta$, a positive undetermined constant.

We note that the system coefficient $\frac{1}{\sin(\pi x)}$ has singularities at the boundaries $x = 0$ and $x = 1$. We will use the same

Table 1: The values of $\eta[\varphi_n]$ for $n = 2, 3 \dots, 10$.

n	2	3	4	5	6	7	8	9	10
$\eta[\varphi_n]$	1/3	0.470215	0.284776	0.265421	0.256374	0.252802	0.251259	0.250575	0.250266

Table 2: The maximal error parameter ME_n.

n	2	3	4	5	6	7	8
ME_n	0.0833333	0.220395	0.0347761	0.015421	0.00637375	0.00280172	0.00125873
n	9	10					
ME_n	0.000575373	0.000266226					

inverse linear operator $L_{0.5,0.5}^{-1}$ for the two sub-BVPs in this example.

By the decomposition of the solution $u(x) = \sum_{n=0}^{\infty} u_n(x)$, the Adomian polynomials for the quadratic nonlinearity $Nu = u^2$ are

$$A_0 = {u_0}^2, \; A_n = \sum_{k=0}^{n} u_{n-k} u_k, \; n \geq 1. \tag{44}$$

In order to calculate the recursion integrals analytically, we express the singular coefficient as

$$\frac{1}{\sin(\pi x)} = \sum_{n=0}^{\infty} a_n \left(x - \frac{1}{2} \right)^{2n}, \tag{45}$$

where

$$a_n = \frac{1}{(2n)!} \left[\frac{d^{2n}}{dx^{2n}} \left(\frac{1}{\sin(\pi x)} \right) \right]_{x=1/2}.$$

Hence

$$\frac{u^2(x)}{\sin(\pi x)} = \sum_{n=0}^{\infty} B_n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k \left(x - \frac{1}{2} \right)^{2k} A_{n-k}, \tag{46}$$

where

$$B_n = \sum_{k=0}^{n} a_k \left(x - \frac{1}{2} \right)^{2k} A_{n-k}. \tag{47}$$

In the subinterval $[0, 0.5]$, from Equation (20) we obtain the equivalent nonlinear Fredholm-Volterra integral equation as

$$\begin{aligned} u(x) &= 2\eta x + \pi^3(1 - 2x) \left[L_{0.5,0.5}^{-1} \frac{u^2(x)}{\sin(\pi x)} \right]_{x=0} \\ &\quad - \pi^3 L_{0.5,0.5}^{-1} \frac{u^2(x)}{\sin(\pi x)}. \end{aligned} \tag{48}$$

The solution components on the subinterval $[0, 0.5]$ are calculated to be

$$\begin{aligned} u_0 &= 2\eta x, \\ u_1 &= \frac{1}{24}\pi^3 x \eta^2 - \frac{1}{3}\pi^3 x^4 \eta^2, \\ u_2 &= \frac{1}{10}\pi^5 x^5 \eta^2 - \frac{1}{15}\pi^5 x^6 \eta^2 + \frac{2}{63}\pi^6 x^7 \eta^3 \\ &\quad + x^4 \left(-\frac{1}{24}\pi^5 \eta^2 - \frac{\pi^6 \eta^3}{72} \right) \\ &\quad + x \left(\frac{\pi^5 \eta^2}{960} + \frac{5\pi^6 \eta^3}{4032} \right), \dots. \end{aligned}$$

In the subinterval $[0.5, 1]$, from Equation (29) we obtain the equivalent nonlinear Fredholm-Volterra integral equation as

$$\begin{aligned} u(x) &= 2\eta(1 - x) + \pi^3(2x - 1) \left[L_{0.5,0.5}^{-1} \frac{u^2(x)}{\sin(\pi x)} \right]_{x=1} \\ &\quad - \pi^3 L_{0.5,0.5}^{-1} \frac{u^2(x)}{\sin(\pi x)}. \end{aligned} \tag{49}$$

The solution components on the subinterval $[0.5, 1]$ are calculated to be

$$\begin{aligned} u_0 &= 2\eta(1 - x), \\ u_1 &= -\frac{7}{24}\pi^3 \eta^2 + \frac{31}{24}\pi^3 x \eta^2 - 2\pi^3 x^2 \eta^2 + \frac{4}{3}\pi^3 x^3 \eta^2 \\ &\quad - \frac{1}{3}\pi^3 x^4 \eta^2, \\ u_2 &= -\frac{7}{960}\pi^5 \eta^2 + \frac{11\pi^6 \eta^3}{576} - \frac{2}{63}\pi^6 x^7 \eta^3 \\ &\quad + x^3 \left(\frac{\pi^5 \eta^2}{2} - \frac{19\pi^6 \eta^3}{18} \right) + x^5 \left(\frac{3\pi^5 \eta^2}{10} - \frac{2\pi^6 \eta^3}{3} \right) \\ &\quad + x \left(\frac{21\pi^5 \eta^2}{320} - \frac{677\pi^6 \eta^3}{4032} \right) + x^6 \left(-\frac{1}{15}\pi^5 \eta^2 \right. \\ &\quad \left. + \frac{2\pi^6 \eta^3}{9} \right) + x^2 \left(-\frac{1}{4}\pi^5 \eta^2 + \frac{7\pi^6 \eta^3}{12} \right) \\ &\quad + x^4 \left(-\frac{13}{24}\pi^5 \eta^2 + \frac{79\pi^6 \eta^3}{72} \right), \dots. \end{aligned}$$

Using a similar technique as used in Example 1, we obtain the sequence of values of $\eta[\varphi_n]$ for $n = 2, 3, \ldots$. In Table 3, we list the values of $\eta[\varphi_n]$ for $n = 2, 3, \ldots, 10$. We note that the true value of η is $0.31831\ldots$.

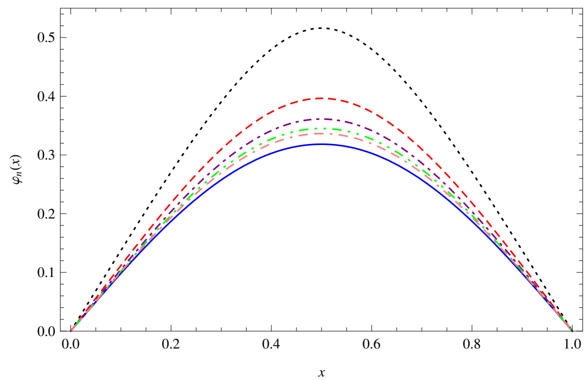

Figure 4: The exact solution $u^*(x)$ (solid line) and the matched nth-stage approximations $\varphi_n(x)$ for $n = 2$ (dot line), $n = 3$ (dashed line), $n = 4$ (dot-side line), $n = 5$ (dot-dot-side line), and $n = 6$ (dot-side-side line).

The curves of the exact solution $u^*(x)$ and the matched nth-stage approximations $\varphi_n(x)$ for $n = 2, 3, 4, 5, 6$ are shown in Figure 4.

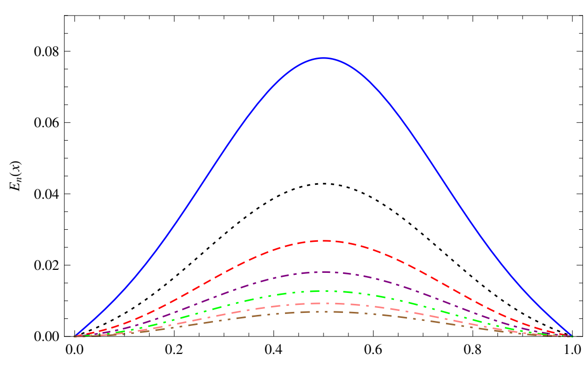

Figure 5: Curves of the error functions $E_n(x)$ for $n = 3$ (solid line), $n = 4$ (dot line), $n = 5$ (dashed line), $n = 6$ (dot-side line), $n = 7$ (dot-dot-side line), $n = 8$ (dot-side-side line), and $n = 9$ (dot-dot-side-side line) .

We have also computed the error functions $E_n(x) = \varphi_n(x) - u^*(x)$ and the maximal error parameters $ME_n = \max_{0 \le x \le 1} |\varphi_n(x) - u^*(x)|$. The curves of the error functions $E_n(x)$ for $n = 3$ through 9 are plotted in Figure 5, which displays a rapid rate of convergence on the domain of validity. The values of the ME_n for $n = 2, 3, \ldots, 10$ are listed in Table 4. The logarithmic plots of the values of ME_4 through ME_{10} are displayed in Figure 6, which

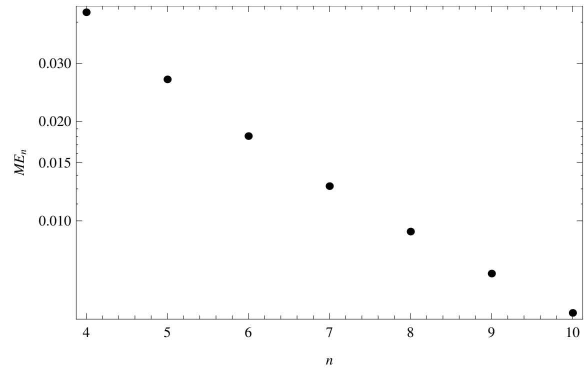

Figure 6: Logarithmic plots of the ME_n for $n = 4$ through 10.

demonstrates an approximately exponential rate of convergence for the obtained decomposition series.

We note that Examples 1 and 2 are taken from reference [2], where Wazwaz has shown that the method of undetermined coefficients in the ADM combined with the Padé approximants technique can also be used to obtain the positive solutions of homogeneous nonlinear BVPs.

Example 3. Consider the nonlinear BVP for the unforced pendulum equation featuring a sinusoidal nonlinearity and subject to homogeneous Dirichlet boundary conditions

$$u''(x) + 4\sin(u(x)) = 0, \ 0 \le x \le 2, \tag{50}$$

$$u(0) = 0, \ u(2) = 0. \tag{51}$$

By the decomposition of the solution $u(x) = \sum_{n=0}^{\infty} u_n(x)$, the Adomian polynomials for the sinusoidal nonlinearity $Nu = \sin(u(x))$ are

$$\begin{aligned} A_0 &= \sin(u_0), \\ A_1 &= u_1 \cos(u_0), \\ A_2 &= u_2 \cos(u_0) - \frac{1}{2} u_1^2 \sin(u_0), \\ A_3 &= \left(u_3 - \frac{1}{6} u_1^3\right) \cos(u_0) - u_1 u_2 \sin(u_0), \end{aligned}$$

$$\begin{aligned} A_4 &= \left(u_4 - \frac{1}{2} {u_1}^2 u_2\right) \cos(u_0) \\ &\quad + \left(\frac{1}{24} {u_1}^4 - u_3 u_1 - \frac{1}{2} {u_2}^2\right) \sin(u_0), \\ A_5 &= \left(u_5 + \frac{1}{120} {u_1}^5 - \frac{1}{2} u_3 {u_1}^2 - \frac{1}{2} {u_2}^2 u_1\right) \cos(u_0) \\ &\quad + \left(\frac{1}{6} u_2 {u_1}^3 - u_4 u_1 - u_2 u_3\right) \sin(u_0), \cdots, \end{aligned}$$

which were first published by Adomian and Rach [60] in 1983.

We partition the interval $[0, 2]$ into two subintervals $[0, 1]$ and $[1, 2]$ and assume that $u(1) = \eta$, which is a positive undetermined constant.

Table 3: The values of $\eta[\varphi_n]$ for $n = 2, 3 \ldots, 10$.

n	2	3	4	5	6	7
$\eta[\varphi_n]$	$16/\pi^3$	0.396413	0.361162	0.345142	0.336376	0.331051
n	8	9	10			
$\eta[\varphi_n]$	0.32759	0.325234	0.323572			

Table 4: The maximal error parameter ME_n.

n	2	3	4	5	6	7
ME_n	0.197715	0.0781033	0.0428517	0.0268319	0.0180662	0.0127409
n	8	9	10			
ME_n	0.0092806	0.00692397	0.00526245			

On the subinterval [0, 1], the solution components are

$$\begin{aligned} u_0 &= \eta x, \\ u_1 &= \frac{4(\sin(\eta x) - x\sin(\eta))}{\eta^2}, \\ u_2 &= \frac{2}{\eta^5}\Big(-16x\sin^2(\eta) + 16\sin(\eta)\sin(\eta x) + \eta\sin(2\eta x) \\ &\quad +6\eta x\sin(\eta)\cos(\eta) - 8\eta x\sin(\eta)\cos(\eta x)\Big), \ \ldots. \end{aligned}$$

On the subinterval [1, 2], the solution components are

$$\begin{aligned} u_0 &= \eta(2-x), \\ u_1 &= \frac{4(x-2)\sin(\eta) - 4\sin(\eta(x-2))}{\eta^2}, \\ u_2 &= \frac{1}{\eta^5}2(6\eta\sin(2\eta) - 3\eta x\sin(2\eta) - 4\eta x\sin(\eta(x-3)) \\ &- 4\eta x\sin(\eta - \eta x) + 8\eta\sin(\eta(x-3)) \\ &+ 8\eta\sin(\eta - \eta x) + \eta\sin(4\eta - 2\eta x) - 8(x-2)\cos(2\eta) \\ &- 8\cos(\eta(x-3)) + 8\cos(\eta - \eta x) + 8x - 16), \ \ldots. \end{aligned}$$

By solving the matching equation for the flux at the interior point $x = 1$ and using a similar technique as used in Example 1, we obtain the values $\eta[\varphi_n]$ in Table 5.

Table 5: The values of the $\eta[\varphi_n]$ for $n = 2, 3, 4, 5$.

n	2	3	4	5
$\eta[\varphi_n]$	1.66080	1.91500	1.89184	1.86686

We note that the solution of the nonlinear BVP (50) and (51) is identical to the solution of the following nonlinear IVP over the interval [0, 2]

$$u''(x) + 4\sin u(x) = 0, \tag{52}$$

$$u(0) = 0, \ u'(0) = \beta, \tag{53}$$

where the positive number β satisfies $K(\beta^2/16) = 2$, which means that the half period of the solution of the IVP (52) and (53) is 2, and where K is the complete elliptic integral of the first kind,

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}. \tag{54}$$

The value of β is 3.20962606955…. The exact solution of both this IVP (52) and (53) and the BVP (50) and (51) can be expressed as

$$u^*(x) = 2\arcsin\left[\frac{\beta}{4}\,\mathrm{sn}\left(2x, \left(\frac{\beta}{4}\right)^2\right)\right], \tag{55}$$

where $\mathrm{sn}(z, k^2)$ is the Jacobi elliptic function [61]. The true value of η is $u^*(1) = 1.86263\ldots$.

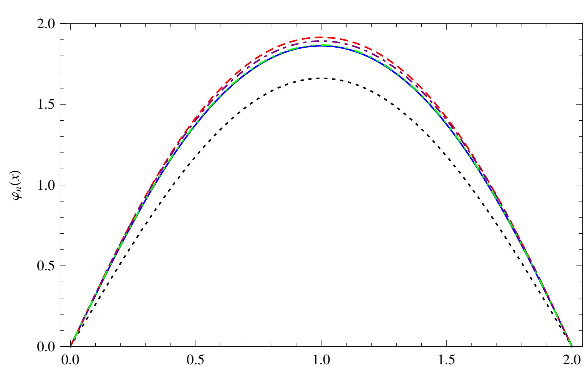

Figure 7: Curves of $u^*(x)$ (solid line) and the matched nth-stage approximations $\varphi_n(x)$ for $n = 2$ (dot line), $n = 3$ (dashed line), $n = 4$ (dot-side line), and $n = 5$ (dot-dot-side line).

We take $\beta = 3.20962607$ and plot the curves of the exact solution $u^*(x)$ and the matched nth-stage approximations $\varphi_n(x)$ for $n = 2, 3, 4, 5$ in Figure 7, where the curves of $u^*(x)$ and $\varphi_5(x)$ overlap. The curves of the error func-

Table 6: The maximal error parameter ME_n.

n	2	3	4	5
ME_n	0.206445	0.0527716	0.0300126	0.00500742

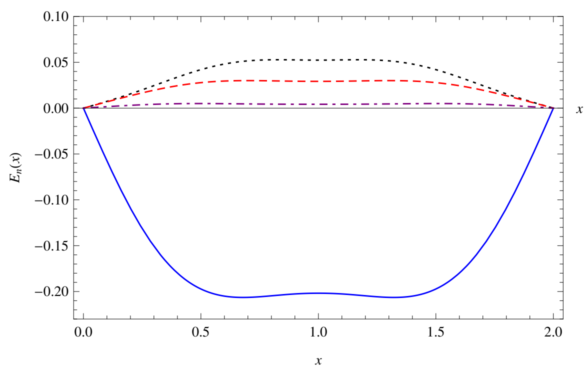

Figure 8: Curves of the error functions $E_n(x)$ for $n = 2$ (solid line), $n = 3$ (dot line), $n = 4$ (dashed line), and $n = 5$ (dot-side line).

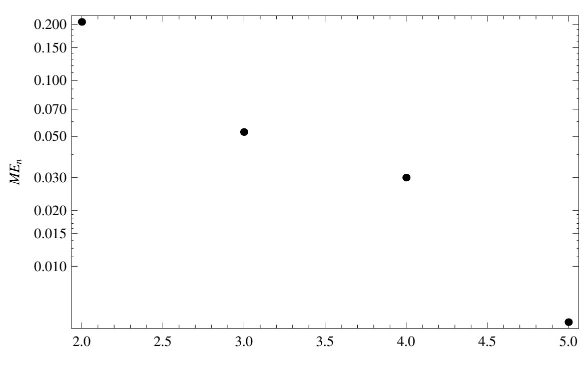

Figure 9: Logarithmic plots of the values of the maximal error parameters ME_n for $n = 2, 3, 4, 5$.

tions $E_n(x) = \varphi_n(x) - u^*(x)$ for $n = 2, 3, 4, 5$ are plotted in Figure 8, which displays a rapid rate of convergence on the domain of validity. The maximal error parameters $ME_n = \max_{0\le x\le 2} |\varphi_n(x) - u^*(x)|$ for $n = 2, 3, 4, 5$ are listed in Table 6, and the logarithmic plots of the values are displayed in Figure 9, where the values of the ME_n decrease in an approximately exponential rate.

Example 4. Consider the nonlinear BVP for the unforced van der Pol equation featuring a product nonlinearity and subject to homogeneous Dirichlet boundary conditions

$$u''(x) + u'(x) + u(x) + u^2(x)u'(x) = 0,$$
$$0 \le x \le 4, \tag{56}$$
$$u(0) = 0,\ u(4) = 0. \tag{57}$$

By the decomposition of the solution $u(x) = \sum_{n=0}^{\infty} u_n(x)$, the first six Adomian polynomials for the product nonlinearity $Nu = u^2(x)u'(x)$ are [17, 62]

$$\begin{aligned}
A_0 &= u_0^2 u_0', \\
A_1 &= 2u_0u_1u_0' + u_0{}^2u_1', \\
A_2 &= u_1{}^2u_0' + 2u_0u_2u_0' + 2u_0u_1u_1' + u_0{}^2u_2', \\
A_3 &= 2u_1u_2u_0' + 2u_0u_3u_0' + u_1{}^2u_1' + 2u_0u_2u_1' \\
&\quad +2u_0u_1u_2' + u_0{}^2u_3',
\end{aligned}$$

$$\begin{aligned}
A_4 &= u_2{}^2u_0' + 2u_1u_3u_0' + 2u_0u_4u_0' + 2u_1u_2u_1' \\
&\quad +2u_0u_3u_1' + u_1{}^2u_2' + 2u_0u_2u_2' + 2u_0u_1u_3' \\
&\quad +u_0{}^2u_4', \\
A_5 &= 2u_2u_3u_0' + 2u_1u_4u_0' + 2u_0u_5u_0' + u_2{}^2u_1' \\
&\quad +2u_1u_3u_1' + 2u_0u_4u_1' \\
&\quad +2u_1u_2u_2' + 2u_0u_3u_2' + u_1{}^2u_3' + 2u_0u_2u_3' \\
&\quad +2u_0u_1u_4' + u_0{}^2u_5',
\end{aligned}$$

which were first published by Adomian in 1994 [17]. Generally, we have

$$A_n = \sum_{m=0}^{n} u_{n-m} \sum_{k=0}^{m} u_{m-k}u_k'. \tag{58}$$

First, we partition the entire interval [0, 4] into two equal-length subintervals [0, 2] and [2, 4]. Let $u(2) = \eta$, which is a positive undetermined constant.

On the subinterval [0, 2], the solution components are calculated by the modified recursion scheme

$$\begin{aligned}
u_0 &= \frac{\eta x}{2}, \\
u_{n+1} &= \frac{x}{2}\left[L_{0,0}^{-1}(u_n' + u_n + A_n)\right]_{x=2} - L_{0,0}^{-1}(u_n' + u_n + A_n), \\
&\quad n \ge 0.
\end{aligned}$$

On the subinterval [2, 4], the solution components are calculated by the modified recursion scheme

$$\begin{aligned}
u_0 &= \frac{\eta(4-x)}{2}, \\
u_{n+1} &= \frac{x-2}{2}\left[L_{2,2}^{-1}(u_n' + u_n + A_n)\right]_{x=4} \\
&\quad -L_{2,2}^{-1}(u_n' + u_n + A_n),\ n \ge 0.
\end{aligned}$$

Solving the matching equation for the continuity of the flux at the interior point $x = 2$, we find that for $n = 2$, the matching equation gives $\eta = 0$, and for $n = 3$ the matching equation becomes $9\eta^3\left(28 + 5\eta^2\right) = 448\eta$, which has one

Table 7: The values of the $\eta[\varphi_n]$ for $n = 3, 4, 5, 6, 7$.

n	3	4	5	6	7
$\eta[\varphi_n]$	1.19100	0.879861	0.747938	0.694222	0.675191

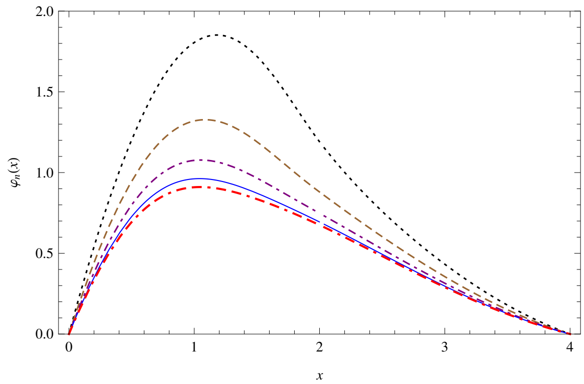

Figure 10: Curves of the matched nth-stage approximations $\varphi_n(x)$ for $n = 3$ (dot line), $n = 4$ (dashed line), $n = 5$ (dot-side line), $n = 6$ (solid line), and $n = 7$ (dot-side-side line).

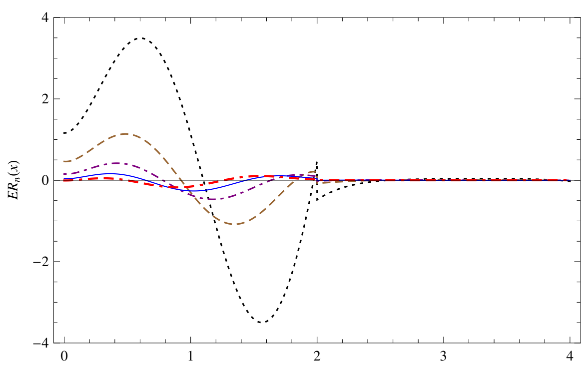

Figure 11: Curves of the error remainder functions $ER_n(x)$ for $n = 3$ (dot line), $n = 4$ (dashed line), $n = 5$ (dot-side line), $n = 6$ (solid line), and $n = 7$ (dot-side-side line).

positive root $\eta[\varphi_3] = 1.19100\ldots$. Similarly, we obtain the subsequent values of $\eta[\varphi_n]$ in Table 7.

The curves of the matched nth-stage approximations $\varphi_n(x)$ for $n = 3, 4, 5, 6, 7$ are plotted in Figure 10, which presents a skewed bell shape in appearance.

Since the exact solution is unknown a priori, we instead investigate the error remainder functions

$$ER_n(x) = \frac{d^2}{dx^2}\varphi_n(x) + \frac{d}{dx}\varphi_n(x) + \varphi_n(x) + (\varphi_n(x))^2\frac{d}{dx}\varphi_n(x), \quad 0 \le x \le 4,\ x \ne 2, \tag{59}$$

and the maximal error remainder parameters

$$MER_n = \sup_{0\le x\le 4,\ x\ne 2} |ER_n(x)|. \tag{60}$$

In this case, we have used the supremum operator sup(·) instead of the maximum operator max(·) in our formula for the MER_n, since the second-order derivative may fail to exist possibly at the interior endpoints of subintervals for the matched approximations $\varphi_n(x)$.

In Appendix D, we present an overview of error analysis formulas tailored to analytic approximate solutions by the ADM, including when the exact solution is unknown in advance.

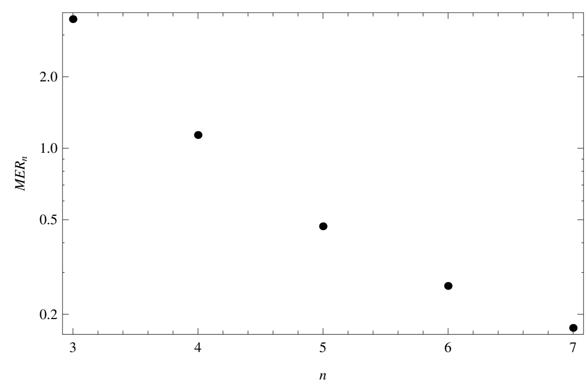

Figure 12: Logarithmic plots of MER_n for $n = 3, 4, 5, 6, 7$.

The curves of the error remainder functions $ER_n(x)$ for $n = 3, 4, 5, 6, 7$ are plotted in Figure 11, which displays a rapid rate of convergence on the domain of validity. The maximal error remainder parameters MER_n for $n = 3, 4, 5, 6, 7$ are listed in Table 8. The logarithmic plots of these values are displayed in Figure 12.

We have also checked the 3-stage and 4-stage matched ADM solutions with the three equal-length subintervals $[0, 4/3]$, $[4/3, 8/3]$, and $[8/3, 4]$ and four equal-length subintervals $[0, 1]$, $[1, 2]$, $[2, 3]$ and $[3, 4]$, respectively. Comparison of their maximal error remainder parameters are displayed in Figure 13, which suggests that we can increase the rate of convergence by increasing the number of stages in the multistage ADM for BVPs for equal-length subintervals.

Next, we partition the interval $[0, 4]$ into three unequal-length subintervals $[0, 1]$, $[1, 2]$ and $[2, 4]$. In order to compare the rate of convergence of the 3-stage ADM solution with equal-length subintervals with the 3-stage ADM solution with unequal-length subintervals, we display the two sets of logarithmic plots of their respective

Table 8: The maximal error remainder parameter MER_n for $n = 3, 4, 5, 6, 7$.

n	3	4	5	6	7
MER_n	3.49215	1.13727	0.46914	0.263193	0.175182

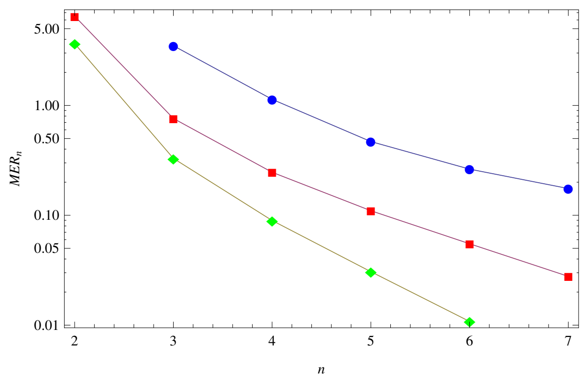

Figure 13: Comparison of the logarithmic plots of MER_n with equal-length subintervals: circle dot for the 2-stage ADM, square dot for the 3-stage ADM, and diamond dot for 4-stage ADM.

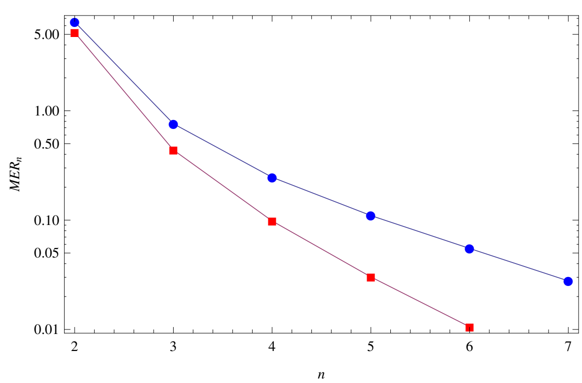

Figure 14: Comparison of the logarithmic plots of MER_n: circle dot for the 3-stage ADM solution with equal-length subintervals and square dot for the 3-stage ADM solution with unequal-length subintervals.

MER_n data sets in Figure 14. The set of logarithmic plots of the MER_n data for the 3-stage decomposition spline with unequal-length subintervals possesses a steeper negative slope than the set of logarithmic plots of the MER_n data for the 3-stage decomposition spline with equal-length subintervals, which demonstrates a more rapid rate of convergence for the 3-stage ADM solution series for unequal-length subintevals than the 3-stage ADM solution series for equal-length subintevals. Note that we have judiciously adjusted the lengths of the subintervals for the unequal-length 3-stage ADM decomposition series to favor the area of the maximum rate of change in the solution, i.e. about the maximum, in order to increase the rate of convergence above that for the equal-length 3-stage ADM decomposition series.

4 Conclusion

We have presented a new approach to systematically compute positive solutions of homogeneous nonlinear BVPs by the multistage ADM for BVPs. We first partition the original interval into two or more subintervals, and next derive the equivalent nonlinear Fredholm-Volterra integral equation for the solution within each subinterval, and the appropriate modified recursion scheme for calculation of the solution components. Next we match each subinterval's approximate solution by applying the continuity condition of the flux at the interior point, or interior points as required. Finally we combine the matched sub-solutions using the boxcar function. The proposed approach yields an easily computable, readily verifiable and rapidly convergent sequence of analytic solution approximations.

In the introduction, we have reviewed the concept of the ADM and the Adomian polynomials for convenient solution of nonlinear differential equations. We have described the proposed approach in Section 2 featuring second-order homogeneous nonlinear BVPs using the multistage ADM for BVPs. We have demonstrated the practicality and efficiency of the approach for computation of positive solutions of homogeneous nonlinear BVPs in Section 3 by four numerical examples for a variety of nonlinearities, including a quadratic nonlinearity in the first-order derivative, a quadratic nonlinearity in the solution with a singular variable system coefficient, a sinusoidal nonlinearity and a product nonlinearity in the first-order derivative and the square of the solution.

We have also emphasized the rapid convergence of our decomposition series solutions by the sequence of the curves of the error functions, or error remainder functions as required, and the logarithmic plots of the maximal error parameters, or maximal error remainder parameters as required. Each of our computed positive solutions for homogeneous nonlinear BVPs exhibits an approximately exponential rate of convergence usually after the first few approximations.

Furthermore, we have demonstrated two novel techniques to accelerate the rate of convergence of our decomposition series solutions by increasing the number of subintervals and adjusting the lengths of subintervals in the multistage ADM for BVPs; specifically we solved Example 4, which has a larger domain than previous examples, using two, three and four equal-length subintervals, and then resolved it using three unequal-length subintervals.

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Appendix A. MATHEMATICA code for the one-variable Adomian polynomials based on Duan's Corollary 3 algorithm [55]

```
AP2[f_, M_] := Module[{c, n, k, j, der},
Table[c[n, k], {n, 1, M}, {k, 1, n}];
der = Table[D[f[Subscript[u, 0]], {Subscript[u, 0], k}], {k, 1, M}];
A[0] = f[Subscript[u, 0]];
For[n = 1, n <= M, n++, c[n, 1] = Subscript[u, n];
For[k = 2, k <= n, k++,
c[n, k] = Expand[1/n* Sum[(j + 1)*Subscript[u, j+1]* c[n-1-j, k-1],
{j, 0, n-k}]] ];
A[n] = Take[der, n].Table[c[n, k], {k, 1, n}] ];
Table[A[n], {n, 0, M}] ]
```

Appendix B. MATHEMATICA code for the multi-variable Adomian polynomials based on Duan's Corollary 1 algorithm [54]

```
APmulti[f_, m_, M_] := Module[{i, j, r},
Subscript[u, 0] = Table[Subscript[u, i, 0], {i, 1, m}];
A[0] = f@@Subscript[u, 0]; Table[T[i, j], {i, 1, M}, {j, 1, i}];
se = Table[_, {m}] /. List -> Sequence;
For[r = 1, r <= M, r++,
T[r, 1]=Table[Subscript[u, i, r]*D[f@@Subscript[u, 0],
Subscript[u, i, 0]], {i, 1, m}];
For[k = 2, k <= r, k++,
T[r, k]=Union[Flatten[Table[D[Map[#*Subscript[u, i, 1]/
(Exponent[#, Subscript[u, i, 1]]+1)&,
T[r-1, k-1]], Subscript[u, i, 0]], {i, 1, m}]]]];
For[k = 2, k <= Floor[r/2], k++,
T[r, k] = T[r, k] \[Union](T[r - k, k] /.
Flatten[Table[Subscript[u, i, j] -> Subscript[u, i, j+1], {i,1,m},
{j, 1, r-2*k+1}]]])];
A[r] = Sum[Total[T[r, k] ], {k, 1, r}];
If[EvenQ[r], Do[T[r/2, k] =., {k, 1, r/2}]] ];
Table[A[r], {r, 0, M}]]
```

Appendix C. The first six two-variable Adomian polynomials for the differential nonlinearity $Nu = f(u(x), u'(x))$

$$
\begin{aligned}
A_0 &= f(u_0, u_0'),\\
A_1 &= u_1' f^{(0,1)}(u_0, u_0') + u_1 f^{(1,0)}(u_0, u_0'),\\
A_2 &= u_2' f^{(0,1)}(u_0, u_0') + \frac{1}{2} u_1'^2 f^{(0,2)}(u_0, u_0') + u_2 f^{(1,0)}(u_0, u_0') + u_1 u_1' f^{(1,1)}(u_0, u_0') + \frac{1}{2} u_1^2 f^{(2,0)}(u_0, u_0'),\\
A_3 &= u_3' f^{(0,1)}(u_0, u_0') + u_1' u_2' f^{(0,2)}(u_0, u_0') + \frac{1}{6} u_1'^3 f^{(0,3)}(u_0, u_0') + u_3 f^{(1,0)}(u_0, u_0') + \left(u_2 u_1' + u_1 u_2'\right) f^{(1,1)}(u_0, u_0')\\
&\quad + \frac{1}{2} u_1 u_1'^2 f^{(1,2)}(u_0, u_0') + u_1 u_2 f^{(2,0)}(u_0, u_0') + \frac{1}{2} u_1^2 u_1' f^{(2,1)}(u_0, u_0') + \frac{1}{6} u_1^3 f^{(3,0)}(u_0, u_0'),\\
A_4 &= u_4' f^{(0,1)}(u_0, u_0') + \left(\frac{1}{2} u_2'^2 + u_1' u_3'\right) f^{(0,2)}(u_0, u_0') + \frac{1}{2} u_1'^2 u_2' f^{(0,3)}(u_0, u_0') + \frac{1}{24} u_1'^4 f^{(0,4)}(u_0, u_0')\\
&\quad + u_4 f^{(1,0)}(u_0, u_0') + \left(u_3 u_1' + u_2 u_2' + u_1 u_3'\right) f^{(1,1)}(u_0, u_0') + \left(\frac{1}{2} u_2 u_1'^2 + u_1 u_1' u_2'\right) f^{(1,2)}(u_0, u_0')\\
&\quad + \frac{1}{6} u_1 u_1'^3 f^{(1,3)}(u_0, u_0') + \left(\frac{1}{2} u_2^2 + u_1 u_3\right) f^{(2,0)}(u_0, u_0') + \left(u_1 u_2 u_1' + \frac{1}{2} u_1^2 u_2'\right) f^{(2,1)}(u_0, u_0')\\
&\quad + \frac{1}{4} u_1^2 u_1'^2 f^{(2,2)}(u_0, u_0') + \frac{1}{2} u_1^2 u_2 f^{(3,0)}(u_0, u_0') + \frac{1}{6} u_1^3 u_1' f^{(3,1)}(u_0, u_0') + \frac{1}{24} u_1^4 f^{(4,0)}(u_0, u_0'),\\
A_5 &= u_5' f^{(0,1)}(u_0, u_0') + \left(u_2' u_3' + u_1' u_4'\right) f^{(0,2)}(u_0, u_0') + \left(\frac{1}{2} u_1' u_2'^2 + \frac{1}{2} u_1'^2 u_3'\right) f^{(0,3)}(u_0, u_0')\\
&\quad + \frac{1}{6} u_1'^3 u_2' f^{(0,4)}(u_0, u_0') + \frac{1}{120} u_1'^5 f^{(0,5)}(u_0, u_0') + u_5 f^{(1,0)}(u_0, u_0') + \Big(u_4 u_1' + u_3 u_2'\\
&\quad + u_2 u_3' + u_1 u_4'\Big) f^{(1,1)}(u_0, u_0') + \left(\frac{1}{2} u_3 u_1'^2 + u_2 u_1' u_2' + \frac{1}{2} u_1 u_2'^2 + u_1 u_1' u_3'\right) f^{(1,2)}(u_0, u_0')\\
&\quad + \left(\frac{1}{6} u_2 u_1'^3 + \frac{1}{2} u_1 u_1'^2 u_2'\right) f^{(1,3)}(u_0, u_0') + \frac{1}{24} u_1 u_1'^4 f^{(1,4)}(u_0, u_0') + (u_2 u_3 + u_1 u_4) f^{(2,0)}(u_0, u_0')\\
&\quad + \left(\frac{1}{2} u_2^2 u_1' + u_1 u_3 u_1' + u_1 u_2 u_2' + \frac{1}{2} u_1^2 u_3'\right) f^{(2,1)}(u_0, u_0') + \left(\frac{1}{2} u_1 u_2 u_1'^2 + \frac{1}{2} u_1^2 u_1' u_2'\right) f^{(2,2)}(u_0, u_0')\\
&\quad + \frac{1}{12} u_1^2 u_1'^3 f^{(2,3)}(u_0, u_0') + \left(\frac{1}{2} u_1 u_2^2 + \frac{1}{2} u_1^2 u_3\right) f^{(3,0)}(u_0, u_0') + \left(\frac{1}{2} u_1^2 u_2 u_1' + \frac{1}{6} u_1^3 u_2'\right) f^{(3,1)}(u_0, u_0')\\
&\quad + \frac{1}{12} u_1^3 u_1'^2 f^{(3,2)}(u_0, u_0') + \frac{1}{6} u_1^3 u_2 f^{(4,0)}(u_0, u_0') + \frac{1}{24} u_1^4 u_1' f^{(4,1)}(u_0, u_0') + \frac{1}{120} u_1^5 f^{(5,0)}(u_0, u_0'),
\end{aligned}
$$

where $f^{(p,q)}(u_0, u_0') = \frac{\partial^{p+q} f(u,v)}{\partial u^p \partial v^q}|_{u=u_0, v=u_0'}$, and $u_n' = \frac{du_n}{dx}$.

Appendix D. Error analysis formulas for approximate solutions by the ADM

We suppose that $u^*(x)$ is the exact solution, which is known a priori, of the nonlinear differential equation

$$Lu(x) + Ru(x) + Nu(x) = g(x), \ a \le x \le b, \qquad (D.61)$$

where $Lu(x) = u''(x)$, $Ru(x) = \alpha_1 u'(x) + \alpha_0 u(x)$, and $Nu(x) = f(u(x), u'(x))$.

In the ADM, the matched analytic approximate solution, or approximate decomposition spline, is denoted by $\varphi_n(x)$. We first list the usual error formulas as follows.

The error function:

$$E_n(x) = \varphi_n(x) - u^*(x). \qquad (D.62)$$

The absolute error function:

$$|E_n(x)| = |\varphi_n(x) - u^*(x)|. \qquad (D.63)$$

The maximal error parameter:

$$ME_n = \max_{a \le x \le b} |\varphi_n(x) - u^*(x)|. \qquad (D.64)$$

However when the exact solution is unknown in advance, which is most often the case for nonlinear engineering model equations, we instead compute the following error remainder formulas for the Adomian approximate solutions.

The error remainder function:

$$ER_n(x) = L\varphi_n(x) + R\varphi_n(x) + N\varphi_n(x) - g(x), \; x \neq x_i. \quad \text{(D.65)}$$

The absolute error remainder function:

$$|ER_n(x)| = |L\varphi_n(x) + R\varphi_n(x) + N\varphi_n(x) - g(x)|, \; x \neq x_i. \quad \text{(D.66)}$$

The maximal error remainder parameter:

$$MER_n = \sup_{a \le x \le b,\; x \neq x_i} \left| L\varphi_n(x) + R\varphi_n(x) + N\varphi_n(x) - g(x) \right|. \quad \text{(D.67)}$$

Whenever $\varphi_n(x)$ is matched on two or more subintervals, x does not assume the values x_i at the interior endpoints of the subintervals, or interpolation nodes, in formulas (D.65), (D.66) and (D.67), since the highest order derivative of $\varphi_n(x)$ may fail to exist possibly at such points in the sequence of the approximate decomposition splines.

In the limit as $n \to \infty$, for convergence, we have $\lim_{n\to\infty} \varphi_n(x) = u^*(x)$, where the exact solution $u^*(x)$ identically satisfies the original nonlinear differential equation and all specified auxiliary conditions, including initial conditions for IVPs, boundary condition equations and conditions of continuity for all interior points for BVPs. Therefore, for convergence, we have

$$\lim_{n\to\infty} ME_n = 0; \; \lim_{n\to\infty} MER_n = 0.$$

Thus the ADM intrinsically provides the applied scientist and engineer with a valuable capacity to refine their model equations whenever the computed solution differs from laboratory measurements allowing for the experimental margins of error. This advantage is unavailable with other solution methodologies that require restrictive assumptions or otherwise alter the model equation for the sake of mathematical tractability over preference to physical fidelity, including linearization or perturbation except in so-called linear regimes, where the nonlinearity becomes irrelevant anyway. Thus modeling and application of the solution procedure should be used interactively in the design process as ably recommended by Adomian on Pages 1 through 5 in [17].

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