Research Article

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Optimal Auxiliary Functions Method for viscous flow due to a stretching surface with partial slip

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Abstract: The viscous flow induced by a stretching surface with partial slip is investigated. The flow is governed by a third-order nonlinear differential equation, corresponding to the planar and axisymmetric stretching. The derived equation of motion is solved analytically by Optimal Auxiliary Functions Method (OAFM). This procedure is highly efficient and it controls the convergence of the approximate solution. A few examples are given, showing the exceptionally good agreement between the analytical and numerical solutions of the nonlinear problem. OAFM is very efficient in practice, ensuring a very rapid convergence after only one iteration.

Keywords: Optimal auxiliary functions method, viscous flow, partial slip, stretching surface.

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1 Introduction

The study of boundary layer viscous flow due to a stretching surface is very important because of its several engineering applications such as: crystal growing, drawing of elastic films, the heat treated materials traveling between a feed roll and the wind-up roll or a conveyor belt poses the features of a moving continuous surface. There exist situations where there may be a partial slip between the fluid and the boundary. In the last few decades, nonlinear prob-

lems are widely used as models to describe complex physical phenomena in various scientific field of science. The boundary layer flow over a continuously stretching surface was first studied by Sakiadis [1, 2]. Crane [3] found a closed form exponential solution for the planar viscous flow of a linear stretching case. Sparrow *et al.* [4, 5] studied a number of flow problems taking the velocity slip into account. In [6] is given an algorithm for the moving boundaries and for the flow of a fluid past a rotating disk. Rajagopal *et al.* [7] obtained numerical solution for the flow of viscoelastic second order fluid past a stretching sheet. Ariel [8] and Anderson [9] have reported the analytical closed form solutions of the second grade fluid and of the fourth order nonlinear differential equations arising due to the MHD flow. Wang [10] studied axisymmetric cases for the viscous flow. Mirogolbabei *et al.* [11] considered a number of boundary layer flows induced by the axisymmetric stretching of a sheet. The unsteady axisymmetric flow and heat transfer of a viscous fluid is treated by Sajid *et al.* [12]. Also, the analytical solution for these problems has been considered by Sajid *et al.* in [13], Miklavčič and Wang [14] first analyzed properties of the flow to a shrinking sheet with suction. Many other researchers denoted themselves on investigating these problems [15–19].

In general in fluid mechanics of viscous fluids, by means of similarity transformations, the set of partial differential equations are reduced to that of ordinary differential equations. Much attention of the researchers was focused in obtaining the analytical solutions for the nonlinear problems. But, in general, the nonlinear problems cannot be solved analytically by means of traditional methods. For the weakly nonlinear problems, many methods exist for approximating the solutions. The most widely applied method by researches is perturbation technique [20] which is based on the existence of small parameters. Others methods for approximating the solutions of nonlinear problems are the modified Lindstedt-Poincare method [21], the Adomian decomposition method [22], the parameter-expansion method [23], the optimal homotopy perturbation method [24–26], the optimal homotopy asymptotic method [27–29].

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Recently, several methods has been used for solving different nonlinear differential equations such as: traveling-wave transformation method [30], Cole-Hopf transformation method [31].

The objective of the present paper is to propose a new and accurate approach to nonlinear differential equation of viscous fluid due to a stretching surface with partial slip, using an analytical technique, namely the optimal auxiliary functions method. The motivation of our approach is to present a new procedure that is effective, explicit and very accurate for nonlinear approximations rapidly convergent to the exact solution using only one iteration. This procedure must provide a rigorous way to control and adjust the convergence of the approximate solutions using some convergence-control parameters. The proposed method is easy, concise and can be applied to other nonlinear problems. The validity of our procedure, which does not imply the presence of a small or large parameter in the equation or into the boundary / initial conditions, is based on the construction and determination of the auxiliary functions, combined with a convenient way to optimally control the convergence of the solutions. The efficiency of the proposed procedure is proven when an accurate solution is explicitly analytically obtained in an iterative way after only one iteration. Our procedure can be applied to a variety of engineering domains as [32–40]. The validity of this method is demonstrated by comparing the results obtained with the numerical solutions.

2 The governing equations

Consider the three dimensional flow of a viscous fluid bounded by a stretching surface. If (u, v, w) is the velocity components in the Cartesian directions (x, y, z) respectively, then the continuity and steady constant property Navier-Stokes equations for viscous fluid flow are [7], [10, 13]:

$$u_x + v_y + w_z = 0 \quad (1)$$

$$uu_x + vu_y + wu_z = -\frac{p_x}{\rho} + \nu \nabla^2 u \quad (2)$$

$$uv_x + vv_y + wv_z = -\frac{p_y}{\rho} + \nu \nabla^2 v \quad (3)$$

$$uw_x + vw_y + ww_z = -\frac{p_z}{\rho} + \nu \nabla^2 w \quad (4)$$

where p is the pressure, ρ is the density and ν is the kinematic viscosity. If a is the stretching constant, $a > 0$, W is

the suction velocity and m is a parameter describing the type of stretching, then the velocity components on the stretching surface are:

$$u = ax, \quad v = a(m-1)y, \quad w = -W. \quad (5)$$

It is known that for $m = 1$, we have planar stretching case, while for $m = 2$ we have axisymmetric stretching case. In order to simplify the governing equations we use the similarity transform [10]:

$$u = ax\Phi'(\eta), \quad v = a(m-1)y\Phi'(\eta), \quad (6)$$

$$w = -m\sqrt{av}\Phi(\eta) \quad \text{and} \quad \eta = z\sqrt{\frac{a}{\nu}},$$

where prime denotes the differentiation with respect to η . Eq. (1) is identically satisfied and Eq. (4) becomes

$$p = \nu\rho w_z - \frac{1}{2}\rho W^2 + C, \quad (7)$$

with C a constant. From the Eqs. (2) and (3) we deduce that:

$$\Phi'''(\eta) + m\Phi(\eta)\Phi''(\eta) - \Phi'(\eta)^2 = 0, \quad (8)$$

as there is no lateral pressure gradient at infinity. If N denotes a slip constant, then on the surface of the stretching sheet, the velocity slip is assumed to be proportional to the local shear stress [41]:

$$u - ax = \rho\nu Nu_z < 0, \quad v - a(m-1)y = \rho\nu Nv_z < 0. \quad (9)$$

From the similarity transform (6) we obtain

$$\Phi'(0) = 1 + \lambda\Phi''(0), \quad (10)$$

with $\lambda = \rho N\sqrt{av} > 0$ a non-dimensional parameter indicating the relative importance of partial slip. If $\lambda = 0$ there is no slip. Given a suction velocity of $-W$ on the stretching surface, we have the boundary condition

$$\Phi(0) = \alpha, \quad (11)$$

where $\alpha = \frac{W}{m\sqrt{av}}$ is a non-dimensional constant which determines the transpiration rate at the surface and $\alpha < 0$ if injection from the surface occurs, $\alpha > 0$ for suction and $\alpha = 0$ for an impermeable sheet. Also, since there is no lateral velocity at infinity, we have yet the condition

$$\Phi'(\infty) = 0. \quad (12)$$

3 Basic ideas of optimal auxiliary functions method

Eq. (8) with the initial / boundary conditions (10), (11) and (12) can be written in a more general form as [42, 43]:

$$L[\Phi(\eta)] + g(\eta) + N[\Phi(\eta)] = 0, \quad (13)$$

where L is a linear operator, g is a known function and N is a given nonlinear operator, η denotes independent variable and $\Phi(\eta)$ is an unknown function. The initial / boundary conditions are

$$B\left(\Phi(\eta), \frac{d\Phi(\eta)}{d\eta}\right) = 0. \quad (14)$$

An exact solution for strongly nonlinear equation (13) and with initial / boundary conditions (14) are frequently scarce. To find an approximate analytic solution of Eqs. (13) and (14), we assume that the approximate solution can be written in the form with only two components:

$$\bar{\Phi}(\eta) = \Phi_0(\eta) + \Phi_1(\eta, C_i), \quad i = 1, 2, \dots, s, \quad (15)$$

where the initial approximation $\Phi_0(\eta)$ and the first approximation $\Phi_1(\eta, C_i)$ will be determined as follows. Substituting Eq. (15) into Eq. (13), it results in:

$$\begin{aligned} L[\Phi_0(\eta)] + L[\Phi_1(\eta, C_i)] + g(\eta) \\ + N[\Phi_0(\eta) + \Phi_1(\eta, C_i)] = 0. \end{aligned} \quad (16)$$

The initial approximation $\Phi_0(\eta)$ can be determined from the linear equation

$$L[\Phi_0(\eta)] + g(\eta) = 0, \quad B\left(\Phi_0(\eta), \frac{d\Phi_0(\eta)}{d\eta}\right) = 0 \quad (17)$$

and the first approximation $\Phi_1(\eta, C_i)$, from the following equation

$$\begin{aligned} L[\Phi_1(\eta, C_i)] + N[\Phi_0(\eta) + \Phi_1(\eta, C_i)] = 0, \\ B\left(\Phi_1(\eta, C_i), \frac{d\Phi_1(\eta, C_i)}{d\eta}\right) = 0. \end{aligned} \quad (18)$$

Now, the nonlinear term from Eq. (18) is expanded in the form

$$\begin{aligned} N[\Phi_0(\eta) + \Phi_1(\eta, C_i)] \\ = N[\Phi_0(\eta)] + \sum_{k=1}^{\infty} \frac{\Phi_1^k(\eta, C_i)}{k!} N^{(k)}[\Phi_0(\eta)]. \end{aligned} \quad (19)$$

In order to avoid the difficulties that appears in solving of the nonlinear differential equation (18) and to accelerate the rapid convergence of the first approximation $\Phi_1(\eta, C_i)$ and implicit of the approximate solution $\bar{\Phi}(\eta)$, instead of the last term arising into Eq. (18), we propose another expression, such that the nonlinear differential equation (18) can be written as a linear differential equation taking into account the Eq. (19), in the form:

$$\begin{aligned} L[\Phi_1(\eta, C_i)] + A_1[\Phi_0(\eta), C_i]N[\Phi_0(\eta)] \\ + A_2[\Phi_0(\eta), C_j] = 0, \end{aligned} \quad (20)$$

$$B\left(\Phi_1(\eta, C_i), \frac{d\Phi_1(\eta, C_i)}{d\eta}\right) = 0, \quad (21)$$

where A_1 and A_2 are two arbitrary auxiliary functions depending on the initial approximation $\Phi_0(\eta)$ and several unknown parameters C_i and C_j , $i = 1, 2, \dots, p$, $j = p+1, p+2, \dots, s$.

The auxiliary functions A_1 and A_2 (called optimal auxiliary functions) are not unique, and are of the same form like $\Phi_0(\eta)$ or of the form of the $N[\Phi_0(\eta)]$ or combinations of the forms of $\Phi_0(\eta)$ and $N[\Phi_0(\eta)]$.

For example, if $\Phi_0(\eta)$ or $N[\Phi_0(\eta)]$ contain polynomial functions, then $A_1[\Phi_0(\eta), C_i]$ and $A_2[\Phi_0(\eta), C_j]$ are sums of polynomial functions. If $\Phi_0(\eta)$ or $N[\Phi_0(\eta)]$ contain exponential functions, then A_1 and A_2 would be sums of the exponential functions. If $\Phi_0(\eta)$ or $N[\Phi_0(\eta)]$ contain trigonometric functions, then A_1 and A_2 would be sums of the trigonometric functions, and so on. In all these sums, the coefficients of the polynomial, exponential, trigonometric and so on functions, are the parameters $C_1, C_2, \dots, C_s$.

If in a special case $N[\Phi_0(\eta)] = 0$, then it is clear that $\Phi_0(\eta)$ is an exact solution of Eqs. (13) and (14). The unknown parameters C_i and C_j can be optimally identified via different method such as: the Galerkin method, the least square method, the collocation method, the Ritz method, the Kantorovich method or by minimizing the square residual error, using:

$$J(C_i, C_j) = \int_a^b R^2(m, \lambda, \alpha, \eta) d\eta, \quad (22)$$

where

$$\begin{aligned} R(m, \lambda, \alpha, \eta) = L[\bar{\Phi}(\eta, C_i, C_j)] + g(\eta) + \\ + N[\bar{\Phi}(\eta, C_i, C_j)], \\ i = 1, 2, \dots, p, \quad j = p+1, p+2, \dots, s, \end{aligned} \quad (23)$$

a and b are two values depending on the given problem.

The unknown parameters C_i, C_j can be optimally identified from the equations

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \dots = \frac{\partial J}{\partial C_p} = \frac{\partial J}{\partial C_{p+1}} = \dots = \frac{\partial J}{\partial C_s} = 0. \quad (24)$$

With these parameters C_i, C_j known, by this novel approach, the approximate solution (15) is well determined.

Our procedure proves to be a powerful tool for solving nonlinear problems not depending on small or large parameters. It should be emphasize that our method contains the optimal auxiliary functions A_1 and A_2 which pro-

vides us with a simple way to adjust and control the convergence of the approximate solutions after only one iteration. Also, it is remarkable that a nonlinear differential problem is transformed into two linear differential problems.

4 Application of OAFM to viscous fluid given by Eqs. (8), (10), (11) and (12)

We introduce the basic ideas of the OAFM by considering Eq. (8) with the initial / boundary conditions given by Eqs. (10), (11) and (12). We choose the linear operator L by the form [43]:

$$L(\Phi(\eta)) = \Phi''' + K\Phi''. \quad (25)$$

We can choose the linear operator in the form $L(\Phi(\eta)) = \Phi''' + K^2\Phi'$, and so on, where $K > 0$ is an unknown parameter.

Eq. (17) becomes ($g(\eta) = 0$):

$$\begin{aligned} \Phi_0'''(\eta) + K\Phi_0''(\eta) &= 0, \quad \Phi_0(0) = \alpha, \\ \Phi_0'(0) &= 1 + \lambda\Phi_0''(0), \quad \Phi_0'(\infty) = 0, \end{aligned} \quad (26)$$

with the solution

$$\Phi_0(\eta) = \alpha + \frac{1 - e^{-K\eta}}{K(1 + \lambda K)}. \quad (27)$$

The nonlinear operator $N(\Phi(\eta))$ is obtained from Eqs. (8) and (25):

$$N(\Phi(\eta)) = -K\Phi''(\eta) + m\Phi(\eta)\Phi''(\eta) - \Phi'(\eta)^2. \quad (28)$$

By means of the Eqs. (27) and (28) it is obtain

$$N(\Phi_0(\eta)) = \frac{\beta e^{-K\eta} + (m-1)e^{-2K\eta}}{(1 + \lambda K)^2}, \quad (29)$$

where

$$\beta = K(K - m\alpha)(1 + \lambda K) - m. \quad (30)$$

Having in view Eqs. (20), (25) and (29), the first approximation is obtained from the equation:

$$\begin{aligned} \Phi_1''' + K\Phi_1'' + A_2(e^{-K\eta}, e^{-2K\eta}, C_j) \\ + A_1(e^{-K\eta}, e^{-2K\eta}, C_i) \frac{\beta e^{-K\eta} + (m-1)e^{-2K\eta}}{(1 + 2\lambda K)^2} = 0, \end{aligned} \quad (31)$$

with the initial / boundary conditions

$$\Phi_1(0) = 0, \quad \Phi_1'(0) = \lambda\Phi_1''(0), \quad \Phi_1'(\infty) = 0. \quad (32)$$

We have the freedom to choose the optimal auxiliary functions A_1 and A_2 in the following forms:

$$A_1[\Phi_0(\eta), C_i] = -(1 + \lambda K)^2(C_1 + C_2\eta), \quad (33)$$

$$\begin{aligned} A_2[\Phi_0(\eta), C_j] &= -(C_3 + C_4\eta)(e^{-2K\eta} + K_1e^{-3K\eta}) \\ &- (C_5 + C_6\eta)(e^{-3K\eta} + K_2e^{-4K\eta}), \end{aligned} \quad (34)$$

or

$$A_1[\Phi_0(\eta), C_i] = 0, \quad (35)$$

$$\begin{aligned} A_2[\Phi_0(\eta), C_j] &= -(C_1 + C_2\eta)e^{-K\eta} \\ &- (C_3 + C_4\eta + C_5\eta^2)e^{-2K\eta}, \end{aligned} \quad (36)$$

or yet

$$A_1[\Phi_0(\eta), C_i] = -(1 + \lambda K)^2 C_1, \quad (37)$$

$$\begin{aligned} A_2[\Phi_0(\eta), C_j] &= -(C_2 + C_3\eta + C_4\eta^2)e^{-2K\eta} \\ &- (C_5 + C_6\eta + C_7\eta^2)e^{-3K\eta}, \end{aligned} \quad (38)$$

and so on.

If the auxiliary functions A_1 and A_2 are given by Eqs. (33) and (34) then Eq. (20) may be written as:

$$\begin{aligned} \Phi_1''' + K\Phi_1'' &= (\beta C_1 + \beta C_2\eta)e^{-K\eta} \\ &+ \left[(mC_2 - C_2 + C_4)\eta + (m-1)C_1 + C_3 \right] e^{-2K\eta} \\ &+ \left[(K_1C_4 + C_6)\eta + K_1C_3 + C_5 \right] e^{-3K\eta} \\ &+ \left[K_2C_6\eta + K_2C_5 \right] e^{-4K\eta}, \end{aligned} \quad (39)$$

with the initial / boundary conditions given in Eq. (32), whose solution is:

$$\begin{aligned} \Phi_1(\eta, C_i) &= M_1 + \left[\frac{\beta C_2}{2K^2}\eta^2 + \left(\frac{\beta C_1}{K^2} + \frac{2\beta C_2}{K^3} \right) \eta \right. \\ &+ M_2 \left. \right] e^{-K\eta} + \left[\frac{(1-m)C_2 - C_4}{4K^3}\eta + \frac{1-m}{4K^3}C_1 + \frac{1-m}{2K^4}C_2 \right. \\ &- \frac{C_3}{4K^3} - \frac{C_4}{2K^4} \left. \right] e^{-2K\eta} - \left[\frac{K_1C_4 + C_6}{18K^3}\eta + \frac{K_1C_3 + C_5}{18K^3} \right. \\ &+ \frac{7(K_1C_4 + C_6)}{108K^4} \left. \right] e^{-3K\eta} - \left[\frac{K_2}{48K^3}C_6\eta + \frac{K_2}{48K^3}C_5 \right. \\ &+ \frac{5K_2}{288K^4}C_6 \left. \right] e^{-4K\eta}, \end{aligned} \quad (40)$$

where

$$\begin{aligned} M_1 &= \frac{1 - m - 4\beta + \lambda K(3 - 3m - 8\beta)}{4K^3(1 + \lambda K)} C_1 \\ &- \frac{9 + 4K_1 + \lambda K(16K_1 + 27)}{36K^3(1 + \lambda K)} C_3 \end{aligned}$$

$$\begin{aligned}
& + \frac{1-m-8\beta+\lambda K(2-2m-12\beta)}{4K^4(1+\lambda K)} C_2 \\
& - \frac{8K_1+27+\lambda K(2K_1-18)}{108K^4(1+\lambda K)} C_4 \\
& - \frac{9K_2+16+\lambda K(45K_2+64)}{144K^3(1+\lambda K)} C_5 \\
& - \frac{27K_2+112+\lambda K(81K_2+208)}{864K^4(1+\lambda K)} C_6, \\
M_2 = & \frac{(2\beta+m-1)(1+2\lambda K)}{2K^3(1+\lambda K)} C_1 \\
& + \frac{8\beta+3m-3+4\lambda K(m-1+3\beta)}{4K^4(1+\lambda K)} C_2 \\
& + \frac{K_1+3+3\lambda K(K_1+2)}{6K^3(1+\lambda K)} C_3 \\
& + \frac{27+5K_1+9\lambda K(K_1+4)}{36K^4(1+\lambda K)} C_4 \\
& + \frac{2+K_2+2\lambda K(3+2K_2)}{12K^3(1+\lambda K)} C_5 \\
& + \frac{20+7K_2+4\lambda K(9+4K_2)}{144K^4(1+\lambda K)} C_6.
\end{aligned}$$

If we consider $m = 1$ and $\beta = 0$ (the planar stretching case for impermeable sheet) into Eq. (29) then we have $N[\Phi_0(\eta)] = 0$ and therefore we can obtain the exact solution of the equation:

$$\begin{aligned}
\Phi''' + \Phi\Phi'' - \Phi'^2 &= 0, \\
\Phi(0) = 0, \quad \Phi'(0) &= 1 + \lambda\Phi''(0), \quad \Phi'(\infty) = 0,
\end{aligned} \quad (41)$$

which is:

$$\Phi(\eta) = \alpha + \frac{1 - e^{-K\eta}}{K(1 + \lambda K)}, \quad (42)$$

where K is obtained from the equation:

$$\lambda K^3 + (1 - \alpha\lambda)K^2 - \alpha K - 1 = 0. \quad (43)$$

Beyond this remarkable case, the approximate analytic solution of the Eqs. (8), (10), (11) and (12) can be obtained from Eqs. (15), (27) and (40).

5 Numerical examples

We illustrate the accuracy of OAFM by comparing obtained approximate solutions with the numerical integration results obtained by means of a fourth-order Runge-Kutta method in combination with the shooting method. In all cases, the unknown parameters are optimally identified via Galerkin method. For this, we use the following eleven weighted functions f_i given by [41]:

$$f_1(\eta) = \gamma e^{-3K\eta} + \eta^2 e^{-2K\eta} + \delta \eta e^{-K\eta}, \quad (44)$$

$$\begin{aligned}
f_2(\eta) &= \eta e^{-K\eta}, \quad f_3(\eta) = \eta^2 e^{-K\eta}, \\
f_4(\eta) &= 1 + \gamma \eta e^{-K\eta}, \quad f_5(\eta) = e^{-2K\eta}, \\
f_6(\eta) &= e^{-K\eta}, \quad f_7(\eta) = \eta e^{-4K\eta}, \quad f_8(\eta) = e^{-4K\eta}, \\
f_9(\eta) &= \delta \eta e^{-2K\eta} + \eta^3 e^{-2K\eta}, \\
f_{10}(\eta) &= e^{-K\eta} + K_1 e^{-2K\eta} + K_3 \eta e^{-3K\eta}, \\
f_{11}(\eta) &= \eta e^{-K\eta} + K_2 e^{-3K\eta} + K_4 e^{-4K\eta},
\end{aligned}$$

where $\gamma, \delta, K_1, K_2, K_3$ and K_4 are unknown parameters.

The parameters $K, K_1, K_2, K_3, K_4, \gamma, \delta, C_1, \dots, C_6$ are determined from equations (Galerkin method):

$$\begin{aligned}
J_j &= \int_0^\infty R(m, \lambda, \alpha, \eta) f_j(\eta) d\eta = 0, \\
i &= 1, \dots, 6, \quad j = 1, \dots, 11,
\end{aligned} \quad (45)$$

where the residual $R(m, \alpha, \lambda, \eta)$ is given by Eq. (23):

$$R(m, \lambda, \alpha, \eta) = \overline{\Phi}'''(\eta) + m\overline{\Phi}(\eta)\overline{\Phi}''(\eta) - \overline{\Phi}'(\eta)^2 \quad (46)$$

and $\overline{\Phi}(\eta)$ is given by Eq. (15) with the initial / boundary conditions (10)-(12).

Example 5.1. Consider planar stretching case with impermeable sheet, $m = 1, \alpha = 0$ and $\lambda = 1$. In this case, from Eqs. (45) and (46) we obtain

$$\begin{aligned}
K &= 0.4631238249, \quad \gamma = 0.6858214854, \\
\delta &= -0.1052314539, \quad K_1 = 0, \quad K_2 = 0, \\
K_3 &= 2.13 \cdot 10^{-6}, \quad K_4 = 0.21 \cdot 10^{-6}, \\
C_1 &= -0.0119658357, \quad C_2 = 0.0007396024, \\
C_3 &= 0.0642359221, \quad C_4 = 0.0326512044, \\
C_5 &= 0.0530417312, \quad C_6 = 0.0102396495.
\end{aligned}$$

The first-order approximate solution given by Eq. (15) can be written in the form

$$\begin{aligned}
\overline{\Phi}(\eta) &= 0.7549045180 + \left(-0.0011830813\eta^2 \right. \\
&+ 0.0280632559\eta - 0.1942637382 \Big) e^{-0.4631238249\eta} \\
&- \left(0.0821765397\eta + 0.5165462911 \right) e^{-0.9262476499\eta} \\
&- 0.0440894886 e^{-1.3893714748\eta}.
\end{aligned} \quad (47)$$

Example 5.2. In the second case for the same planar stretching case with impermeable sheet $m = 1, \alpha = 0$ but $\lambda = 5$, from Eqs. (45) the values of the parameters are

$$\begin{aligned}
K &= 0.5252370049, \quad \gamma = 4.8796805607, \\
\delta &= -1.2890550305, \quad K_1 = 0, \quad K_2 = 0,
\end{aligned}$$

Table 1: Values of $\Phi''(0)$ and $\Phi(\infty)$ in the case of planar stretching case with impermeable sheet $m = 1$, $\alpha = 0$

λ	1	5	10
$\Phi''_{\text{numeric}}(0)$	-0.4301597090	-0.1448401942	-0.0812419798
$\Phi''_{\text{OAFM}}(0)$	-0.4301595579	-0.1448401939	-0.0812419808
$\Phi_{\text{numeric}}(\infty)$	0.7548776662	0.5251657154	0.4331053011
$\Phi_{\text{OAFM}}(\infty)$	0.7549045179	0.5251657158	0.4331053009

$$\begin{aligned}
 K_3 &= 1.24 \cdot 10^{-6}, \quad K_4 = 1.02 \cdot 10^{-6}, \\
 C_1 &= -0.0279185716, \quad C_2 = -1.9918945423 \cdot 10^{-6}, \\
 C_3 &= 6.2493581461 \cdot 10^{-12}, \\
 C_4 &= 3.90211197551 \cdot 10^{-12}, \\
 C_5 &= -9.621041312 \cdot 10^{-12}, \quad C_6 = 0.
 \end{aligned}$$

Approximate solution (15) becomes:

$$\begin{aligned}
 \bar{\Phi}(\eta) &= 0.5251657155 \\
 &+ \left(-1.3352013705 \cdot 10^{-9} \eta^2 - 0.0000374387 \eta \right. \\
 &\quad \left. - 0.5251657155 \right) e^{-0.5252370049 \eta} \\
 &+ \left(6.7324778915 \cdot 10^{-12} \eta + 1.4853702952 \cdot 10^{-11} \right) \\
 &\times e^{-1.0504740099 \eta} + \left(9.1284670601 \cdot 10^{-13} \eta \right. \\
 &\quad \left. + 5.7165228355 \cdot 10^{-12} \right) e^{-1.5757110148 \eta}.
 \end{aligned} \quad (48)$$

Example 5.3. In the third case for the planar stretching case with impermeable sheet $m = 1$, $\alpha = 0$ and with sleep parameter $\lambda = 10$, from Eqs. (45) we obtain

$$\begin{aligned}
 K &= 0.4331821853, \quad \gamma = 3.0639340595, \\
 \delta &= -0.5722383481, \quad K_1 = 0, \quad K_2 = 0, \\
 K_3 &= 1.03 \cdot 10^{-6}, \quad K_4 = 2.11 \cdot 10^{-6}, \\
 C_1 &= -0.0125089730, \quad C_2 = -9.6275057188 \cdot 10^{-7}, \\
 C_3 &= 6.6010897191 \cdot 10^{-12}, \\
 C_4 &= -3.360346796 \cdot 10^{-12}, \\
 C_5 &= -1.0095931161 \cdot 10^{-11}, \\
 C_6 &= -2.0561054881 \cdot 10^{-12}.
 \end{aligned}$$

The first-order approximate solution (15) is

$$\begin{aligned}
 \bar{\Phi}(\eta) &= 0.4331053011 \\
 &+ \left(-1.2809679965 \cdot 10^{-9} \eta^2 - 0.0000332989 \eta \right. \\
 &\quad \left. - 0.4331053011 \right) e^{-0.433182185 \eta} \\
 &+ \left(1.0336892501 \cdot 10^{-11} \eta + 2.7423124117 \cdot 10^{-11} \right) \\
 &\times e^{-0.8663643706 \eta} + \left(+0.932245039 \cdot 10^{-12} \eta \right. \\
 &\quad \left. + 1.065716162 \cdot 10^{-11} \right) e^{-1.2995465559 \eta}.
 \end{aligned} \quad (49)$$

The values of $\Phi''(0)$ and $\Phi(\infty)$ are given in Table 1 for the same planar stretching case with impermeable sheet, calculated by means OAFM, and by numerical integration.

Example 5.4. In this case, we consider planar stretching case but suction sheet, $m = 1$, $\alpha = 3$ and with slip parameter $\lambda = 1$. The parameters obtained by means of Eqs. (45) are:

$$\begin{aligned}
 K &= 1.0265261151, \quad \gamma = 3.011278740, \\
 \delta &= -0.9721507438, \quad K_1 = 0, \quad K_2 = 0, \\
 K_3 &= 1.002 \cdot 10^{-6}, \quad K_4 = 0.52 \cdot 10^{-6}, \\
 C_1 &= 2.6781920761 \cdot 10^{-11}, \\
 C_2 &= -4.2525109956 \cdot 10^{-12}, \\
 C_3 &= -4.655928821 \cdot 10^{-9}, \\
 C_4 &= 1.6964107211 \cdot 10^{-9}, \\
 C_5 &= -6.9684341491 \cdot 10^{-10}, \\
 C_6 &= 0.0001204043
 \end{aligned}$$

and the first-order approximate solution (15) becomes

$$\begin{aligned}
 \bar{\Phi}(\eta) &= 3.0795956234 \\
 &+ \left(-2.3525169361 \cdot 10^{-12} \eta^2 + 2.0134806302 \cdot 10^{-11} \eta \right. \\
 &\quad \left. - 4.9211079655 \cdot 10^{-11} \right) \times e^{-1.0265261151 \eta} \\
 &+ \left(8.7180902556 \cdot 10^{-11} \eta \right. \\
 &\quad \left. - 6.9432001142 \cdot 10^{-11} \right) e^{-2.0530522302 \eta} \\
 &+ \left(1.3753249143 \cdot 10^{-6} \eta \right. \\
 &\quad \left. - 0.0795955933 \right) e^{-3.0795783454 \eta}.
 \end{aligned} \quad (50)$$

Example 5.5. For planar stretching case and suction sheet, $m = 1$, $\alpha = 3$ and slip parameter $\lambda = 5$, we have

$$\begin{aligned}
 K &= 1.0016112565, \quad \gamma = 2.9079023283, \\
 \delta &= -0.9038557714, \quad K_1 = 0, \quad K_2 = 0, \\
 K_3 &= 0.55 \cdot 10^{-6}, \quad K_4 = 0.62 \cdot 10^{-6}, \\
 C_1 &= 1.3658235079 \cdot 10^{-6}, \\
 C_2 &= -2.1653508962 \cdot 10^{-7},
 \end{aligned}$$

Table 2: Values of $\Phi''(0)$ and $\Phi(\infty)$ in the case of planar stretching case with suction sheet $m = 1, \alpha = 3$

λ	1	5	10
$\Phi''_{numeric}(0)$	-0.7548776662	-0.1875797976	-0.0967852665
$\Phi''_{OAFM}(0)$	-0.7548776659	-0.1875797939	-0.0967852020
$\Phi_{numeric}(\infty)$	3.0795956235	3.0205594824	3.0106792212
$\Phi_{OAFM}(\infty)$	3.0795956233	3.0205594411	3.0106777729

$$C_3 = -0.0005801616156, \quad C_4 = 0.0002087511, \\ C_5 = -0.9806268698, \quad C_6 = 0.0158334825$$

and therefore, the first-order approximate solution (15) has the form

$$\begin{aligned} \bar{\Phi}(\eta) = & 3.0205594824 \\ & + \left(-5.4255801019 \cdot 10^{-7} \eta^2 + 4.6777709040 \cdot 10^{-6} \eta \right. \\ & - 0.0000118225 \Big) e^{-1.0016112565\eta} + \left(0.0000200454\eta \right. \\ & - 0.0000156837 \Big) e^{-2.0032225131\eta} + \left(0.0003400695\eta \right. \\ & \left. - 0.0205318961 \right) e^{-3.0048337696\eta}. \end{aligned} \quad (51)$$

Example 5.6. The planar stretching case and suction sheet, $m = 1, \alpha = 3$ but slip parameter $\lambda = 10$ we obtain from Eqs. (45):

$$\begin{aligned} K &= 1.1395297367, \quad \gamma = 2.9650196892, \\ \delta &= -0.9588570845, \quad K_1 = 0, \quad K_2 = 0, \\ K_3 &= 1.002 \cdot 10^{-6}, \quad K_4 = 1.32 \cdot 10^{-6}, \\ C_1 &= 0.0000646091, \quad C_2 = -0.0000113762, \\ C_3 &= -0.0863263871, \quad C_4 = 0.02944192315, \\ C_5 &= -0.2391591505, \quad C_6 = -0.06102791273 \end{aligned}$$

and the first-order approximate solution (15) can be written as

$$\begin{aligned} \bar{\Phi}(\eta) = & 3.0106792212 + \left(-0.0000661253\eta^2 \right. \\ & + 0.0005189796\eta - 0.0011722736 \Big) e^{-1.1395297367\eta} \\ & + \left(0.0027526798\eta - 0.0032398505 \right) e^{-2.2790594735\eta} \\ & + \left(-0.0012678992\eta - 0.0062671170 \right) e^{-3.4185892103\eta}. \end{aligned} \quad (52)$$

In Table 2, we present the values of $\Phi''(0)$ and $\Phi(\infty)$ in the case of planar stretching case with suction sheet.

Example 5.7. Now, we consider the axisymmetric flow with impermeable sheet, $m = 2, \alpha = 0$ and with slip parameter $\lambda = 1$. From Eqs. (45) we obtain the following set of parameters

$$K = 1.2284861855, \quad \gamma = -1.2107679860,$$

$$\begin{aligned} \delta &= 5.9462560050, \quad K_1 = K_2 = 0.7335772195, \\ K_3 &= 1.11 \cdot 10^{-6}, \quad K_4 = 1.06 \cdot 10^{-6}, \\ C_1 &= -0.0272717347, \quad C_2 = -0.0098884303, \\ C_3 &= -0.0130978383, \quad C_4 = -0.1950919615, \\ C_5 &= -0.06294893217, \quad C_6 = 0.0018821549. \end{aligned}$$

In this case, the first-order approximate solution (15) is

$$\begin{aligned} \bar{\Phi}(\eta) = & 0.5509446955 + \left(-0.0077420177\eta^2 \right. \\ & - 0.0679124177\eta - 0.6421304231 \Big) e^{-1.2284861855\eta} \\ & + \left(0.0469289624\eta + 0.0831404603 \right) e^{-2.4569723711\eta} \\ & + \left(0.0041898146\eta + 0.0075368933 \right) e^{-3.6854585567\eta} \\ & + \left(0.0000154947\eta + 0.0005083699 \right) e^{-4.9139447423\eta}. \end{aligned} \quad (53)$$

Example 5.8. For the case of the axisymmetric flow with impermeable sheet, $m = 2, \alpha = 0$ and with slip parameter $\lambda = 5$, from Eqs. (45) we have

$$\begin{aligned} K &= 0.8408529662, \quad \gamma = -2.5936764722, \\ \delta &= 12.6879512489, \quad K_1 = K_2 = 0.5953831575, \\ K_3 &= 2.03 \cdot 10^{-6}, \quad K_4 = 1.01 \cdot 10^{-6}, \\ C_1 &= -0.0053548861, \quad C_2 = -0.0012448162, \\ C_3 &= -0.00821412956, \quad C_4 = -0.03031382709, \\ C_5 &= -0.01091079627, \quad C_6 = 0.00176755942 \end{aligned}$$

and therefore, the first-order approximate solution (15) becomes

$$\begin{aligned} \bar{\Phi}(\eta) = & 0.3780173693 + \left(-0.0023588678\eta^2 \right. \\ & - 0.0315158198\eta - 0.4396463038 \Big) e^{-0.8408529662\eta} \\ & + \left(0.0208604015\eta + 0.0573797361 \right) e^{-1.6817059324\eta} \\ & + \left(0.001423055078\eta + 0.0040581045 \right) e^{-2.5225588986\eta} \\ & + \left(-0.0000368811\eta + 0.0001911237 \right) e^{-3.3634118648\eta}. \end{aligned} \quad (54)$$

Example 5.9. In this case we consider the axisymmetric flow with impermeable sheet, $m = 2, \alpha = 0$ and with slip

Table 3: Values of $\Phi''(0)$ and $\Phi(\infty)$ in the case of stretching flow with impermeable sheet $m = 2$, $\alpha = 0$

λ	1	5	10
$\Phi''_{numeric}(0)$	-0.4625096500	-0.1493933452	-0.0829116367
$\Phi''_{OAFM}(0)$	-0.4625096441	-0.1493933439	-0.0829118109
$\Phi_{numeric}(\infty)$	0.5509502913	0.37801840311	0.31065126857
$\Phi_{OAFM}(\infty)$	0.5509446952	0.3780173713	0.31063735490

parameter $\lambda = 10$. From Eqs. (45) yields

$$\begin{aligned} K &= 0.5239724222, \quad \gamma = 1.6648535124, \\ \delta &= -1.8794890289, \quad K_1 = K_2 = -3.4855129921, \\ K_3 &= 0.43 \cdot 10^{-6}, \quad K_4 = 0.31 \cdot 10^{-6}, \\ C_1 &= 0.0112870692, \quad C_2 = -0.0004699006, \\ C_3 &= 0.0366309932, \quad C_4 = -0.0059152635, \\ C_5 &= -0.0310033957, \quad C_6 = -0.00815982977. \end{aligned}$$

The first-order approximate solution can be written in the form

$$\begin{aligned} \bar{\Phi}(\eta) &= 0.3106373548 + \left(-0.0006102510\eta^2 \right. \\ &+ 0.0246579539\eta - 0.3170175340 \left. \right) e^{-0.5239724222\eta} \\ &+ \left(-0.0247341994\eta + 0.0442009306 \right) e^{-1.0479448445\eta} \\ &+ \left(-0.0157948648\eta - 0.0156201635 \right) e^{-1.5719172668\eta} \\ &+ \left(-0.0041191012\eta - 0.0222006138 \right) e^{-2.095889689\eta}. \end{aligned} \quad (55)$$

In Table 3 we present the values of $\Phi''(0)$ and $\Phi(\infty)$ for the stretching flow with impermeable sheet.

Example 5.10. For the axisymmetric flow with suction sheet $m = 2$, $\alpha = 3$ and slip parameter $\lambda = 1$ we obtain the following values of the parameters:

$$\begin{aligned} K &= 1.5574274146, \quad \gamma = -0.7860056759, \\ \delta &= 3.6690514687, \quad K_1 = K_2 = -0.0507748232, \\ K_3 &= 1.003 \cdot 10^{-6}, \quad K_4 = 0.32 \cdot 10^{-6}, \\ C_1 &= 0.0107133300, \quad C_2 = -0.0024557002, \\ C_3 &= -9.2340545611, \quad C_4 = 4.93780472001, \\ C_5 &= -2.72171778619, \quad C_6 = 23.18498897742928, \end{aligned}$$

with the first-order approximate solution obtained from (15):

$$\begin{aligned} \bar{\Phi}(\eta) &= 3.0235364880 + \left(0.0026339287\eta^2 \right. \\ &+ 0.0162169332\eta - 0.0279371463 \left. \right) e^{-1.5574274146\eta} \\ &+ \left(0.0864947892\eta - 0.0510827083 \right) e^{-3.1148548292\eta} \end{aligned} \quad (56)$$

$$\begin{aligned} &+ \left(0.0937682076\eta + 0.0527720186 \right) e^{-4.6722822438\eta} \\ &+ \left(0.0064922112\eta + 0.0027118480 \right) e^{-6.2297096585\eta}. \end{aligned}$$

Example 5.11. For the case of the axisymmetric flow with suction sheet $m = 2$, $\alpha = 3$ and slip parameter $\lambda = 5$, the parameters are

$$\begin{aligned} K &= 1.7780169491, \quad \gamma = -0.5957717503, \\ \delta &= 2.8222930255, \quad K_1 = K_2 = -0.0131160498, \\ K_3 &= 1.013 \cdot 10^{-6}, \quad K_4 = 1.22 \cdot 10^{-6}, \\ C_1 &= 0.0000862400, \quad C_2 = -0.0000222658, \\ C_3 &= -0.3512786202, \quad C_4 = 0.1994379874, \\ C_5 &= -1.6374819712, \quad C_6 = 1.6614059805 \end{aligned}$$

and therefore, the first-order approximate solution (15) has the form

$$\begin{aligned} \bar{\Phi}(\eta) &= 3.0053612258 + \left(-0.0001065840\eta^2 \right. \\ &+ 0.0005858601\eta - 0.0008994531 \left. \right) e^{-1.7780169491\eta} \\ &+ \left(0.0035222493\eta - 0.0022439786 \right) e^{-3.55603389829\eta} \\ &+ \left(0.0065444365\eta - 0.0021759449 \right) e^{-5.3340508474\eta} \\ &+ \left(0.0000797662\eta - 0.0000397491 \right) e^{-7.1120677965\eta}. \end{aligned} \quad (57)$$

Example 5.12. In the last case we consider the axisymmetric flow with suction sheet $m = 2$, $\alpha = 3$ and slip parameter $\lambda = 10$, such that the parameters are

$$\begin{aligned} K &= 1.8412120648, \quad \gamma = -0.554362126, \\ \delta &= 2.6330245292, \quad K_1 = K_2 = -0.0066382243, \\ K_3 &= 1.004 \cdot 10^{-6}, \quad K_4 = 0.52 \cdot 10^{-6}, \\ C_1 &= 9.5831454966 \cdot 10^{-6}, \\ C_2 &= -2.5525898638 \cdot 10^{-6}, \\ C_3 &= -0.0821658733, \quad C_4 = 0.04718459612, \\ C_5 &= -0.8534171847, \quad C_6 = 0.5288479009. \end{aligned}$$

The first-order approximate solution (15) can be written as

$$\bar{\Phi}(\eta) = 3.0027282856 + \left(-0.0000243991\eta^2 \right. \quad (58)$$

Table 4: Values of $\Phi''(0)$ and $\Phi(\infty)$ in the case of stretching flow with suction sheet $m = 2$, $\alpha = 3$

λ	1	5	10
$\Phi''_{numeric}(0)$	-0.8578597370	-0.1935567447	0.0163824516
$\Phi''_{OAFM}(0)$	-0.8578597339	-0.1935567450	0.0163824517
$\Phi_{numeric}(\infty)$	3.0235285642	3.005309966	3.0027282376
$\Phi_{OAFM}(\infty)$	3.0235359880	3.0053592258	3.0027282799

$$\begin{aligned}
& + 0.0001301957\eta - 0.0001939768) e^{-1.8412120648\eta} \\
& + (0.0008131418\eta - 0.0005329181) e^{-3.6824241297\eta} \\
& + (0.0020278091\eta - 0.0019878153) e^{-5.5236361946\eta} \\
& + (0.0000116973\eta - 0.0000135853) e^{-7.3648482595\eta}.
\end{aligned}$$

In Table 4 are presented the values of $\Phi''(0)$ and $\Phi(\infty)$ for stretching flow with suction sheet.

From Tables 1-4 we deduced that there exist an excellent agreement between the numerical results and the results obtained by means of OAFM.

On the other hand, considering the effect of slip parameter on the velocity $\bar{\Phi}(\eta)$ in both flows, Figures 1-4 have been displayed. Figure 1 shows the variation of $\bar{\Phi}$ for planar flow and impermeable sheet. In Figure 2 the variation of $\bar{\Phi}$ for planar flow and suction sheet is plotted.

Figure 3 shows the variation of $\bar{\Phi}$ for axisymmetric flow and impermeable sheet. Figure 4 shows the variation of $\bar{\Phi}$ for axisymmetric flow and suction sheet. It is clear that the velocity components decrease with an increase in the slip parameter for all cases.

In Figures 5-7 the planar cases for every value of slip parameter λ have been plotted and in Figures 8-10 the

Figure 1: Variation of $\bar{\Phi}$ by increasing the slip parameter λ for planar flow and impermeable sheet ($m = 1$, $\alpha = 0$):
— numerical solution; OAFM solution

Figure 3: Variation of $\bar{\Phi}$ by increasing the slip parameter λ for axisymmetric flow and impermeable sheet ($m = 2$, $\alpha = 0$):
— numerical solution; OAFM solution

Figure 2: Variation of $\bar{\Phi}$ by increasing the slip parameter λ for planar flow and suction sheet ($m = 1$, $\alpha = 3$):
— numerical solution; OAFM solution

Figure 4: Variation of $\bar{\Phi}$ by increasing the slip parameter λ for axisymmetric flow and suction sheet ($m = 2$, $\alpha = 3$):
— numerical solution; OAFM solution

Figure 5: Variation of $\bar{\Phi}$ by increasing the coefficient α for planar flow ($m = 1$, $\lambda = 1$):
— numerical solution; OAFM solution

Figure 8: Variation of $\bar{\Phi}$ by increasing the coefficient α for axisymmetric flow ($m = 2$, $\lambda = 1$):
— numerical solution; OAFM solution

Figure 6: Variation of $\bar{\Phi}$ by increasing the coefficient α for planar flow ($m = 1$, $\lambda = 5$):
— numerical solution; OAFM solution

Figure 9: Variation of $\bar{\Phi}$ by increasing the coefficient α for axisymmetric flow ($m = 2$, $\lambda = 5$):
— numerical solution; OAFM solution

Figure 7: Variation of $\bar{\Phi}$ by increasing the coefficient α for planar flow ($m = 1$, $\lambda = 10$):
— numerical solution; OAFM solution

Figure 10: Variation of $\bar{\Phi}$ by increasing the coefficient α for axisymmetric flow ($m = 2$, $\lambda = 10$):
— numerical solution; OAFM solution

Figure 11: The residual $R(1, 1, 0, \eta)$ for Eq. (47) obtained by OAFM

Figure 15: The residual $R(1, 5, 3, \eta)$ for Eq. (51) obtained by OAFM

Figure 12: The residual $R(1, 5, 0, \eta)$ for Eq. (48) obtained by OAFM

Figure 16: The residual $R(1, 10, 3, \eta)$ for Eq. (52) obtained by OAFM

Figure 13: The residual $R(1, 10, 0, \eta)$ for Eq. (49) obtained by OAFM

Figure 17: The residual $R(2, 1, 0, \eta)$ for Eq. (53) obtained by OAFM

Figure 14: The residual $R(1, 1, 3, \eta)$ for Eq. (50) obtained by OAFM

Figure 18: The residual $R(2, 5, 0, \eta)$ for Eq. (54) obtained by OAFM

Figure 19: The residual $R(2, 10, 0, \eta)$ for Eq. (55) obtained by OAFM

Figure 20: The residual $R(2, 1, 3, \eta)$ for Eq. (56) obtained by OAFM

Figure 21: The residual $R(2, 5, 3, \eta)$ for Eq. (57) obtained by OAFM

Figure 22: The residual $R(2, 10, 3, \eta)$ for Eq. (58) obtained by OAFM

stretching cases for different values of λ have been plotted. It is evident that the velocity is less for the axisymmetric flow when compared with the planar case.

Finally, the residual functions obtained for the approximate analytic solutions given by Eqs. (47)-(58) are plotted in Figures 11-22.

6 Conclusions

In this work, the Optimal Auxiliary Functions Method (OAFM) is employed to propose an analytic approximate solution to the boundary nonlinear problem of the viscous flow induced by a stretching surface with partial slip. Our procedure is valid even if the nonlinear equations of motion do not contain any small or large parameters. The proposed approach is mainly based on a new construction of the solutions and especially on the auxiliary function. These auxiliary functions lead to an excellent agreement of the solutions with numerical results. This technique is very effective, explicit and accurate for non-linear approximations rapidly converging to the exact solution after only one iteration. Also OAFM provides a simple but rigorous way to control and adjust the convergence of the solution by means of some convergence-control parameters. Our construction is different from other approaches especially referring to the linear operator L depending of an optimal parameter K and to the auxiliary convergence-control function A_1 and A_2 which ensure a fast convergence of the solutions. In a special case OAFM lead to choice of the exact solution. Numerical results are obtained using the shooting method in combination with the fourth-order Runge-Kutta method, using Wolfram Mathematica 9.0 software.

Further important contributions in the area of this paper can be: OAFM allows us to obtain a very good results for both cases: planar and axisymmetric flow for any value of the slip parameter. We point out, also the influence of the coefficient α in the case of impermeable or suction sheet. The values $\bar{\Phi}''(0)$ and $\bar{\Phi}(\infty)$ are given for both flow situations, and found in excellent agreement with numerical results.

The proposed method is straightforward, effective, concise and can be applied to other different nonlinear problems.

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