Research Article

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Development and refinement of the Variational Method based on Polynomial Solutions of Schrödinger Equation

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Abstract: The variational method is known as a powerful and preferred technique to find both analytical and numerical solutions for numerous forms of anharmonic oscillator potentials. In the present study, we considered certain conditions for the choice of the trial wave function. The current form of the trial wave function is based on the possible polynomial solutions of the Schrödinger equation. The advantage of our modified variational method is its ability to reduce the calculation steps and hence computation time. Also, we compared the results provided by our modified method with the results obtained by different methods in general but particularly Numerov method for the same problem.

Keywords: Variational Methods; Anharmonic Oscillator; Energy Eigenvalues; Polynomial Solutions; Schrödinger Equation

1 Introduction

The precise solution of the Schrodinger equation is possible only in few cases such as infinite square well and harmonic oscillator potentials. However, the complete spectra of the anharmonic oscillators are not fully solved yet. In most cases, the conventional approximate methods discussed in most standard textbooks are either unsatisfactory or computationally complicated. Several techniques of approximations have been used over the years to determine the spectral energies of various anharmonic oscillators. These approximations lead to developing many models for the study of many problems in physics which are te-

dious computationally. However, we observed some very simple and effective models in literature for the same purpose [1].

Here, we present some insight from available literature about variational principle together with appropriate approximations for the electron-electron interactions which are the basis for most practical approaches to solving the Schrödinger equation in condensed matter physics.

For the generalized anharmonic oscillator in D dimensions, Popescu *et al.* [2, 3] used approximation method along with variational method for the calculation of the ground energy state and first even-parity excited state of a single-well and found improved results. The energy levels of one dimensional quartic anharmonic oscillator were obtained by using neural network system [4], however, the analytical solutions were given by the triconfluent Heun functions [5]. Later on, Popescu *et al.* [6] considered a different form of the successive variational method based on a solution of a differential equation. They successfully combined the variational method which uses variational global parameter with the finite element method for the study of the generalized anharmonic oscillator in D dimensions [7]. Further, Cooper *et al.* [8] in another work used a newly suggested algorithm of Gozzi, Reuter and, Thacker to determine the excited states of one-dimensional systems. They determined approximated eigenvalues and eigenfunctions of the anharmonic oscillator. While Karl & Novikov [9] calculated the energies of excited states for two- and three-particle systems with arbitrary blocking potential within the framework of a simple variational approach. In another work, Mei, W. N. [10] used variational method and analytical wave functions which have extremely accurate expectation values for the quartic or sextic oscillators. The Variational Method was also applied within the context of Super-symmetric Quantum Mechanics [11–13] to provide information to Morse and Hulthén potentials for several diatomic molecules and the results were in agreement with established results.

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Borges *et al.* [14] suggested a method for constructing trial eigenfunctions for excited states to be used in the variational method. This method is a generalization of the one that uses super-potential to obtain the trial functions for the ground state. The first four eigenvalues for a quartic double-well potential were calculated at different values of the potential parameter.

By means of a collocation approach based on little Sinc functions (LSF), Amore and Fernández [15] obtained accurate eigenvalues and eigenfunctions of the stationary Schrödinger equation for systems of coupled oscillators. Gribov and Prokof'eva [16] proposed a variational method of the solutions of anharmonic problems in the theory of molecular vibrations in curvilinear coordinates taking into account the kinematic anharmonicity. VEGA and FLORES [17] used the variational method and supersymmetric quantum mechanics to calculate in an approximate way, the eigenvalues, eigenfunctions and wave functions at the origin of the Cornell potential. Payandeh and Mohammadpour used the Delta method to evaluate the energy of ground and excited stationary states in quantum mechanics. The advantage of the Delta method compared to the variational method is its simplicity and reduction of the calculation procedures [18].

P. M. Gaiki and P. M. Gade [19] demonstrated how a freeware, SAGE, can be employed for the variational solution of simple and complex Hamiltonians in one dimension to estimate the ground state energy. F M Fernández and J Garcia [20] considered Rayleigh-Ritz variational computations with non-orthogonal basic sets with the correct asymptotic behavior. This approach is illustrated by the construction of appropriate basis sets for one-dimensional models such as the two double-well oscillators recently examined by other authors. The convergence rate of the variational method is considerably greater than that of orthogonal.

S. Khuri and A. Wazwaz [21] applied an amended variational scheme for the solution of a second-order nonlinear boundary value problem. However, the variational iteration method was used for solving linear and nonlinear ODEs and scientific models with variable coefficients [22, 23] and the asymptotic iteration method was applied to certain quasinormal modes and non Hermitian systems [24].

In this paper, we start by formulating the problem in Sec. 2. Then, we show in Sec. 3 and 4, that under certain conditions, the harmonic plus linear term and the sextic anharmonic potentials energy is easily solvable by the variational method. In Sec. 5, we explore the non-polynomial exactly solvable quartic potential. The development of this variational method is explained in Sec. 6. Sec. 7 is devoted to some applications and discussions about anharmonic

potentials. Finally, the conclusion of the work is presented in Sec. 8.

2 Problem formulation

2.1 Schrödinger equation

Let us consider the one-dimensional time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi_n(x)}{dx^2} + V(x) \psi_n(x) = E_n \psi_n(x) \quad (1)$$

Where E_n is the system's energy and ψ_n is the wave function (n^{th} eigenstates). $V(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \dots + \alpha_N x^N$, $\alpha_2 = \frac{1}{2} k > 0$, $k = m\omega^2$, m is the particle mass and ω the angular frequency. $V(x)$ is anharmonic potential energy, the coefficients $\alpha_{p < N}$ are real and α_N is positive. This potential energy form was chosen because using the Lagrange interpolation, we can approach any potential energy with high accuracy in each continuous potential energy to polynomial potential energy [25]. Dividing Eq. (1) by $\hbar\omega$, putting $\lambda = \frac{\hbar}{2\omega m}$ and $E_n = \frac{E_n}{\hbar\omega}$ (to express the energy parameters in the unit of $\hbar\omega_0$) and moving to the variable, $y = \left(\frac{2\omega m}{\hbar}\right)^{1/2} x$, we obtain the dimensionless equation:

$$\frac{d^2 \psi_n(y)}{dy^2} + \left[E_n - \left(\frac{\alpha_1}{\hbar\omega} \lambda^{1/2} y + \frac{\alpha_2}{\hbar\omega} \lambda y^2 + \frac{\alpha_3}{\hbar\omega} \lambda^{3/2} y^3 + \frac{\alpha_4}{\hbar\omega} \lambda^2 y^4 + \dots + \frac{\alpha_N}{\hbar\omega} \lambda^{N/2} y^N \right) \right] \psi_n(y) = 0,$$

and

$$\frac{d^2 \psi_n(y)}{dy^2} + [E_n - v(y)] \psi_n(y) = 0 \quad (2)$$

where $v(y) = b_1 y + b_2 y^2 + b_3 y^3 + b_4 y^4 + \dots + b_i y^i + \dots + b_N y^N$, $b_i = \frac{\alpha_i}{\hbar\omega} \lambda^{i/2}$ and $i = 1, 2, \dots, N$

The Hamiltonian system is consequently:

$$H = -\frac{d^2}{dy^2} + v(y) \quad (4)$$

Under certain conditions applied on the parameters of the potentials as described in earlier work of Maiz *et al.* [26], some potentials are exactly solvable. However, if there is no exact solution, then we applied an approximation approach like perturbation theory, variational and WKB methods and numerical method such as Numerov method, Airy function approach, and the asymptotic iteration method. The variational method (VM) will be modified to carry out calculations in this work.

To explore the conditions of the existence of polynomial solutions, we use the trial normalized wavefunctions

in the form of:

$$\psi_n(y) = Af_n(y) \exp(-h(y)) \quad (5)$$

The expectation value of the energy is $E = \langle \Psi H \Psi \rangle$. The substitution of the wave function ψ_n leads to the following relation:

$$E = \left\langle Af_n(y) \exp(-h(y)) \left| -\frac{d^2}{dy^2} + v(y) \right| Af_n(y) \exp(-h(y)) \right\rangle \quad (6)$$

Applying the Hamiltonian gives:

$$E = \left\langle Af_n(y) e^{(-h(y))} \left| \left(-\frac{d^2 f(y)}{dy^2} + 2 \frac{dh(y)}{dy} \frac{df(y)}{dy} + f_n(y) \left(\frac{d^2 h(y)}{dy^2} - \left(\frac{dh(y)}{dy} \right)^2 + v(y) f_n(y) \right) \right| A e^{(-h(y))} \right\rangle \quad (7)$$

Eq. 7 states the conditions of the polynomial solution existence; it depends on the potential energy expression.

2.2 The variational method review

The variational method provides a simple way to place an upper bound on the ground state energy of any quantum system and it is particularly useful when trying to demonstrate that existence of bound states. In some cases, it can also be used to estimate higher energy levels. We start with a quantum system with Hamiltonian H , which has a discrete spectrum: $H|n\rangle = E_n|n\rangle$ with $n = 0, 1, 2, \dots$, the energy eigen values are ordered such that $E_n \leq E_{n+1}$. The simplest application of the variational method places an upper bound is on the value of the ground state energy E_0 . If we consider an arbitrary normalized state $|\Psi\rangle$, ($\langle \Psi | \Psi \rangle = 1$), the expectation value of the energy obeys the inequality: $E = \langle \Psi | H | \Psi \rangle \geq E_0$.

Let's consider a family of normalized states, $|\Psi(\alpha)\rangle$, depending on some number of parameters α . The expectation value of energy, $E(\alpha)$ obeys $E(\alpha) \geq E_0$ for all α .

The minimum value of $E(\alpha)$ which is $E(\alpha_{min})$ with $\left(\frac{dE(\alpha)}{d\alpha} \right)_{\alpha=\alpha_{min}} = 0$,

offers the upper bound $E_0 \leq E(\alpha_{min})$. This is the soul of the variational method. But the variational method does not give the difference between the founded and exact values of energies $\Delta E = E(\alpha_{min}) - E_0$. This difference disappears when the chosen trial wave function coincides with the exact problem solution. This means that the choice of the trial wave function is decisive and, in this case, the variational method gives remarkably accurate results.

3 The harmonic oscillator potential

Considering the well-known case of harmonic oscillator potential energy plus linear term $v(y) = b_1 y + b_2 y^2$, the coefficient b_1 is real and b_2 is positive.

The node-less eigenfunction (ground state) is given by $\psi_0(y) = Af_0(y) \exp(-h(y))$, with $f_0(y) = 1$ and $h(y) = \sum_{p=1}^{2N} a_p y^p$. Introducing this solution in Eq. 7 leads to the equation:

$$E = \left(-4a_2^2 + b_2 \right) \langle \psi_0 | y^2 | \psi_0 \rangle + (-4a_1 a_2 + b_1) \langle \psi_0 | y | \psi_0 \rangle + (2a_2 - a_1^2) \quad (8)$$

Note that for

$$a_2 = \frac{\sqrt{b_2}}{2}, \quad \text{and} \quad a_1 = \frac{b_1}{2\sqrt{b_2}}, \quad E = (2a_2 - a_1^2)$$

and the eigenenergy value is constant and equal to the exact value of energy

$$E_e = \sqrt{b_2} - \frac{b_1^2}{4b_2}$$

however, the ground state function is $\psi_0(y) = A \exp\left(-\frac{b_1}{2\sqrt{b_2}}y - \frac{\sqrt{b_2}}{2}y^2\right)$.

These results were found to be the same as previously published for the same potential energy and level [27].

4 The closely solvable sextic anharmonic oscillator potential energy

Considering the case of sextic anharmonic oscillator potential $v(y) = b_1 y + b_2 y^2 + b_3 y^3 + b_4 y^4 + b_5 y^5 + b_6 y^6$, the coefficients $b_{i < 6}$ is real and b_6 is positive.

The node-less eigenfunction (ground state) is given by $\psi_0(y) = Af_0(y) \exp(-h(y))$, with $f_0(y) = 1$ and $h(y) = \sum_{p=1}^{2N} a_p y^p$. Introducing this solution in Eq. 7 offers the energy expectation value:

$$E = \left(b_6 - 16a_4^2 \right) \langle \psi_0 | y^6 | \psi_0 \rangle + (b_5 - 24a_3 a_4) \langle \psi_0 | y^5 | \psi_0 \rangle + \left(b_4 - 16a_2 a_4 - 9a_3^2 \right) \langle \psi_0 | y^4 | \psi_0 \rangle + (b_3 - 12a_2 a_3 - 8a_1 a_4) \langle \psi_0 | y^3 | \psi_0 \rangle + \left(b_2 + 12a_4 - 4a_2^2 - 6a_1 a_3 \right) \langle \psi_0 | y^2 | \psi_0 \rangle + (b_1 + 6a_3 - 4a_1 a_2) \langle \psi_0 | y | \psi_0 \rangle + (2a_2 - a_1^2) \quad (9)$$

For $a_4 = \frac{b_1^{1/2}}{4}$, $a_3 = \frac{b_5}{6b_6^{1/2}}$, $a_2 = \frac{b_4}{4b_6^{1/2}} - \frac{b_5^2}{16b_6^{3/2}}$, and $a_1 = \frac{b_3}{2b_6^{1/2}} - \frac{b_4b_5}{4b_6^{3/2}} + \frac{b_5^3}{16b_6^{5/2}}$ and the two conditions:

$$b_2 = 4a_2^2 + 6a_1a_3 - 12a_4 \text{ and } b_1 = 4a_1a_2 - 6a_3,$$

The expectation value of energy remains $E_e = (2a_2 - a_1^2)$, it is constant and constitutes an exact solution. The polynomial coefficients $a_{i=1,2,3,4}$ are the only function of the potential parameters $b_{i=3,4,5,6}$ and not the function of the two firsts parameters $b_{i=2,3}$. Furthermore, for any quadruplet of real ($a_1, a_2, a_3, a_4 > 0$), the sextic anharmonic potential energy:

$v_{6e}(y) = (4a_1a_2 - 6a_3)y + (4a_2^2 + 6a_1a_3 - 12a_4)y^2 + (12a_2a_3 + 8a_1a_4)y^3 + (9a_3^2 + 16a_2a_4)y^4 + 24a_3a_4y^5 + 16a_4^2y^6$, admits an exact ground state energy $E_0 = (2a_2 - a_1^2)$. For non-closely solvable sextic anharmonic oscillator potential energy, we use quartic anharmonic potential, this will be explained in the forthcoming section.

5 The quartic anharmonic potential

Studying the case of quartic anharmonic oscillator potential $v(y) = b_1y + b_2y^2 + b_3y^3 + b_4y^4$, the coefficients $b_{i<4}$ is real and b_4 is positive. It is well known that this potential has no exactly polynomial solutions [26]. To estimate the energy expectation value, we apply the normalized wave function: $\psi_n(y) = Af_n(y) \exp(-h(y))$.

The node-less eigenfunction is given by $\psi_0(y) = Af_0(y) \exp(-h(y))$, with $f_0(y) = 1$ and $h(y) = \sum_{p=1}^{2N} a_p y^p$. Eq. (7) leads to the following expression:

$$\begin{aligned} E_0 = & (-16a_4^2) \langle \psi_0 | y^6 | \psi_0 \rangle \\ & + (-24a_3a_4) \langle \psi_0 | y^5 | \psi_0 \rangle \\ & + (b_4 - 16a_2a_4 - 9a_3^2) \langle \psi_0 | y^4 | \psi_0 \rangle \\ & + (b_3 - 12a_2a_3 - 8a_1a_4) \langle \psi_0 | y^3 | \psi_0 \rangle \\ & + (b_2 + 12a_4 - 4a_2^2 - 6a_1a_3) \langle \psi_0 | y^2 | \psi_0 \rangle \\ & + (b_1 + 6a_3 - 4a_1a_2) \langle \psi_0 | y | \psi_0 \rangle + (2a_2 - a_1^2) \end{aligned} \quad (10)$$

For

$$a_4 = \frac{1}{3}a_2^2 + \frac{1}{3}a_2a_1^2 - \frac{1}{12}a_1b_1 - \frac{1}{12}b_2, \quad (11)$$

$$a_3 = \frac{2}{3}a_1a_2 - \frac{1}{6}b_1 \quad (12)$$

and the two conditions:

$$b_4 = \frac{28}{3}a_1^2a_2^2 - \frac{10}{3}a_1a_2b_1 + \frac{1}{4}b_1^2 + \frac{16}{3}a_2^3 - \frac{4}{3}a_2b_2 \quad (13)$$

$$b_3 = \frac{32}{3}a_1a_2^2 - 2a_2b_1 + \frac{8}{3}a_1^3a_2 - \frac{2}{3}a_1^2b_1 - \frac{2}{3}a_1b_2, \quad (14)$$

The expectation value of energy is

$$\begin{aligned} E_0 = & (-16a_4^2) \langle \psi_0 | y^6 | \psi_0 \rangle + (-24a_3a_4) \langle \psi_0 | y^5 | \psi_0 \rangle \\ & + (2a_2 - a_1^2). \end{aligned}$$

Equations (11-12) show the expressions of the coefficients a_3 and a_4 as a function of the coefficients a_1, a_2 and the two firsts potential energy parameters b_1 and b_2 . It should be noted here that the coefficients a_1 and a_2 are the acceptable solutions of equations (13-14), they are a function of the potential's parameters. The minimum of E_0 in the vicinity of the coefficients $a_{i(i=1,2,3,4)}$ values give a good estimation of the ground state energy value. This minimum is noted E_{vm} and equal to

$$\begin{aligned} & (-16a_4^{*2}) \langle \varphi_0 | y^6 | \varphi_0 \rangle + (-24a_3^*a_4^*) \langle \varphi_0 | y^5 | \varphi_0 \rangle \\ & + (2a_2^* - a_1^{*2}) \end{aligned}$$

where the associated wave function is $\varphi(y) = A \exp(-h(y))$, with $h(y) = \sum_{p=1}^{2N} a_p^* y^p$. For each integer p , the value of the coefficient a_p^* is in the vicinity of the a_p one.

6 The variational method development

As we noted in section 2.a, using the Lagrange interpolation, a given potential energy $v(y)$ may be approached with high accuracy to a polynomial potential energy $v_n(y)$ of degree n [25]:

$$v(y) \rightarrow v_n(y) = \sum_{i=1}^n b_i y^i$$

For non-exactly solvable energy potential, we considered just the fourth firsts terms for a given potential energy, $v_4(y)$, which is the corresponding quartic anharmonic potential energy. As mentioned in the preceding paragraph, for the ground state energy the expectation value of energy is:

$$\begin{aligned} E_0 = & (-16a_4^2) \langle \psi_0 | y^6 | \psi_0 \rangle \\ & + (-24a_3a_4) \langle \psi_0 | y^5 | \psi_0 \rangle + (2a_2 - a_1^2). \end{aligned}$$

The coefficients $a_{i(i=1,2,3,4)}$ were derived in paragraph 4. Using the variational method for the given potential energy $v(y)$, the expectation energy level value is $E =$

$\langle \varphi | H | \varphi \rangle$. The trial normalized wave function is $\varphi(y) = A \exp(-h(y))$, with $h(y) = \sum_{p=1}^{2N} a_p^* y^p$. For the given potential $v(y)$, E may be obtained by differentiating this previous expectation value of energy with respect to the four coefficients $a_{i=1,2,3,4}^*$. Canceling different equations leads to a system of four equations on various $a_{i=1,2,3,4}^*$:

$$\begin{cases} \frac{\partial E}{\partial a_1^*} = 0 \\ \frac{\partial E}{\partial a_2^*} = 0 \\ \frac{\partial E}{\partial a_3^*} = 0 \\ \frac{\partial E}{\partial a_4^*} = 0 \end{cases}$$

To calculate the ground state energy value E for the given potential energy $v(y)$, we solve this previous system for the coefficients $a_{i=1,2,3,4}^*$ in the vicinity of the previous coefficients $a_{i=1,2,3,4}$ for the corresponding quartic anharmonic potential energy. By limiting the calculations in the vicinity of the previous coefficients $a_{i=1,2,3,4}$, we were successful in reducing computation time. Finally, we deduce the estimated ground state energy value: $E_{vm} = E_{min}(a_{i=1,2,3,4}^*)$ and compare this result with available ones for the same potential energy [27].

7 Results and discussions

We applied our developed variational method (DVM) to study three classes of potential designed by the following eight potentials energies:

$$\begin{aligned} v_1(y) &= y + y^2, \quad v_2(y) = y^2 + y^4 + y^6, \\ v_3(y) &= y^2 - y^3 + y^4 - y^5 + y^6, \\ v_4(y) &= y + y^2 + y^3 + y^4 + y^5 + y^6, \\ v_5(y) &= -y + 2y^2 - 3y^3 + 4y^4 - 5y^5 + 6y^6, \\ v_6(y) &= y + y^2 + y^3 + y^4, \\ v_7(y) &= y^2 + y^3 + y^4, \quad \text{and } v_8(y) = y^3 + y^4. \end{aligned}$$

The well-known harmonic plus linear term potential constitutes the first class; it is exactly solvable potential as described in paragraph 3. It was chosen to make the first test to our developed variational method. The second class is presented by the four sextic anharmonic potentials, while the last quartic formed the last class of potential energy. Profiles of these potentials energy are shown in Figure 1.

Figure 1: Studied potentials energy profiles

7.1 Harmonic plus linear term

The harmonic plus linear potential energy is expressed by $v_1(y) = y + y^2$, it is exactly solvable as shown in paragraph 3. The ground state energy value is $E_e = \sqrt{b_2} - \frac{b_1^2}{4b_2}$, which numerically equal to 0.75. The corresponding exactly solvable potential energy is clearly the same: $v_e(y) = v_1(y) = y + y^2$. The associated normalized wavefunction is written as $\varphi(y) = A \exp(-h(y))$, with $h(y) = \sum_{p=1}^{2N} a_p^* y^p$, and $N = 1$. The coefficients a_1^* and a_2^* values are obtained which are nearly equal as 0.5000000002 and 0.4999999999 respectively. Furthermore, the obtained ground state energy value is $E_{vm} = 0.749999$ while Numerov method gives $E_{nm} = 0.750009$. These values are in agreement with the exact solution derived in paragraph 3. The relative energy difference between them $\Delta E/E$ is in the range of 10^{-4} . Physical parameters for the potential energy $v_1(y) = y + y^2$ are collected in Table 1.

7.2 Sextic potential energy

Some sextic potential energy is exactly solvable. This depends on relations between the potential energy parameters [26]. Sextic potential were found reliable as a potential model for quark confinement in quantum chromodynamics [28]. Furthermore, this model is very important when trying to understand many theories including molecular spectroscopy, quantum-tunneling time, and field theories [29]. In paragraph 4, the study of the sextic anharmonic potential energy demonstrate that under some conditions on its parameters, it may be exactly solvable. In

Table 1: Physical parameters for the potential energy $v_1(y) = y + y^2$

potential energy		polynomial coefficients		ground state energy values			$10^5 \Delta E/E$
$v_1(y)$	$y + y^2$	$a_1^* = 0.500000$	$a_2^* = 0.499999$	E_{vm}	E_{nm}	E_e	1.3
$v_e(y)$	$y + y^2$	$a_1 = 0.5$	$a_2 = 0.5$	0.749999	0.750009	0.750	

Table 2: Physical parameters for the potential energy $v_2(y) = y^2 + y^4 + y^6$

potential energy		polynomial coefficients				ground state energy values			$\frac{10^3 \Delta E}{E}$
$v_2(y)$	$a_1^* = 0$	$a_2^* = 0.715$	$a_3^* = 0$	$a_4^* = 0.087$	E_{vm}	E_{nm}	E_e	2	
$v_e(y)$	$a_1 = 0$	$a_2 = \frac{19439}{27166}$	$a_3 = 0$	$a_4 = \frac{11955}{136873}$	1.615237	1.614889	1.431127		

Table 3: Physical parameters for the potential energy $v_3(y) = x^2 - x^3 + x^4 - x^5 + x^6$

potential energy		polynomial coefficients				ground state energy values			$\frac{10^3 \Delta E}{E}$
$v_3(y)$	$a_1^* = -\frac{9484}{41429}$	$a_2^* = \frac{1195}{1756}$	$a_3^* = -\frac{14584}{93615}$	$a_4^* = \frac{17239}{86619}$	E_{vm}	E_{nm}	E_e	1	
$v_e(y)$	$a_1 = -\frac{9484}{41429}$	$a_2 = \frac{1195}{1756}$	$a_3 = -\frac{3559}{34268}$	$a_4 = \frac{2320}{27977}$	1.472721	1.471141	1.308642		

general, we applied numerical and approximated methods to solve the Schrödinger equation. To verify our developed variational method for the non-exactly solvable anharmonic sextic potential energy, we propose to study four examples of potential energy. We start with the symmetric sextic potential as $v_2(y) = x^2 + x^4 + x^6$. The corresponding exactly solvable potential energy is:

$$v_e(y) = x^2 + x^4 + \frac{12826}{105077}x^6 \quad \text{with} \\ (a_1, a_2, a_3, a_4) = \left(0, \frac{19439}{27166}, 0, \frac{11955}{136873}\right)$$

and $E_e = 1.431127$. For the potential energy $v_2(y)$, the associated normalized wavefunction is $\varphi(y) = A \exp(-h(y))$, with $h(y) = \sum_{p=1}^{2N} a_p y^p$, and $N = 2$. The coefficients a_1^* and a_3^* values are null because of the potential's symmetry. The coefficients a_2^* and a_4^* values are obtained as $\frac{11313}{14116}$ and $\frac{2081}{12474}$, respectively.

Furthermore, the obtained ground state energy value is $E_{vm} = 1.615237$ while, Numerov method offers $E_{nm} = 1.614889$. These values agree very closely and the relative energy difference is a less than 10^{-3} . Physical parameters for the potential energy $v_2(y) = y^2 + y^4 + y^6$ are collected in Table 2.

Secondly, we studied the anharmonic sextic potential $v_3(y) = y^2 - y^3 + y^4 - y^5 + y^6$. The corresponding potential energy exactly solvable is:

$$v_e(y) = y^2 - 0.999999y^3 + 0.999999y^4 - \frac{11300}{54669}y^5 \\ + \frac{10237}{93042}y^6,$$

where the exact ground state energy value is $E_e = 1.308642$ and the associated wavefunction: $\psi_n(y) = A f_n(y) \exp(-h(y))$, with $h(y) = \sum_{p=1}^{2N} a_p y^p$, and $(a_1, a_2, a_3, a_4) = (-\frac{9484}{41429}, \frac{1195}{1756}, -\frac{3559}{34268}, \frac{2320}{27977})$.

For the potential energy $v_3(y)$, the associated normalized wavefunction is: $\varphi(y) = A \exp(-h(y))$, with $h(y) = \sum_{p=1}^{2N} a_p y^p$. The quadruplet $(a_1^*, a_2^*, a_3^*, a_4^*)$ is equal to $(-\frac{9484}{41429}, \frac{1195}{1756}, -\frac{14584}{93615}, \frac{17239}{86619})$.

Furthermore, the obtained ground state energy value is $E_{vm} = 1.472721$ while, Numerov method offers $E_{nm} = 1.471141$. These values agree very closely and the relative energy difference is approximately equal to 10^{-3} . Table 3. shows these parameters.

Thirdly, we explore the anharmonic sextic potential defined by: $v_4(y) = y + y^2 + y^3 + y^4 + y^5 + y^6$. The corresponding potential energy exactly solvable is:

$$v_e(y) = y + y^2 + 0.999y^3 + y^4 + \frac{13811}{85343}y^5 + \frac{2091}{16849}y^6,$$

where the exact ground state energy value is $E_e = 1.049869$. Additionally, the obtained ground state energy value is equal to $E_{vm} = 1.230$ although, the numerical Numerov method gives: $E_{nm} = 1.220$. The relative energy difference is in the range of 10^{-3} (see Table 4).

Finally, we investigate the anharmonic sextic potential: $v_5(y) = -y + 2y^2 - 3y^3 + 4y^4 - 5y^5 + 6y^6$. The corresponding potential energy exactly solvable as:

$$v_e(y) = -y + 2y^2 - 2.999999y^3 + 3.999999y^4 - \frac{53061}{56984}y^5 \\ + \frac{35442}{42859}y^6,$$

Table 4: Physical parameters for the potential energy $v_4(y) = y + y^2 + y^3 + y^4 + y^5 + y^6$

potential energy	polynomial coefficients				ground state energy values			$10^3 \frac{\Delta E}{E}$
$v_4(y)$	$a_1^* = \frac{55558}{102365}$	$a_2^* = \frac{19850}{29529}$	$a_3^* = \frac{19850}{29529}$	$a_4^* = \frac{8038}{46095}$	E_{vm}	E_{nm}	E_e	2
$v_e(y)$	$a_1 = \frac{55558}{102365}$	$a_2 = \frac{19850}{29529}$	$a_3 = \frac{19850}{29529}$	$a_4 = \frac{13321}{151254}$	1.223769	1.220971	1.049869	

Table 5: Physical parameters for the potential energy $v_5(y) = -y + 2y^2 - 3y^3 + 4y^4 - 5y^5 + 6y^6$

potential energy	polynomial coefficients				ground state energy values			$10^3 \frac{\Delta E}{E}$
$v_5(y)$	$a_1^* = -\frac{4440}{10019}$	$a_2^* = \frac{26827}{23733}$	$a_3^* = -\frac{18498}{60217}$	$a_4^* = \frac{107987}{250000}$	E_{vm}	E_{nm}	E_e	1
$v_e(y)$	$a_1 = -\frac{80994}{164489}$	$a_2 = \frac{31752}{30899}$	$a_3 = -\frac{26735}{156656}$	$a_4 = \frac{24910}{109571}$	2.045453	2.043194	1.812756	

Table 6: Physical parameters for the potential energy $v_6(y) = y + y^2 + y^3 + y^4$

potential energy	polynomial coefficients				ground state energy values			$10^4 \frac{\Delta E}{E}$
$v_6(y)$	$a_1^* = \frac{67843}{125000}$	$a_2^* = \frac{33605}{49991}$	$a_3^* = \frac{13459}{140633}$	$a_4^* = \frac{8807}{100000}$	E_{vm}	E_{nm}	E_e	1
$v_e(y)$	$a_1 = \frac{55558}{102365}$	$a_2 = \frac{19850}{29529}$	$a_3 = \frac{7758}{101329}$	$a_4 = \frac{13321}{151254}$	1.035545	1.034035	1.049869897	

Table 7: Physical parameters for the potential energy $v_7(y) = y^2 + y^3 + y^4$

potential energy	polynomial coefficients				ground state energy values			$10^4 \frac{\Delta E}{E}$
$v_7(y)$	$a_1^* = \frac{20603}{100000}$	$a_2^* = \frac{14078}{20687}$	$a_3^* = \frac{2921}{31250}$	$a_4^* = \frac{7082}{94891}$	E_{vm}	E_{nm}	E_e	5
$v_e(y)$	$a_1 = \frac{9484}{41429}$	$a_2 = \frac{1195}{1756}$	$a_3 = \frac{3559}{34268}$	$a_4 = \frac{2320}{27977}$	1.311036	1.31026	1.308642	

where the exact ground state energy value is $E_e = 1.812756$. Additionally, the obtained ground state energy value is $E_{vm} = 2.045453$ although, the numerical Numerov method gives $E_{nm} = 2.045895$. The relative energy difference between these results is less than 10-3. The results are collected in Table 5.

7.3 Quartic potential energy

Anharmonic octic potential energy has no exactly polynomials solutions. For this class of potentials, we start by studying the given potential energy: $v_6(y) = y + y^2 + y^3 + y^4$. The corresponding potential energy is the exactly solvable potential as:

$$v_e(y) = y + y^2 + 0.999y^3 + y^4 + \frac{13811}{85343}y^5 + \frac{2091}{16849}y^6,$$

where the exact ground state energy value is $E_e = 1.049869897$. Moreover, the developed variational and the Numerov methods evaluate, the ground state energy

as $E_{vm} = 1.034086$ and $E_{nm} = 1.034035$ respectively. The relative energy difference between these results is less than 10^{-4} . We have shown our findings in Table 6.

The non-exactly polynomials solutions potential energy: $v_7(y) = y^2 + y^3 + y^4$ is also investigated. $v_e(y) = y^2 + 0.999999y^3 + 0.999999y^4 + \frac{11300}{54669}y^5 + \frac{10237}{93042}y^6$ presents the corresponding potential energy. For this potential, the exact ground state energy is evaluated to be, $E_e = 1.308642$. In addition, the developed variational and the Numerov methods estimate the ground state energy is equal to $E_{vm} = 1.311036$ and $E_{nm} = 1.31026$ respectively. Table 7 shows our estimated results.

Finally, we explore the non-exactly polynomials solutions potential energy given by: $v_8(y) = y^3 + y^4$. Here the corresponding potential energy is found to be:

$$v_e(y) = 10^{-9}y^2 + 0.999999y^3 + 0.999999y^4 + \frac{12113}{40787}y^5 + \frac{8306}{45161}y^6.$$

The exact ground state energy value is closed to $E_e = 0.912247$. Moreover, the use of approximated and numerical methods offers the two following values for the ground

Table 8: Physical parameters for the potential energy $v_8(y) = y^3 + y^4$

potential energy	polynomial coefficients				ground state energy values			$10^4 \frac{AE}{E}$
$v_8(y)$	$a_1^* = \frac{24603}{81013}$	$a_2^* = \frac{19748}{38491}$	$a_3^* = \frac{7934}{76381}$	$a_4^* = \frac{6210}{64357}$	E_{vm}	E_{nm}	E_e	2
$v_e(y)$	$a_1 = \frac{2769}{8206}$	$a_2 = \frac{13381}{26081}$	$a_3 = \frac{4871}{42204}$	$a_4 = \frac{50284}{469003}$	0.907941	0.905322	0.912247	

state energy: $E_{vm} = 0.905322$ and $E_{nm} = 0.905322$. Physical parameters values are collected in Table 8.

8 Conclusion

The polynomial solutions of the Schrödinger equation for some anharmonic potential energy helped us to our modified and improved variational method. This study shows that under appropriate conditions on the potential's parameters, the choice of the suitable trial wave function becomes easy. Focusing on the anharmonic potential problem, we have also presented a comparison between the solutions provided by this developed variational method and the results for the same problem by different methods such as the Numerov method. The obtained results for the ground state energy values are found to be accurate. Finally, our results agreed well with those available in literature [27, 30, 31]. We believe that this study will encourage the researchers and scientists for further investigation of general polynomial potentials.

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