Research Article

Manuel Cabral Morais*

An ARL-Unbiased np -Chart

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Abstract: We usually assume that counts of nonconforming items have a binomial distribution with parameters (n, p) , where n and p represent the sample size and the fraction nonconforming, respectively. The non-negative, discrete and usually skewed character and the target mean (np_0) of this distribution may prevent the quality control engineer to deal with a chart to monitor p with: a pre-specified in-control average run length (ARL), say α^{-1} ; a positive lower control limit; the ability to control not only increases but also decreases in p in an expedient fashion. Furthermore, as far as we have investigated, the np - and p -charts proposed in the Statistical Process Control literature are ARL-biased, in the sense that they take longer, in average, to detect some shifts in the fraction nonconforming than to trigger a false alarm. Having all this in mind, this paper explores the notions of uniformly most powerful unbiased tests with randomization probabilities to eliminate the bias of the ARL function of the np -chart and to bring its in-control ARL exactly to α^{-1} .

Keywords: Average Run Length, Uniformly Most Powerful Unbiased Tests, Randomization Probabilities

MSC 2010: 62P30

1 Charts for Nonconforming Items

The np -chart with 3-sigma limits has been historically used to detect whether the process quality deteriorates or improves. This chart triggers an alarm at sample t if its control statistic X_t ($t = 1, 2, \dots$), the number of nonconforming items in the t -th sample of size n , is beyond the lower and upper control limits (LCL and UCL). Since we are dealing with a non-negative integer-valued control statistic, it is more appropriate to define these control limits as the following ceiling and floor functions of the known target fraction of nonconforming units (p_0):

$$\text{LCL} = \left\lceil \max\{0, np_0 - 3\sqrt{np_0(1-p_0)}\} \right\rceil, \quad \text{UCL} = \left\lfloor np_0 + 3\sqrt{np_0(1-p_0)} \right\rfloor.$$

In the Statistical Process Control (SPC) literature, we usually assume that the random variables (r.v.) X_t , $t = 1, 2, \dots$, are independent and identically distributed (i.i.d.) to the r.v. $X \sim \text{Binomial}(n, p)$. Moreover, the process is said to be in-control (resp. out-of-control) if $p = p_0$ (resp. $p \neq p_0$); if $p \in (0, p_0)$ (resp. $p \in (p_0, 1)$) then a downward (resp. upward) shift has occurred and the quality control engineer ought to be alerted to the improvement (resp. deterioration) of the process quality.

The performance of the np -chart is usually assessed in terms of the expected number of samples taken until a signal is triggered, the average run length (ARL). Unsurprisingly, it is desirable that false alarms (resp. valid signals) are rarely triggered (resp. emitted as quickly as possible), corresponding to a large in-control (resp. small out-of-control) ARL.

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*Corresponding author: Manuel Cabral Morais: Department of Mathematics & CEMAT (Center for Computational and Stochastic Mathematics), Instituto Superior Técnico, University of Lisbon, Av. Rovisco Pais, 1049-001 Lisboa, Portugal, e-mail: maj@math.ist.utl.pt

The run length of this Shewhart chart has a geometric distribution with parameter

$$\xi(p) = P(\text{emission of a signal} \mid p) = 1 - \sum_{x=LCL}^{UCL} P_{n,p}(x),$$

where $P_{n,p}(x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, \dots, n$. Hence $ARL(p) = 1/\xi(p)$.

Since in most practical applications we expect p_0 to be quite small, the LCL is often equal to 0, namely when $p_0 < 9/(9+n)$ (see [18, p. 181]). Consequently, $ARL(p) > ARL(p_0)$ for any $p \in (0, p_0)$, that is, it would take the np -chart longer, in average, to detect any reduction in the fraction nonconforming than to trigger a false alarm.

To overcome this undesirable feature, Ryan [18, p. 182] recommends selecting the sample size so that $np_0 - 3\sqrt{np_0(1-p_0)} > 0$, for instance the smallest integer n verifying $n > 9(1-p_0)/p_0$. Regretfully, this requirement can lead to impractical sample sizes, as shown by Ryan [18, Table 8.3, p. 187], in particular for small values of p_0 .

Another problem arises from the fact that the 3-sigma control limits presume the adequacy of the normal approximation to the binomial distribution, as put by Ryan [18, p. 181] and Ryan and Schwertman [19]. However, they add that it is known that the normal approximation to the binomial distribution can perform poorly even if the rule of thumb that both np and $n(1-p)$ should exceed 5 is met, with the accuracy of the approximation highly dependent upon the value of p , according to Schader and Schmid [20].

In case that approximation is good, we would expect the in-control lower tail area $P(X < LCL \mid p = p_0)$ and in-control upper tail area $P(X > UCL \mid p = p_0)$ to be both close to 0.00135, the corresponding value for an $\bar{X}$ -chart when normality is assumed. However, the binomial distribution is skewed to the right for $p_0 \in (0, 0.5)$, hence the in-control lower (resp. upper) tail area is in general smaller (resp. larger) than the desired value 0.00135, as exemplified by Ryan [18, Table 8.1, p. 184] for a large constellation of values of (n, p_0) . Expectedly, the in-control ARL of the np -chart with 3-sigma limits can significantly differ from $(2 \times 0.00135)^{-1} \approx 370.4$.

In addition, the probability of a point falling below the LCL of the np -chart with 3-sigma limits may be rather small in the presence of downward shifts in the fraction nonconforming, leading to an unacceptable detection ability of this chart for indicating quality improvement.

All the variants of the np -chart with 3-sigma limits, which we describe in Section 1.1, have been proposed to mitigate these two types of problems. These variants can be broadly divided into two categories, according to the adopted approach: 1) transforming data; 2) modifying control limits.

1.1 Some Variants of the np -Chart with 3-Sigma Limits

The rationale behind a better method for obtaining the LCL is to transform the binomial data (x) so that the transformed data (y) are approximately normal, and then use 3-sigma limits for the transformed data [18, p. 182]. The description of three such charts can be found in [18, pp. 182–186]. They are associated with arcsin transformations that can be traced back to

- Freeman and Tukey [3], $y = 0.5[\arcsin \sqrt{x/(n+1)} + \arcsin \sqrt{(x+1)/(n+1)}]$;
- Hald [6, p. 685], $y = \arcsin \sqrt{x/n}$;
- Johnson and Kotz [9, p. 65], $y = \arcsin \sqrt{(x+3/8)/(n+3/4)}$.

As stated by Ryan [18, p. 186], these three transformations differ only slightly and lead to similar performance results; the minimum sample size required to deal with a positive LCL is roughly one-fourth the size needed for the traditional np -chart with 3-sigma limits, as illustrated in [18, Table 8.3, p. 187]. In addition to this, Ryan and Schwertman [19] add that the resulting tail areas will be closer to the ones under the normal assumption.

The Q chart proposed by Quesenberry [17] for the binomial parameter p also falls in the transforming data approach, and led to the conclusion that the arcsin transformation (resp. Q chart) gives a better approximation to the in-control lower (resp. upper) tail area [19].

Feeling that it would be preferable to plot the untransformed statistic of interest and come closer to the nominal tail areas, Ryan [18, p. 189] suggested the use of an np -chart with modified LCL and UCL given by $np_0 - 3\sqrt{np_0(1-p_0)} + 1.25$ and $np_0 + 3\sqrt{np_0(1-p_0)} + 1.15$ (respectively), to account for the skewness of the binomial distributions when p is small.

Following once again the second approach, Ryan and Schwertman [19] proposed to search for values of n that would permit the obtention of modified control limits associated with in-control tail areas very close to the nominal value of 0.00135. The authors regressed these limits against np_0 and $\sqrt{np_0}$, found an accurate fitting and the new set of control limits is $LCL = 2.9529 + 1.01956np_0 - 3.2729\sqrt{np_0}$ and $UCL = 0.6195 + 1.00523np_0 + 2.983\sqrt{np_0}$ for $p_0 \in (0, 0.03]$. In [19, Table 7] (resp. [19, Table 1]), we can find the values of n , LCL and UCL for $p_0 = 0.01, 0.03$ (resp. $p_0 = 0.005, 0.01, 0.02, 0.03, 0.04, 0.05, 0.1$), such that $1/P(X < LCL \mid p = p_0)$ and $1/P(X > UCL \mid p = p_0)$ are each within 5% (resp. 6%) of the nominal value $1/0.00135$.

It is important to bear in mind that all these charts are ARL-biased control charts, that is, they take longer, in average, to detect some shifts in the process mean than to trigger a false alarm. The reader should note that the term *ARL-unbiased* is due to Pignatiello et al. [16], who used it to coin any control chart for which all out-of-control ARL values are smaller than the in-control ARL.

1.2 A Brief Review on ARL-Unbiased Charts

The first attempt to obtain control limits that correct the bias of the ARL function of the np -chart is attributed to Acosta-Mejía [1].

By differentiating the probability of triggering a signal with respect to p and conditioning this derivative to be equal to zero when p is equal to its target value p_0 , Acosta-Mejía [1] derived the equation

$$\frac{p_0^{LCL-1}(1-p_0)^{n-LCL}}{\Gamma(n-LCL+1)\Gamma(LCL)} = \frac{p_0^{UCL}(1-p_0)^{n-UCL-1}}{\Gamma(n-UCL)\Gamma(UCL+1)}, \quad (1)$$

where LCL and UCL represent the desired lower and upper control limits and Γ denotes the gamma function. Equation (1) defines what Acosta-Mejía [1] called the unbiased performance line (UPL) and relating the control limits of an np -chart with an ARL-unbiased function. Since this equation leads in general to non-integer control limits, Acosta-Mejía [1] suggested the adoption of the pair of integers closest to the intersection point of the UPL and the iso-ARL curve that defines all pairs (LCL, UCL) having the same desired in-control ARL. The resulting chart is ARL-biased, nonetheless in [1] it was termed *nearly* ARL-unbiased np -chart.

Considerable attention has been given to ARL-unbiased charts for parameters of absolutely continuous quality characteristics; we refer the reader to [10, 11] for a partial list, and also to papers such as [2, 4, 5, 7, 8, 13, 23].

As discrete quality characteristics are concerned, besides the nearly unbiased geometric and the cumulative count conforming (CCC) charts by L. Zhang et al. [25] and C. W. Zhang et al. [24], we ought to mention that an ARL-unbiased EWMA- p control chart has been recently proposed by Yang and Arnold [22] to monitor the variance for process data with non-normal or unknown distributions. Moreover, Paulino et al. [14, 15] proposed an ARL-unbiased c -chart to monitor the expected number of nonconformities in a sample with fixed size; the approach was inspired in randomized tests, rarely used in an SPC setting, and in uniformly most powerful unbiased (UMPU) tests.

Surprisingly, no ARL-unbiased np -chart has been proposed, as far as we have investigated. Thus, in the remainder of this paper, we shall adapt the approach by Paulino et al. [14, 15] to the binomial distribution to define and illustrate an np -chart associated with: an ARL profile with a maximum when the process is in-control; and a pre-specified false alarm rate equal to α .

2 Eliminating the Bias of the ARL Function

Knoth and Morais [11] refer that the concept of an ARL-unbiased Shewhart-type chart is related to the notion of *unbiased test*. Recall that a size α test for $H_0 : p = p_0$ against $H_1 : p \neq p_0$, with power function $\xi(p)$, is said to be *unbiased* if $\xi(p_0) \leq \alpha$ and $\xi(p) \geq \alpha$, for $p \neq p_0$, that is, in ARL terms:

$$\text{ARL}(p_0) \geq \alpha^{-1}, \quad \text{ARL}(p) \leq \alpha^{-1}, \quad p \neq p_0.$$

To fulfill these two conditions we can capitalize on the UMPU test, derived by Lehmann [12, Example 1, pp. 128–129], for the parameter p of the binomial distribution. However, the two equations defining this test (see [12, p. 128]) are not sufficient to define the two control limits, LCL and UCL, and the randomization probabilities, say γ_{LCL} and γ_{UCL} . As a consequence, we follow [15] and adopt alternative control limits based on quantiles of the binomial distribution and combine them with the randomization probabilities satisfying the two equations found in [12, p. 128], as we describe below.

Definition. The ARL-unbiased np -chart, with in-control ARL equal to α^{-1} , uses what Paulino et al. [15] termed *quantile based control limits*; these limits are denoted by LCL and UCL and are defined as the largest and smallest non-negative integers verifying

$$P(X < \text{LCL} \mid p = p_0) \leq \alpha_{\text{LCL}}, \quad P(X > \text{UCL} \mid p = p_0) \leq \alpha_{\text{UCL}},$$

where $\alpha_{\text{LCL}} + \alpha_{\text{UCL}} = \alpha$. This chart emits a signal at sample t with

- *probability one* if the observed number of nonconforming items, x_t , is below LCL or above UCL;
- *probability γ_{LCL} (resp. γ_{UCL})* if x_t is equal to LCL (resp. UCL), where γ_{LCL} and γ_{UCL} are obtained by solving the system of *linear* equations

$$\gamma_{\text{LCL}} \times P_{n,p_0}(\text{LCL}) + \gamma_{\text{UCL}} \times P_{n,p_0}(\text{UCL}) = \alpha - \left[1 - \sum_{x=\text{LCL}}^{\text{UCL}} P_{n,p_0}(x) \right], \quad (2)$$

$$\gamma_{\text{LCL}} \times \text{LCL} \times P_{n,p_0}(\text{LCL}) + \gamma_{\text{UCL}} \times \text{UCL} \times P_{n,p_0}(\text{UCL}) = \alpha \times np_0 - \left[np_0 - \sum_{x=\text{LCL}}^{\text{UCL}} x \times P_{n,p_0}(x) \right]. \quad (3)$$

We ought to mention the following:

- (a) For testing $H_0 : p = p_0$ against $H_1 : p \neq p_0$, we make use of the critical function

$$\phi(x) = P(\text{Reject } H_0 \mid X = x) = \begin{cases} 1 & \text{if } x < \text{LCL or } x > \text{UCL}, \\ \gamma_{\text{LCL}} & \text{if } x = \text{LCL}, \\ \gamma_{\text{UCL}} & \text{if } x = \text{UCL}, \\ 0 & \text{if } \text{LCL} < x < \text{UCL}, \end{cases}$$

where X is a sufficient statistic for p and γ_{LCL} and γ_{UCL} are determined by

$$E_{n,p_0}[\phi(X)] = \alpha, \quad E_{n,p_0}[X\phi(X)] = \alpha \times E_{n,p_0}(X).$$

- (b) Equation (2) guarantees that $\text{ARL}(p_0) = \alpha^{-1}$, whereas equation (3) ensures that the np -chart is indeed ARL-unbiased.
- (c) The second equation defining the randomizing probabilities in [12, p. 128] is nothing but (3) rewritten by taking advantage of the identity $x \times P_{n,p_0}(x) = np_0 \times P_{n-1,p_0}(x-1)$.
- (d) The quantile based control limits ensure that the in-control ARL is not smaller than the requested nominal value α^{-1} and the adoption the randomization increases the probability of triggering a signal, thus bringing down the in-control ARL to its desired value.
- (e) A description of how to obtain the control limits (LCL, UCL) and the associated and admissible randomization probabilities (γ_{LCL} , γ_{UCL}), satisfying (2) and (3), can be found in Appendix A; we propose a linear search of (LCL, UCL) in a grid of points defined by (5), until the solution (γ_{LCL} , γ_{UCL}) given by (4) belongs to the unit square $(0, 1)^2$.

- (f) The randomization of the emission of the signal – whenever the observed number of nonconforming items is equal to LCL (resp. UCL) – can be done in practice by incorporating the generation of a pseudo-random number from a Bernoulli distribution with parameter γ_{LCL} (resp. γ_{UCL}) in the software used to monitor the data fed from the production line.
- (g) The conversion to the corresponding ARL-unbiased p -chart is evidently made by dividing the quantile based control limits by n .

Finally, it is worth noting that the signal is triggered by the ARL-unbiased np -chart with probability

$$\xi_{\text{unbiased}}(p) = \left[1 - \sum_{x=LCL}^{UCL} P_{n,p}(x) \right] + \gamma_{LCL} \times P_{n,p}(LCL) + \gamma_{UCL} \times P_{n,p}(UCL)$$

and the corresponding ARL function is given by $1/\xi_{\text{unbiased}}(p)$.

3 Illustrations

The results presented in this section refer to existing np -charts and, obviously, to the ARL-unbiased np -chart we have just proposed.

Example 1. In the first illustration of this section, we consider $n = 1000$ and $p_0 = 0.01$, and plot the ARL functions of the np -charts associated with what Acosta-Mejía [1, Section 7] determined to be the integer pairs of control limits (or points), (LCL, UCL), closest to the intersection of the UPL and the iso-ARL curve for an in-control ARL equal to $\alpha^{-1} = 300$: $B = (2, 19)$; $C = (3, 20)$; $D = (3, 21)$.

Figure 1 confirms that all these three charts are ARL-biased. The maximum and the relative bias of these ARL functions, defined as $\max_{p \in (0,1)} 1/\xi(p)$ and $[p_0 - \arg\max_{p \in (0,1)} 1/\xi(p)]/p_0 \times 100\%$, respectively, along with the in-control ARL values can be found in Table 1.

The smallest relative bias in Table 1 corresponds to $C = (3, 20)$ and is associated with what Acosta-Mejía [1] termed the *nearly* ARL-unbiased np -chart for $ARL(p_0) = 300$. Regrettably, it is the np -chart with the in-control ARL furthest from the pre-specified in-control ARL. More importantly, since that in-control ARL is smaller than the desired value, it is futile to complement these control limits with some randomization probabilities to bring the in-control ARL precisely to 300 and obtain an ARL-unbiased np -chart.

Now, it is time to derive the ARL-unbiased np -chart with a false alarm rate equal to $\alpha = 1/300$.

Its control limits and randomization probabilities are given by $[LCL, UCL] = [2, 21]$ and $(\gamma_{LCL}, \gamma_{UCL}) = (0.673094, 0.853994)$. Furthermore, Figure 2 substantiates our allegations that the ARL-unbiased np -chart has its in-control ARL equal to the pre-specified value of $\alpha^{-1} = 300$, and the chart requires, in average, less time to trigger a valid signal in the presence of any shift in p than to trigger a false alarm.

Interestingly enough, the np -chart associated $D = (3, 21)$, whose in-control ARL is very close to 300 and whose ARL function is positively biased, outperforms (resp. is outperformed by) the ARL-unbiased np -chart when it comes to the detection of decreases (resp. increases) in the fraction nonconforming.

Figure 2 also includes the graph of the ARL function of the ARL-unbiased np -chart, when we shrink the sample size to a tenth of its original value. In this case, we got $[LCL, UCL] = [0, 5]$, randomization probabilities $(\gamma_{LCL}, \gamma_{UCL}) = (0.007621, 0.003168)$ and an expected decrease in the detection power of the np -chart due to the sample size reduction.

Having said that, we ought to mention that we are not dealing with an upper one-sided np -chart, even though $LCL = 0$ for $n = 100$. This appealing feature is guaranteed by the randomization of our decision to trigger a signal whenever the observed number of nonconforming items is equal to $LCL = 0$ or $UCL = 5$.

Example 2. We start by confronting the ARL functions of: the np -chart with 3-sigma control limits; the np -chart with Ryan and Schwertman's *optimal* control limits and the *smallest absolute total deviation* between the reciprocal of the nominal tail areas and $1/0.00135$ (resp. *smallest sample size*) taken from [19, Table 7]; the ARL-unbiased np -chart. The two target values of the fraction nonconforming are $p_0 = 0.01, 0.03$, and the false alarm rate is obviously $\alpha = 0.0027$.

Figure 1. ARL(p) curves associated with the pairs of control limits $B = (2, 19)$, $C = (3, 20)$, and $D = (3, 21)$, $n = 1000$, $p_0 = 0.01$, $\alpha = 1/300$ and $p \in [0.5p_0, 1.5p_0]$.

(LCL, UCL)	Maximum of ARL	Relative bias of the ARL	In-control ARL
$B = (2, 19)$	458.698	-10.901%	265.421
$C = (3, 20)$	241.056	+1.237%	239.469
$D = (3, 21)$	336.472	+5.219%	300.187

Table 1. LCL, UCL, maximum, relative bias and in-control ARL values, $n = 1000$, $p_0 = 0.01$ and $\alpha = 1/300$.

Figure 2. ARL(p) curves of the np -chart with $D = (\text{LCL}, \text{UCL}) = (3, 21)$ and $n = 1000$, and the ARL-unbiased np -charts ($n = 1000, 100$), $p_0 = 0.01$, $\alpha = 1/300$ and $p \in [0.5p_0, 1.5p_0]$.

Not at all surprisingly, the graphs in Figures 3 and 4 and the results in Tables 2 and 3 lead to the conclusion that the np -charts with 3-sigma control limits or the ones proposed by Ryan and Schwertman [19] are all ARL-biased. However, judging by the bottom graphs of Figures 3 and 4, the np -charts with Ryan and Schwertman's *optimal* control limits are *nearly* ARL-unbiased, for $(n, p_0) = (0.01, 661), (0.03, 356)$.

The elimination of the bias of the ARL function is due to the adoption of the control limits and the randomization probabilities in Table 4.

We can add that the ARL-unbiased np -charts have indeed a maximum of $\alpha^{-1} \approx 370.4$ at $p = p_0$, as depicted by Figures 3 and 4.

The parameters defining the ARL-unbiased np -chart, for $\alpha = 0.0027$ and a wider constellation of values of (n, p_0) and $(\lfloor n/10 \rfloor, p_0)$, can be found in Table 5 in Appendix A. This table is surely useful for the quality control practitioner willing to implement an ARL-unbiased np -chart and to reduce the sample size to practical levels.

Figure 3. $ARL(p)$ curves of the: np -chart with 3-sigma control limits; np -chart associated with Ryan and Schwertman's optimal limits and the smallest absolute total deviation; ARL-unbiased np -chart, $(n, p_0) = (1267, 0.01)$ (top, $p \in [0.5p_0, 1.5p_0]$) and $(n, p_0) = (1541, 0.03)$ (bottom, $p \in [p_0 - 0.001, p_0 + 0.001]$).

Figure 4. $ARL(p)$ curves of the: np -chart with 3-sigma control limits; np -chart associated with Ryan and Schwertman's optimal limits and the smallest sample size; ARL-unbiased np -chart, $(n, p_0) = (661, 0.01)$ (top), $(n, p_0) = (356, 0.03)$ (bottom) and $p \in [0.5p_0, 1.5p_0]$.

p_0	n	[LCL, UCL]	Maximum of ARL	Relative bias of the ARL	In-control ARL
0.01	1267	[3, 23]	650.419	−10.723%	327.976
		[4, 24]	381.718	−1.449%	376.811
0.03	1541	[27, 66]	410.361	−2.553%	353.028
		[28, 67]	367.444	−0.316%	366.590

Table 2. LCL, UCL, maximum, relative bias and in-control ARL values, listed in order corresponding to the np -charts with 3-sigma control limits and Ryan and Schwertman's *optimal* control limits with the *smallest absolute total deviation*.

p_0	n	[LCL, UCL]	Supremum of ARL	Relative bias of the ARL	In-control ARL
0.01	661	[0, 14]	$+\infty$	−100%	305.807
		[1, 15]	394.459	−2.588%	386.156
0.03	356	[2, 20]	715.586	−12.648%	319.097
		[3, 21]	368.641	−1.203%	365.847

Table 3. LCL, UCL, supremum, relative bias and in-control ARL values, listed in order corresponding to the np -charts with 3-sigma control limits and Ryan and Schwertman's *optimal* control limits with the *smallest sample size*.

p_0	n	[LCL, UCL]	$(\gamma_{LCL}, \gamma_{UCL})$
0.01	1267	[4, 25]	(0.076400, 0.713818)
	661	[1, 16]	(0.040135, 0.704839)
0.03	1541	[28, 68]	(0.079969, 0.771860)
	356	[3, 22]	(0.036328, 0.744955)

Table 4. Quantile based control limits and randomization probabilities of the ARL-unbiased np -chart.

4 Final Thoughts

We came a long way since Walter A. Shewhart proposed the p -chart in the 1920s (see [21]).

In this paper we described a method to obtain an np -chart such that it has a pre-specified in-control ARL, as opposed to the np -chart with 3-sigma control limits or the ones proposed by Ryan and Schwertman [19] and Acosta-Mejía [1].

We strongly believe that the method to obtain it is much simpler than the one proposed by Acosta-Mejía [1], not to mention the fact that the resulting chart can be defined for any values of p_0 unlike the *optimal* control limits derived by Ryan and Schwertman [19].

Moreover, the np -chart we described is undoubtedly ARL-unbiased – in contrast to the ones derived by Ryan and Schwertman [19] and Acosta-Mejía [1] –, in the sense that any shift in p leads to a valid signal triggered in less time, in average, than a false alarm. Therefore the ARL-unbiased np -chart can be recommended as a tool for continuous improvement.

By relying on randomization probabilities to trigger a signal when the observed number of nonconforming items is equal to LCL or UCL, the ARL-unbiased np -chart can definitively tackle small sample sizes and the associated *curse of the null LCL*, and still detect decreases in p in a timely fashion, as opposed to any np -charts proposed so far.

Randomizing the emission of the signal should not constitute a problem, for it can be easily incorporated in the statistical quality control procedure linked on-line to operate together with the automatic data acquisition and monitoring system.

It would be very interesting to derive an ARL-unbiased version of the cumulative sum (CUSUM) chart for binomial data, in order to improve the detection of small-to-moderate shifts in the fraction nonconforming. However, since the control statistics of the CUSUM chart are dependent r.v., the probability of triggering a signal at sample t is time-varying, quantile based control limits are not reasonable and we have to resort to

search methods to determine alternative control limits, as rightly noted by Paulino et al. [15]. In addition, since it is not possible to derive a pair of equations such as (2) and (3) to define the randomization probabilities, we have to explore a simultaneous search such as the two-dimensional secant rule in order to deal with a CUSUM chart for binomial output that is ARL-unbiased. Needless to say that it would be useful to compare the performance of this CUSUM chart with the one of the ARL-unbiased EWMA- p chart proposed by Yang and Arnold [22] to control the process variance while dealing with non-normal or unknown distributions.

Please note that simple programs for *Mathematica* (<http://reference.wolfram.com/legacy/v5/>) were written to produce all the results and graphs and will be made available to those who are interested and request them from the author.

A Appendix

One of the purposes of this appendix is to describe the obtention of the control limits and the admissible values of γ_{LCL} and γ_{UCL} which characterize the ARL-unbiased np -chart.

Firstly, note that the system of linear equations (2) and (3) is associated with a square matrix whose determinant is equal to

$$P_{n,p_0}(LCL) \times UCL \times P_{n,p_0}(UCL) - P_{n,p_0}(UCL) \times LCL \times P_{n,p_0}(LCL);$$

thus, for $LCL, UCL \in \{0, 1, \dots, n\}$ and $LCL < UCL$, the matrix is singular if and only if $LCL = UCL$ which is out of the question. Consequently, there is one solution for that system:

$$\gamma_{LCL} = \frac{de - bf}{ad - bc} \quad \text{and} \quad \gamma_{UCL} = \frac{af - ce}{ad - bc}, \quad (4)$$

where

$$\begin{aligned} a &= P_{n,p_0}(LCL), \quad b = P_{n,p_0}(UCL), \quad c = LCL \times P_{n,p_0}(LCL), \quad d = UCL \times P_{n,p_0}(UCL), \\ e &= \alpha - 1 + \sum_{x=LCL}^{UCL} P_{n,p_0}(x), \quad f = \alpha \times np_0 - np_0 + \sum_{x=LCL}^{UCL} x \times P_{n,p_0}(x). \end{aligned}$$

Secondly, we have to check whether or not this solution belongs to the unit square $(0, 1)^2$. Thus, consider $\eta \in [0, 1]$ and let

- $LCL(\eta)$ be the largest non-negative integer LCL satisfying $F_{n,p_0}(LCL - 1) = P(X < LCL \mid p = p_0) \leq \eta$, that is,

$$LCL(\eta) = \begin{cases} F_{n,p_0}^{-1}(\eta), & \text{if } F_{n,p_0}[F_{n,p_0}^{-1}(\eta)] > \eta, \\ F_{n,p_0}^{-1}(\eta) + 1, & \text{otherwise,} \end{cases}$$

where $F_{n,p_0}^{-1}(\eta)$ denotes the quantile of order η with respect to the target binomial distribution;

- $\alpha_{LCL(\eta)} = F_{n,p_0}[LCL(\eta) - 1]$ be the lower tail in-control area associated with the lower control limit $LCL(\eta)$;
- $UCL_{LCL(\eta)} = F_{n,p_0}^{-1}[1 - (\alpha - \alpha_{LCL(\eta)})]$ be the corresponding upper control limit.

Thirdly, since giving protection to decreases (resp. increases) in p means a lower (resp. upper) control limit as large (resp. small) as possible, the search for admissible values of γ_{LCL} and γ_{UCL} , in all the cases considered in this paper, has been restricted to the set of control limits

$$\begin{aligned} &\{(LCL(\alpha), UCL_{LCL(\alpha)}), (LCL(\alpha), UCL_{LCL(\alpha)} + 1), (LCL(\alpha) - 1, UCL_{LCL(\alpha)-1}), \\ &(LCL(\alpha) - 1, UCL_{LCL(\alpha)-1} + 1), \dots, (0, UCL_0), (0, UCL_0 + 1)\}; \end{aligned} \quad (5)$$

and the search routine ends as soon as the first solution of the linear system of equations (2) and (3) is found in the unit square $(0, 1)^2$.

Finally, we present Table 5 with the parameters of the ARL-unbiased np -charts not only for the pairs (n, p_0) in [19, Table 1], but also for $(\lfloor n/10 \rfloor, p_0)$.

p_0	n	[LCL, UCL]	$(\gamma_{LCL}, \gamma_{UCL})$	$\lfloor n/10 \rfloor$	[LCL, UCL]	$(\gamma_{LCL}, \gamma_{UCL})$
0.005	1324	[1, 16]	(0.039089, 0.642052)	132	[0, 5]	(0.004567, 0.554449)
0.01	664	[1, 16]	(0.045716, 0.646175)	66	[0, 5]	(0.004573, 0.599389)
	1267	[4, 25]	(0.076399, 0.713818)	126	[0, 6]	(0.007775, 0.141892)
0.02	533	[3, 22]	(0.017480, 0.683500)	53	[0, 6]	(0.006534, 0.655018)
	708	[5, 27]	(0.017478, 0.712931)	70	[0, 6]	(0.008927, 0.008561)
0.03	357	[3, 22]	(0.045553, 0.691577)	35	[0, 6]	(0.006507, 0.795807)
	474	[5, 27]	(0.059043, 0.716629)	47	[0, 6]	(0.009037, 0.027439)
	874	[13, 43]	(0.051330, 0.737303)	87	[0, 9]	(0.027701, 0.485802)
	923	[14, 45]	(0.089674, 0.865971)	92	[0, 9]	(0.031914, 0.228522)
0.04	218	[2, 19]	(0.038876, 0.702542)	21	[0, 5]	(0.005362, 0.269840)
	268	[3, 22]	(0.062363, 0.772698)	26	[0, 6]	(0.006481, 0.966599)
	393	[6, 29]	(0.029994, 0.744246)	39	[0, 7]	(0.010418, 0.655596)
	620	[12, 41]	(0.084557, 0.672194)	62	[0, 9]	(0.024866, 0.906530)
	755	[15, 48]	(0.990580, 0.771784)	75	[0, 10]	(0.040897, 0.969212)
	893	[20, 55]	(0.071036, 0.844692)	89	[0, 11]	(0.070106, 0.978386)
0.05	175	[2, 19]	(0.056816, 0.741418)	17	[0, 5]	(0.005422, 0.299790)
	315	[6, 29]	(0.064331, 0.804571)	31	[0, 7]	(0.010397, 0.784981)
	345	[7, 31]	(0.034659, 0.759024)	34	[0, 7]	(0.011903, 0.379684)
	466	[11, 39]	(0.082700, 0.734550)	46	[0, 8]	(0.021011, 0.218476)
	606	[16, 48]	(0.092161, 0.756650)	60	[0, 9]	(0.041372, 0.026530)
0.1	104	[3, 21]	(0.041042, 0.732947)	10	[0, 5]	(0.006285, 0.243150)
	139	[5, 26]	(0.071442, 0.81354)	13	[0, 6]	(0.008391, 0.569880)
	154	[6, 28]	(0.030245, 0.745744)	15	[0, 6]	(0.010116, 0.158061)
	229	[11, 38]	(0.050072, 0.800832)	22	[0, 8]	(0.020094, 0.784673)
	299	[16, 47]	(0.086104, 0.864793)	29	[0, 9]	(0.039811, 0.404286)
	339	[19, 52]	(0.076982, 0.853738)	33	[0, 10]	(0.059491, 0.760501)

Table 5. Quantile based control limits and randomization probabilities of the ARL-unbiased np -charts, for $p_0 = 0.005, 0.01, 0.02, 0.03, 0.04, 0.05, 0.1$.

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