Research Article

Mustafa Ç. Korkmaz and Haitham M. Yousof*

The One-Parameter Odd Lindley Exponential Model: Mathematical Properties and Applications

DOI: 10.1515/eqc-2017-0008

Received March 1, 2017; accepted June 20, 2017

Abstract: In this article, an exponential model with only one shape parameter, which can be used in modeling survival data, reliability problems and fatigue life studies, is studied. We derive explicit expressions for some of its statistical and mathematical quantities including the ordinary moments, generating function, incomplete moments, order statistics, moment of residual life and reversed residual life. The model parameter is estimated by using the maximum likelihood method. A real data application is given to illustrate the flexibility of the model. We assess the performance of the maximum likelihood estimators in terms of biases and mean squared errors by means of a simulation study.

Keywords: Odd Lindley-G Family, Exponential Distribution, Simulation Study

MSC 2010: 62-XX

1 Introduction

Among the parametric distributions, the exponential distribution is perhaps the most widely applied statistical model in several fields. One of the reasons for its importance is that the exponential model has constant failure rate function. Additionally, this model was the first lifetime model for which statistical methods were extensively developed in the life testing literature. The exponential distribution is used for the waiting time until the first event in a random process where events are occurring at a given rate. It is a relatively simple distribution; a random variable having this distribution is necessarily positive, and it is one of the more important distributions among those used for positive random variables. The probability density function (PDF) and the cumulative distribution function (CDF) of a random variable X with exponential distribution are

$$g(x; \lambda) = \lambda \exp(-\lambda x) \quad \text{and} \quad G(x; \lambda) = 1 - \exp(-\lambda x), \quad (1.1)$$

respectively, where $\lambda > 0$ and $x > 0$. The moments, the moment generating function (mgf) and several other properties of this distribution can be expressed in terms of elementary functions; see, for example, [25, Chapter 19], [12] and [30, Chapter 8].

In this paper, an alternative distribution with one parameter of exponential (Exp) distribution is presented on the basis of the odd Lindley-G (OL-G) family of distributions which was introduced in [40] with one positive scale parameter α . In this article we consider the special case OL-G with scale parameter $\alpha = 1$. Then

Mustafa Ç. Korkmaz: Department of Measurement and Evaluation in Education, Artvin Çoruh University, Artvin, Turkey, e-mail: mcagatay@artvin.edu.tr

***Corresponding author: Haitham M. Yousof:** Department of Statistics, Mathematics and Insurance, Benha University, Benha, Egypt, e-mail: haitham.youssof@fcom.bu.edu.eg

the PDF and CDF of the OL-G family of distribution are reduced to

$$f(x; \xi) = \frac{g(x; \xi)}{2\bar{G}(x; \xi)^3} \exp\left[-\frac{G(x; \xi)}{\bar{G}(x; \xi)}\right] \quad (1.2)$$

and

$$F(x; \xi) = 1 - \frac{1 + \bar{G}(x; \xi)}{2\bar{G}(x; \xi)} \exp\left[-\frac{G(x; \xi)}{\bar{G}(x; \xi)}\right], \quad (1.3)$$

respectively. To this end, we use equations (1.1), (1.2) and (1.3) to obtain the so-called Type II Odd Lindley Exponential model (TIIOLExp) PDF (for $x \geq 0$)

$$f_{\text{TIIOLExp}}(x; \lambda) = \frac{\lambda}{2} \exp[1 + 2\lambda x - \exp(\lambda x)], \quad x \geq 0. \quad (1.4)$$

The corresponding CDF of (1.1) is given by

$$F_{\text{TIIOLExp}}(x; \lambda) = 1 - \frac{1 + \exp(-\lambda x)}{2 \exp(-\lambda x)} \exp[1 - \exp(\lambda x)], \quad x \geq 0, \quad (1.5)$$

and the hazard rate function (HRF) is given by

$$h(x; \lambda) = \frac{\lambda \exp(\lambda x)}{1 + \exp(-\lambda x)}.$$

The quantile function of the TIIOLExp distribution is given as follows: if U has a uniform random number on $U(0, 1)$, then

$$X = -\frac{1}{\lambda} \log[-(1 + W_{-1}\{2(U - 1) \exp(-2)\})^{-1}]$$

has random number on the TIIOLExp distribution, where W_{-1} denotes the negative branch of the Lambert W-function. We note that the Lambert W-function is defined as $W(z) \exp[W(z)] = z$. The Lambert W-function has two real branches. The lower branch, W_{-1} , is defined in the interval $[-\exp(-1), 1]$ and the upper branch, W_0 , is defined in the interval $[-\exp(-1), \infty]$ (see [40]). Also, the random number from the TIIOLExp distribution can be obtained with the solution of its CDF inverse.

The TIIOLExp density function can be expressed as an infinite mixture of exponentiated-exponential (Exp-Exp) density functions

$$f(x) = \sum_{j,k=0}^{\infty} v_{j,k} \pi_{j+k+1}(x), \quad (1.6)$$

where

$$v_{j,k} = \frac{(-1)^k \Gamma(j+k+3)}{2(j+k+1)j!k!\Gamma(k+3)}$$

and

$$\pi_{\gamma}(x) = \gamma \frac{\lambda e^{-\lambda x}}{g(x; \lambda)} \frac{[1 - \exp(-\lambda x)]^{\gamma-1}}{[G(x; \lambda)]^{\gamma-1}}$$

represents the Exp-Exp density with power parameter $\gamma > 0$. The CDF of the new model can be given by integrating (1.4) as

$$F(x) = \sum_{j,k=0}^{\infty} v_{j,k} \Pi_{j+k+1}(x),$$

where

$$\Pi_{\gamma}(x) = \frac{[1 - \exp(-\lambda x)]^{\gamma}}{[G(x; \lambda)]^{\gamma}}$$

is the CDF of the Exp-Exp model with power parameter γ . The properties of the Exp-Exp distribution have been studied by many authors. The original paper [19] provided expressions for the survival function, the hazard rate function, the shapes of the PDF and the hazard rate function, stochastic orders, the moment generating function, moments, L-moments, the mean, the variance, the distribution of a sum of random variables following the Exp-Exp model, the distribution of extreme values, the maximum likelihood estimation including the case of censoring, the Fisher information matrix and tests of hypotheses. Since this seminal paper many authors have derived other properties, see [1, 11, 20–22, 26–28, 35–37, 39, 41, 47, 48].

Many extensions for the Exp model can be cited, such as the two-sided generalized Exp model by Korkmaz and Genç [27], the transmuted exponentiated generalized Exp model by Yousof, Afify, Alizadeh, Butt, Hamedani and Ali [42], the Kumaraswamy transmuted exponentiated Exp model by Nofal, Afify, Yousof, Granzotto and Louzada [34], the transmuted geometric Exp model by Afify, Alizadeh, Yousof, Aryal and Ahmad [2], the Kumaraswamy transmuted Exp model by Afify, Cordeiro, Yousof, Alzaatreh and Nofal [4], the complementary geometric transmuted Exp model by Afify, Cordeiro, Nadarajah, Yousof, Ozel, Nofal and Altun [3], the beta transmuted Exp model by Afify, Yousof and Nadarajah [5], the transmuted Weibull Exp model by Alizadeh, Rasekhi, Yousof and Hamedani [6], the complementary generalized transmuted Poisson Exp model by Alizadeh, Yousof, Afify, Cordeiro and Mansoor [7], the exponentiated transmuted Exp model by Merovci, Alizadeh, Yousof and Hamedani [31], the Burr X Exp model by Yousof, Afify, Hamedani and Aryal [44], the Topp–Leone generated Exp model by Aryal, Ortega, Hamedani and Yousof [9], the exponentiated generalized-Exp Poisson model by Aryal and Yousof [10], the Type I general exponential Exp model by Hamedani, Yousof, Rasekhi, Alizadeh and Najibi [23], the generalized transmuted Exp model by Nofal, Afify, Yousof and Cordeiro [33], the exponentiated Weibull Exp model by Cordeiro, Afify, Yousof, Pescim and Aryal [15], the Burr XII Exp model by Cordeiro, Yousof, Ramires and Ortega [16], the Weibull generalized Exp model by Yousof, Majumder, Jahanshahi, Ali and Hamedani [45], the beta Weibull Exp model by Yousof, Rasekhi, Afify, Alizadeh, Ghosh and Hamedani [46] and the Topp–Leone odd log-logistic Exp model by de Brito, Cordeiro, Yousof, Alizadeh and Silva [17], among others.

Let Y be a lifetime random variable having the Exp distribution $G(x; \lambda)$ in (1.1). The odds ratio that an individual (or component) following the lifetime Y will die (failure) at time x is $[\exp(\lambda x) - 1]$. Consider that the variability of this odds of death is represented by the random variable X and assume that it follows the Lindley model with scale = 1. We can write

$$\Pr(Y \leq x) = \Pr\{X \leq [\exp(\lambda x) - 1]\} = F_{\text{TIOLExp}}(x; \lambda)$$

which is given by (1.5).

The paper is outlined as follows. In Section 2, some shapes for the TIOLExp model are provided. In Section 3, we derive some mathematical properties of the new distribution. In Section 4, the model parameter is estimated by using the maximum likelihood (ML) method. A real data application is given in Section 5 to illustrate the flexibility of the TIOLExp model. We assess the performance of the maximum likelihood estimators in terms of biases and mean squared errors by means of a simulation study in Section 6. Finally, some concluding remarks are presented in Section 7.

2 Shapes

For the PDF of the TIOLExp distribution, the first and the second derivatives of $f(x; \lambda)$ are

$$\begin{aligned}\frac{\partial}{\partial x} f(x; \lambda) &= \frac{\lambda}{2} [2\lambda - \lambda \exp(\lambda x)] \exp[1 + 2\lambda x - \exp(\lambda x)], \\ \frac{\partial^2}{\partial x^2} f(x; \lambda) &= \lambda^3 \left\{ \frac{\exp(\lambda x)}{2} [\exp(\lambda x) - 5] + 2 \right\} \exp[1 + 2\lambda x - \exp(\lambda x)].\end{aligned}$$

Let x^* be the critical point of $\frac{\partial f(x; \lambda)}{\partial x}$. Then its solution is $x^* = \lambda^{-1} \log 2$. We obtain $\frac{\partial^2 f(x; \lambda)}{\partial x^2} \big|_{x=x^*} < 0$. Hence, it has a local maximum point and the TIOLExp distribution is unimodal. Also, for $x < x^*$ the PDF is increasing, and for $x > x^*$ the PDF is decreasing. The mode of the distribution is given by x^* . We note that $\lim_{x \rightarrow 0} f(x; \lambda) = \frac{\lambda}{2}$ and $\lim_{x \rightarrow \infty} f(x; \lambda) = 0$.

For the HRF of the TIOLExp distribution, the first derivative of $h(x; \lambda)$ is

$$\frac{\partial}{\partial x} h(x; \lambda) = \left[\frac{\lambda}{1 + \exp(-\lambda x)} \right]^2 [2 + \exp(\lambda x)] > 0 \quad \text{for all } x > 0, \lambda > 0.$$

It follows that the TIOLExp distribution has increasing HRF. We also note that $\lim_{x \rightarrow 0} h(x; \lambda) = \frac{\lambda}{2}$ and $\lim_{x \rightarrow \infty} h(x; \lambda) = \infty$. We sketched the PDF and HRF of the TIOLExp distribution in Figure 1 for selected parameter values. These figures are consistent with the above expressions.

Figure 1. Plots of the TIIOLE PDF and HRF for some parameter values.

3 Some Mathematical Properties

The r th moment of the TIIOLE model can be written as

$$\mu_r' = \Gamma(1+r) \sum_{i,j,k=0}^{\infty} \Upsilon_{i,j,k}^{(j+k+1,r)} \quad \text{for all } r > -1,$$

where

$$\Upsilon_{i,j,k}^{(j+k+1,r)} = \nu_{j,k} \frac{(j+k+1)(-1)^i}{\lambda^r(i+1)^{1+r}} \zeta(j+k, i)$$

and

$$\zeta(j+k, i) = \frac{(j+k) \cdots (j+k-i)}{i!}.$$

The r th incomplete moment of X , say $I_r(t)$, is given by

$$I_r(t) = \int_0^t x^r f(x) dx.$$

Using equation (1.6), we obtain

$$I_r(t) = \gamma\left(1+r, \frac{\lambda}{t}\right) \sum_{i,j,k=0}^{\infty} \Upsilon_{i,j,k}^{(j+k+1,r)} \quad \text{for all } r > -1, \quad (3.1)$$

where

$$\gamma(\zeta, x) = \int_0^x x^{\zeta-1} e^{-x} dx$$

is the incomplete gamma function. The first incomplete moment of X , denoted by $I_1(t)$, is simply determined from (3.1) by setting $r = 1$. The first incomplete moment has important applications related to the Bonferroni and Lorenz curves and the mean residual life and the mean waiting time.

The n th moment of the residual life, say $z_n(t) = E[(X-t)^n | X > t]$, $n = 1, 2, \dots$, uniquely determines $F(x)$. The n th moment of the residual life of X is given by

$$z_n(t) = \frac{1}{1-F(t)} \int_t^{\infty} (x-t)^n dF(x).$$

Therefore,

$$z_n(t) = \frac{\gamma(1+n, \frac{\lambda}{t})}{1-F(t)} \sum_{i,j,k=0}^{\infty} \Upsilon_{i,j,k}^{(j+k+1,n)*},$$

where

$$\Upsilon_{i,j,k}^{(j+k+1,n)*} = v_{j,k} \sum_{r=0}^n \binom{n}{r} (-t)^{n-r}.$$

The mean residual life (MRL) function or the life expectation at age t is defined by

$$z_1(t) = E[(X-t) | X > t],$$

which represents the expected additional life length for a unit which is alive at age t . The MRL of X can be obtained by setting $n = 1$ in the last equation. The n th moment of the reversed residual life, say $Z_n(t) = E[(t-X)^n | X \leq t]$, for $t > 0$ and $n = 1, 2, \dots$, uniquely determines $F(x)$. We obtain

$$Z_n(t) = \frac{1}{F(t)} \int_0^t (t-x)^n dF(x).$$

Then the n th moment of the reversed residual life of X becomes

$$Z_n(t) = \frac{\gamma(1+n, \frac{\lambda}{t})}{F(t)} \sum_{i,j,k=0}^{\infty} \Upsilon_{i,j,k}^{(j+k+1,n)**},$$

where

$$\Upsilon_{i,j,k}^{(j+k+1,n)**} = v_{j,k} \sum_{r=0}^n (-1)^r \binom{n}{r} t^{n-r}.$$

The mean inactivity time (MIT), also called the mean reversed residual life function, is given by

$$Z_1(t) = E[(t-X) | X \leq t],$$

and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$.

Let $X_1, \dots, X_n$ be a random sample from the THOLExp model of distributions and let $X_{1:n}, \dots, X_{n:n}$ be the corresponding order statistics. The PDF of the i th order statistic, say $X_{i:n}$, can be expressed as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} F(x)^{i-1} [1-F(x)]^{n-i}, \quad (3.2)$$

where $B(\cdot, \cdot)$ is the beta function. Substituting (1.1) and (1.2) into (3.2), we obtain

$$f_{i:n}(x) = \sum_{m,p=0}^{\infty} \sum_{j=0}^{k+n-i} v_{j,m,p} \pi_{j+m+p}(x),$$

where

$$v_{j,m,h} = \sum_{k=0}^{i-1} \frac{(-1)^{k+m} 2^{-(j+1)} \binom{j+m+p}{j+m} \binom{k+n-1}{j} \binom{i-1}{k}}{m! B(i, n-i+1) (j+m+p+1)}.$$

Then the q th moment of $X_{i:n}$ is given by

$$E(X_{i:n}^q) = \Gamma(1+q) \sum_{m,p=0}^{\infty} \sum_{j=0}^{k+n-i} \Upsilon_{j,m,p,w}^{(j+m+p,q)} \quad \text{for all } q > -1,$$

where

$$\Upsilon_{j,m,p,w}^{(j+m+p,q)} = v_{j,m,p} \Upsilon_w^{(j+m+p,q)}.$$

4 Estimation

Given a random sample $X_1, X_2, \dots, X_n$ of size n from the **TIIOLE**Exp distribution with the PDF (1.4), the log-likelihood function is given by

$$\ell(\lambda; x_1, x_2, \dots, x_n) = \ell = n \log \frac{\lambda}{2} + n + 2\lambda \sum_{i=1}^n x_i - \sum_{i=1}^n \exp(\lambda x_i). \quad (4.1)$$

The ML estimate $\hat{\lambda}$ of λ is the solution of the non-linear equation

$$\frac{1}{\lambda} + 2\bar{x} - \frac{1}{n} \sum_{i=1}^n x_i \exp(\lambda x_i) = 0, \quad (4.2)$$

where $\bar{x}$ is the sample mean. To obtain the MLE of λ , we can maximize (4.1) directly with respect to λ or we can solve the non-linear equation (4.2). Note that ML estimate of the λ cannot be solved analytically; numerical iteration techniques, such as the Newton–Raphson algorithm, are adopted to solve the log-likelihood equation for which (4.1) is maximized.

The **TIIOLE**Exp distribution satisfies all the regularity conditions (see [24, Chapter 6]). Therefore applying the usual large sample approximation, the estimators $\hat{\lambda}$ are treated as being approximately normal with mean λ and variance I^{-1} , where I is the Fisher information which is given by

$$I(\lambda) = E\left(-\frac{\partial^2 \ell}{\partial \lambda^2}\right) = \frac{n}{\lambda^2} + nE(X^2 \exp(\lambda X)) = \frac{n}{\lambda^2} + \frac{n}{2\lambda^2} \int_1^{\infty} (u \log u)^2 \exp(1-u) du. \quad (4.3)$$

Using MAPLE, the integral in (4.3) is obtained as

$$\int_1^{\infty} (u \log u)^2 \exp(1-u) du = 2 \exp(1) \left[\frac{\pi^2}{6} - \frac{5}{4} + \left(\frac{3}{2} - \gamma\right)^2 - \frac{1}{27} {}_3F_3([3, 3, 3], [4, 4, 4], -1) \right],$$

where γ is Euler's constant which is approximately $0.5772156649 \dots$ and ${}_pF_q([n], [d], \Delta)$ is the generalized hypergeometric function. This function is also known as Barnes's extended hypergeometric function. The definition of ${}_pF_q([n], [d], \Delta)$ is given by

$${}_pF_q([n], [d], \Delta) = \sum_{j=0}^{\infty} \frac{\Delta^j \prod_{i=1}^p \Gamma(n_i + j) \Gamma^{-1}(n_i)}{\Gamma(1+j) \prod_{i=1}^q \Gamma(d_i + j) \Gamma^{-1}(d_i)},$$

where $n = [n_1, n_2, \dots, n_p]$, p is the number of operands of n , $d = [d_1, d_2, \dots, d_q]$, q is the number of operands of d and $\Gamma(\cdot)$ is the gamma function. The generalized hypergeometric function is quickly evaluated and readily available in standard software, such as MAPLE and MATHEMATICA. Now, the asymptotic $100(1 - \alpha)\%$ confidence interval for $\hat{\lambda}$ is given by

$$\hat{\lambda} \mp z_{\alpha/2} \sqrt{\frac{I^{-1}(\hat{\lambda})}{n}},$$

where $z_{\alpha/2}$ is the quantile with $1 - \frac{\alpha}{2}$ of the standard normal distribution.

5 Real Data Modeling

In this section, we provide an application to see the data modeling ability of the **TIIOLE**Exp model. For the **TIIOLE**Exp model, we analyzed the stress-rupture life of kevlar 49/epoxy strands which were subjected to constant sustained pressure at the 70 % stress level until all had failed. This data set was studied by Barlow, Toland and Freeman [13], Andrews and Herzberg [8] and Cooray and Ananda [14]. The failure times in

hours are 1051, 1337, 1389, 1921, 1942, 2322, 3629, 4006, 4012, 4063, 4921, 5445, 5620, 5817, 5905, 5956, 6068, 6121, 6473, 7501, 7886, 8108, 8546, 8666, 8831, 9106, 9711, 9806, 10205, 10396, 10861, 11026, 11214, 11362, 11604, 11608, 11745, 11762, 11895, 12044, 13520, 13670, 14110, 14496, 15395, 16179, 17092, 17568, 17568. Using this data set, we fit the **TIIOLExp**, the exponential (E), the Lindley (L) (Lindley [29]), the generalized exponential (GE) (Gupta and Kundu [19]), the Nadarajah and Haghighi exponential (NH) (Nadarajah and Haghighi [32]), the extended exponential (EE) (Gomez, Bolfarine and Gomez [18]) and the Xgamma (XG) (Sen, Maiti and Chandra [38]) distribution models. The CDFs of the GE, NH, L, EE and XG models are respectively (for $x > 0$, $\alpha, \lambda > 0$)

$$\begin{aligned} F_{GE}(x; \alpha, \lambda) &= [1 - \exp(-\lambda x)]^\alpha, \\ F_{NH}(x; \alpha, \lambda) &= 1 - \exp[1 - (1 + \lambda x)^\alpha], \\ F_L(x; \lambda) &= 1 - (1 + \lambda + \lambda x)(1 + \lambda)^{-1} \exp(-\lambda x), \\ F_{EE}(x; \alpha, \lambda) &= 1 - (\alpha + \lambda + \alpha \lambda x)(\alpha + \lambda)^{-1} \exp(-\lambda x), \\ F_{XG}(x; \lambda) &= 1 - \left(1 + \lambda + \lambda x + \frac{\lambda^2}{2} x^2\right)(1 + \lambda)^{-1} \exp(-\lambda x). \end{aligned}$$

We compare the **TIIOLExp** model with the above models under the estimated log-likelihood ($\hat{\ell}$) value, the Kolmogorov–Smirnov (K-S) statistics, the Akaike information criteria (AIC), the consistent Akaike information criteria (CAIC), the Bayesian information criteria (BIC) and the Hannan–Quinn information criteria (HQIC). Note that AIC, CAIC, BIC and HQIC are by given by

$$\begin{aligned} \text{AIC} &= -2\hat{\ell} + 2p, \\ \text{CAIC} &= -2\hat{\ell} + 2np(n - p - 1)^{-1}, \\ \text{BIC} &= -2\hat{\ell} + p \log n, \\ \text{HQIC} &= -2\hat{\ell} + 2p \log(\log n), \end{aligned}$$

where p is the number of the estimated model parameters and n is the sample size. The distribution with the smallest AIC, CAIC, BIC, HQIC and K-S values and the biggest log-likelihood and p values of the K-S statistics is chosen as the best model. All calculations are obtained by maxLik routine in the R programme.

The results of this application are listed in Table 1. These results show that the **TIIOLExp** distribution has the lowest AIC, CAIC, BIC, HQIC and K-S values and has the biggest estimated log-likelihood and p -value of the K-S statistics among all the fitted models. So it could be chosen as the best model under these criteria. The estimated CDFs of the application models are displayed in Figure 2. It is clear from this figure that the **TIIOLExp** model provides the best fit to this data set as compared to other models. The 95 % confidence interval for the **TIIOLExp** parameter λ is then computed as $[6.6620 \times 10^{-6}, 0.00011]$.

Model	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	K-S (p -value)
TIIOLExp	–	9.7192×10^{-5} (6.6621×10^{-6})	–480.7314	963.4628	963.5480	965.3547	964.1806	0.1041 (0.6630)
GE	2.8822 (0.6295)	0.0002 (0.00002)	–484.0703	972.1406	972.4014	975.9242	973.5761	0.1184 (0.4988)
EE	26.0101 (4.1943)	0.00022 (0.00002)	–484.4839	972.9679	973.2287	976.7515	974.4034	0.1288 (0.3904)
NH	14.6917 (0.00765)	0.000005 (0.00001)	–485.0580	974.1160	974.3768	977.8996	975.5515	0.1786 (0.0877)
XG	–	0.0003 (0.00002)	–483.2144	968.4289	968.5140	970.3207	969.1466	0.1116 (0.5751)
L	–	0.00022 (0.00002)	–484.4858	970.9717	971.0568	972.8635	971.6894	0.1288 (0.3904)
E	–	0.0001 (0.00001)	–494.0745	990.1491	990.2342	992.0409	990.8668	0.2366 (0.0083)

Table 1. The MLEs (standard errors within the parentheses), $\hat{\ell}$, AIC, CAIC, BIC, HQIC and K-S statistics (p -value) for kevlar data.

Figure 2. Fitted CDFs on empirical CDF.

6 Simulation Study

In this section, we perform the simulation study using the **TIIOLExp** distribution. To see the performance of MLEs of this distribution, we generate 1,000 samples of sizes 20, 50 and 100 from **TIIOLExp** using the inverse of its CDF. We also compute the biases and mean squared errors (MSE) of the MLEs with

$$\text{Bias}_{\hat{\lambda}} = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\lambda}_i - \lambda)$$

and

$$\text{MSE}_{\hat{\lambda}} = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\lambda}_i - \lambda)^2$$

respectively. To get the inverse of the CDF, we use the uniroot routine in the R programme for random generation and use the optim routine for MLEs. The results of the simulation are reported in Table 2. From this table, we observe that when the sample size increases, the empirical mean, the standard deviations (SD), the biases and the MSEs decrease in all the cases, as expected. We also note that the SDs, biases and MSEs increase when λ increases.

7 Conclusions

In this article, an exponential model with only one shape parameter, which can be used in modeling survival data, reliability problems and fatigue life studies, was studied. We derived explicit expressions for some of its statistical and mathematical quantities including the ordinary moments, the generating function, incomplete moments, order statistics, the moment of residual life and reversed residual life. The model parameter was estimated by using the maximum likelihood method. A real data application was given to illustrate the flexibility of the model. We assessed the performance of the maximum likelihood estimators in terms of biases and mean squared errors by means of a simulation study. We hope that the new distribution will attract wider applications in engineering, reliability and other areas of research. Estimation of the **TIIOLExp** model

n	Parameter λ	$\hat{\lambda}$	SD	Bias $_{\hat{\lambda}}$	MSE $_{\hat{\lambda}}$
20	0.25	0.2564	0.0302	0.0064	0.0009
	0.5	0.5135	0.0577	0.0135	0.0035
	1	1.0279	0.1188	0.0279	0.0149
	1.5	1.5498	0.1787	0.0498	0.0344
	3	3.0903	0.3798	0.0903	0.1522
	5	5.1528	0.6108	0.1528	0.3960
	50	51.5325	5.8259	1.5324	36.2560
50	0.25	0.2530	0.0173	0.0030	0.0003
	0.5	0.5056	0.0347	0.0056	0.0012
	1	1.0103	0.0704	0.0102	0.0050
	1.5	1.5163	0.1059	0.0163	0.0114
	3	3.0265	0.2117	0.0265	0.0454
	5	5.0505	0.3442	0.0505	0.1209
	50	50.5831	3.4350	0.5831	12.1274
100	0.25	0.2517	0.0124	0.0017	0.0001
	0.5	0.5024	0.0228	0.0024	0.0005
	1	1.0036	0.0474	0.0036	0.0022
	1.5	1.5105	0.0731	0.0105	0.0054
	3	3.0115	0.1497	0.0115	0.0225
	5	5.0162	0.2428	0.0162	0.0591
	50	50.3917	2.5278	0.3917	6.5371

Table 2. Empirical mean, SD, Bias and MSE of the estimator $\hat{\lambda}$.

parameters under the Bayesian paradigm is currently underway and will be reported in a separate article elsewhere. However, we must make a note of the fact that under the Bayesian setting, a non-informative prior approach is essentially the maximum likelihood estimation under the classical approach. In the absence of an appropriate conjugate prior, the choice of prior will be a challenge in such a setting. As a future work we will consider the bivariate and multivariate extensions of the THOLExp distribution, in particular with the copula based construction method, the trivariate reduction, etc.

References

- [1] A. H. Abdel-Hamid and E. K. Al-Hussaini, Estimation in step-stress accelerated life tests for the exponentiated exponential distribution with type-I censoring, *Comput. Statist. Data Anal.* **53** (2009), no. 4, 1328–1338.
- [2] A. Z. Afify, M. Alizadeh, H. M. Yousof, G. Aryal and M. Ahmad, The transmuted geometric-G family of distributions: Theory and applications, *Pakistan J. Statist.* **32** (2016), no. 2, 139–160.
- [3] A. Z. Afify, G. M. Cordeiro, S. Nadarajah, H. M. Yousof, G. Ozel, Z. M. Nofal and E. Altun, The complementary geometric transmuted-G family of distributions: Model, properties and applications, *Hacet. J. Math. Stat.* (2017), DOI 10.15672/HJMS.2017.439.
- [4] A. Z. Afify, G. M. Cordeiro, H. M. Yousof, Z. M. Nofal and A. Alzaatreh, The Kumaraswamy transmuted-G family of distributions: Properties and applications, *J. Data Sci.* **14** (2016), no. 2, 245–270.
- [5] A. Z. Afify, H. M. Yousof and S. Nadarajah, The beta transmuted-H family for lifetime data, *Stat. Interface* **10** (2017), no. 3, 505–520.
- [6] M. Alizadeh, M. Rasekhi, H. M. Yousof and G. G. Hamedani, The transmuted Weibull G family of distributions, *Hacet. J. Math. Stat.* (2017), DOI 10.15672/HJMS.2017.440.
- [7] M. Alizadeh, H. M. Yousof, A. Z. Afify, G. M. Cordeiro and M. Mansoor, The complementary generalized transmuted Poisson-G family, *Austrian J. Stat.*, to appear.
- [8] D. F. Andrews and A. M. Herzberg, *Data. A Collection of Problems from Many Fields for the Student and Research Worker*, Springer Ser. Statist., Springer, New York, 1985.
- [9] G. R. Aryal, E. M. Ortega, G. G. Hamedani and H. M. Yousof, The Topp Leone generated Weibull distribution: Regression model, characterizations and applications, *Int. J. Statist. Probab.* **6** (2017), 126–141.

- [10] G. R. Aryal and H. M. Yousof, The exponentiated generalized-G Poisson family of distributions, *Stochastics Quality Control* (2017), DOI 10.1515/eqc-2017-0004.
- [11] M. Aslam, D. Kundu and M. Ahmad, Time truncated acceptance sampling plans for generalized exponential distribution, *J. Appl. Stat.* **37** (2010), no. 3-4, 555–566.
- [12] N. Balakrishnan and A. P. Basu, *The Exponential Distribution*, Gordon and Breach Publishers, Amsterdam, 1995.
- [13] R. E. Barlow, R. H. Toland and T. Freeman, A Bayesian analysis of stress-rupture life of kevlar 49/epoxy spherical pressure vessels, in: *Proceedings of the Canadian Conference in Application Statistics*, Marcel Dekker, New York (1984).
- [14] K. Cooray and M. M. A. Ananda, A generalization of the half-normal distribution with applications to lifetime data, *Comm. Statist. Theory Methods* **37** (2008), no. 8–10, 1323–1337.
- [15] G. M. Cordeiro, A. Z. Afify, H. M. Yousof, R. R. Pescim and G. R. Aryal, The exponentiated Weibull-H family of distributions: Theory and Applications, *Mediterr. J. Math.*, to appear.
- [16] G. M. Cordeiro, H. M. Yousof, T. G. Ramires and E. M. M. Ortega, The Burr XII system of densities: Properties, regression model and applications, *J. Stat. Comput. Simul.*, to appear.
- [17] E. de Brito, G. M. Cordeiro, H. M. Yousof, M. Alizadeh and G. O. Silva, Topp–Leone odd log-logistic family of distributions, *J. Stat. Comput. Simul.*, to appear.
- [18] Y. M. Gómez, H. Bolfarine and H. W. Gómez, A new extension of the exponential distribution, *Rev. Colombiana Estadíst.* **37** (2014), no. 1, 25–34.
- [19] R. D. Gupta and D. Kundu, Generalized exponential distributions, *Aust. N. Z. J. Stat.* **41** (1999), no. 2, 173–188.
- [20] R. D. Gupta and D. Kundu, Exponentiated exponential family: An alternative to gamma and Weibull distributions, *Biom. J.* **43** (2001), no. 1, 117–130.
- [21] R. D. Gupta and D. Kundu, Generalized exponential distribution: different method of estimations, *J. Statist. Comput. Simulation* **69** (2001), no. 4, 315–337.
- [22] R. D. Gupta and D. Kundu, Generalized exponential distribution: existing results and some recent developments, *J. Statist. Plann. Inference* **137** (2007), no. 11, 3537–3547.
- [23] G. G. Hamedani, H. M. Yousof, M. Rasekhi, M. Alizadeh and S. M. Najibi, Type I general exponential class of distributions, *Int. J. Appl. Exp. Math.*, to appear.
- [24] R. V. Hogg, J. W. McKean and A. T. Craig, *Introduction to Mathematical Statistics*, 6th ed., Pearson Education, Upper Saddle River, 2005.
- [25] N. L. Johnson, S. Kotz and N. Balakrishnan, *Continuous Univariate Distributions. Vol. 1*, 2nd ed., Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics, John Wiley & Sons, New York, 1994.
- [26] C. S. Kakade and D. T. Shirke, Tolerance interval for exponentiated exponential distribution based on grouped data, *Int. J. Agric. Stat. Sci.* **3** (2007), 625–631.
- [27] M. C. Korkmaz and A. I. Genç, Two-sided generalized exponential distribution, *Comm. Statist. Theory Methods* **44** (2015), no. 23, 5049–5070.
- [28] D. Kundu and B. Pradhan, Bayesian inference and life testing plans for generalized exponential distribution, *Sci. China Ser. A* **52** (2009), no. 6, 1373–1388.
- [29] D. V. Lindley, Fiducial distributions and Bayes' theorem, *J. Roy. Statist. Soc. Ser. B* **20** (1958), 102–107.
- [30] A. W. Marshall and I. Olkin, *Life Distributions*, Springer Ser. Statist., Springer, New York, 2007.
- [31] F. Merovci, M. Alizadeh, H. M. Yousof and G. G. Hamedani, The exponentiated transmuted-G family of distributions: Theory and applications, *Comm. Statist. Theory Methods* (2016), DOI 10.1080/03610926.2016.1248782.
- [32] S. Nadarajah and F. Haghighi, An extension of the exponential distribution, *Statistics* **45** (2011), no. 6, 543–558.
- [33] Z. M. Nofal, A. Z. Afify, H. M. Yousof and G. M. Cordeiro, The generalized transmuted-G family of distributions, *Comm. Statist. Theory Methods* **46** (2017), no. 8, 4119–4136.
- [34] Z. M. Nofal, A. Z. Afify, H. M. Yousof, D. C. T. Granzotto and F. Louzada, Kumaraswamy transmuted exponentiated additive Weibull distribution, *Int. J. Statist. Probab.* **5** (2016), no. 2, 78–99.
- [35] M. Z. Raqab, Inferences for generalized exponential distribution based on record statistics, *J. Statist. Plann. Inference* **104** (2001), 339–350.
- [36] M. Z. Raqab and M. Ahsanullah, Estimation of the location and scale parameters of generalized exponential distribution based on order statistics, *J. Statist. Comput. Simulation* **69** (2001), no. 2, 109–123.
- [37] A. M. Sarhan, Analysis of incomplete, censored data in competing risks models with generalized exponential distributions, *IEEE Trans. Reliab.* **56** (2007), 132–138.
- [38] S. Sen, S. S. Maiti and N. Chandra, The Xgamma distribution: statistical properties and application, *J. Mod. Appl. Statist. Methods* **15** (2016), 774–788.
- [39] D. T. Shirke, R. R. Kumbhar and D. Kundu, Tolerance intervals for exponentiated scale family of distributions, *J. Appl. Stat.* **32** (2005), no. 10, 1067–1074.
- [40] F. S. G. Silva, A. Percontini, E. de Brito, M. W. Ramos, R. Venancio and G. M. Cordeiro, The odd Lindley-G family of distributions, *Austrian J. Stat.* **46** (2017), no. 1, 65–87.
- [41] H. M. Srivastava, S. Nadarajah and S. Kotz, Some generalizations of the Laplace distribution, *Appl. Math. Comput.* **182** (2006), 223–231.

- [42] H. M. Yousof, A. Z. Afify, M. Alizadeh, N. S. Butt, G. G. Hamedani and M. M. Ali, The transmuted exponentiated generalized-G family of distributions, *Pak. J. Stat. Oper. Res.* **11** (2015), no. 4, 441–464.
- [43] H. M. Yousof, A. Z. Afify, G. M. Cordeiro, A. Alzaatreh and M. Ahsanullah, A new four-parameter Weibull model for lifetime data, *J. Stat. Theory Appl.*, to appear.
- [44] H. M. Yousof, A. Z. Afify, G. G. Hamedani and G. Aryal, The Burr X generator of distributions for lifetime data, *J. Statist. Theory Appl.*, to appear.
- [45] H. M. Yousof, M. Majumder, S. M. A. Jahanshahi, M. M. Ali and G. G. Hamedani, A new Weibull class of distributions: Theory, characterizations and applications, *J. Stat. Res. Iran*, to appear.
- [46] H. M. Yousof, M. Rasekhi, A. Z. Afify, M. Alizadeh, I. Ghosh and G. G. Hamedani, The beta Weibull-G family of distributions: Theory, characterizations and applications, *Pakistan J. Statist.* **33** (2017), no. 2, 95–116.
- [47] G. Zheng, On the Fisher information matrix in type II censored data from the exponentiated exponential family, *Biom. J.* **44** (2002), 353–357.
- [48] G. Zheng and S. Park, A note on time savings in censored life testing, *J. Statist. Plann. Inference* **124** (2004), no. 2, 289–300.