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RESEARCH PAPER

OPERATIONAL METHOD FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS WITH THE LEFT- AND RIGHT-HAND SIDED ERDÉLYI-KOBER FRACTIONAL DERIVATIVES

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Abstract

In this paper, we first provide a survey of some basic properties of the left- and right-hand sided Erdélyi-Kober fractional integrals and derivatives and introduce their compositions in form of the composed Erdélyi-Kober operators. Then we derive a convolutional representation for the composed Erdélyi-Kober fractional integral in terms of its convolution in the Dimovski sense. For this convolution, we also determine the divisors of zero. These both results are then used for construction of an operational method for solving an initial value problem for a fractional differential equation with the left- and right-hand sided Erdélyi-Kober fractional derivatives defined on the positive semi-axis. Its solution is obtained in terms of the four-parameters Wright function of the second kind. The same operational method can be employed for other fractional differential equation with the left- and right-hand sided Erdélyi-Kober fractional derivatives.

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Key Words and Phrases: operational method; fractional differential equations; left- and right-hand sided Erdélyi-Kober integrals; left- and right-hand sided Erdélyi-Kober derivatives; composed Erdélyi-Kober integrals and derivatives; convolutions; integral transforms; Wright type functions

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1. **Introduction**

Fractional calculus of variations is one of the recent and important research areas in Fractional Calculus (FC). It is a natural extension of the calculus of variations to the case of functionals that depend not only on integer order derivatives, but also on fractional derivatives ([3, 23]). As in the case of conventional calculus of variations, a necessary condition for a function that solves an optimization problem for a functional depending on some fractional derivatives can be formulated in terms of a suitably modified Euler-Lagrange equation ([3]). The Euler-Lagrange equations of the fractional calculus of variations are very special fractional differential equations that as a rule contain both the left- and right-hand sided fractional derivatives as well as their compositions. The standard methods for analytical treatment of the fractional differential equations were worked out for equations with either only left- or only right-hand sided fractional derivatives. That is why they do not work for the fractional Euler-Lagrange equations and thus some new techniques are needed for their analytical (and numerical) treatment. The case of the fractional differential equations with the left- and right-hand sided fractional derivatives defined on a finite interval has been considered in [14]. However, to the best knowledge of the authors, no methods for analytical treatment of these equations defined on an infinite interval, say, on the positive real semi-axis, are yet known (the methods from [14] cannot be applied in the case of an infinite interval).

In this paper, we suggest an operational method for solving the initial-value problems for some fractional differential equations with both the left- and right-hand sided Erdélyi-Kober fractional derivatives defined on the positive semi-axis. The main idea behind this method is to reduce these problems to the integro-differential equations that contain certain compositions of the left- and right-hand sided Erdélyi-Kober fractional derivatives and integrals that we call the composed Erdélyi-Kober fractional integrals. The composed Erdélyi-Kober fractional integrals are the so called generating operators for a special integral transform that was introduced by the authors of this paper in [2]. A one-parametric family of convolutions for this integral transform has been also constructed in [2]. It turns our that these convolutions can be interpreted as convolutions for the composed Erdélyi-Kober fractional integrals in the Dimovski sense ([4, 5]). Moreover, the composed Erdélyi-Kober fractional integrals possess a convolutional representation with the power law functions. Thus, the integro-differential equations that contain the composed Erdélyi-Kober fractional integrals are of convolutional type and can be solved following the standard procedure by applying the associated integral transform, i.e., by using an operational relation connecting the composed Erdélyi-Kober fractional integrals with their associated integral transforms.

Of course, another possible approach would be to develop an operational calculus of Mikusiński type for the left-inverse operator to the composed Erdélyi-Kober fractional integral, i.e., for the composed Erdélyi-Kober fractional derivative. The operational calculi of Mikusiński type for the left-hand sided Erdélyi-Kober fractional derivative as well as for compositions of a finite number of such derivatives have been developed in [1, 22, 29]. These calculi were used for analytical treatment of the initial-value problems for the fractional differential equations with the Riemann-Liouville fractional derivatives [20], the Caputo fractional derivatives [18], the Hilfer derivatives [10], the Caputo-type modifications of the Erdélyi-Kober fractional derivatives [9], and the multiple Erdélyi-Kober fractional derivatives [1, 22, 29]. We also refer to [17] for a general overview of the operational calculi for different fractional differential operators. In [7], a Mikusiński type operational calculus for the Riemann-Liouville fractional derivatives was used for solving the generalized Abel integral equations of the second kind. Thus, the operational calculi for the fractional derivatives can be applied both for analytical treatment of the initial-value problems for the fractional differential equations and for deriving the closed form formulas for solutions of the fractional integral equations. In this paper, we apply and extend the results presented in [2] towards an operational calculus of Mikusiński type for the composed Erdélyi-Kober fractional derivatives. This calculus as well as its applications to analytical treatment of the initial-value problems for some fractional differential equations with both the left- and right-hand sided Erdélyi-Kober fractional derivatives will be considered elsewhere.

The rest of this paper is organized as follows. In Section 2, we remind the readers of the relevant properties of the Erdélyi-Kober fractional integrals and derivatives and define their suitable compositions that we call the composed Erdélyi-Kober fractional integrals and derivatives. Some basic properties of the composed Erdélyi-Kober fractional integrals and derivatives are discussed and an associated integral transform is introduced. An important operational relation that connects this integral transform with the composed Erdélyi-Kober fractional integrals is also mentioned. Section 3 is devoted to a discussion of further properties of the composed Erdélyi-Kober fractional integrals that play an essential role both in our operational method as well as in the Mikusiński type operational calculus for the composed Erdélyi-Kober fractional derivative that will be considered elsewhere. In particular, a convolutional representation of the composed Erdélyi-Kober fractional integrals via the convolutions introduced in [2] is derived. The divisors of zero of these convolutions are discussed, too. In

Section 4, the operational relations deduced in Section 3 are applied to derive a closed form formula for solution of an initial-value problem for a sample equation containing both the left- and right-hand sided Erdélyi-Kober fractional derivatives. The obtained solution is expressed in terms of the four-parameters Wright function of the second kind. The same operational method can be also employed for analytical treatment of the initial-value problems for other fractional differential equations with both the left- and right-hand sided Erdélyi-Kober fractional derivatives.

2. The Erdélyi-Kober fractional integrals and derivatives and their basic properties

The Erdélyi-Kober fractional integrals and derivatives, their properties, and applications were discussed in detail in many publications (see, e.g. [11, 12, 21, 25, 26, 29]). In this section, we remind the readers their basic properties that are used in further discussions.

The left- and right-hand sided Erdélyi-Kober fractional integrals of order δ or α, respectively, are defined as follows:

$$(I_\beta^{\gamma,\delta} f)(x) = \frac{\beta}{\Gamma(\delta)} x^{-\beta(\gamma+\delta)} \int_0^x (x^\beta - t^\beta)^{\delta-1} t^{\beta(\gamma+1)-1} f(t)\, dt,\ \ \delta, \beta > 0, \gamma \in \mathbb{R}, \tag{2.1}$$

$$(K_\beta^{\tau,\alpha} f)(x) = \frac{\beta}{\Gamma(\alpha)} x^{\beta\tau} \int_x^\infty (t^\beta - x^\beta)^{\alpha-1} t^{-\beta(\tau+\alpha-1)-1} f(t)\, dt,\ \ \alpha, \beta > 0, \tau \in \mathbb{R}. \tag{2.2}$$

For $\delta = 0$ or $\alpha = 0$, respectively, these operators are defined as the identity operators:

$$(I_\beta^{\gamma,0} f)(x) = f(x),\ \ (K_\beta^{\tau,0} f(x) = f(x).$$

For our aims it is convenient to use the following representations of the Erdélyi-Kober fractional integrals:

$$(I_\beta^{\gamma,\delta} f)(x) = \frac{1}{\Gamma(\delta)} \int_0^1 (1-t)^{\delta-1} t^\gamma f\left(x t^{\frac{1}{\beta}}\right) dt,\ \ \delta, \beta > 0, \gamma \in \mathbb{R}, \tag{2.3}$$

$$(K_\beta^{\tau,\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} t^\tau f\left(x t^{-\frac{1}{\beta}}\right) dt,\ \ \alpha, \beta > 0, \tau \in \mathbb{R}, \tag{2.4}$$

which are obtained from (2.1) and (2.2), respectively, by the simple variables substitutions.

The left- and right-hand sided Erdélyi-Kober fractional derivatives of order δ or α, respectively, are defined as follows ($n-1 < \delta \le n,\ n \in \mathbb{N}$ and $m-1 < \alpha \le m,\ m \in \mathbb{N}$):

$$(D_\beta^{\gamma,\delta} f)(x) = \prod_{k=0}^{n-1} (1 + \gamma + k + \frac{1}{\beta} x \frac{d}{dx})(I_\beta^{\gamma+\delta, n-\delta} f)(x), \tag{2.5}$$

$$(P_\beta^{\tau,\alpha} f)(x) = \prod_{k=0}^{m-1} (\tau + k - \frac{1}{\beta} x \frac{d}{dx})(K_\beta^{\tau+\alpha,m-\alpha} f)(x), \tag{2.6}$$

where the operators $I_\beta^{\gamma,\delta}$ and $K_\beta^{\tau,\alpha}$ are the left- and right-hand sided Erdélyi-Kober fractional integrals (2.3) and (2.4).

In [21], the Caputo type modifications of the Erdélyi-Kober fractional derivatives were introduced and discussed. The main advantage of these modifications over the conventional Erdélyi-Kober fractional derivatives is that the initial conditions for the fractional differential equations with the Caputo type modifications of the Erdélyi-Kober fractional derivatives can be formulated in terms of the conventional derivatives while the initial conditions for the fractional differential equations with the Erdélyi-Kober fractional derivatives require initial values involving fractional integrals and derivatives. For this reason, in this paper we deal with the Caputo type modifications of the left- and right-hand sided Erdélyi-Kober fractional derivatives defined as follows ($n-1 < \delta \le n,\ n \in \mathbb{N}$ and $m-1 < \alpha \le m,\ m \in \mathbb{N}$):

$$({}_*D_\beta^{\gamma,\delta} f)(x) = (I_\beta^{\gamma+\delta,n-\delta} \prod_{k=0}^{n-1} \left(1 + \gamma + k + \frac{1}{\beta} t \frac{d}{dt}\right) f)(x), \tag{2.7}$$

$$({}_*P_\beta^{\tau,\alpha} f)(x) = (K_\beta^{\tau+\alpha,m-\alpha} \prod_{k=0}^{m-1} \left(\tau + k - \frac{1}{\beta} t \frac{d}{dt}\right) f)(x). \tag{2.8}$$

In the formulas (2.7) and (2.8), the operators $I_\beta^{\gamma,\delta}$ and $K_\beta^{\tau,\alpha}$ are the left- and right-hand sided Erdélyi-Kober fractional integrals of orders δ and α, respectively, defined by (2.3) and (2.4). In what follows, we call the operators defined by (2.7) and (2.8) just the Erdélyi-Kober fractional derivatives.

The mapping properties of the Erdélyi-Kober fractional integrals and derivatives were studied for several different spaces of functions in [11, 12, 21, 26, 29] to mention only few of relevant publications. The main focus of this paper is in derivation of solution formulas for some fractional differential equations that contain both the left- and right-hand sided Erdélyi-Kober fractional derivatives. We do this in some convenient spaces of functions, where both the left- and right-hand sided Erdélyi-Kober fractional integrals and derivatives are well defined, namely, in the class of functions that are continuous on the semi-axis $]0, \infty[$ and can be represented as the convergent power series with the power functions weights in some neighborhoods $U_{\epsilon_1}(0)$ and $U_{\epsilon_2}(+\infty)$ of the points $x = 0$ and $x = +\infty$, respectively, i.e., in the form

$$f(x) = x^\alpha \sum_{k=0}^{\infty} a_k (x^\rho)^k,\ \rho > 0,\ x \in U_{\epsilon_1}(0), \tag{2.9}$$

and

$$f(x) = x^{\beta} \sum_{k=0}^{\infty} b_k (x^{-\sigma})^k, \ \sigma > 0, \quad x \in U_{\epsilon_2}(+\infty). \tag{2.10}$$

This space of functions will be denoted by $\mathfrak{O}$. The functions from $\mathfrak{O}$ have a power law asymptotic behavior at the points 0 and $+\infty$ that appears to be an appropriate asymptotics for solutions of the fractional differential equations that contain both the left- and right-hand sided Erdélyi-Kober fractional derivatives. Because of the well-known properties:

$$(I_{\beta}^{\gamma,\delta} t^p)(x) = \frac{\Gamma\left(\gamma + 1 + \frac{p}{\beta}\right)}{\Gamma\left(\delta + \gamma + 1 + \frac{p}{\beta}\right)} x^p, \quad \gamma + 1 + \frac{p}{\beta} > 0, \tag{2.11}$$

$$({}_*D_{\beta}^{\gamma,\delta} t^p)(x) = \frac{\Gamma\left(\delta + \gamma + 1 + \frac{p}{\beta}\right)}{\Gamma\left(\gamma + 1 + \frac{p}{\beta}\right)} x^p, \quad \gamma + 1 + \frac{p}{\beta} > 0, \tag{2.12}$$

$$(K_{\beta}^{\tau,\alpha} t^p)(x) = \frac{\Gamma\left(\tau - \frac{p}{\beta}\right)}{\Gamma\left(\alpha + \tau - \frac{p}{\beta}\right)} x^p, \quad \tau - \frac{p}{\beta} > 0, \tag{2.13}$$

$$({}_*P_{\beta}^{\tau,\alpha} t^p)(x) = \frac{\Gamma\left(\alpha + \tau - \frac{p}{\beta}\right)}{\Gamma\left(\tau - \frac{p}{\beta}\right)} x^p, \quad \tau - \frac{p}{\beta} > 0, \tag{2.14}$$

the power law functions are eigenfunctions of the Erdélyi-Kober fractional integrals and derivatives and thus they translate the power series from (2.9) or (2.10), respectively, into series of the same form.

Like in the case of other spaces of functions, the following properties are valid for the functions from $\mathfrak{O}$ under evident restrictions on the parameters that follow from the formulas (2.11) and (2.13):

$$(I_{\beta}^{\gamma,\delta} x^{\lambda\beta} f)(x) = x^{\lambda\beta} (I_{\beta}^{\gamma+\lambda,\delta} f)(x), \tag{2.15}$$

$$(I_{\beta}^{\gamma,\delta} I_{\beta}^{\gamma+\delta,\alpha} f)(x) = (I_{\beta}^{\gamma,\delta+\alpha} f)(x), \tag{2.16}$$

$$(I_{\beta}^{\gamma,\delta} I_{\beta}^{\alpha,\eta} f)(x) = (I_{\beta}^{\alpha,\eta} I_{\beta}^{\gamma,\delta} f)(x), \tag{2.17}$$

$$(K_{\beta}^{\tau,\alpha} x^{\lambda\beta} f)(x) = x^{\lambda\beta} (K_{\beta}^{\tau-\lambda,\alpha} f)(x), \tag{2.18}$$

$$(K_{\beta}^{\tau,\alpha} K_{\beta}^{\tau+\alpha,\delta} f)(x) = (K_{\beta}^{\tau,\alpha+\delta} f)(x), \tag{2.19}$$

$$(K_{\beta}^{\gamma,\delta} K_{\beta}^{\alpha,\eta} f)(x) = (K_{\beta}^{\alpha,\eta} K_{\beta}^{\gamma,\delta} f)(x). \tag{2.20}$$

On the space $\mathfrak{O}$, the left-hand sided Erdélyi-Kober fractional derivative is a left-inverse operator to the left-hand sided Erdélyi-Kober fractional integral and the right-hand sided Erdélyi-Kober fractional derivative is a left-inverse operator to the right-hand sided Erdélyi-Kober fractional integral (for a proof that was provided for another space of functions but is also valid for the space $\mathfrak{O}$ see, e.g., [21]):

$$({}_*D_\beta^{\gamma,\delta}\, I_\beta^{\gamma,\delta} f)(x) = f(x), \tag{2.21}$$

$$({}_*P_\beta^{\tau,\alpha}\, K_\beta^{\tau,\alpha} f)(x) = f(x). \tag{2.22}$$

As in the case of the first order derivative and the definite integral, the Erdélyi-Kober fractional derivatives are not right-inverse operators to the Erdélyi-Kober fractional integrals. A closed form formula for the composition of the left-hand sided Erdélyi-Kober fractional integral and the left-hand sided Erdélyi-Kober fractional derivative was derived in [21]:

$$(I_\beta^{\gamma,\delta}\, {}_*D_\beta^{\gamma,\delta} f)(x) \;=\; f(x) - \sum_{k=0}^{n-1} p_k x^{-\beta(1+\gamma+k)}, \tag{2.23}$$

$$p_k = \lim_{x\to 0} x^{\beta(1+\gamma+k)} \prod_{i=k+1}^{n-1} \left(1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx}\right),\; k = 0,\dots,n-1. \tag{2.24}$$

A similar formula is valid for a composition of the right-hand sided Erdélyi-Kober fractional integral and the right-hand sided Erdélyi-Kober fractional derivative, but in this paper we will need only the formulas (2.23) and (2.24).

In the rest of this section, we discuss some suitable compositions of the left- and right-hand sided Erdélyi-Kober fractional integrals and derivatives that were introduced in [2]. These compositions will be employed for solving the fractional differential equations with the left- and right-hand sided Erdélyi-Kober fractional derivatives.

Let $\mu > 0$ and $a > b > 0$. The composition

$$(L_\mu f)(x) = x^\mu \left(I_{1/a}^{-\alpha,a\mu}\, {}_*P_{1/b}^{\beta-b\mu,b\mu}\, f\right)(x) \tag{2.25}$$

of the right-hand sided Erdélyi-Kober fractional derivative ${}_*P_{1/b}^{\beta-b\mu,b\mu}$ and the left-hand sided Erdélyi-Kober fractional integral $I_{1/a}^{-\alpha,a\mu}$ is called the composed Erdélyi-Kober fractional integral and the composition

$$(D_\mu f)(x) = x^{-\mu} \left(K_{1/b}^{\beta,b\mu}\, {}_*D_{1/a}^{-\alpha-a\mu,a\mu}\, f\right)(x) \tag{2.26}$$

of the left-hand sided Erdélyi-Kober fractional derivative ${}_*D_{1/a}^{-\alpha-a\mu,a\mu}$ and the right-hand sided Erdélyi-Kober fractional integral $K_{1/b}^{\beta,b\mu}$ is called the composed Erdélyi-Kober fractional derivative.

By definition, the composed Erdélyi-Kober fractional integrals and derivatives are integro-differential operators. Because of the condition $a > b >$

0, the composed Erdélyi-Kober fractional integrals behave more like integral operators whereas the composed Erdélyi-Kober fractional derivatives possess several typical properties of the (fractional) derivatives. The composed Erdélyi-Kober fractional integrals and derivatives are well defined, say, for the functions from the space $\mathfrak{O}$ under the additional condition that the fractional derivatives ${}_*P_{1/b}^{\beta-b\mu,b\mu} f$ or ${}_*D_{1/a}^{-\alpha-a\mu,a\mu} f$, respectively, also belong to the space $\mathfrak{O}$.

In [2], some useful properties of the composed Erdélyi-Kober fractional integrals and derivatives were derived. In what follows, we shortly mention the formulas from [2] that will be used in the further discussions.

For $\mu > 0$ and $a > b > 0$, the composed Erdélyi-Kober fractional integral (2.25) and the composed Erdélyi-Kober fractional derivative (2.26) can be represented as the Mellin-Barnes integrals:

$$(L_\mu f)(x) = x^\mu \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(s)}{\Phi(s-\mu)} f^*(s) x^{-s} ds, \tag{2.27}$$

$$(D_\mu f)(x) = x^{-\mu} \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(s)}{\Phi(s+\mu)} f^*(s) x^{-s} ds, \tag{2.28}$$

respectively, with the kernel function

$$\Phi(s) = \Gamma(1-\alpha-as)\Gamma(\beta+bs). \tag{2.29}$$

In the formulas above, we denoted by $f^*(s)$ the Mellin integral transform of a function f:

$$F(s) = \mathfrak{M}\{f(x);s\} = f^*(s) = \int_0^{+\infty} f(x)x^{s-1}\,dx. \tag{2.30}$$

These representations were used in [2] for derivation of the following important result: Let $\mu > 0$ and $a > b > 0$. The composed Erdélyi-Kober fractional derivative (2.26) is a left-inverse operator to the composed Erdélyi-Kober fractional integral (2.25), i.e.,

$$(D_\mu\, L_\mu\, f)(x) = f(x). \tag{2.31}$$

Another useful property is a simple closed form formula for the compositions of n composed Erdélyi-Kober fractional integrals or derivatives: For $\mu > 0$, $a > b > 0$, and $n \in \mathbb{N}$, the representations

$$(L_\mu^n\, f)(x) = (L_\mu \ldots L_\mu f)(x) = (L_{n\mu} f)(x), \tag{2.32}$$

$$(D_\mu^n\, f)(x) = (D_\mu \ldots D_\mu f)(x) = (D_{n\mu} f)(x) \tag{2.33}$$

hold true.

The formulas (2.32) and (2.33) can be extended to arbitrary powers of the composed Erdélyi-Kober fractional integrals and derivatives as follows:

$$(L_\mu^\theta\, f)(x) = (L_{\theta\mu} f)(x),\ \theta > 0,$$

$$(D_\mu^\theta f)(x) = (D_{\theta\mu} f)(x),\ \theta > 0.$$

It is easy to see that the semigroup properties for the fractional powers of the composed Erdélyi-Kober fractional integrals and derivatives are valid:

$$(L_\mu^{\theta_1} L_\mu^{\theta_2} f)(x) = (L_\mu^{\theta_1+\theta_2} f)(x),$$

$$(D_\mu^{\theta_1} D_\mu^{\theta_2} f)(x) = (D_\mu^{\theta_1+\theta_2} f)(x).$$

In [2], an integral transform associated with the composed Erdélyi-Kober fractional integral was introduced in form of a Mellin-Barnes integral:

$$(T_{\Phi,\lambda} f)(x) = \frac{x^\lambda}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Phi(s)\, f^*(s) x^{-s} ds, \tag{2.34}$$

where the kernel function Φ is defined by (2.29).

Another, more suitable for our aims form of the integral transform T_Φ that is valid under the condition

$$\frac{\alpha - 1}{a} < \frac{\beta}{b} \tag{2.35}$$

is as follows (see [2] for a derivation of this formula):

$$(T_{\Phi,\lambda} f)(x) = x^\lambda \int_0^\infty \int_0^\infty e^{-\tau_1-\tau_2} \tau_1^{-\alpha} \tau_2^{\beta-1} f\left(\frac{x\tau_1^a}{\tau_2^b}\right) d\tau_1 \tau_2. \tag{2.36}$$

In [2], an important operational relation for the integral transform $T_{\Phi,\lambda}$ and the composed Erdélyi-Kober fractional integral was derived: For $\mu > 0$ and $a > b > 0$, the operational relation

$$(T_{\Phi,\lambda} L_\mu f)(x) = x^\mu (T_{\Phi,\lambda} f)(x) \tag{2.37}$$

holds true, i.e., the integral transform $T_{\Phi,\lambda}$ translates the composed Erdélyi-Kober fractional integral into multiplication with a power law function in the frequency domain.

3. Convolutional representation of the composed Erdélyi-Kober fractional integral

In this section, we present some new properties of the composed Erdélyi-Kober fractional integral and the integral transform $T_{\Phi,\lambda}$ that are the basic components of the operational method for solving the initial value problems for the fractional differential equations with the left- and right-hand sided Erdélyi-Kober fractional derivatives on the positive semi-axis.

We start with the following important result:

THEOREM 3.1. *Let the condition* (2.35) *be satisfied. On the subspace of the space* $\mathfrak{O}$ *that is invariant with respect to the Erdélyi-Kober fractional integrals and derivatives from the formulation of this theorem, the* *composed Erdélyi-Kober fractional integral has the following convolutional representation:*

$$(L_\mu f)(x) = (f \overset{\lambda}{*} h_{\mu,\lambda})(x), \quad h_{\mu,\lambda}(x) = \frac{x^{\mu-\lambda}}{\Gamma(1-\alpha+a(\mu-\lambda))\Gamma(\beta-b(\mu-\lambda))}, \tag{3.1}$$

where the convolution $\overset{\lambda}{*}$ *is defined by* ([2])

$$(f \overset{\lambda}{*} g)(x) = (I_{1/a}^{1-2\alpha-a\lambda,\alpha+a\lambda-1} {}_*P_{1/b}^{\beta,\beta+b\lambda} f \circ g)(x) \tag{3.2}$$

with

$$(f \circ g)(x) = x^\lambda \int_0^1 \int_0^1 \tau_1^{-\alpha}(1-\tau_1)^{-\alpha} \tau_2^{\beta-1}(1-\tau_2)^{\beta-1} \tag{3.3}$$
$$\times f\left(\frac{x\tau_1^a}{\tau_2^b}\right) g\left(\frac{x(1-\tau_1)^a}{(1-\tau_2)^b}\right) d\tau_1 d\tau_2.$$

P r o o f. The proof of the theorem is straightforward, i.e., we transform the right-hand side of the formula (3.1), i.e., $(f \overset{\lambda}{*} h_{\mu,\lambda})(x)$, to the form of the left-hand side of the formula using the properties of the Erdélyi-Kober fractional integrals and derivatives. We start with the expression $(f \circ h_{\mu,\lambda})(x)$:

$$(f \circ h_{\mu,\lambda})(x) = x^\lambda \int_0^1 \int_0^1 \tau_1^{-\alpha}(1-\tau_1)^{-\alpha} \tau_2^{\beta-1}(1-\tau_2)^{\beta-1}$$
$$\times f\left(\frac{x\tau_1^a}{\tau_2^b}\right) \frac{\left(\frac{x(1-\tau_1)^a}{(1-\tau_2)^b}\right)^{\mu-\lambda}}{\Gamma(1-\alpha+a(\mu-\lambda))\Gamma(\beta-b(\mu-\lambda))} d\tau_1 d\tau_2$$
$$= x^\mu \int_0^1 \int_0^1 f\left(\frac{x\tau_1^a}{\tau_2^b}\right) \frac{\tau_1^{-\alpha}(1-\tau_1)^{a(\mu-\lambda)-\alpha} \tau_2^{\beta-1}(1-\tau_2)^{\beta-1-b(\mu-\lambda)}}{\Gamma(1-\alpha+a(\mu-\lambda))\Gamma(\beta-b(\mu-\lambda))} d\tau_1 d\tau_2$$
$$= x^\mu \left(K_{1/b}^{\beta,b-b(\mu-\lambda)} I_{1/a}^{-\alpha,1-\alpha+a(\mu-\lambda)} f\right)(x),$$

as immediately follows from the representations (2.3) and (2.4) of the left- and right-hand sided Erdélyi-Kober fractional integrals.

Thus we get the formula

$$(f \overset{\lambda}{*} h_{\mu,\lambda})(x) = (I_{1/a}^{1-2\alpha-a\lambda,\alpha+a\lambda-1} {}_*P_{1/b}^{\beta,\beta+b\lambda} x^\mu K_{1/b}^{\beta,b-b(\mu-\lambda)} I_{1/a}^{-\alpha,1-\alpha+a(\mu-\lambda)} f)(x). \tag{3.4}$$

On the subspace of the space $\mathfrak{O}$ that is invariant with respect to the Erdélyi-Kober fractional integrals and derivatives from the formulation of this theorem, the properties (2.15)-(2.20) are valid not only for the left- and right-hand sided Erdélyi-Kober fractional integrals, but also for the left- and right-hand sided Erdélyi-Kober fractional derivatives as can be easily proved using the formulas (2.11)-(2.14). Moreover, on this subspace the left- and right-hand sided Erdélyi-Kober fractional integrals and derivatives commute and thus we can change their order in any composition of these operators. Employing (2.15)-(2.20) and the formulas of the same form for the left- and right-hand sided Erdélyi-Kober fractional derivatives, we get the relations

$$ {}_*P_{1/b}^{\beta,\beta+b\lambda} x^\mu \, K_{1/b}^{\beta,b-b(\mu-\lambda)} = x^\mu \, {}_*P_{1/b}^{\beta-b\mu,\beta+b\lambda} K_{1/b}^{\beta,b-b(\mu-\lambda)}, $$

$$ {}_*P_{1/b}^{\beta-b\mu,\beta+b\lambda} = {}_*P_{1/b}^{\beta-b\mu,b\mu} \, {}_*P_{1/b}^{\beta,\beta-b(\mu-\lambda)}, $$

leading to the formula

$$ {}_*P_{1/b}^{\beta,\beta+b\lambda} x^\mu \, K_{1/b}^{\beta,b-b(\mu-\lambda)} = x^\mu \, {}_*P_{1/b}^{\beta-b\mu,b\mu}, $$

because the right-hand sided Erdélyi-Kober fractional derivative is a left-inverse operator to the right-hand sided Erdélyi-Kober fractional integral. Thus we arrived at the representation

$$ (f \overset{\lambda}{*} h_{\mu,\lambda})(x) = (I_{1/a}^{1-2\alpha-a\lambda,\alpha+a\lambda-1} \, x^\mu \, {}_*P_{1/b}^{\beta-b\mu,b\mu} \, I_{1/a}^{-\alpha,1-\alpha+a(\mu-\lambda)} \, f)(x). $$

Now we change the order of the right-hand sided Erdélyi-Kober fractional derivative and the second of the left-hand sided Erdélyi-Kober fractional integrals and employ the formulas

$$ (I_{1/a}^{1-2\alpha-a\lambda,\alpha+a\lambda-1} \, x^\mu \, g)(x) = x^\mu \, (I_{1/a}^{1-2\alpha-a\lambda+a\mu,\alpha+a\lambda-1} \, g)(x), $$

$$ I_{1/a}^{1-2\alpha-a\lambda+a\mu,\alpha+a\lambda-1} \, I_{1/a}^{-\alpha,1-\alpha+a(\mu-\lambda)} = I_{1/a}^{-\alpha,a\mu}, $$

to get the final expression

$$ (f \overset{\lambda}{*} h_{\mu,\lambda})(x) = x^\mu (I_{1/a}^{-\alpha,a\mu} \, {}_*P_{1/b}^{\beta-b\mu,b\mu} \, f)(x), $$

that is exactly the left-hand side of the formula (3.1). □

It is worth mentioning that according to the results derived in [2], the operation $\overset{\lambda}{*}$ defined by (3.2) is bilinear, commutative, and associative and satisfies the property

$$ (T_{\Phi,\lambda}(f \overset{\lambda}{*} g))(x) = (T_{\Phi,\lambda} f)(x)(T_{\Phi,\lambda} g)(x), \tag{3.5} $$

i.e., it is a convolution of the integral transform $T_{\Phi,\lambda}$ in the usual sense. Moreover, $\overset{\lambda}{*}$ is a convolution for the composed Erdélyi-Kober fractional integral in the Dimovski sense ([4, 5]), i.e., the relation

$$ L_\mu \, (f \overset{\lambda}{*} g) = (L_\mu f) \overset{\lambda}{*} g \tag{3.6} $$

holds true.

Let us now discuss an important issue regarding the divisors of zero of the convolution $\overset{\lambda}{*}$. They are given in the following theorem.

THEOREM 3.2. *Let $m-1 < \beta + b\lambda \le m,\ m \in \mathbb{N}$. Then the solutions to the equation $(f \overset{\lambda}{*} g)(x) \equiv 0,\ x > 0$ are given by the formula*

$$f(x) = x^p,\ g(x) = x^q,\ \lambda + p + q = \frac{1}{b}(\beta + k),\ k = 0, ..., m-1. \tag{3.7}$$

P r o o f. We start by noting that the operation $\circ$ defined by (3.3) is a modified two-dimensional Laplace convolution that has no divisors of zero on the space of continuous on the positive semi-axis functions (we remind that the functions from our space $\mathfrak{O}$ are continuous on $\mathbb{R}_+$) according to the famous theorem of Mikusiński and Ryll-Nardzewski from [24].

Another component of the proof is the statement that the null-space of the left-hand sided Erdélyi-Kober fractional integral consists just of the null-function (function that is identically equal to zero on the positive semi-axis). This immediately follows from the fact that this integral operator has a left-inverse operator, namely, the left-hand sided Erdélyi-Kober fractional derivative (see (2.21)).

On the other hand, the null-space of the right-hand sided Erdélyi-Kober fractional derivative ${}_*P_\beta^{\tau,\alpha}$ of the order $\alpha,\ m-1 < \alpha \le m,\ m \in \mathbb{N}$ evidently consists of the functions $y(t) = C_k\, t^{\beta(\tau+k)},\ C_k \in \mathbb{R},\ k = 0, \ldots, m-1$ that are solutions to the following differential equations (see (2.8)):

$$\left(\tau + k - \frac{1}{\beta} t \frac{d}{dt}\right) y(t) = 0, \quad k = 0, \ldots, m-1.$$

These arguments and the representation (3.2) of the convolution $\overset{\lambda}{*}$ lead to the following description of the divisors of zero of $\overset{\lambda}{*}$:

$$(f \overset{\lambda}{*} g)(x) \equiv 0 \ \Leftrightarrow\ (f \circ g)(x) \equiv x^{\frac{1}{b}(\beta+k)}, k = 0, \ldots, m-1, \tag{3.8}$$

where $m \in \mathbb{N}$ is determined by the inequalities $m-1 < \beta + b\lambda \le m$.

Let us now calculate $(t^p \circ t^q)(x)$ under the conditions $\frac{\alpha-1}{a} < p < \frac{\beta}{b},\ \frac{\alpha-1}{a} < q < \frac{\beta}{b}$ that ensure existence of the convolution $(t^p \circ t^q)(x)$:

$$(t^p \circ t^q)(x) = x^\lambda \int_0^1 \int_0^1 \tau_1^{-\alpha}(1-\tau_1)^{-\alpha}\, \tau_2^{\beta-1}(1-\tau_2)^{\beta-1} \tag{3.9}$$

$$\times \left(\frac{x\tau_1^a}{\tau_2^b}\right)^p \left(\frac{x(1-\tau_1)^a}{(1-\tau_2)^b}\right)^q d\tau_1 d\tau_2$$

$$= x^{\lambda+p+q} \int_0^1 \int_0^1 \tau_1^{-\alpha+ap}(1-\tau_1)^{-\alpha+aq}\, \tau_2^{\beta-bp-1}(1-\tau_2)^{\beta-bq-1}\, d\tau_1 d\tau_2$$

$$= x^{\lambda+p+q} \frac{\Gamma(-\alpha+ap+1)\Gamma(-\alpha+aq+1)}{\Gamma(-2\alpha+a(p+q)+2)} \frac{\Gamma(\beta-bp)\Gamma(\beta-bq)}{\Gamma(2\beta-b(p+q))}.$$

Comparing the last formula with the right-hand side of the statement (3.8) completes the proof of the theorem. □

4. **Fractional differential equations with the left- and right-hand sided Erdélyi-Kober derivatives**

In this section, we suggest an operational method for analytical treatment of the initial-value problems for some fractional differential equations with the left- and right-hand sided Erdélyi-Kober derivatives on the positive semi-axis. To illustrate the schema, let us consider the following sample equation ($a > b > 0$, $n-1 < a\mu \le n$, $n \in \mathbb{N}$):

$$({}_*D_{1/a}^{-\alpha-a\mu,a\mu}\, y)(x) \,+\, \rho\, x^{\mu}({}_*P_{1/b}^{\beta-b\mu,b\mu}\, y)(x) \,=\, f(x),\ x > 0,\ \rho > 0 \quad (4.1)$$

subject to the initial conditions ($k = 0, \ldots, n-1$)

$$\lim_{x\to 0} x^{\frac{1}{a}(1-\alpha-a\mu+k)} \prod_{i=k+1}^{n-1} \left(1-\alpha-a\mu+i+ax\frac{d}{dx}\right) y(x) \,=\, c_k. \quad (4.2)$$

It is worth mentioning that the initial conditions in form (4.2) are quite natural for the equation (4.1) because they are determined by the projector operator of the left-hand sided Erdélyi-Kober fractional integral $I_{1/a}^{-\alpha-a\mu,a\mu}$:

$$(P\,y)(x) = y(x) - \,(I_{1/a}^{-\alpha-a\mu,a\mu}\ {}_*D_{1/a}^{-\alpha-a\mu,a\mu}\, y)(x) = \sum_{k=0}^{n-1} c_k x^{-\frac{1}{a}(1-\alpha-a\mu+k)}, \quad (4.3)$$

$$c_k = \lim_{x\to 0} x^{\frac{1}{a}(1-\alpha-a\mu+k)} \prod_{i=k+1}^{n-1} \left(1-\alpha-a\mu+i+ax\frac{d}{dx}\right) y(x), \quad (4.4)$$

as immediately follows from the formula (2.23).

We start with a formal derivation of a solution and then proceed with a verification that it is really the (unique) solution to the initial-value problem (4.1)-(4.2).

First step of our procedure is application of the left-hand sided Erdélyi-Kober fractional integral $I_{1/a}^{-\alpha-a\mu,a\mu}$ to the equation (4.1). Taking into account the formulas (4.3)-(4.4) for the projector operator of $I_{1/a}^{-\alpha-a\mu,a\mu}$, we obtain the equation

$$y(x) - \sum_{k=0}^{n-1} c_k x^{-\frac{1}{a}(1-\alpha-a\mu+k)} + \rho\,(I_{1/a}^{-\alpha-a\mu,a\mu}\ x^{\mu}\ {}_*P_{1/b}^{\beta-b\mu,b\mu}\, y)(x) \,=\, g(x),\ x > 0, \quad (4.5)$$

where $g(x) = (I_{1/a}^{-\alpha-a\mu,a\mu} f)(x)$. Applying the property (2.15) of the left-hand sided Erdélyi-Kober fractional integral to the third term of the left-hand side of the equation (4.5), we can rewrite this equation in the form

$$y(x) - \sum_{k=0}^{n-1} c_k x^{-\frac{1}{a}(1-\alpha-a\mu+k)} + \rho\,(L_\mu\, y)(x) \;=\; g(x),\;\; x > 0, \tag{4.6}$$

where the composed Erdélyi-Kober fractional integral L_μ is defined by (2.25).

Because the null-space of the left-hand sided Erdélyi-Kober fractional integral consists just of the null-function, the equation (4.6) is equivalent to the initial-value problem (4.1)-(4.2). Moreover, the equation (4.6) incorporates the initial conditions and can be interpreted as a convolutional type integral equation of the second kind (see Theorem 3.1). Its solution strategy is straightforward: We apply the associated integral transform $T_{\Phi,\lambda}$ and use the operational relation (2.37) to deduce the equation

$$(T_{\Phi,\lambda} y)(x) - (T_{\Phi,\lambda} \sum_{k=0}^{n-1} c_k t^{-\frac{1}{a}(1-\alpha-a\mu+k)})(x) + \rho\, x^\mu (T_{\Phi,\lambda} y)(x) \;=\; h(x),\;\; x > 0, \tag{4.7}$$

where $h(x) = (T_{\Phi,\lambda} g)(x)$. The representation (2.36) of the integral transform $T_{\Phi,\lambda}$ allows us to calculate the image of the power function x^p in the frequency domain

$$(T_{\Phi,\lambda}\, t^p)(x) = x^\lambda \int_0^\infty \int_0^\infty e^{-\tau_1-\tau_2} \tau_1^{-\alpha} \tau_2^{\beta-1} \left(\frac{x\tau_1^a}{\tau_2^b}\right)^p d\tau_1 \tau_2 = C\, x^{\lambda+p}, \tag{4.8}$$

where $C = \Gamma(1-\alpha+ap)\Gamma(\beta-bp)$ that is valid under the condition $\frac{\alpha-1}{a} < p < \frac{\beta}{b}$ (compare to the condition (2.35)). Thus the equation (4.7) can be rewritten in the following form:

$$(T_{\Phi,\lambda} y)(x) - \sum_{k=0}^{n-1} d_k x^{\lambda+\mu-\frac{1}{a}(1-\alpha+k)} + \rho\, x^\mu (T_{\Phi,\lambda} y)(x) \;=\; h(x),\;\; x > 0, \tag{4.9}$$

where $d_k = c_k\, \Gamma(a\mu - k)\Gamma\left(\beta + \frac{b}{a}(1-\alpha-a\mu+k)\right),\; k = 0, \ldots, n-1$. This equation can be easily solved with respect to the unknown function y in the frequency domain:

$$(T_{\Phi,\lambda} y)(x) = \frac{h(x)}{1+\rho\, x^\mu} + \frac{1}{1+\rho\, x^\mu} \sum_{k=0}^{n-1} d_k x^{\lambda+\mu-\frac{1}{a}(1-\alpha+k)},\;\; x > 0. \tag{4.10}$$

It turns out that the inverse transform $T_{\Phi,\lambda}^{-1}$ of the right-hand side of the last equation - and thus the unknown function y - can be calculated in explicit form in terms of the four-parameters Wright function of the second kind. First we formally derive a formula for the inverse transform and then prove its validity. We start with the well-known formula

$$\frac{1}{1+\rho\, x^{\mu}} = \sum_{i=0}^{\infty}(-\rho\, x^{\mu})^i$$

that is valid only for $|\rho\, x^{\mu}| < 1$. However, we formally substitute this series representation into the equation (4.10) and thus arrive at the formula

$$(T_{\Phi,\lambda} y)(x) = h(x) \sum_{i=0}^{\infty}(-\rho\, x^{\mu})^i + \sum_{k=0}^{n-1} d_k x^{\lambda+\mu-\frac{1}{a}(1-\alpha+k)} \sum_{i=0}^{\infty}(-\rho\, x^{\mu})^i,\ x > 0. \tag{4.11}$$

Using the formula (4.8) and linearity of the integral transform $T_{\Phi,\lambda}$, we can determine the inverse transform $T^{-1}_{\Phi,\lambda}$ of a power law function:

$$(T^{-1}_{\Phi,\lambda}\, t^q)(x) = \frac{x^{q-\lambda}}{\Gamma(1-\alpha+a(q-\lambda))\Gamma(\beta-b(q-\lambda))}. \tag{4.12}$$

Applying this formula to the second power series at the right-hand side of (4.11) (the one that contains the initial values of the problem under consideration) term by term, we get the following correspondences ($k = 0, \ldots, n-1$):

$$\sum_{k=0}^{n-1} d_k x^{\lambda+\mu-\frac{1}{a}(1-\alpha+k)} \sum_{i=0}^{\infty}(-\rho\, x^{\mu})^i \quad \stackrel{T^{-1}_{\Phi,\lambda}}{\longrightarrow} \quad \sum_{k=0}^{n-1} c_k y_k(x) \tag{4.13}$$

with

$$y_k(x) = \Gamma(a\mu-k)\Gamma\left(\beta+\frac{b}{a}(1-\alpha-a\mu+k)\right) x^{\mu-\frac{1}{a}(1-\alpha+k)} \tag{4.14}$$

$$\times\phi((a\mu, a\mu-k), (-b\mu, \beta+\frac{b}{a}(1-\alpha-a\mu+k)); -\rho\, x^{\mu}),$$

where the four-parameters Wright function ϕ is defined by the convergent series

$$\phi((\mu,\alpha),(\nu,\beta);z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha+\mu k)\Gamma(\beta+\nu k)},\quad \mu+\nu>0,\ \alpha,\ \beta\in\mathbb{C}. \tag{4.15}$$

Wright himself investigated this function in the cases $\mu > 0,\ \nu > 0$ in [27] and $\beta = \nu = 1,\ -1 < \mu < 0$ in [28]. Later on, this function was studied in detail and employed as a kernel of an integral transform in [6]. It is worth mentioning that the four-parameters Wright function ϕ is a particular case of the so-called multi-index (or vector index) Mittag-Leffler function (see [13, 17, 18]).

For the parameters values $0 < -\mu < \nu \le 2$, this function was introduced, investigated, and applied to several different FC problems in

[8, 15, 16, 19]. In what follows, we refer to the four-parameters Wright function as to the function of the first kind if both μ and ν are positive and as to the function of the second kind if one of the parameters μ and ν is positive and another one is negative. The properties of the Wright functions of the first and the second kind are very different. For the FC applications, mainly the Wright functions of the second kind are relevant. In particular, it it the case for the problem we deal with in this paper.

Now we proceed with the first power series at the right-hand side of (4.11) that we denote by T_{y_f} and first represent it in the form

$$T_{y_f}(x) = h(x)\sum_{i=0}^{\infty}(-\rho\, x^{\mu})^i = h(x) + h(x)\sum_{i=1}^{\infty}(-\rho\, x^{\mu})^i. \tag{4.16}$$

Now we remind that $h(x) = (T_{\Phi,\lambda}g)(x)$ and use the operational relation (3.5) along with the formula (4.8) to get the representation

$$T_{y_f}(x)=(T_{\Phi,\lambda}g)(x)+\sum_{i=1}^{\infty}(-\rho)^i\, T_{\Phi,\lambda}\left(g \overset{\lambda}{*} \frac{x^{\mu i-\lambda}}{\Gamma(1-\alpha+a(\mu i-\lambda))\Gamma(\beta-b(\mu i-\lambda)}\right). \tag{4.17}$$

Due to the linearity of both the integral transform $T_{\Phi,\lambda}$ and the convolution $\overset{\lambda}{*}$, the right-hand side of (4.17) can be rewritten as

$$T_{y_f}(x) = (T_{\Phi,\lambda}g)(x) + (T_{\Phi,\lambda}\,(g \overset{\lambda}{*} y_{\phi}))(x) \tag{4.18}$$

with

$$y_{\phi}(x) = \rho\, x^{\mu-\lambda}\phi((a\mu, 1-\alpha+a(\mu-\lambda)), (-b\mu, \beta-b(\mu-\lambda)); -\rho\, x^{\mu}))(x). \tag{4.19}$$

Now we can apply the inverse transform $T^{-1}_{\Phi,\lambda}$ to the right-hand side of the formula (4.18) and get the relation

$$y_f(x) := (T^{-1}_{\Phi,\lambda}T_{y_f})(x) = g(x) + (g \overset{\lambda}{*} y_{\phi})(x), \tag{4.20}$$

where the function y_{ϕ} is given by the formula (4.19) in terms of the four-parameters Wright function.

Thus we deduced the unique formal solution to the initial-value problem (4.1)-(4.2) (it is unique because all steps in the procedure presented above are invertible that leads to uniqueness of the formal solution). In the following theorem we prove that the obtained formal solution is also solution in the conventional sense.

Theorem 4.1. *Let* $a > b > 0$, $n-1 < a\mu \le n$, $n \in \mathbb{N}$, $f \in \mathfrak{O}$, *and the condition* (2.35) *be satisfied. Then the initial-value problem* (4.1)-(4.2) *possesses a unique solution on the space* $\mathfrak{O}$ *in the form*

$$y(x) = \sum_{k=0}^{n-1} c_k y_k(x) + y_f(x), \tag{4.21}$$

where the functions y_k *are defined by* (4.14) *and the function* y_f *is given by* (4.20) *with* $g(x) = (I_{1/a}^{-\alpha-a\mu,a\mu} f)(x)$.

The function y_f *satisfies the inhomogeneous equation* (4.1) *and homogeneous initial conditions, wheres the functions* $y_k,\ k = 0,\dots,n-1$ *satisfy the homogeneous equation* (4.1) *(*$f(x) \equiv 0,\ x > 0$*) and the initial conditions (*$k = 0,\dots,n-1,\ j = 0,\dots,n-1$*)*

$$\lim_{x\to 0} x^{\frac{1}{a}(1-\alpha-a\mu+j)} \prod_{i=j+1}^{n-1} \left(1 - \alpha - a\mu + i + ax\frac{d}{dx}\right) y_k(x) = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases} \tag{4.22}$$

P r o o f. The proof of the theorem is a very technical one and thus we restrict ourselves to a short description of the most important steps and ideas and do not present all (in part very long) calculations in detail.

We start with the remark that because the formal solution (4.21) is unique, there exist at most one solution in conventional sense. As soon as we check that (4.21) is a solution to the initial-value problem (4.1)-(4.2), uniqueness of solution is also established. Another observation is that the four-parameters Wright function ϕ defined by the series (4.15) belongs to the class $\mathfrak{O}$ of functions if restricted to the negative real semi-axis (and also on the positive real semi-axis under some additional conditions, see Remark 4.2). Indeed, this function can be represented in form of the following Mellin-Barnes integral or the Fox H-function ([15]):

$$\phi((\mu,\alpha),(\nu,\beta);-x) = \frac{1}{2\pi i} \int_L \frac{\Gamma(1-s)\Gamma(s)}{\Gamma(\alpha-\mu s)\Gamma(\beta-\nu s)}\, x^{-s}\, ds,\ \mu+\nu > 0,\ x > 0 \tag{4.23}$$

with the contour L as in the definition of the Fox H-function ([11, 29]). Evaluating this integral using the Cauchy residue theorem via the residuals $s = -k,\ k = 0, 1, \dots$ of $\Gamma(s)$ leads to the power series (4.15) (compare to (2.9)) whereas doing this with via the residuals $s = 1 + k,\ k = 0, 1, \dots$ of $\Gamma(1-s)$ leads to the (asymptotic) power series (compare to (2.10))

$$\phi((\mu,\alpha),(\nu,\beta);-x) = \sum_{k=0}^{\infty} \frac{z^{-1-k}}{\Gamma(\alpha-\mu-\mu k)\Gamma(\beta-\nu-\nu k)}, \quad x \to +\infty.$$

Because both y_k and y_ϕ (see the formula (4.20)) are products of the power law functions with some four-parameters Wright functions, they belong to the space $\mathfrak{O}$. The function y_f is also from $\mathfrak{O}$ as convolution of two functions from $\mathfrak{D}$. Thus the function y defined by the formula (4.21) belongs to the space $\mathfrak{D}$.

To prove that the function (4.21) is a solution to the initial-value problem (4.1)-(4.2), we check that it satisfies the equation (4.6) that is equivalent to the fractional differential equation (4.1) together with the initial conditions (4.2). Thus we substitute (4.21) in the form

$$y(x) = \sum_{k=0}^{n-1} c_k y_k(x) + g(x) + (g \overset{\lambda}{*} y_\phi)(x)$$

into the equation (4.6) and have to prove that the function

$$F(x) := \sum_{k=0}^{n-1} c_k y_k(x) + (g \overset{\lambda}{*} y_\phi)(x) - \sum_{k=0}^{n-1} c_k x^{-\frac{1}{a}(1-\alpha-a\mu+k)} \tag{4.24}$$
$$+\rho \left(L_\mu \left(\sum_{k=0}^{n-1} c_k y_k(t) + g(t) + (g \overset{\lambda}{*} y_\phi)(t) \right) \right)(x)$$

is identically equal to zero for $x > 0$. First we represent F as follows:

$$F(x) = F_1(x) + F_2(x), \tag{4.25}$$

where the function

$$F_1(x) := \sum_{k=0}^{n-1} c_k y_k(x) - \sum_{k=0}^{n-1} c_k x^{-\frac{1}{a}(1-\alpha-a\mu+k)} + \rho(L_\mu \sum_{k=0}^{n-1} c_k y_k(t))(x) \tag{4.26}$$

is connected with the initial-values (4.2) and the function

$$F_2(x) := (g \overset{\lambda}{*} y_\phi)(x) + \rho \left(L_\mu \left(g(t) + (g \overset{\lambda}{*} y_\phi(t) \right) \right)(x) \tag{4.27}$$

involves the right-hand side of the fractional differential equation (4.1).

It turns out that both $F_1(x)$ and $F_2(x)$ are identically equal to zero for $x > 0$ and thus the function (4.21) is a solution to the initial-value problem (4.1)-(4.2). We start with the function F_1 defined by (4.26) and first calculate the last term at the right-hand side of the formula, i.e., the operator L_μ applied to some power series. A term by term application of the operator L_μ is not allowed because of divergence of the integral part of the right-hand side Erdélyi-Kober fractional derivative from L_μ. To overcome this problem, we employ the technique of the Mellin-Barnes integrals and use the representation (4.23) of the four-parameters Wright function (it can be interpreted as the inverse Mellin integral transform of the quotient of the products of the Gamma-functions under the integral sign) to deduce its Mellin integral transform:

$$\mathfrak{M}\{\phi((\mu,\alpha),(\nu,\beta);-x);s\} = \phi^*(s) = \frac{\Gamma(1-s)\Gamma(s)}{\Gamma(\alpha-\mu s)\Gamma(\beta-\nu s)},\ 0<\Re(s)<1. \tag{4.28}$$

To calculate the Mellin integral transform of the functions $y_k,\ k=0,\ldots,n-1$, we employ the formula (4.28) and the basic properties of the Mellin integral transform (by $\stackrel{\mathfrak{M}}{\leftrightarrow}$ the juxtaposition of a function f with its Mellin transform f^* is denoted)

$$f(at) \stackrel{\mathfrak{M}}{\leftrightarrow} a^{-s}f^*(s),\ a>0, \tag{4.29}$$

$$t^\alpha f(t) \stackrel{\mathfrak{M}}{\leftrightarrow} f^*(s+\alpha), \tag{4.30}$$

$$f(t^\alpha) \stackrel{\mathfrak{M}}{\leftrightarrow} \frac{1}{|\alpha|}f^*(s/\alpha),\ \alpha\neq 0. \tag{4.31}$$

Finally, we use the Mellin-Barnes representation (2.27) of the operator L_μ that we sum up in the form

$$(L_\mu f)(x) = x^\mu \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1-\alpha-as)\Gamma(\beta+bs)}{\Gamma(1-\alpha+a\mu-as)\Gamma(\beta-b\mu+bs)} f^*(s)x^{-s}ds \tag{4.32}$$

to represent $(L_\mu\, y_k(t))(x)$ as a Mellin-Barnes integral and then the formulas (4.28)-(4.31) to get the final result:

$$\rho(L_\mu\, y_k(t))(x) = \Gamma(a\mu-k)\Gamma\left(\beta+\frac{b}{a}(1-\alpha-a\mu+k)\right)\rho x^{2\mu-\frac{1}{a}(1-\alpha+k)}$$

$$\times\ \phi((a\mu,2a\mu-k),(-b\mu,\beta+\frac{b}{a}(1-\alpha-2a\mu+k));-\rho\, x^\mu). \tag{4.33}$$

Comparing the series in this formula with the series in the formula (4.14) for y_k, we see that the last series is the series in (4.14) with the index shift $i\to i+1$ and the term $-x^{-\frac{1}{a}(1-\alpha-a\mu+k)}$ is the first term of the series in (4.14) with a negative sign. Thus all terms in the sum of the series are annulled and we arrive at the identity $(k=0,\ldots,n-1)$

$$y_k(x) - x^{-\frac{1}{a}(1-\alpha-a\mu+k)} + \rho(L_\mu\, y_k(t))(x) \equiv 0,\ x>0$$

that leads to

$$F_1(x)\equiv 0,\ x>0, \tag{4.34}$$

where the function F_1 is defined by (4.26).

To prove the identity

$$F_2(x)\equiv 0,\ x>0 \tag{4.35}$$

with the function F_2 defined by (4.27), we first apply Theorem 3.1 to represent it in the form

$$F_2(x) = (g \overset{\lambda}{*} y_\phi)(x) + \rho((g + (g \overset{\lambda}{*} y_\phi)) \overset{\lambda}{*} h_{\mu,\lambda})(x),$$

where the power law function $h_{\mu,\lambda}$ is defined as in (3.1). Using the properties of the convolution $\overset{\lambda}{*}$, we can factor our the function g and obtain the formula

$$F_2(x) = (g \overset{\lambda}{*} (y_\phi + \rho h_{\mu,\lambda} + \rho y_\phi \overset{\lambda}{*} h_{\mu,\lambda})(x). \tag{4.36}$$

It turns out that

$$Y_\phi(x) := y_\phi + \rho h_{\mu,\lambda} + \rho(y_\phi \overset{\lambda}{*} h_{\mu,\lambda})(x) \equiv 0, \ x > 0 \tag{4.37}$$

that means the validity of the identity (4.35). Indeed, we again use Theorem 3.1 and get the formula

$$Y_\phi(x) = y_\phi + \rho h_{\mu,\lambda} + \rho(L_\mu y_\phi)(x).$$

The function y_ϕ is a product of a four-parameters Wright function and a power law function. Thus we use exactly the same technique of the Mellin integral transform as was employed for the functions y_k and arrive at the identity (4.37) that leads to validity of (4.35). Putting together (4.25), (4.34), and (4.35) completes the proof of the first part of this theorem.

The validity of the formulas (4.22) are proved by simple direct calculations. This time, we are allowed to differentiate the power series term by term using the well-known formula for the derivative of the power law function that directly leads to the formulas (4.22). □

REMARK 4.1. In the equation (4.1), the parameter ρ can be also negative. Then the result of Theorem (4.1) remains valid under the additional condition $a/3 < b$. To show this, we use the same reasoning as in the proof of Theorem 4.1 along with the asymptotic behavior of the four-parameters Wright function of the second kind on the positive semi-axis (see [19]): Let $0 < \nu/3 < -\mu < \nu \le 2,\ L = 0, 1, 2, \dots,\ P = 0, 1, 2, \dots.$ Then

$$\phi((\mu, \alpha), (\nu, \beta); x) = \sum_{i=0}^{L-1} \frac{x^{(\alpha-1-i)/(-\mu)}}{(-\mu)\Gamma(i+1)\Gamma(\beta + \nu(a-1-i)/(-\mu))} \tag{4.38}$$

$$- \sum_{k=1}^{P} \frac{x^{-k}}{\Gamma(\beta - \nu k)\Gamma(\alpha - \mu k)} + O(x^{(a-1-L)/(-\mu)}) + O(x^{-1-P}), \quad x \to +\infty.$$

REMARK 4.2. The procedure that was employed in this section for analytical treatment of the initial-value problem (4.1)-(4.2) can be applied for derivation of the closed form formulas for solutions of other equations that contain several different left- and right-hand sided Erdélyi-Kober fractional derivatives with the suitable initial conditions in form (4.2). In particular, our method works for equations in the form

$$y(x) - \sum_k c_k x^{\alpha_k} + \sum_k \rho_k \, (L_{\mu k} \, y)(x) \; = \; g(x), \; x > 0,$$

that are obtained after application of the suitable left-hand sided Erdélyi-Kober fractional integral to the corresponding initial-value problem for an equation with several different left- and right-hand sided Erdélyi-Kober fractional derivatives.

Remark 4.3. As already mentioned in Introduction, another powerful technique for analytical treatment of the initial-value problems for the equations that contain the left- and right-hand sided Erdélyi-Kober fractional derivatives would be an operational calculus of Mikusiński type for the composed Erdélyi-Kober fractional derivative. This calculus as well as its applications will be considered elsewhere.

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