RESEARCH PAPER

ERROR ANALYSIS OF NONLINEAR TIME FRACTIONAL
MOBILE/IMMOBILE ADVECTION-DIFFUSION
EQUATION WITH WEAKLY SINGULAR SOLUTIONS

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Abstract

In this paper, a weighted and shifted Grünwald-Letnikov difference (WSGD) Legendre spectral method is proposed to solve the two-dimensional nonlinear time fractional mobile/immobile advection-dispersion equation. We introduce the correction method to deal with the singularity in time, and the stability and convergence analysis are proven. In the numerical implementation, a fast method is applied based on a globally uniform approximation of the trapezoidal rule for the integral on the real line to decrease the memory requirement and computational cost. The memory requirement and computational cost are $O(Q)$ and $O(QK)$, respectively, where K is the number of the final time step and Q is the number of quadrature points used in the trapezoidal rule. Some numerical experiments are given to confirm our theoretical analysis and the effectiveness of the presented methods.

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Key Words and Phrases: two-dimensional nonlinear time fractional mobile/immobile advection-dispersion equation; correction method; stability and convergence analysis; fast method

1. Introduction

In this paper, we consider the following two-dimensional nonlinear time fractional mobile/immobile advection-diffusion equation [33, 42],

$$\begin{cases} u_t + \beta {}^C_0 D_t^\alpha u = -V\nabla u + D\Delta u + F(u) + f(x, y, t), & x, y \in \Omega, t \in I, \\ u(x, y, 0) = u_0(x, y), & x, y \in \Omega, \\ u(x, y, t) = 0, & x, y \in \partial\Omega, t \in I, \end{cases} \quad (1.1)$$

where $\Omega = (a, b) \times (c, d)$, $I = (0, T]$, u denotes the solute concentration in the total (mobile+immobile) phase, and β is the fractional capacity coefficient. Here V and D are the velocity and dispersion coefficient for the mobile phase (and hence may be directly measured). The time drift term u_t is added to describe the motion time and thus helps to distinguish the status of particles conveniently (see also the discussion in [2]). $F(u)$ is a nonlinear term satisfies the local Lipschitz condition. ${}^C_0 D_t^\alpha$ ($0 < \alpha < 1$) is the Caputo fractional derivative of order α that can be defined [31] as follows,

$${}^C_0 D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)ds}{(t-s)^\alpha}. \quad (1.2)$$

Many researchers have studied the nonlinear time fractional mobile/immobile advection-diffusion equation (1.1). A RBF meshless approach was introduced for modeling a fractal mobile/immobile transport model [22]. Zhang et al. [40] proposed a novel numerical method for the time variable fractional order mobile/immobile advection-dispersion model. For a more detailed description of this aspect, we refer the readers to [27, 42]. A variety of numerical methods have been used to approximating the fractional partial differential equations [1, 5, 6, 7, 11, 12, 13, 14, 15, 18, 20, 21, 30]. Among them, spectral method has attracted more and more attentions [3, 9, 17, 36, 37] because of its high-order accuracy. In this paper, we develop a WSGD Legendre spectral method for the considered model.

In practical applications, analytical solutions of time fractional differential equations may have strong singularity at $t = 0$ [28, 35]. In order to solve the singularity, some researchers [16, 32] adopted the graded mesh method. For the considered equation, we introduce some correction terms [38, 39] to deal with the non-smooth solutions. The stability and convergence analysis of the correction method are also proven.

We can rewrite the model as

$$u_t + \beta {}^{RL}_0 D_t^\alpha (u - u_0) = -V\nabla u + D\Delta u + F(u) + f(x, y, t), \quad (1.3)$$

here ${}^{RL}_0 D_t^\alpha$ ($0 < \alpha < 1$) is the Riemann-Liouville fractional derivative of order α [31]. Using WSGD method yields the following discrete convolution [25] as

$${}_0^{RL}D_t^\alpha u(t_n) = \frac{1}{\tau^\alpha} \sum_{l=0}^n \lambda_{n-l}^{(\alpha)} u^l, \quad 0 \leq n \leq K, \quad (1.4)$$

where τ is the time step size, $\lambda_{n-l}^{(\alpha)}$ are the convolution quadrature weights. The implementation of (1.4) requires $O(K)$ active memory and $O(K^2)$ arithmetic operations by direct computation. So the direct calculation of (1.4) becomes computational expensive when it is used to discretize time fractional partial differential equations. Some works [19, 24, 39] have been presented to reduce the memory requirements and computational cost of the discrete convolution for approximating the Riemann-Liouville fractional derivative. In this numerical implementation, we apply a fast method, in which the Hankel contour is used to express the quadrature weight $\lambda_n^{(\alpha)}$ as an integral on the half line, see later for details.

The highlights of this paper are: (i) we introduce some correction terms to solve the two-dimensional nonlinear time fractional mobile/immobile advection-diffusion equation with non-smooth solutions; (ii) stability and convergence analysis of correction method are proven; (iii) a fast method is applied to decrease the memory requirement and computational cost.

The paper is organized as follows. Section 2 develops the WSGD Legendre spectral method to solve the two-dimensional nonlinear time fractional mobile/immobile advection-diffusion equation. And some correction terms are introduced to deal with the non-smooth solutions. The stability and convergence analysis of the correction method are presented in Section 3. In Section 4, a fast algorithm for numerical implementation is proposed. Numerical examples are provided in Section 5. Finally, some conclusions and discussions are summarized in Section 6.

2. Numerical method

For the temporal discretization, we divide the interval $[0, T]$ into K equal subintervals with a time-step size $\tau = T/K$. Let $t_n = n\tau$, $u^n = u(x, y, t_n)$, $0 \leq n \leq K$. For convenience, we introduce the following notations:

$$\delta_t u^{n+1/2} = \frac{u^{n+1} - u^n}{\tau}, \quad u^{n+1/2} = \frac{u^n + u^{n+1}}{2}.$$

We approximate the fractional derivative as (1.3),

$$\lambda_0^{(\alpha)} = \left(1 + \frac{\alpha}{2}\right) g_0^{(\alpha)}, \quad \lambda_j^{(\alpha)} = \left(1 + \frac{\alpha}{2}\right) g_j^{(\alpha)} - \frac{\alpha}{2} g_{j-1}^{(\alpha)}, \quad j \geq 1, \quad (2.1)$$

and $g_j^{(\alpha)} = (-1)^k \binom{\alpha}{j}$ for $j \geq 0$, and they can be evaluated recursively by

$$g_0^{(\alpha)} = 1, \quad g_j^{(\alpha)} = \left(1 - \frac{\alpha + 1}{j}\right) g_{j-1}^{(\alpha)}, \quad j = 1, 2, \dots \quad (2.2)$$

If $\alpha = 0$, we set $g_0^{(\alpha)} = 1$, $g_j^{(\alpha)} = 0$ for $j \geq 1$. It is obvious that

$$\begin{aligned} & \delta_t u^{n+1/2} + \beta D_\tau^{\alpha, n+1/2} \tilde{u} \\ &= -V \nabla u^{n+1/2} + D \Delta u^{n+1/2} + F(u^{n+1/2}) + f^{n+1/2} + E^n, \end{aligned} \quad (2.3)$$

where $\tilde{u}^n = u^n - u_0$, $D_\tau^{\alpha, n} u = \frac{1}{\tau^\alpha} \sum_{l=0}^n \lambda_{n-l}^{(\alpha)} u^l$, and E^n is the truncation error satisfying $|E^n| \leq O(\tau^2)$.

The WSGD scheme can not preserve second-order accuracy in time approximation when u has a strong singularity. The key assumption is that the analytical solution $u(x, y, t)$ satisfies the following form [4, 26],

$$u - u_0 = \sum_{j=1}^m c_j(x, y) t^{\sigma_j} + c_{m+1}(x, y) t^{\sigma_{m+1}} + \dots, \quad (2.4)$$

where $1 \leq \sigma_j < \sigma_{j+1}$. We develop the correction method to deal with the non-smooth solutions [38],

$${}_0^{RL} D_t^\alpha u^n = D_\tau^{\alpha, n} u + \frac{1}{\tau^\alpha} \sum_{j=1}^{m_1} W_{n,j}^\alpha (u^j - u^0) + R_1^n = D_\tau^{\alpha, n, m_1} u + R_1^n, \quad (2.5)$$

where $m_1 \leq m$, $W_{n,j}^\alpha$ are the starting weights satisfying

$$D_\tau^{\alpha, n} u + \frac{1}{\tau^\alpha} \sum_{j=1}^{m_1} W_{n,j}^\alpha (u^j - u^0) = \frac{\Gamma(1 + \sigma_j)}{\Gamma(1 + \sigma_j - \alpha)} t_n^{\sigma_j - \alpha}, \quad (2.6)$$

in which $u = t^{\sigma_j}$. We can get $W_{n,j}^\alpha$ ($1 \leq j \leq m$) from (2.6) and $W_{n,j}^\alpha$ is independent of τ for $n > 0$. For the first-order time derivative, it has

$$\partial_t u^{n+1/2} = \delta_t u^{n+1/2} + \frac{1}{\tau} \sum_{j=1}^{m_2} W_{n,j} (u^j - u^0) + R_2^n = \delta_t^{m_2} u^{n+1/2} + R_2^n, \quad (2.7)$$

where $m_2 \leq m$, $W_{n,j}$ can be obtained by the following equation

$$\frac{\sigma_j}{2} (t_{n+1}^{\sigma_j-1} + t_n^{\sigma_j-1}) = \delta_t u^{n+1/2} + \frac{1}{\tau} \sum_{j=1}^{m_2} W_{n,j} (u^j - u^0), \quad (2.8)$$

for $u = t^{\sigma_j}$.

Based on boundary conditions, we use the following basis functions [37] in the x and y directions for the spatial discretization,

$$\phi_i(x) = L_i(\hat{x}) - L_{i+2}(\hat{x}), \quad \varphi_i(y) = L_i(\hat{y}) - L_{i+2}(\hat{y}), \quad (2.9)$$

where $i = 0, 1, \dots, N-2$ and $L_i(\hat{z})$ is the Legendre polynomial [34], and $\hat{x} \in (-1, 1)$, $x = \frac{(b-a)\hat{x}+a+b}{2} \in (a, b)$, $\hat{y} \in (-1, 1)$, $y = \frac{(d-c)\hat{y}+c+d}{2} \in (c, d)$. The function space S_N can be specified as follows,

$$S_N = \text{span}\{\phi_i(x)\varphi_j(y), \quad i, j = 0, 1, \dots, N-2\}.$$

Combing (2.3) and the above results leads to the following fully discrete form,

$$\begin{aligned} & (\delta_t^{m_2} u_N^{n+1/2}, v) + (D_\tau^{\alpha, n+1/2, m_1} \tilde{u}_N, v) + V(\nabla u_N^{n+1/2}, v) + D(\nabla u_N^{n+1/2}, \nabla v) \\ &= \left(F(u_N^{n+1/2}), v \right) + (f^{n+1/2}, v), \quad \forall v \in S_N, \end{aligned} \quad (2.10)$$

and

$$u_N^0 = \Pi_N^{1,0} u_0, \quad (2.11)$$

where $\Pi_N^{1,0}$ denotes the orthogonal projection operator [37]: $H_0^1(\Omega) \rightarrow S_N$,

$$(\nabla(u - \Pi_N^{1,0} u), \nabla v) = 0, \quad \forall v \in S_N. \quad (2.12)$$

We rewrite the equation as

$$G^n(v) = \left(1 + \frac{\beta \tau^{1-\alpha} \lambda_0^{(\alpha)}}{2} \right) (u_N^{n+1}, v) + \frac{\tau V}{2} (\nabla u_N^{n+1}, v) + \frac{\tau D}{2} (\nabla u_N^{n+1}, \nabla v). \quad (2.13)$$

Next, we derive the matrix representation. The numerical solution can be denoted by

$$u_N^{n+1} = \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} c_{ij}^{n+1} \phi_i(x) \varphi_j(y). \quad (2.14)$$

Letting $v = \phi_h \varphi_l$ ($h, l = 0, 1, \dots, N-2$), we have

$$\begin{aligned} & \sum_{i=0}^{N-2} \sum_{j=0}^{N-2} \left\{ \left(1 + \frac{\beta \tau^{1-\alpha} \lambda_0^{(\alpha)}}{2} \right) (\phi_i \varphi_j, \phi_h \varphi_l) + \frac{\tau V}{2} (\partial_x \phi_i \varphi_j, \phi_h \varphi_l) \right. \\ & \left. + \frac{\tau V}{2} (\phi_i \partial_y \varphi_j, \phi_h \varphi_l) + \frac{\tau D}{2} (\partial_x \phi_i \varphi_j, \partial_x \phi_h \varphi_l) + \frac{\tau D}{2} (\phi_i \partial_y \varphi_j, \phi_h \partial_y \varphi_l) \right\} c_{ij}^{n+1} \\ &= G^n(\phi_h \varphi_l). \end{aligned} \quad (2.15)$$

The matrix representation is shown as

$$\begin{aligned} & \left(\left(1 + \frac{\beta \tau^{1-\alpha} \lambda_0^{(\alpha)}}{2} \right) M_x \otimes M_y + \frac{\tau V}{2} (P_x \otimes M_y + M_x \otimes P_y) \right. \\ & \left. + \frac{\tau D}{2} (S_x \otimes M_y + M_x \otimes S_y) \right) C^{n+1} = G^n, \end{aligned} \quad (2.16)$$

where

$$\begin{aligned}
 (M_x)_{h,i} &= (\phi_i, \phi_h), & (M_y)_{l,j} &= (\varphi_j, \varphi_l), \\
 (P_x)_{h,i} &= (\partial_x \phi_i, \phi_h), & (P_y)_{l,j} &= (\partial_y \varphi_i, \varphi_j), \\
 (S_x)_{h,i} &= (\partial_x \phi_i, \partial_x \phi_h), & (S_y)_{l,j} &= (\partial_y \varphi_j, \partial_y \varphi_l), \\
 (C^{n+1})_{i,j} &= c_{ij}^{n+1}, & i, j &= 0, 1, \dots, N-2.
 \end{aligned}$$

REMARK 2.1. When this model has non-homogeneous boundary conditions, we use the following basis functions [37] in the x and y directions for the spatial discretization,

$$\begin{aligned}
 \phi_1(x) &= \frac{L_0(\hat{x}) - L_1(\hat{x})}{2}, & \varphi_1(y) &= \frac{L_0(\hat{y}) - L_1(\hat{y})}{2}, \\
 \phi_i(x) &= L_{i-2}(\hat{x}) - L_i(\hat{x}), & \varphi_i(y) &= L_{i-2}(\hat{y}) - L_i(\hat{y}), \\
 \phi_{N+1}(x) &= \frac{L_0(\hat{x}) + L_1(\hat{x})}{2}, & \varphi_{N+1}(y) &= \frac{L_0(\hat{y}) + L_1(\hat{y})}{2},
 \end{aligned} \tag{2.17}$$

where $i = 2, 3, \dots, N$.

3. Theoretical analysis of the correction method

3.1. **Preliminaries.** The following lemmas are given used in the later proof.

LEMMA 3.1 ([23]). *As for (1.4), there exists any positive integer L and real vector $(u_0, u_1, u_2, \dots, u_L)^T \in \mathbb{R}^{L+1}$, it holds that*

$$\sum_{n=0}^L \left(\sum_{j=0}^n \lambda_j^{(\alpha)} u_{n-j} \right) u_n \geq 0.$$

LEMMA 3.2 ([37]). *Let τ, g and a_k, γ_k , for integers $k > 0$, be nonnegative numbers such that*

$$a_n \leq \tau \sum_{k=0}^{n-1} \gamma_k a_k + g, \quad n \geq 0.$$

Then,

$$a_n \leq g \exp \left(\tau \sum_{k=0}^{n-1} \gamma_k \right).$$

LEMMA 3.3 ([38]). *Let m_1 be a positive integer and $W_{n,j}^\alpha$ ($j = 1, 2, \dots, m_1$) be defined by (2.6). Suppose that σ_j are a sequence of increasing positive numbers. We have*

$$W_{n,j}^\alpha = O(n^{\sigma_1-2-\alpha}) + O(n^{\sigma_2-2-\alpha}) + \dots + O(n^{\sigma_{m_1}-2-\alpha}).$$

LEMMA 3.4 ([38]). *Let $u(t) = t^\sigma$ ($\sigma \geq 0$) and α be a real number. There exists a positive and bounded constant C independent of n and τ such that the error R_1^n defined in (2.5) has the following estimate,*

$$|R_1^n| \leq CS_{m_1}^\sigma \tau^{\sigma-\alpha} n^{\sigma_{\max}-\alpha-2},$$

where $S_{m_1}^\sigma = \Pi_{k=1}^{m_1} |\sigma - \sigma_k|$, $\sigma_{\max} = \max\{\sigma, \sigma_1, \dots, \sigma_{m_1}\}$.

LEMMA 3.5 ([37]). *For any $u \in S_N$, there exists a positive constant C independent on n , τ , and N such that the inverse inequality is obtained,*

$$\|u\|_{L^\infty} \leq CN\|u\|.$$

LEMMA 3.6 ([41]). *Suppose that $u \in H_0^r(\Omega)$, there exists a positive constant C independent on n , τ , and N such that*

$$\|u - \Pi_N^{1,0} u\|_s \leq CN^{s-r} \|u\|_r, \quad s = 0, 1, \quad r \geq 1.$$

3.2. Stability and convergence analysis. We first present the following stability result.

Suppose that u_N^k has the error $\hat{u}_N^k$. We obtain the error equation,

$$\begin{aligned} & (\delta_t^{m_2} \hat{u}_N^{k+1/2}, v) + (D_\tau^{\alpha, k+1/2, m_1} \hat{u}_N^k, v) + V(\nabla \hat{u}_N^{k+1/2}, v) + D(\nabla \hat{u}_N^{k+1/2}, \nabla v) \\ &= \left(\hat{F}_N^{k+1/2}, v \right) + \left(\hat{f}^{k+1/2}, v \right), \end{aligned} \tag{3.1}$$

where $\hat{F}_N^k = F(u_N^k + \hat{u}_N^k) - F(u_N^k)$. Let C be an appropriate positive constant and Θ be a suitable domain. Suppose that $F'(y)$ is bounded in the appropriate domain Θ . Denoting

$$u_{N \max} = \max_{0 \leq k \leq L} \|u_N^k\|_\infty, \quad F_{N \max} = \max_{y \in \Theta} |F'(y)|. \tag{3.2}$$

We assume

$$\|\hat{u}_N^k\|_\infty \leq C_0, \quad 0 \leq k \leq n \leq m, \quad k, m \in Z^+. \tag{3.3}$$

The following assumption can be given

$$\|\hat{u}_N^{k+1}\|_\infty \leq C_1, \quad \text{if } \|\hat{u}_N^k\|_\infty \leq C_0, \tag{3.4}$$

where C_1 is a positive constant independent of τ , N and k .

Letting $v = \hat{u}_N^{k+1/2}$ in (2.10) yields,

$$\begin{aligned}
 & \frac{\|\hat{u}_N^{k+1}\|^2 - \|\hat{u}_N^k\|^2}{2\tau} + \beta(D_\tau^{\alpha, k+1/2} \hat{u}_N, \hat{u}_N^{k+1/2}) + D\|\nabla \hat{u}_N^{k+1/2}\|^2 \\
 = & - \sum_{j=1}^{m_2} W_{k,j} \left((\hat{u}_N^j - \hat{u}_N^0)/\tau, \hat{u}_N^{k+1/2} \right) + \beta(D_\tau^{\alpha, k+1/2} \hat{u}_N^0, \hat{u}_N^{k+1/2}) \\
 & - \frac{\tau^{1-\alpha}}{2} \sum_{j=1}^{m_1} (W_{k,j}^\alpha + W_{k+1,j}^\alpha) \left((\hat{u}_N^j - \hat{u}_N^0)/\tau, \hat{u}_N^{k+1/2} \right) \\
 & + V(\hat{u}_N^{k+1/2}, \nabla \hat{u}_N^{k+1/2}) + (\hat{F}_N^{k+1/2}, \hat{u}_N^{k+1/2}) + (\hat{f}^{k+1/2}, \hat{u}_N^{k+1/2}).
 \end{aligned} \tag{3.5}$$

Based on the result in [38] and Lemma 3.3, we obtain

$$\begin{aligned}
 q_k &= \sum_{l=0}^k \lambda_l^{(\alpha)} = O(k^{-\alpha}), \\
 |W_{k,j}^\alpha| &\leq Ck^{\sigma_{m_1}-2-\alpha}, \\
 |W_{k,j}| &\leq Ck^{\sigma_{m_2}-3}.
 \end{aligned} \tag{3.6}$$

Combining Cauchy-Schwarz inequality and Lemma 3.1, we can easily get

$$\begin{aligned}
 & \frac{\|\hat{u}_N^{k+1}\|^2 - \|\hat{u}_N^k\|^2}{2\tau} + \frac{D}{2} \|\nabla \hat{u}_N^{k+1/2}\|^2 \\
 \leq & \sum_{j=1}^{m_2} |W_{k,j}| \left(\|(\hat{u}_N^j - \hat{u}_N^0)/\tau\|^2 + \|\hat{u}_N^{k+1/2}\|^2 \right) + C\|\hat{u}_N^{k+1/2}\|^2 \\
 & + \frac{\tau^{1-\alpha}}{2} \sum_{j=1}^{m_1} |W_{k,j}^\alpha + W_{k+1,j}^\alpha| \left(\|(\hat{u}_N^j - \hat{u}_N^0)/\tau\|^2 + \|\hat{u}_N^{k+1/2}\|^2 \right) \\
 & + \beta\tau^{-\alpha} |q_k| (\|\hat{u}_N^0\|^2 + \|\hat{u}_N^{k+1/2}\|^2) + \|\hat{F}_N^{k+1/2}\|^2 + \|\hat{f}^{k+1/2}\|^2.
 \end{aligned} \tag{3.7}$$

Since $F(u)$ satisfies the local Lipschitz condition, it is shown that

$$\|\hat{F}_N^k\| = \|F(u_N^k + \hat{u}_N^k) - F(u_N^k)\| \leq C\|\hat{u}_N^k\|. \tag{3.8}$$

Summing n in (3.7) from 0 to n gives

$$\begin{aligned}
& \|\hat{u}_N^{n+1}\|^2 + D\tau \sum_{k=0}^n \|\nabla \hat{u}_N^{k+1/2}\|^2 \\
& \leq \|\hat{u}_N^0\|^2 + C\tau \sum_{k=0}^n \sum_{j=1}^{m_2} |W_{k,j}| \left(\|(\hat{u}_N^j - \hat{u}_N^0)/\tau\|^2 + \|\hat{u}_N^{k+1/2}\|^2 \right) \\
& \quad + C\tau^{2-\alpha} \sum_{k=0}^n \sum_{j=1}^{m_1} |W_{k,j}^\alpha + W_{k+1,j}^\alpha| \left(\|(\hat{u}_N^j - \hat{u}_N^0)/\tau\|^2 + \|\hat{u}_N^{k+1/2}\|^2 \right) \quad (3.9) \\
& \quad + C\tau^{1-\alpha} \sum_{k=0}^n |q_k| (\|\hat{u}_N^0\|^2 + \|\hat{u}_N^{k+1/2}\|^2) + C\tau \sum_{k=0}^n \|\hat{u}_N^{k+1/2}\|^2 \\
& \quad + C\tau \sum_{k=0}^n \|\hat{f}^{k+1/2}\|^2.
\end{aligned}$$

Using (3.6), it infers that,

$$\begin{aligned}
& \|\hat{u}_N^{n+1}\|^2 + D\tau \sum_{k=0}^n \|\nabla \hat{u}_N^{k+1/2}\|^2 \\
& \leq \|\hat{u}_N^0\|^2 + C \sum_{k=1}^{n+1} \left(\tau k^{\sigma_{m_2}-3} + \tau^{2-\alpha} k^{\sigma_{m_1}-2-\alpha} + \tau^{1-\alpha} k^{-\alpha} + \tau \right) \|\hat{u}_N^k\|^2 \\
& \quad + C\tau^{1-\alpha} \sum_{k=0}^n |q_k| \|\hat{u}_N^0\|^2 + C\tau \sum_{k=1}^n |W_{k,j}| \sum_{j=1}^{m_2} \|\delta_t \hat{u}_N^{j-1/2}\|^2 \\
& \quad + C\tau^{2-\alpha} \sum_{k=1}^n |W_{k,j}^\alpha + W_{k+1,j}^\alpha| \sum_{j=1}^{m_1} \|\delta_t \hat{u}_N^{j-1/2}\|^2 + C\tau \sum_{k=0}^{n+1} \|\hat{f}^k\|^2. \quad (3.10)
\end{aligned}$$

According to the assumption, we have

$$\begin{aligned}
A_1^n &= \sum_{k=1}^n |W_{k,j}^\alpha + W_{k+1,j}^\alpha| \leq C \sum_{k=1}^n k^{\sigma_{m_1}-2-\alpha} \leq C \max\{1, n^{\sigma_{m_1}-1-\alpha}\}, \\
A_2^n &= \sum_{k=1}^n |W_{k,j}| \leq C \sum_{k=0}^n k^{\sigma_{m_2}-3} \leq C \max\{1, n^{\sigma_{m_2}-2}\}. \quad (3.11)
\end{aligned}$$

For sufficiently small τ , we can obtain the following inequality easily by (3.6) and (3.11),

$$\|\hat{u}_N^{n+1}\|^2 \leq \rho^n + C \sum_{k=1}^n b_k \|\hat{u}_N^k\|^2, \quad (3.12)$$

where

$$\begin{aligned}\rho^n = & C \left(\|\hat{u}_N^0\|^2 + A_2^n \sum_{j=1}^{m_2} \|\delta_t \hat{u}_N^{j-1/2}\|^2 \right. \\ & \left. + \tau^{1-\alpha} A_1^n \sum_{j=1}^{m_1} \|\delta_t \hat{u}_N^{j-1/2}\|^2 + \tau \sum_{k=0}^{n+1} \|\hat{f}^k\|^2 \right), \\ b_k = & \tau k^{\sigma_{m_2}-3} + \tau^{2-\alpha} k^{\sigma_{m_1}-2-\alpha} + \tau^{1-\alpha} n^{-\alpha} + \tau.\end{aligned}$$

Based on the condition $\sigma_{m_1}, \sigma_{m_2} \leq 3$, we have

$$\sum_{k=1}^n b_k \leq CT^{2-\alpha}. \quad (3.13)$$

THEOREM 3.1. Suppose that u_N^{n+1} ($n = 0, 1, \dots, K-1$) are the solutions of numerical scheme (2.10). $F(u)$ satisfies the local Lipschitz condition, C, C_0, C_1 and $\hat{C}$ are appropriate positive constants independent of n, τ , and N . Suppose $\|\hat{u}_N^{k+1}\|_\infty \leq C_1$ under the condition $\|\hat{u}_N^k\|_\infty \leq C_0$ ($0 \leq k \leq n$). If $\sigma_{m_1}, \sigma_{m_2} \leq 3$, $\rho^n \leq \left(\frac{C_0}{\hat{C}N}\right)^2 \exp(-CT^{2-\alpha})$, then

$$\|\hat{u}_N^{n+1}\|^2 \leq \rho^n \exp(CT^{2-\alpha}). \quad (3.14)$$

P r o o f. We use the mathematical induction for performing this proof. The inequality (3.14) can be easily proved for $n = 0$. Then we verify that the inequality (3.14) holds for any $0 \leq n \leq K-1$. Assume that $\|\hat{u}_N^k\|_\infty \leq C_0$ ($0 \leq k \leq m$),

$$\|\hat{u}_N^{n+1}\|^2 \leq \rho^n + C \sum_{k=1}^n b_k \|\hat{u}_N^k\|^2, \quad 0 \leq n \leq m. \quad (3.15)$$

From Gronwall inequality (see Lemma 3.2),

$$\|\hat{u}_N^{n+1}\|^2 \leq \rho^n \exp(CT^{2-\alpha}), \quad 0 \leq n \leq m. \quad (3.16)$$

Next, we prove that (3.14) holds for $n = m+1$, by Lemma 3.5

$$\begin{aligned}\|\hat{u}_N^{m+1}\|_\infty^2 & \leq (\hat{C}N)^2 \|\hat{u}_N^{m+1}\|^2 \\ & \leq (\hat{C}N)^2 \rho^m \exp(CT^{2-\alpha}) \\ & \leq (C_0)^2.\end{aligned} \quad (3.17)$$

We get $\|\hat{u}_N^k\|_\infty \leq C_0$ ($0 \leq k \leq m+1$). Thus, it infers that,

$$\begin{aligned}\|\hat{u}_N^{m+2}\|^2 & \leq \rho^{m+1} + C \sum_{k=1}^{m+1} b_k \|\hat{u}_N^k\|^2 \\ & \leq \rho^{m+1} \exp(CT^{2-\alpha}).\end{aligned} \quad (3.18)$$

The inequality (3.14) is held for $n = m+1$. The proof is completed. $\square$

Next, the convergence analysis of the numerical method can be presented.

THEOREM 3.2. *Assume that $r \geq 2$, let u and u_N^{n+1} ($0 \leq n \leq K - 1$) be the solutions of (1.1) and (2.10), respectively. If $r \leq N + 1$, $u \in C^3(I; H^r(\Omega))$, $F(u) \in L^2(I; H^r(\Omega))$ satisfies the local Lipschitz condition, $u_0 \in H^r(\Omega)$, and $m_1, m_2 \leq m$ with $\sigma_{m_1}, \sigma_{m_2} \leq 3$, for small enough τ , there exists a positive constant C independent of n, τ , and N such that*

$$\|u_N^{n+1} - u(\cdot, t_{n+1})\| \leq C \left(N^{-r} + \tau^{\min\{2, \sigma_{m_1+1}+0.5-\alpha, \sigma_{m_2+1}-0.5\}} \right). \quad (3.19)$$

P r o o f. Setting $u_* = \Pi_N^{1,0} u$, $e = u_N - u_*$ and $\eta = u_* - u$. We can obtain the following error equation:

$$\begin{aligned} & (\delta_t^{m_2} e^{k+1/2}, v) + (D_\tau^{\alpha, k+1/2, m_1} e, v) + V(\nabla e^{k+1/2}, v) + D(\nabla e^{k+1/2}, \nabla v) \\ &= \left(F(u_N^{k+1/2}) - F(u_*^{k+1/2}), v \right) + (H^k, v), \end{aligned} \quad (3.20)$$

where

$$\begin{aligned} H^n &= -\delta_t^{m_2} \eta^{k+1/2} - D_\tau^{\alpha, k+1/2, m_1} \eta - V \nabla \eta^{k+1/2} \\ &\quad + F(u_*^{k+1/2}) - F(u^{k+1/2}) + O(\tau^2 t_n^{\sigma_{m_1+1}-2-\alpha}) \\ &\quad + O(\tau^2 t_n^{\sigma_{m_2+1}-3}). \end{aligned}$$

Taking $v = e^{k+1/2}$. Similar to the process of Theorem 3.1,

$$\|e^{n+1}\|^2 \leq C \left(\sum_{j=1}^{m_1} \|\delta_t e^{j-1/2}\|^2 + \sum_{j=1}^{m_2} \|\delta_t e^{j-1/2}\|^2 \right) + C\tau \sum_{k=1}^n \|H^k\|^2. \quad (3.21)$$

From Lemma 3.1, Lemma 3.4, and Lemma 3.6

$$\|H^k\| \leq C \left(N^{-r} + \tau^{\sigma_{m_1+1}-\alpha} k^{\sigma_{m_1+1}-2-\alpha} + \tau^{\sigma_{m_2+1}-1} k^{\sigma_{m_2+1}-3} \right).$$

Based on the above results, we obtain

$$\begin{aligned} & \|e^{n+1}\|^2 \\ & \leq C \left(\sum_{j=1}^{m_2} \|\delta_t e^{j-1/2}\|^2 + \sum_{j=1}^{m_1} \|\delta_t \hat{u}_N^{j-1/2}\|^2 + N^{-2r} \right. \\ & \quad \left. + \tau^{2\sigma_{m_1+1}-2\alpha+1} \sum_{k=1}^n k^{2(\sigma_{m_1+1}-2-\alpha)} + \tau^{2\sigma_{m_2+1}-1} \sum_{k=1}^n k^{2(\sigma_{m_2+1}-3)} \right). \end{aligned} \quad (3.22)$$

Moreover,

$$\begin{aligned} & \sum_{j=1}^{m_1} \|\delta_t e^{j-1/2}\|^2 + \sum_{j=1}^{m_2} \|\delta_t e^{j-1/2}\|^2 \\ & \leq C \left(N^{-2r} + \tau^{\min\{4, 2\sigma_{m_1+1}+1-2\alpha, 2\sigma_{m_2+1}-1\}} \right). \end{aligned}$$

It can be derived that

$$\|e^{n+1}\| \leq C \left(N^{-r} + \tau^{\min\{2, \sigma_{m_1+1}+0.5-\alpha, \sigma_{m_2+1}-0.5\}} \right).$$

Therefore, (3.19) is proven, which ends the proof. $\square$

4. Fast method

In this section, we introduce a fast method for approximating the discrete convolution for discretizing the time fractional derivative to reduce the memory requirement and computational cost.

The convolution weights $\lambda_n^{(\alpha)}$ is expressed by

$$\lambda_n^{(\alpha)} = \tau^{1+\alpha} \frac{\sin \alpha \pi}{\pi} \int_0^\infty w^\alpha (1+w\tau)^{-1-n} \hat{F}_\lambda(-w) dw, \quad (4.1)$$

where

$$\hat{F}_\lambda(w) = (\tau w)^{-\alpha} \lambda^{(\alpha)}(1-\tau w), \quad (4.2)$$

in which $\lambda^{(\alpha)}$ are the generalized formula of order p defined by

$$\lambda^{(\alpha)}(z) = (1-z)^\alpha \sum_{l=1}^2 \hat{g}_{l-1} (1-z)^{l-1} = \sum_{l=0}^\infty \lambda_l^{(\alpha)} z^l, \quad (4.3)$$

the values of $\hat{g}_i$ ($i = 0, 1, \dots, 5$) can be seen in [8]. Let $w = \exp(z)$ [29], Eq. (4.1) becomes

$$\lambda_n^{(\alpha)} = \tau^{1+\alpha} \int_{-\infty}^\infty \psi_n(z) dz, \quad (4.4)$$

where

$$\psi_n(z) = (1 + e^z \tau)^{-1-n} \psi(z), \quad \psi(z) = \frac{\sin(\alpha \pi)}{\pi} e^{(1+\alpha)z} \hat{F}_\lambda(-e^z). \quad (4.5)$$

It can be observed that $\psi_n(z)$ decays exponentially as $|z| \rightarrow \infty$. Using the exponentially convergent trapezoidal rule [29] to approximate the integral $\int_{-\infty}^\infty \psi_n(z) dz$,

$$\lambda_n^{(\alpha)} = \tau^{1+\alpha} \sum_{j=-\infty}^\infty p_j (1 + w_j \tau)^{-1-n}, \quad (4.6)$$

where $w_j = e^{z_j}$, $z_j = j \Delta z$, $\Delta z > 0$, $Q > 1$ is a positive number, $p_j = \Delta z \psi(z_j)$. Determination of p_j and w_j can be found in [10]. We derive the following equation by truncating (4.6),

$$\lambda_n^{(\alpha)} = \tau^{1+\alpha} \sum_{j=0}^{Q-1} p_j (1 + w_j \tau)^{-1-n}. \quad (4.7)$$

The discrete convolution $\frac{1}{\tau^\alpha} \sum_{l=0}^n \lambda_{n-l}^{(\alpha)} u^l$ is decomposed into

$$\frac{1}{\tau^\alpha} \sum_{l=0}^n \lambda_{n-l}^{(\alpha)} u^l = \frac{1}{\tau^\alpha} \sum_{l=n-n_0}^n \lambda_{n-l}^{(\alpha)} u^l + \frac{1}{\tau^\alpha} \sum_{l=0}^{n-n_0-1} \lambda_{n-l}^{(\alpha)} u^l, \quad (4.8)$$

where $\frac{1}{\tau^\alpha} \sum_{l=n-n_0}^n \lambda_{n-l}^{(\alpha)} u^l$ can be computed directly. For $\frac{1}{\tau^\alpha} \sum_{l=0}^{n-n_0-1} \lambda_{n-l}^{(\alpha)} u^l$, we can have

$$\begin{aligned} \frac{1}{\tau^\alpha} \sum_{l=0}^{n-n_0-1} \lambda_{n-l}^{(\alpha)} u^l &= \tau \sum_{l=0}^{n-n_0-1} u^l \sum_{j=0}^{Q-1} p_j (1 + w_j \tau)^{-1-(n-l)} \\ &= (1 + w_j \tau)^{-n_0-1} \sum_{j=0}^{Q-1} p_j q_{n-n_0}^{(j)}, \end{aligned} \quad (4.9)$$

where $q_{n-n_0}^{(j)} = \tau \sum_{l=0}^{n-n_0-1} (1 + w_j \tau)^{-(n-n_0-l)} u^l$ satisfies

$$q_n^{(j)} = \frac{1}{1 + w_j \tau} \left(q_{n-1}^{(j)} + \tau u^{n-1} \right), \quad q_0^{(j)} = 0. \quad (4.10)$$

Therefore, the fully discrete form using fast method can be obtained,

$$\begin{aligned} &(\delta_t^{m_2} u_N^{n+1/2}, v) + \frac{1}{2\tau^\alpha} \left(\sum_{l=n-n_0}^n \lambda_{n-l}^{(\alpha)} + \sum_{l=n-n_0}^{n+1} \lambda_{n+1-l}^{(\alpha)} \right) (u^l, v) \\ &+ \frac{(1 + w_j \tau)^{-n_0-1}}{2} \left(\sum_{j=0}^{Q-1} p_j q_{n-n_0}^{(j)}, v \right) + \frac{(1 + w_j \tau)^{-n_0-1}}{2} \left(\sum_{j=0}^{Q-1} p_j q_{n+1-n_0}^{(j)}, v \right) \\ &+ V(\nabla u_N^{n+1/2}, v) + D(\nabla u_N^{n+1/2}, \nabla v) \\ &= \left(F(u_N^{n+1/2}), v \right) + (f^{n+1/2}, v) - \frac{1}{2\tau^\alpha} \sum_{j=1}^{m_1} (W_{n,j}^\alpha + W_{n+1,j}^\alpha) (u^j - u^0, v). \end{aligned} \quad (4.11)$$

The memory requirement and computational cost are $O(Q)$ and $O(QK)$.

5. Numerical experiments

In this section, we perform numerical experiments to verify the effectiveness and robustness of our methods.

The convergence orders in time and space in the L^2 -norm sense are defined as

$$\text{order} = \begin{cases} \frac{\log(\|error(\tau_1)\|/\|error(\tau_2)\|)}{\log(\tau_1/\tau_2)} & \text{in time,} \\ \frac{\log(\|error(N_1)\|/\|error(N_2)\|)}{\log(N_2/N_1)} & \text{in space,} \end{cases} \quad (5.1)$$

where $error = \|u(t_n) - u_N^n\|$ is the error equation, $\tau_1 \neq \tau_2$, and $N_1 \neq N_2$.

EXAMPLE 5.1. The following equation with non-smooth solution is presented,

$$u_t + {}^C_0 D_t^{0.1} u = -\nabla u + \Delta u + u^2 + f, \quad (x, y, t) \in \Omega \times I, \quad (5.2)$$

where $\Omega = (0, 2)^2$, $I = (0, 1]$, and

$$\begin{aligned} f = & \sum_{k=1}^6 \left(\frac{(1 + 0.1(k-1))t^{0.1(k-1)}}{k+1} + \frac{\Gamma(2 + 0.1(k-1))t^{1+0.1(k-2)}}{(k+1)\Gamma(2 + 0.1(k-2))} \right. \\ & \left. + \frac{2\pi^2 t^{1+0.1(k-1)}}{k+1} \right) \sin \pi x \sin \pi y - \left(\sum_{k=1}^6 \frac{t^{1+0.1(k-1)}}{k+1} \sin \pi x \sin \pi y \right) \\ & + \pi \sum_{k=1}^6 \frac{t^{1+0.1(k-1)}}{k+1} (\cos \pi x \sin \pi y + \sin \pi x \cos \pi y), \end{aligned} \quad (5.3)$$

with the following initial-boundary conditions

$$\begin{aligned} u(x, y, 0) &= 0, \quad (x, y) \in \Omega, \\ u(\cdot, t) &= 0, \quad (x, y) \in \partial\Omega, t \in I. \end{aligned} \quad (5.4)$$

The exact solution of this problem is $u = \sum_{k=1}^6 \frac{t^{1+0.1(k-1)}}{k+1} \sin \pi x \sin \pi y$.

TABLE 5.1. The errors and order of L^2 -norm versus τ for Example 5.1.

$1/\tau$	$m = 0$		$m = 1$		$m = 3$	
	Error	Order	Error	Order	Error	Order
10	4.4286e-04	—	5.5752e-04	—	4.9461e-04	—
20	1.4475e-04	1.6133	1.3051e-04	2.0949	8.9802e-05	2.4615
40	1.8065e-04	-0.3196	4.5476e-05	1.5210	1.6803e-05	2.4180
80	1.7468e-04	0.0485	2.4136e-05	0.9139	2.5634e-06	2.7126
160	1.6002e-04	0.1264	1.6798e-05	0.5229	3.1461e-07	3.0265

Figure 5.1 shows the numerical solution and the exact solution of Example 5.1 when $N = 50$ and $\tau = 0.01$, the numerical solution fits well with the exact solution. According to the regularity of solution, we choose

TABLE 5.2. The errors and order of L^∞ -norm versus τ for Example 5.1.

$1/\tau$	$m = 0$		$m = 1$		$m = 3$	
	Error	Order	Error	Order	Error	Order
10	5.3630e-04	—	7.6964e-04	—	6.3155e-04	—
20	2.1836e-04	1.2963	1.8287e-04	2.0733	1.1189e-04	2.4968
40	2.6904e-04	-0.3011	6.4215e-05	1.5099	2.0676e-05	2.4360
80	2.5587e-04	0.0724	3.5178e-05	0.8682	3.5882e-06	2.5266
160	2.3311e-04	0.1344	2.4467e-05	0.5238	5.6949e-07	2.6555

FIGURE 5.1. The numerical solution and the exact solution for Example 5.1.

FIGURE 5.2. (a): The computational time of the fast method and direct method for Example 5.1; (b): The difference between the numerical solutions of the fast method and direct method for Example 5.1.

FIGURE 5.3. The numerical solution and the exact solution for Example 5.2.

$N = 32$, $m_1 = m_2 = m$ and $\sigma_j = 1 + 0.1(j - 1)$, ($j = 1, 2, \dots$). Table 5.1 and Table 5.2 exhibit the L^2 errors of the method (2.10). We can see that the adding of correction terms increases the accuracy and second-order (even higher) accuracy when three correction terms are used. By selecting the correction term appropriately, we can give the numerical solution with higher accuracy.

Let $Q = 256$ and $\alpha = 0.3$, we apply the fast method to solve the problem. Figure 5.2(a) displays the computational time of the fast method (4.11) and the direct method (2.10). Obviously, the computational time of the fast method increases linearly, while the computational time of the fast method increases superlinearly (quadratic complexity). Figure 5.2(b) shows the difference of the fast method solution and the direct method solution. We observe that machine precision is observed. It is also shown that the error caused by the trapezoidal rule in the fast method is independent of the time stepsize τ and the order of the polynomial space N , which is also almost independent of the fractional order α .

EXAMPLE 5.2. Consider the following equation

$$u_t + {}^C_0 D_t^\alpha u = -\nabla u + \Delta u + u - u^3 + f, \quad (x, y, t) \in \Omega \times I, \quad (5.5)$$

with the following initial-boundary conditions

$$\begin{aligned} u(x, y, 0) &= 0, \quad (x, y) \in \Omega, \\ u(0, y, t) &= \sum_{k=0}^4 t^k e^y, \quad u(1, y, t) = \sum_{k=0}^4 t^k e^{1+y}, \quad y \in \partial\Omega, t \in I, \\ u(x, 0, t) &= \sum_{k=0}^4 t^k e^x, \quad u(x, 1, t) = \sum_{k=0}^4 t^k e^{x+1}, \quad x \in \partial\Omega, t \in I. \end{aligned} \quad (5.6)$$

In this example, we choose

$$f = \sum_{k=1}^4 \left(kt^{k-1} + \frac{\Gamma(k+1)t^{k-\alpha}}{\Gamma(k+1-\alpha)} - t^k \right) e^{x+y} + \left(\sum_{k=0}^4 t^k e^{x+y} \right)^3$$

such that the exact solution of this problem is $u = \sum_{k=0}^4 t^k e^{x+y}$, and it has $\Omega = (0, 1)^2$ and $I = (0, 1]$.

FIGURE 5.4. (a): The computational time of the fast method and direct method for Example 5.2; (b): The difference between the numerical solutions of the fast method and direct method for Example 5.2.

Taking $\alpha = 0.3$, $\tau = 0.01$ and $N = 32$. The numerical solution and the exact solution of Example 5.2 are shown in Fig. 5.3. From the convergence analysis of Theorem 3.2, if the solution $u(x, y, t)$ is sufficiently smooth in time, the convergence rate is $O(\tau^2)$ by considering $m_1 = 2$ and $m_2 = 0$. When $m_1 = 0$, the convergence rate is $O(\tau^{1.5-\alpha})$; when $m_1 = 1$, the convergence rate is $O(\tau^{2.5-\alpha})$ for $\alpha > 1/2$, $O(\tau^2)$ for $\alpha \leq 1/2$. Table 5.2 displays the L^2 -errors at $t = 1$ when $m_2 = 0$. It can be seen that second-order accuracy is observed for $m_1 = 2$ or $m_1 = 1$ ($\alpha \leq 1/2$). When $m_1 = 0$, we can not obtain second-order accuracy. These numerical results are agree with the above theoretical analysis.

Similar to Example 5.1, let $Q = 256$ and $\alpha = 0.3$. The numerical solution is obtained based on the fast method. Fig. 5.4(a) presents the computational time of the fast method and direct method. Fig. 5.4(b) gives the difference between the fast method and the direct method for different α . The validity of this fast method is further illustrated.

6. Conclusions

In this paper, we propose a numerical method for the two-dimensional nonlinear time fractional mobile/immobile advection-diffusion equation.

TABLE 5.3. The errors and order of L^2 -norm versus τ for Example 5.2.

m_1	$1/\tau$	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.9$	
		Error	Order	Error	Order	Error	Order
0	20	1.5857e-03	—	2.9847e-03	—	3.4156e-03	—
	40	5.0423e-04	1.6530	8.0024e-04	1.4228	1.4503e-03	1.2358
	80	1.4490e-04	1.7990	2.6358e-04	1.6022	5.8713e-04	1.3046
	160	4.1056e-05	1.8194	7.9546e-05	1.7284	2.1351e-04	1.4594
	320	1.1090e-05	1.8883	2.3088e-05	1.7846	7.3409e-05	1.5403
1	20	8.8367e-04	—	1.0844e-03	—	2.3154e-03	—
	40	2.6639e-04	1.7300	3.4332e-04	1.6593	7.7491e-04	1.5792
	80	7.6374e-05	1.8024	1.0032e-04	1.7749	2.4815e-04	1.6428
	160	2.0547e-05	1.8942	2.7740e-05	1.8546	7.6899e-05	1.6902
	320	5.3126e-06	1.9514	7.2367e-06	1.9386	2.2647e-05	1.7636
2	20	9.2044e-04	—	1.0625e-03	—	2.9242e-03	—
	40	2.3686e-04	1.9583	2.8836e-04	1.8815	7.8566e-04	1.8961
	80	5.8651e-05	2.0138	7.3227e-05	1.9774	2.0268e-04	1.9547
	160	1.3883e-05	2.0788	1.8218e-05	2.0070	5.1397e-05	1.9794
	320	3.2168e-06	2.1096	4.4093e-06	2.0228	1.3539e-05	1.9900

The WSGD difference scheme is developed for the time stepping, while we apply the Legendre spectral method for the space discretization. The correction scheme is introduced to deal with the non-smooth solutions, and the stability and convergence analysis are proven. In the numerical implementation, a fast method is used based on a globally uniform approximation of the trapezoidal rule for the integral on the real line to decrease the memory requirement and computational cost. Some numerical results are provided to confirm our theoretical analysis and the effectiveness of the presented methods.

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References

- [1] L. Banjai, M. López-Fernández, Efficient high order algorithms for fractional integrals and fractional differential equations. *Numer. Math.* **141**, No 2 (2019), 289–317; DOI: 10.1007/s00211-018-1004-0.
- [2] D.A. Benson, M.M. Meerschaert, A simple and efficient random walk solution of multi-rate mobile/immobile mass transport equations. *Adv. Water. Resources* **32** (2009), 532–539; DOI: 10.1016/j.advwatres.2009.01.002.
- [3] F. Chen, J. Shen, H.J. Yu, A new spectral element method for pricing European options under the Black-Scholes and Merton jump diffusion models. *J. Sci. Comput.* **52** (2012), 499–518; DOI: 10.1007/s10915-011-9556-5.
- [4] K. Diethelm, *The Analysis of Fractional Differential Equations*. Springer-Verlag, Berlin (2010).
- [5] K. Diethelm, J.M. Ford, N.J. Ford, M. Weilbeer, Pitfalls in fast numerical solvers for fractional differential equations. *J. Comput. Appl. Math.* **186**, No 2 (2006), 482–503; DOI: 10.1016/j.cam.2005.03.023.
- [6] W. Fan, F. Liu, X. Jiang, I. Turner, A novel unstructured mesh finite element method for solving the time-space fractional wave equation on a two-dimensional irregular convex domain. *Fract. Calc. Appl. Anal.* **20**, No 2 (2017), 352–383; DOI: 10.1515/fca-2017-0019; <https://www.degruyter.com/view/journals/fca/20/2/fca.20.issue-2.xml>.
- [7] L. Feng, F. Liu, I. Turner, L. Zheng, Novel numerical analysis of multi-term time fractional viscoelastic non-Newtonian fluid models for simulating unsteady and MHD and Couette flow of a generalized Oldroyd-B fluid. *Fract. Calc. Appl. Anal.* **21**, No 4 (2018), 867–868; DOI: 10.1515/fca-2018-0058; <https://www.degruyter.com/view/journals/fca/21/4/fca.21.issue-4.xml>.
- [8] L. Galeone, R. Garrappa, Fractional Adams-Moulton methods. *Math. Comput. Simulation* **79** (2008), 1358–1367; DOI: 10.1016/j.matcom.2008.03.008.
- [9] B.Y. Guo, T.J. Wang, Composite Laguerre-Legendre spectral method for fourth-order exterior problems. *J. Sci. Comput.* **44** (2010), 255–285; DOI: 10.1007/s10915-010-9367-0.
- [10] L. Guo, F.H. Zeng, I. Turner, K. Burrage, G.E. Karniadakis, Efficient multistep methods for tempered fractional calculus: Algorithms and simulations. *SIAM J. Sci. Comput.* **41**, No 4 (2019), A2510–A2535; DOI: 10.1137/18M1230153.

- [11] P. Guo, C. Zeng, C. Li, Y. Chen, Numerics for the fractional Langevin equation driven by the fractional Brownian motion. *Fract. Calc. Appl. Anal.* **16**, No 1 (2013), 123–141; DOI: 10.2478/s13540-013-0009-8; <https://www.degruyter.com/view/journals/fca/16/1/fca.16.issue-1.xml>.
- [12] M. Ilic, F. Liu, I. Turner, V. Anh, Numerical approximation of a fractional-in-space diffusion equation (I). *Fract. Calc. Appl. Anal.* **8**, No 3 (2005), 323–341; at <http://www.math.bas.bg/complan/fcaa>.
- [13] M. Ilic, F. Liu, I. Turner, V. Anh, Numerical approximation of a fractional-in-space diffusion equation (II): With nonhomogeneous boundary conditions. *Fract. Calc. Appl. Anal.* **9**, No 4 (2006), 333–349; at <http://www.math.bas.bg/complan/fcaa>.
- [14] H. Jafari, H. Tajadodi, D. Baleanu, A modified variational iteration method for solving fractional Riccati differential equation by Adomian polynomials. *Fract. Calc. Appl. Anal.* **16**, No 1 (2013), 109–122; DOI: 10.2478/s13540-013-0008-9; <https://www.degruyter.com/view/journals/fca/16/1/fca.16.issue-1.xml>.
- [15] R. Lazarov, P. Vabishchevich, A numerical study of the homogeneous elliptic equation with fractional boundary conditions. *Fract. Calc. Appl. Anal.* **20**, No 2 (2017), 337–351; DOI: 10.1515/fca-2017-0018; <https://www.degruyter.com/view/journals/fca/20/2/fca.20.issue-2.xml>.
- [16] C. Li, Q. Yi, A. Chen, Finite difference methods with non-uniform meshes for nonlinear fractional differential equations. *J. Comput. Phys.* **316** (2016), 614–631; DOI: 10.1016/j.jcp.2016.04.039.
- [17] C. Li, F. Zeng, F. Liu, Spectral approximations to the fractional integral and derivative. *Fract. Calc. Appl. Anal.* **15**, No 3 (2012), 383–406; DOI: 10.2478/s13540-012-0028-x; <https://www.degruyter.com/view/journals/fca/15/3/fca.15.issue-3.xml>.
- [18] C.P. Li, Q. Yi, Finite difference method for two-dimensional nonlinear time-fractional subdiffusion equation. *Fract. Calc. Appl. Anal.* **21**, No 4 (2018), 1046–1072; DOI: 10.1515/fca-2018-0057; <https://www.degruyter.com/view/journals/fca/21/4/fca.21.issue-4.xml>.
- [19] J.R. Li, A fast time stepping method for evaluating fractional integrals. *SIAM J. Sci. Comput.* **31** (2010), 4696–4714; DOI: 10.1137/080736533.
- [20] H.L. Liao, T. Tang, T. Zhou, A second-order and nonuniform time-stepping maximum-principle preserving scheme for time-fractional

- Allen-Cahn equations. *J. Comput. Phys.* **414** (2020), ID 109473; DOI: 10.1016/j.jcp.2020.109473.
- [21] F. Liu, M.M. Meerschaert, R. McGough, P. Zhuang, Q. Liu, Numerical methods for solving the multi-term time fractional wave equations. *Fract. Calc. Appl. Anal.* **16**, No 1 (2013), 9–25; DOI: 10.2478/s13540-013-0002-2; <https://www.degruyter.com/view/journals/fca/16/1/fca.16.issue-1.xml>.
- [22] Q. Liu, F. Liu, I. Turner, V. Anh, Y.T. Gu, A RBF meshless approach for modeling a fractal mobile/immobile transport model. *Appl. Math. Comput.* **226** (2014), 336–347; DOI: 10.1016/j.amc.2013.10.008.
- [23] Y. Liu, Y.W. Du, H. Li, J.F. Wang, A two-grid finite element approximation for a nonlinear time-fractional Cable equation. *Nonlinear Dyn.* **85** (2016), 2535–2548; DOI: 10.1007/s11071-016-2843-9.
- [24] M. López-Fernández, C. Lubich, A. Schädle, Adaptive, fast, and oblivious convolution in evolution equations with memory. *SIAM J. Sci. Comput.* **30** (2008), 1015–1037; DOI: 10.1137/060674168.
- [25] C. Lubich, Discretized fractional calculus. *SIAM J. Math. Anal.* **17** (1986), 704–719; DOI: 10.1137/0517050.
- [26] Y. Luchko, Initial-boundary-value problems for the generalized multi-term time fractional diffusion equation. *J. Math. Anal. Appl.* **374** (2011), 538–548; DOI: 10.1016/j.jmaa.2010.08.048.
- [27] H.P. Ma, Y.B. Yang, Jacobi spectral collocation method for the time variable-order fractional mobile-immobile advection-dispersion solute transport model. *East Asian J. Appl. Math.* **6** (2016), 337–352; DOI: 10.4208/eajam.141115.060616a.
- [28] W. McLean, K. Mustapha, A second-order accurate numerical method for a fractional wave equation. *Numer. Math.* **105** (2007), 481–510; DOI: 10.1007/s00211-006-0045-y.
- [29] W. McLean, Exponential Sum Approximations for $t^{-\beta}$. In: J. Dick, F. Kuo, H. Woźniakowski (Eds). *Contemporary Computational Mathematics—A Celebration of the 80th Birthday of Ian Sloan*. Springer, Cham (2018).
- [30] T.B. Nguyen, B. Jang, A high-order predictor-corrector method for solving nonlinear differential equations of fractional order. *Fract. Calc. Appl. Anal.* **20**, No 2 (2017), 447–476; DOI: 10.1515/fca-2017-0023; <https://www.degruyter.com/view/journals/fca/20/2/fca.20.issue-2.xml>.
- [31] I. Podlubny, *Fractional Differential Equations*. Academic Press, San Diego (1999).

- [32] A. Schädle, M. López-Fernández, C. Lubich, Fast and oblivious convolution quadrature; *SIAM J. Sci. Comput.* **28** (2006), 421–438; DOI: 10.1137/050623139.
- [33] R. Schumer, D.A. Benson, M.M. Meerschaert, B. Baeumer, Fractal mobile/immobile solute transport. *Water Resour. Res.* **39**, (2003), 1296–1307; DOI: 10.1029/2003WR002141.
- [34] J. Shen, T. Tang, L.L. Wang, *Spectral Methods: Algorithms, Analysis and Applications*. Springer Ser. Comput. Math. Springer, Heidelberg (2011).
- [35] M. Stynes, E. O’Riordan, J. Gracia, Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation. *SIAM J. Numer. Anal.* **55** (2017), 1057–1079; DOI: 10.1137/16m1082329.
- [36] L.L. Wang, Analysis of spectral approximations using prolate spheroidal wave functions. *Math. Comput.* **79** (2010), 807–827; DOI: 10.1090/S0025-5718-09-02268-6.
- [37] F. Zeng, F. Liu, C. Li, K. Burrage, I. Turner, V. Anh, Crank-Nicolson ADI spectral method for the two-dimensional Riesz space fractional nonlinear reaction-diffusion equation. *SIAM J. Numer. Anal.* **52** (2014), 2599–2622; DOI: 10.1137/130934192.
- [38] F. Zeng, Z. Zhang, G.E. Karniadakis, Second-order numerical methods for multi-term fractional differential equations: Smooth and non-smooth solutions. *Comput. Methods Appl. Mech. Engrg.* **327** (2017), 478–502; DOI: 10.1016/j.cma.2017.08.029.
- [39] F. Zeng, I. Turner, K. Burrage, A stable fast time-stepping method for fractional integral and derivative operators. *J. Sci. Comput.* **77** (2018), 283–307; DOI: 10.1007/s10915-018-0707-9.
- [40] H. Zhang, F. Liu, M.S. Phanikumar, M.M. Meerschaert, A novel numerical method for the time variable fractional order mobile-immobile advection-dispersion model. *Comput. Math. Appl.* **66** (2013), 693–701; DOI: 10.1016/j.camwa.2013.01.031.
- [41] J. Zhang, C.J. Xu, Finite difference/spectral approximations to a water wave model with a nonlocal viscous term. *Appl. Math. Model.* **38** (2014), 4912–4925; DOI: 10.1016/j.apm.2014.03.051.
- [42] Y. Zhang, D.A. Benson, D.M. Reeves, Time and space nonlocalities underlying fractional-derivative models: Distinction and literature review of field applications. *Adv. Water. Resources* **32** (2009), 561–581; DOI: 10.1016/j.advwatres.2009.01.008.

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