RESEARCH PAPER

CHARACTERIZATIONS OF VARIABLE MARTINGALE HARDY SPACES VIA MAXIMAL FUNCTIONS

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*Dedicated to 80th anniversary
of Professor Stefan Samko*

Abstract

We introduce a new type of dyadic maximal operators and prove that under the log-Hölder continuity condition of the variable exponent $p(\cdot)$, it is bounded on $L_{p(\cdot)}$ if $1 < p_- \leq p_+ \leq \infty$. Moreover, the space generated by the $L_{p(\cdot)}$ -norm (resp. the $L_{p(\cdot),q}$ -norm) of the maximal operator is equivalent to the Hardy space $H_{p(\cdot)}$ (resp. to the Hardy-Lorentz space $H_{p(\cdot),q}$). As special cases, our maximal operator contains the usual dyadic maximal operator and four other maximal operators investigated in the literature.

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Key Words and Phrases: variable exponent; variable Hardy spaces; variable Hardy-Lorentz spaces; atomic decomposition; dyadic maximal operators

1. Introduction

For a measurable function $p(\cdot)$, the variable Lebesgue space $L_{p(\cdot)}$ consists of all measurable functions f for which $\int_0^1 |f(x)|^{p(x)} dx < \infty$. If $p(\cdot)$ is a constant, we get back the usual L_p space. This topic needs essentially new ideas and is investigated very intensively in the literature nowadays (see e.g. Cruz-Uribe and Fiorenza [6], Diening et al. [7], Kokilashvili et al. [16, 17], Kováčik and Rákosník [18], Cruz-Uribe et al. [4, 3, 2], Nakai and Sawano [22, 31], Kempka and Vybíral [15], Rafeiro et al. [24, 27], Samko

[8, 28, 29], Jiao et al. [11, 12, 14], Yan et al. [40], Liu et al. [19, 20]). Interest in the variable Lebesgue spaces has increased since the 1990s because of their use in a variety of applications (see the references in Jiao et al. [11]).

Usually, we suppose that $p(\cdot)$ or $1/p(\cdot)$ satisfy the log-Hölder continuity condition. The classical Hardy-Littlewood maximal operator is bounded on the variable $L_{p(\cdot)}$ spaces if the exponent function $p(\cdot)$ is log-Hölder continuous and $1 < p_- \leq p_+ < \infty$, where p_- denotes the infimum and p_+ the supremum of $p(\cdot)$ (see for example Cruz-Uribe et.al [4] and Nekvinda [23]). Moreover, under the log-Hölder continuity condition of $1/p(\cdot)$, the maximal operator is bounded on $L_{p(\cdot)}$ when $1 < p_- \leq p_+ \leq \infty$ (see e.g. Cruz-Uribe and Fiorenza [6] and Diening et al. [7]). The fractional integral operator was investigated in Ephremidze et al. [8], Rafeiro and Samko [27, 25, 26, 30] and, for martingales, in Hao et al. [9, 13].

The boundedness result for the usual dyadic (or martingale) maximal operator on the $L_{p(\cdot)}$ spaces was proved in [11, 12] if $1 < p_- \leq p_+ < \infty$. In [36], we generalized this result to $1 < p_- \leq p_+ \leq \infty$. In [11, 35], we investigated four more dyadic maximal operators and show that they are bounded on $L_{p(\cdot)}$ if $1 < p_- \leq p_+ < \infty$. The boundedness of these operators was the key point in proving the boundedness of the maximal Fejér, Cesàro and Riesz operators of the Walsh-Fourier series from the variable Hardy space $H_{p(\cdot)}$ to $L_{p(\cdot)}$ (see [11, 33]).

Nakai and Sawano [22] first introduced the Hardy space $H_{p(\cdot)}(\mathbb{R})$ with a variable exponent $p(\cdot)$ and established the atomic decompositions. Independently, Cruz-Uribe and Wang [5] also investigated the variable Hardy space $H_{p(\cdot)}(\mathbb{R})$. Sawano [31] improved the results in [22]. Ho [10] studied weighted Hardy spaces with variable exponents. Recently, Yan et al. [40] introduced the variable weak Hardy space $H_{p(\cdot),\infty}(\mathbb{R})$ and characterized these spaces via radial maximal functions. The Hardy-Lorentz spaces $H_{p(\cdot),q}(\mathbb{R})$ were investigated by Jiao et al. in [14]. Similar results for the anisotropic Hardy spaces $H_{p(\cdot)}(\mathbb{R})$ and $H_{p(\cdot),q}(\mathbb{R})$ can be found in Liu et al. [19, 20]. Martingale Musielak-Orlicz Hardy spaces were investigated in Xie et al. [37, 38, 39]. Very recently, these results are generalized for martingale Hardy spaces with variable exponent in Jiao et al. [11].

In this paper, we introduce a common generalization of the usual dyadic and the other four maximal operators, denoted by $U_{\gamma,s}$, where γ and s are positive parameters. We prove that if $1/p(\cdot)$ satisfy the log-Hölder continuity condition, $\frac{1}{p_-} - \frac{1}{p_+} < \gamma + s$ and $1 < p_- \leq p_+ \leq \infty$, then $U_{\gamma,s}$ is bounded on $L_{p(\cdot)}$. We also verify that under the same conditions, $\|U_{\gamma,s}f\|_{L_{p(\cdot)}}$ is equivalent to $\|f\|_{H_{p(\cdot)}}$ and $\|U_{\gamma,s}f\|_{L_{p(\cdot),q}}$ is equivalent

to $\|f\|_{H_{p(\cdot),q}}$, where $H_{p(\cdot)}$ and $H_{p(\cdot),q}$ denote the variable Hardy and Hardy-Lorentz spaces. Moreover, we show with a counterexample that the condition $\frac{1}{p_-} - \frac{1}{p_+} < \gamma + s$ is important, the results do not hold without this condition.

2. Variable Lebesgue and Lorentz spaces

In this section, we recall some basic notations on variable Lebesgue spaces and give some elementary and necessary facts about these spaces. Our main references are Cruz-Uribe and Fiorenza [6] and Diening et al. [7].

For a constant p , the L_p space is equipped with the quasi-norm

$$\|f\|_p := \left(\int_0^1 |f(x)|^p dx \right)^{1/p} \quad (0 < p < \infty),$$

with the usual modification for $p = \infty$. Here we integrate with respect to the Lebesgue measure λ .

We are going to generalize these spaces. A measurable function $p(\cdot) : [0, 1) \rightarrow (0, \infty]$ is called a variable exponent. For a measurable set $A \subset [0, 1)$, we denote

$$p_-(A) := \operatorname{ess\,inf}_{x \in A} p(x), \quad p_+(A) := \operatorname{ess\,sup}_{x \in A} p(x)$$

and for convenience

$$p_- := p_-([0, 1)), \quad p_+ := p_+([0, 1)).$$

Denote by $\mathcal{P}$ the collection of all variable exponents $p(\cdot)$ such that

$$0 < p_- \leq p_+ \leq \infty.$$

In what follows, we use the symbol

$$\underline{p} = \min\{p_-, 1\}.$$

We define the modular functional by

$$\rho(f) = \int_{[0,1) \setminus \Omega_\infty} |f(x)|^{p(x)} dP + \|f\|_{L_\infty(\Omega_\infty)},$$

where $\Omega_\infty = \{x \in [0, 1) : p(x) = \infty\}$. The variable Lebesgue space $L_{p(\cdot)}$ is the collection of all measurable functions f for which there exists $\nu > 0$ such that

$$\rho(f/\nu) < \infty.$$

This becomes a quasi-Banach function space when it is equipped with the quasi-norm

$$\|f\|_{p(\cdot)} := \inf\{\nu > 0 : \rho(f/\nu) \leq 1\}.$$

If $p(\cdot) = p$ is a constant, then we get back the definition of the usual L_p spaces. For any $f \in L_{p(\cdot)}$, we have $\rho(f) \leq 1$ if and only if $\|f\|_{p(\cdot)} \leq 1$ (see Cruz-Uribe and Fiorenza [6]). It is known that $\|\nu f\|_{p(\cdot)} = |\nu| \|f\|_{p(\cdot)}$,

$$\| |f|^s \|_{p(\cdot)} = \|f\|_{sp(\cdot)}^s$$

and

$$\|f + g\|_{p(\cdot)}^p \leq \|f\|_{p(\cdot)}^p + \|g\|_{p(\cdot)}^p,$$

where $p(\cdot) \in \mathcal{P}$, $s \in (0, \infty)$, $\rho \in \mathbb{C}$ and $f, g \in L_{p(\cdot)}$. Details can be found in the monographs Cruz-Uribe and Fiorenza [6] and Diening et al. [7]. The variable exponent $p'(\cdot)$ is defined pointwise by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in [0, 1].$$

The next two lemmas are well known, see Cruz-Uribe and Fiorenza [6] or Diening et al. [7].

LEMMA 2.1. *Let $p(\cdot) \in \mathcal{P}$ with $1 \leq p_- \leq p_+ \leq \infty$. For all $f \in L_{p(\cdot)}$ and $g \in L_{p'(\cdot)}$,*

$$\int_0^1 |fg| \, d\lambda \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}.$$

LEMMA 2.2. *Let $p(\cdot) \in \mathcal{P}$ with $1 \leq p_- \leq p_+ \leq \infty$. Then*

$$\|f\|_{p(\cdot)} \sim \sup \left| \int_0^1 fg \, dP \right|,$$

where the supremum is taken over all $g \in L_{p'(\cdot)}$ with $\|g\|_{p'(\cdot)} \leq 1$.

We denote by $C^{\log}$ the set of all functions $p(\cdot) \in \mathcal{P}$ satisfying the so-called *globally log-Hölder continuous condition*, namely, there exists a positive constant $C_{\log}(p)$ such that, for any $x, y \in [0, 1]$,

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{\log(e + 1/|x - y|)}. \quad (2.1)$$

The following two lemmas were proved in Cruz-Uribe and Fiorenza [6] (see also Hao and Jiao [9]).

LEMMA 2.3. *If $1/p(\cdot) \in C^{\log}$, then there exists a constant $0 < \beta < 1$ such that for all intervals $I \subset [0, 1]$,*

$$\lambda(I)^{1/p_+(I) - 1/p_-(I)} \leq \frac{1}{\beta}. \quad (2.2)$$

LEMMA 2.4. If $1/p(\cdot) \in C^{\log}$, then for any interval $I \subset [0, 1)$,

$$\lambda(I)^{1/p_-(I)} \sim \lambda(I)^{1/p(x)} \sim \lambda(I)^{1/p_+(I)} \sim \|\chi_I\|_{p(\cdot)} \quad (\forall x \in I),$$

where $\sim$ denotes the equivalence of the numbers.

Note that under the condition $0 < p_- \leq p_+ < \infty$, $p(\cdot) \in C^{\log}$ if and only if $1/p(\cdot) \in C^{\log}$.

For general martingale Hardy spaces, instead of the log-Hölder continuity condition, we supposed in [9, 11, 13, 33, 36] the slightly more general condition (2.2) for all atoms of the σ -algebras. All results of this paper remain true if, instead of (2.1), we suppose (2.2) for all dyadic intervals $I \subset [0, 1)$. By a dyadic interval, we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k, n \in \mathbb{N}$, $0 \leq k < 2^n$.

REMARK 2.1. There exist a lot of functions $p(\cdot)$ satisfying (2.1) and (2.2). For concrete examples we mention the function $a + cx$ for parameters a and c such that the function is positive ($x \in [0, 1)$). All positive Lipschitz functions with order $0 < \alpha \leq 1$ also satisfy (2.1) and (2.2).

The following lemma can be found in [36]. A first version of this lemma was proved in Jiao et al. [12, 11].

LEMMA 2.5. Let $p(\cdot) \in \mathcal{P}$ satisfy (2.2) and $1 \leq p_- \leq p_+ \leq \infty$. Suppose that $f \in L_{p(\cdot)}$ with $\|f\|_{p(\cdot)} \leq 1$, $f = f\chi_{|f|>1}$ and $\text{supp } f \subset \Omega_\infty^c$. Then, for any interval $I \subset [0, 1)$ with $\lambda(I \cap \Omega_\infty^c) > 0$ and for any $p_-(I) \leq r \leq p_+(I)$ ($r < \infty$),

$$\left(\frac{\beta^t}{P(I)} \int_I |f(y)| dy \right)^r \leq \frac{1}{P(I)} \int_I |f(y)|^{p(y)} dy.$$

The *variable Lorentz spaces* were introduced and investigated by Kempka and Vybíral [15]. $L_{p(\cdot), q}$ is defined to be the space of all measurable functions f such that

$$\|f\|_{p(\cdot), q} := \begin{cases} \left(\int_0^\infty \rho^q \|\chi_{\{x \in [0, 1): |f(x)| > \rho\}}\|_{p(\cdot)}^q \frac{d\rho}{\rho} \right)^{1/q}, & \text{if } 0 < q < \infty, \\ \sup_{\rho \in (0, \infty)} \rho \|\chi_{\{x \in [0, 1): |f(x)| > \rho\}}\|_{p(\cdot)}, & \text{if } q = \infty \end{cases}$$

is finite. If $p(\cdot)$ is a constant, we get back the classical Lorentz spaces (see Lorentz [21] or Bergh and Löfström [1]).

In this paper the constants C are absolute constants and the constants $C_{p(\cdot)}$ are depending only on $p(\cdot)$ and may denote different constants in different contexts. For two positive numbers A and B , we use also the

notation $A \lesssim B$, which means that there exists a constant C such that $A \leq CB$.

3. Variable martingale Hardy spaces

Let $\mathcal{F}_n$ be the σ -algebra

$$\mathcal{F}_n = \sigma\{[k2^{-n}, (k+1)2^{-n}) : 0 \leq k < 2^n\},$$

where $\sigma(\mathcal{H})$ denotes the σ -algebra generated by an arbitrary set system $\mathcal{H}$. The conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E_n . An integrable sequence $f = (f_n)_{n \in \mathbb{N}}$ is said to be a martingale if f_n is $\mathcal{F}_n$ -measurable for all $n \in \mathbb{N}$ and $E_n f_m = f_n$ in case $n \leq m$. Martingales with respect to $(\mathcal{F}_n, n \in \mathbb{N})$ are called dyadic martingales. It is easy to show (see e.g. Weisz [34]) that the sequence $(\mathcal{F}_n, n \in \mathbb{N})$ is regular, i.e., $f_n \leq Rf_{n-1}$ for all non-negative dyadic martingales.

For a dyadic martingale $f = (f_n)_{n \in \mathbb{N}}$, the maximal function is defined by

$$Mf := \sup_{n \in \mathbb{N}} |f_n|.$$

Now we can define the *variable martingale Hardy spaces* by

$$H_{p(\cdot)} := \left\{ f = (f_n)_{n \in \mathbb{N}} : \|f\|_{H_{p(\cdot)}} := \|M(f)\|_{p(\cdot)} < \infty \right\}.$$

The *variable martingale Hardy-Lorentz spaces* can be defined similarly:

$$H_{p(\cdot),q} := \left\{ f = (f_n)_{n \in \mathbb{N}} : \|f\|_{H_{p(\cdot),q}} := \|M(f)\|_{p(\cdot),q} < \infty \right\}.$$

The Hardy spaces can also be defined via equivalent norms. In [11], we have shown that the Hardy spaces defined the square function and the conditional square function are equivalent. In this paper, we give other equivalent characterization of the variable Hardy spaces via new maximal functions.

The atomic decomposition is a useful characterization of the Hardy spaces. A measurable function a is called a $p(\cdot)$ -atom if there exists a stopping time τ such that

- (a) $E_n(a) = 0$ for all $n \leq \tau$,
- (b) $\|M(a)\|_\infty \leq \|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}^{-1}$.

The atomic decomposition of the spaces $H_{p(\cdot)}$ and $H_{p(\cdot),q}$ were proved in Jiao et al. [11, 12]. The classical case can be found in [34].

THEOREM 3.1. *Let $p(\cdot) \in C^{\log}$, $0 < p_- \leq p_+ < \infty$ and $0 < q \leq \infty$. Then the martingale $f = (f_n)_{n \in \mathbb{N}} \in H_{p(\cdot)}$ or $f = (f_n)_{n \in \mathbb{N}} \in H_{p(\cdot),q}$,*

respectively, if and only if there exists a sequence $(a^k)_{k \in \mathbb{Z}}$ of $p(\cdot)$ -atoms such that for every $n \in \mathbb{N}$,

$$f_n = \sum_{k \in \mathbb{Z}} \mu_k E_n a^k \quad \text{almost everywhere,} \quad (3.1)$$

where $\mu_k = 3 \cdot 2^k \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}$ and τ_k is the stopping time associated with the $p(\cdot)$ -atom a^k . Moreover,

$$\begin{aligned} \|f\|_{H_{p(\cdot)}} &\sim \inf \left\| \left(\sum_{k \in \mathbb{Z}} \left(\frac{\mu_k \chi_{\{\tau_k < \infty\}}}{\|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}} \right)^t \right)^{1/t} \right\|_{p(\cdot)}, \\ \|f\|_{H_{p(\cdot),q}} &\sim \inf \left(\sum_{k \in \mathbb{Z}} 2^{kq} \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}^q \right)^{1/q}, \end{aligned}$$

respectively, where $0 < t \leq \underline{p}$ is fixed and the infimum is taken over all decompositions of the form (3.1).

4. The boundedness of maximal operators on $L_{p(\cdot)}$

The following result was proved in Jiao et al. [11, 12] if $p_+ < \infty$ and by the author [36] if $p_+ \leq \infty$. For the boundedness of the classical Hardy-Littlewood maximal operator see e.g. Cruz-Uribe and Fiorenza [6] and Diening et al. [7].

THEOREM 4.1. *If $1/p(\cdot) \in C^{\log}$ and $1 < p_- \leq p_+ \leq \infty$, then*

$$\|Mf\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \quad (f \in L_{p(\cdot)}).$$

We generalize the preceding dyadic maximal operator. For a martingale $f = (f_n)$, let us introduce

$$U_{\gamma,s} f(x) := \sup_{x \in I} \sum_{m=0}^n \sum_{j=0}^m 2^{(j-n)\gamma} \sum_{i=j}^m 2^{(j-i)s} \frac{1}{\lambda(I^{j,i})} \left| \int_{I^{j,i}} f_n d\lambda \right|,$$

where I is a dyadic interval with length 2^{-n} , γ, s are positive constants and

$$I^{j,i} := I \dot{+} [0, 2^{-i}) \dot{+} 2^{-j-1}.$$

Here $\dot{+}$ denotes the dyadic addition (see e.g. Schipp, Wade, Simon and Pál [32]). Of course, if $f \in L_1$, then we can write in the definition f instead of f_n . Let us define $I_{k,n} := [k2^{-n}, (k+1)2^{-n})$ with $0 \leq k < 2^n$, $n \in \mathbb{N}$. The preceding definition can be rewritten to

$$U_{\gamma,s}f := \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}} \sum_{m=0}^n \sum_{j=0}^m 2^{(j-n)\gamma} \sum_{i=j}^m 2^{(j-i)s} \frac{1}{\lambda(I_{k,n}^{j,i})} \left| \int_{I_{k,n}^{j,i}} f_n d\lambda \right|.$$

We proved the following theorem in [33].

THEOREM 4.2. *For all $1 < p \leq \infty$ and all $0 < \gamma, s < \infty$, we have*

$$\|U_{\gamma,s}f\|_p \leq C_p \|f\|_p \quad (f \in L_p).$$

Now we generalize this theorem to variable Lebesgue spaces.

THEOREM 4.3. *Let $1/p(\cdot) \in C^{\log}$, $1 < p_- \leq p_+ \leq \infty$ and $0 < \gamma, s < \infty$.*

If

$$\frac{1}{p_-} - \frac{1}{p_+} < \gamma + s, \quad (4.1)$$

then

$$\|U_{\gamma,s}f\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \quad (f \in L_{p(\cdot)}).$$

P r o o f. By homogeneity, it is enough to show the theorem for $\|f\|_{p(\cdot)} = 1$. We may suppose that f is non-negative. We decompose f as $f^1 + f^2$, where

$$f^1 = f \chi_{\{f > 1\}}, \quad f^2 = f \chi_{\{f \leq 1\}}.$$

Then $\|f^i\|_{p(\cdot)} \leq 1$ and $\rho(f^i) \leq 1$, $i = 1, 2$. Since

$$U_{\gamma,s}f^2 \leq \frac{2^s}{2^s - 1} \frac{2^{2\gamma}}{(2^\gamma - 1)^2} := C_s C_\gamma,$$

we get by convexity that

$$\begin{aligned} \rho(\alpha U_{\gamma,s}f) &\leq \frac{1}{2} \rho(2\alpha U_{\gamma,s}f^1) + \frac{1}{2} \rho(2\alpha U_{\gamma,s}f^2) \\ &\leq \frac{1}{2} \int_{\Omega_\infty^c} (2\alpha U_{\gamma,s}f^1(x))^{p(x)} dx \\ &\quad + \alpha \|U_{\gamma,s}f^1\|_{L_\infty(\Omega_\infty)} + 2\alpha C_s C_\gamma, \end{aligned} \quad (4.2)$$

where $8\alpha C_s C_\gamma < \beta$. For a fixed k, n , let us denote by $\Lambda_{k,n}$ those pairs (j, i) for which $0 \leq j \leq n, 0 \leq i \leq n$ and

$$\frac{\beta}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \leq 1.$$

Then,

$$\begin{aligned}
 & \frac{1}{2} \int_{\Omega_{\infty}^c} (2\alpha U_{\gamma,s} f^1(x))^{p(x)} dx \\
 & \leq \frac{1}{4} \int_{\Omega_{\infty}^c} \left(4\alpha \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \right. \\
 & \quad \times \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)\gamma} 2^{(j-i)s} \frac{\chi_{\Lambda_{k,n}}(j,i)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \Big)^{p(x)} dx \\
 & \quad + \frac{1}{4} \int_{\Omega_{\infty}^c} \left(4\alpha \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \right. \\
 & \quad \times \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)\gamma} 2^{(j-i)s} \frac{\chi_{\Lambda_{k,n}^c}(j,i)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \Big)^{p(x)} dx \\
 & =: (A_1) + (A_2).
 \end{aligned}$$

It is easy to see that

$$(A_1) \leq \frac{1}{4} \int_{\Omega_{\infty}^c} \left(\frac{4\alpha C_s C_{\gamma}}{\beta} \right)^{p(x)} dx \leq \frac{1}{4} \frac{4\alpha C_s C_{\gamma}}{\beta} \leq 1. \quad (4.3)$$

We denote by $I_{k,n,j,i,1}$ (resp. $I_{k,n,j,i,2}$) those points $x \in I_{k,n}$ for which $p(x) \leq p_-(I_{k,n}^{j,i})$ (resp. $p(x) > p_-(I_{k,n}^{j,i})$). Then

$$\begin{aligned}
 (A_2) & \leq \frac{1}{8} \sum_{l=1}^2 \int_{\Omega_{\infty}^c} \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)\gamma} 2^{(j-i)s} \right. \\
 & \quad \times \frac{8\alpha \chi_{\Lambda_{k,n}^c}(j,i) \chi_{I_{k,n,j,i,l}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \Big)^{p(x)} dx \\
 & =: (A_{21}) + (A_{22}).
 \end{aligned}$$

Let $q(x) := p(x)/p_0 > 1$ for some $1 < p_0 < p_-$. Note that the sets $I_{k,n}$ are disjoint for a fixed n . By convexity,

$$\begin{aligned}
 (A_{21}) &\leq \frac{1}{8} \int_{\Omega_\infty^c} \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \left(\sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m \frac{2^{(j-n)\gamma} 2^{(j-i)s}}{C_s C_\gamma} \right. \right. \\
 &\quad \times \left. \left. \frac{8\alpha C_s C_\gamma \chi_{\Lambda_{k,n}^c}(j,i) \chi_{I_{k,n,j,i,1}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \right)^{q(x)} \right)^{p_0} dx \\
 &\leq \frac{1}{8} \int_{\Omega_\infty^c} \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m \frac{2^{(j-n)\gamma} 2^{(j-i)s}}{C_s C_\gamma} \right. \\
 &\quad \times \left. \left(\frac{\beta \chi_{\Lambda_{k,n}^c}(j,i) \chi_{I_{k,n,j,i,1}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \right)^{q(x)} \right)^{p_0} dx. \quad (4.4)
 \end{aligned}$$

Using that $q(x) \leq q_-(I_{k,n}^j)$ on $I_{k,n,j,K,1}$ and

$$\frac{\beta}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt > 1,$$

we get that

$$\begin{aligned}
 (A_{21}) &\leq \frac{1}{8(C_s C_\gamma)^{p_0}} \int_{\Omega_\infty^c} \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)\gamma} 2^{(j-i)s} \right. \\
 &\quad \times \left. \left(\frac{\beta \chi_{\Lambda_{k,n}^c}(j,i) \chi_{I_{k,n,j,i,1}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \right)^{q_-(I_{k,n}^{j,i})} \right)^{p_0} dx. \quad (4.5)
 \end{aligned}$$

Since $\rho(f^1) \leq 1$ implies that $\text{supp } f^1 \subset \Omega_\infty^c$, we can apply Lemma 2.5 and Theorem 4.2 to obtain

$$\begin{aligned}
 (A_{21}) &\leq \frac{1}{8(C_s C_\gamma)^{p_0}} \int_{\Omega_\infty^c} \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)\gamma} 2^{(j-i)s} \right. \\
 &\quad \times \left. \frac{1}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f^1(t)|^{q(t)} dt \right)^{p_0} dx \\
 &\leq \frac{1}{8(C_s C_\gamma)^{p_0}} \left\| U_{\gamma,s}((f^1)^{q(\cdot)}) \right\|_{p_0}^{p_0} \leq C_1 \left\| |f|^{q(\cdot)} \right\|_{p_0}^{p_0} \leq C_1. \quad (4.6)
 \end{aligned}$$

Choosing $0 < \gamma_0 < \gamma$ and $0 < r < s + \gamma_0$, we obtain

$$\begin{aligned}
 & (A_{22}) \\
 & \leq \frac{1}{8} \int_{\Omega_{\infty}^c} \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \left(\sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m \frac{2^{(j-n)(\gamma-\gamma_0)} 2^{(j-i)(\gamma_0+s-r)}}{C_{\gamma-\gamma_0} C_{\gamma_0+s-r}} \right. \right. \\
 & \quad \times \left. \frac{8\alpha C_{\gamma-\gamma_0} C_{\gamma_0+s-r} 2^{(j-i)r} \chi_{\Lambda_{k,n}^c}(j,i) \chi_{I_{k,n,j,i,2}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \right)^{q(x)} \Big)^{p_0} dx \\
 & \leq \frac{1}{8(C_{\gamma-\gamma_0} C_{\gamma_0+s-r})^{p_0}} \int_{\Omega_{\infty}^c} \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \right. \\
 & \quad \times \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)(\gamma-\gamma_0)} 2^{(j-i)(\gamma_0+s-r)} \\
 & \quad \times \left. \left(\beta^{p-} 2^{(j-i)r} \frac{\chi_{\Lambda_{k,n}^c}(j,i) \chi_{I_{k,n,j,i,2}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \right)^{q(x)} \right)^{p_0} dx, \quad (4.7)
 \end{aligned}$$

whenever $8\alpha C_{\gamma-\gamma_0} C_{\gamma_0+s-r} \leq \beta^{p-}$. By Hölder's inequality,

$$\begin{aligned}
 & (A_{22}) \\
 & \leq \frac{1}{8(C_{\gamma-\gamma_0} C_{\gamma_0+s-r})^{p_0}} \int_{\Omega_{\infty}^c} \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \right. \\
 & \quad \times \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)(\gamma-\gamma_0)} 2^{(j-i)(\gamma_0+s-r)} 2^{(j-i)r q(x)} \beta^{p-q(x)} \\
 & \quad \times \left. \left(\frac{\chi_{\Lambda_{k,n}^c}(j,i) \chi_{I_{k,n,j,i,2}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f^1(t)|^{q-(I_{k,n}^{j,i})} dt \right)^{q(x)/q-(I_{k,n}^{j,i})} \right)^{p_0} dx \\
 & \leq \frac{1}{8(C_{\gamma-\gamma_0} C_{\gamma_0+s-r})^{p_0}} \int_{\Omega_{\infty}^c} \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \right. \\
 & \quad \times \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)(\gamma-\gamma_0)} 2^{(j-i)(\gamma_0+s-r)} 2^{(j-i)r q(x)} \beta^{p-q(x)} 2^{i q(x)/q-(I_{k,n}^{j,i})} \\
 & \quad \times \chi_{I_{k,n,j,i,2}}(x) \left. \left(\int_{I_{k,n}^{j,i}} |f^1(t)|^{q-(I_{k,n}^{j,i})} dt \right)^{q(x)/q-(I_{k,n}^{j,i})} \right)^{p_0} dx. \quad (4.8)
 \end{aligned}$$

Observe that $\rho(f^1) \leq 1$ implies that $\text{supp } f^1 \subset \Omega_\infty^c$. Since $f^1 > 1$ or $f^1 = 0$, $q(x) > q_-(I_{k,n}^{j,i})$ on $I_{k,n,j,K,2}$, $q_-(I_{k,n}^{j,i}) \leq q(t) < p(t)$ for all $t \in I_{k,n}^{j,i}$ and

$$\int_{I_{k,n}^{j,i}} |f^1(t)|^{q_-(I_{k,n}^{j,i})} dt \leq \int_{\Omega_\infty^c} |f^1(t)|^{p(t)} dt \leq 1, \quad (4.9)$$

we can see that

$$\begin{aligned} (A_{22}) &\leq \frac{1}{8(C_{\gamma-\gamma_0}C_{\gamma_0+s-r})^{p_0}} \int_{\Omega_\infty^c} \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \right. \\ &\quad \times \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)(\gamma-\gamma_0)} 2^{(j-i)(\gamma_0+s-r)} 2^{(j-i)r q(x)} 2^{i q(x)/q_-(I_{k,n}^{j,i})-i} \\ &\quad \times \beta^{p-q(x)} \frac{\chi_{I_{k,n,j,i,2}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)|^{q_-(I_{k,n}^{j,i})} dt \Big)^{p_0} dx. \end{aligned}$$

For fixed k, n let J_j denote the dyadic interval with length 2^{-j} and $I_{k,n} \subset J_j$. Then $I_{k,n}^{j,i} \subset J_j + 2^{-j-1} = J_j$. Inequality (2.2) implies that

$$2^{-j/q(x)+j/q_-(I_{k,n}^{j,i})} \leq \left(\frac{1}{\beta} \right)^{p_0} \leq \left(\frac{1}{\beta} \right)^{p_-}, \quad (4.10)$$

thus

$$\beta^{p-q(x)} \leq 2^{j-jq(x)/q_-(I_{k,n}^{j,i})}$$

for $x \in I_{k,n}$. Hence,

$$\begin{aligned} &2^{(j-i)r q(x)} 2^{i q(x)/q_-(I_{k,n}^{j,i})-i} \beta^{p-q(x)} \\ &\leq 2^{(j-i)r q(x)} 2^{i q(x)/q_-(I_{k,n}^{j,i})-i} 2^{j-jq(x)/q_-(I_{k,n}^{j,i})} \\ &= 2^{(j-i) \left(r q(x) - \frac{q(x)}{q_-(I_{k,n}^{j,i})} + 1 \right)}. \end{aligned} \quad (4.11)$$

Observe that

$$r q(x) - \frac{q(x)}{q_-(I_{k,n}^j)} + 1 \geq q(x) \left(r - \frac{1}{q_-} \right) + 1 > 0,$$

whenever

$$\frac{1}{q_-} - \frac{1}{q_+} < r. \quad (4.12)$$

In this case the expression in (4.11) can be estimated by 1. Hence

$$\begin{aligned}
 (A_{22}) &\leq \frac{1}{8(C_{\gamma-\gamma_0}C_{\gamma_0+s-r})^{p_0}} \int_{\Omega_\infty^c} \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m \right. \\
 &\quad \times \left. 2^{(j-n)(\gamma-\gamma_0)} 2^{(j-i)(\gamma_0+s-r)} \frac{1}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)|^{q(t)} dt \right)^{p_0} dx \\
 &\leq \frac{1}{8(C_{\gamma-\gamma_0}C_{\gamma_0+s-r})^{p_0}} \left\| U_{\gamma-\gamma_0, \gamma_0+s-r}(|f|^{q(\cdot)}) \right\|_{p_0}^{p_0} \\
 &\leq C_2 \left\| |f|^{q(\cdot)} \right\|_{p_0}^{p_0} \leq C_2,
 \end{aligned} \tag{4.13}$$

whenever (4.12) holds. Since r can be arbitrarily near to $s + \gamma_0$ and γ_0 to γ , (4.12) gives (4.1).

Now, we consider the second term of (4.2). Notice that if Ω_∞ has positive measure, then $p_+ = \infty$. We have

$$\begin{aligned}
 \alpha U_{\gamma,s} f^1(x) &\leq \alpha \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \\
 &\quad \times \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)\gamma} 2^{(j-i)s} \frac{\chi_{\Lambda_{k,n}}(j,i)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \\
 &\quad + \alpha \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \\
 &\quad \times \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)\gamma} 2^{(j-i)s} \frac{\chi_{\Lambda_{k,n}^c}(j,i)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \\
 &=: (B_1)(x) + (B_2)(x),
 \end{aligned}$$

where $x \in \Omega_\infty$. Thus

$$(B_1) \leq \frac{\alpha C_s C_\gamma}{\beta} \leq 1. \tag{4.14}$$

For $1 < u < \infty$, we denote by $\Gamma_{k,n,u}$ those pairs (j,i) for which $u \leq p_-(I_{k,n}^{j,i})$. Then

$$\begin{aligned}
(B_2)(x) &\leq \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)\gamma} 2^{(j-i)s} \\
&\quad \times \frac{\alpha \chi_{\Lambda_{k,n}^c}(j, i) \chi_{\Gamma_{k,n,u}}(j, i)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \\
&\quad + \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)\gamma} 2^{(j-i)s} \\
&\quad \times \frac{\alpha \chi_{\Lambda_{k,n}^c}(j, i) \chi_{\Gamma_{k,n,u}^c}(j, i)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \\
&=: (B_{21})(x) + (B_{22})(x).
\end{aligned}$$

For a fixed $x \in \Omega_\infty$ there exist $n \in \mathbb{N}$ and $0 \leq k < 2^n$ such that

$$\begin{aligned}
(B_{21})(x) &\leq 2 \chi_{I_{k,n}}(x) \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)\gamma} 2^{(j-i)s} \\
&\quad \times \frac{\alpha \chi_{\Lambda_{k,n}^c}(j, i) \chi_{\Gamma_{k,n,u}}(j, i)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt =: (B'_{21})(x).
\end{aligned}$$

Similarly to (4.4) and (4.5), we obtain

$$\begin{aligned}
(B'_{21})^u(x) &\leq \left(\sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m \frac{2^{(j-n)\gamma} 2^{(j-i)s}}{C_s C_\gamma} \right. \\
&\quad \times \left. \frac{2 \alpha C_s C_\gamma \chi_{\Lambda_{k,n}^c}(j, i) \chi_{\Gamma_{k,n,u}}(j, i)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \right)^u,
\end{aligned}$$

and further,

$$\begin{aligned}
(B'_{21})^u(x) &\leq \frac{1}{C_s C_\gamma} \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)\gamma} 2^{(j-i)s} \\
&\quad \times \left(\frac{\beta \chi_{\Lambda_{k,n}^c}(j, i) \chi_{\Gamma_{k,n,u}}(j, i)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \right)^u \\
&\leq \frac{1}{C_s C_\gamma} \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)\gamma} 2^{(j-i)s} \\
&\quad \times \left(\frac{\beta \chi_{\Lambda_{k,n}^c}(j, i) \chi_{\Gamma_{k,n,u}}(j, i)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \right)^{p-(I_{k,n}^{j,i})}.
\end{aligned}$$

Note that $\text{supp } f^1 \subset \Omega_\infty^c$ and we use the convention $0^\infty = 0$. So the following holds also if $p_-(I_{k,n}^{j,i}) = \infty$. By Lemma 2.5 and (4.9), we get that

$$\begin{aligned} (B'_{21})^u(x) &\leq \frac{1}{C_s C_\gamma} \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)\gamma} 2^{(j-i)s} \frac{1}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f^1(t)|^{p(t)} dt \\ &\leq \frac{1}{C_s C_\gamma} \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)\gamma} 2^{(j-i)s} 2^i \leq 2^n. \end{aligned}$$

In other words, $(B'_{21})(x) \leq 2^{n/u}$. Since this holds for all $1 < u < \infty$, we conclude that

$$(B_{21})(x) \leq (B'_{21})(x) \leq 1 \quad (x \in \Omega_\infty). \quad (4.15)$$

Similarly, for a fixed $x \in \Omega_\infty$ there exist $n \in \mathbb{N}$ and $0 \leq k < 2^n$ such that

$$\begin{aligned} (B_{22})(x) &\leq 2\chi_{I_{k,n}}(x) \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)\gamma} 2^{(j-i)s} \\ &\quad \times \frac{\alpha \chi_{\Lambda_{k,n}^c}(j,i) \chi_{\Gamma_{k,n,u}^c}(j,i)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt =: (B'_{22})(x). \end{aligned}$$

We can see as in (4.7) and (4.8) that

$$\begin{aligned} &(B'_{22})^u(x) \\ &\leq \left(\sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m \frac{2^{(j-n)(\gamma-\gamma_0)} 2^{(j-i)(\gamma_0+s-r)}}{C_{\gamma-\gamma_0} C_{\gamma_0+s-r}} \right. \\ &\quad \times \left. \frac{2\alpha C_{\gamma-\gamma_0} C_{\gamma_0+s-r} 2^{(j-i)r} \chi_{\Lambda_{k,n}^c}(j,i) \chi_{\Gamma_{k,n,u}^c}(j,i)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \right)^u \\ &\leq \frac{1}{C_{\gamma-\gamma_0} C_{\gamma_0+s-r}} \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)(\gamma-\gamma_0)} 2^{(j-i)(\gamma_0+s-r)} \\ &\quad \times \left(\frac{\beta 2^{(j-i)r} \chi_{\Lambda_{k,n}^c}(j,i) \chi_{\Gamma_{k,n,u}^c}(j,i)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} f^1(t) dt \right)^u \\ &\leq \frac{1}{C_{\gamma-\gamma_0} C_{\gamma_0+s-r}} \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)(\gamma-\gamma_0)} 2^{(j-i)(\gamma_0+s-r)} 2^{(j-i)ru} \\ &\quad \times \beta^u \left(\frac{\chi_{\Lambda_{k,n}^c}(j,i) \chi_{\Gamma_{k,n,u}^c}(j,i)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f^1(t)|^{p_-(I_{k,n}^{j,i})} dt \right)^{u/p_-(I_{k,n}^{j,i})}. \end{aligned}$$

Inequality (4.9) implies

$$\begin{aligned}
& (B'_{22})^u(x) \\
& \leq \frac{1}{C_{\gamma-\gamma_0}C_{\gamma_0+s-r}} \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)(\gamma-\gamma_0)} 2^{(j-i)(\gamma_0+s-r)} 2^{(j-i)ru} \\
& \quad \times \beta^u 2^{iu/p_-(I_{k,n}^{j,i})} \chi_{\Gamma_{k,n,u}^c}(j,i) \left(\int_{I_{k,n}^{j,i}} |f^1(t)|^{p_-(I_{k,n}^{j,i})} dt \right)^{u/p_-(I_{k,n}^{j,i})} \\
& \leq \frac{1}{C_{\gamma-\gamma_0}C_{\gamma_0+s-r}} \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)(\gamma-\gamma_0)} 2^{(j-i)(\gamma_0+s-r)} \\
& \quad \times 2^{(j-i)ru} \beta^u 2^{iu/p_-(I_{k,n}^{j,i})} \chi_{\Gamma_{k,n,u}^c}(j,i) \int_{I_{k,n}^{j,i}} |f^1(t)|^{p_-(I_{k,n}^{j,i})} dt.
\end{aligned}$$

Since $x \in \Omega_\infty \cap I_{k,n}$, we have $p_+(I_{k,n}) = \infty$. If $p_-(I_{k,n}^{j,i}) = \infty$, then $\int_{I_{k,n}^{j,i}} |f^1(t)|^{p_-(I_{k,n}^{j,i})} dt = 0$. Otherwise, we obtain

$$2^{-j/u+j/p_-(I_{k,n}^{j,i})} \leq \frac{1}{\beta} \quad \text{and} \quad \beta^u \leq 2^{j-ju/p_-(I_{k,n}^{j,i})}$$

for all $p_-(I_{k,n}^{j,i}) < u < \infty$ as in (4.10). So

$$\begin{aligned}
2^{(j-i)ru} 2^{iu/p_-(I_{k,n}^{j,i})-i} \beta^u & \leq 2^{(j-i)ru} 2^{iu/q_-(I_{k,n}^{j,i})-i} 2^{j-ju/p_-(I_{k,n}^{j,i})} \\
& = 2^{(j-i) \left(ru - \frac{u}{p_-(I_{k,n}^{j,i})} + 1 \right)} \leq 1.
\end{aligned}$$

Indeed,

$$ru - \frac{u}{p_-(I_{k,n}^{j,i})} + 1 > 0$$

whenever

$$\frac{1}{p_-} < r. \quad (4.16)$$

Recall that $p_+ = \infty$. This and inequality (4.10) imply

$$\begin{aligned}
(B'_{22})^u(x) & \leq \frac{1}{C_{\gamma-\gamma_0}C_{\gamma_0+s-r}} \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)(\gamma-\gamma_0)} 2^{(j-i)(\gamma_0+s-r)} 2^i \\
& \quad \times \int_{I_{k,n}^{j,i}} |f^1(t)|^{p_-(I_{k,n}^{j,i})} dt \\
& \leq \frac{1}{C_{\gamma-\gamma_0}C_{\gamma_0+s-r}} \sum_{m=0}^n \sum_{j=0}^m \sum_{i=j}^m 2^{(j-n)(\gamma-\gamma_0)} 2^{(j-i)(\gamma_0+s-r)} 2^i \leq 2^n
\end{aligned}$$

and so

$$(B_{22})(x) \leq (B'_{22})(x) \leq 1 \quad (x \in \Omega_\infty). \quad (4.17)$$

Since r can be arbitrarily near to $s + \gamma_0$ and γ_0 to γ , conditions (4.16) and (4.1) are the same. Taking into account (4.2), (4.3), (4.6), (4.13), (4.14), (4.15) and (4.17), we obtain $\rho(\alpha U_{\gamma,s}f) \leq C$, where $C = 5 + C_1 + C_2$. By convexity,

$$\rho\left(\frac{U_{\gamma,s}f}{C/\alpha}\right) \leq \frac{1}{C}\rho(\alpha U_{\gamma,s}f) \leq 1,$$

which means that

$$\|U_{\gamma,s}f\|_{p(\cdot)} \leq \frac{C}{\alpha} = \frac{C}{\alpha} \|f\|_{p(\cdot)},$$

whenever (4.1) is satisfied. This finishes the proof of Theorem 4.3. $\square$

REMARK 4.1. Inequality (4.1) and Theorem 4.3 hold if $p_- > \max(1/(\gamma+s), 1)$.

COROLLARY 4.1. Let $1/p(\cdot) \in C^{\log}$ satisfy (4.1), $1 < p_- \leq p_+ \leq \infty$ and $0 < \gamma, s < \infty$. Then $\|f\|_{p(\cdot)} \sim \|U_{\gamma,s}f\|_{p(\cdot)}$ and

$$\|Mf\|_{p(\cdot)} \leq \|U_{\gamma,s}f\|_{p(\cdot)} \leq C_{p(\cdot)} \|Mf\|_{p(\cdot)} \quad (f \in L_{p(\cdot)}).$$

P r o o f. If $j = i = n$, then $I^{j,i} = I$, hence $Mf \leq U_{\gamma,s}f$. Theorem 4.3 completes the proof:

$$\|f\|_{p(\cdot)} \leq \|Mf\|_{p(\cdot)} \leq \|U_{\gamma,s}f\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \quad (f \in L_{p(\cdot)}).$$

$\square$

In special cases, we proved in [11, 35] that the condition (4.1) is important, the theorems are not true without this condition.

5. Some special maximal operators

In this section we consider five special cases of the maximal operator $U_{\gamma,s}$. Theorem 4.1 holds for the operators in Examples 5.1 and 5.2 and Theorem 4.3 holds for Examples 5.3, 5.4 and 5.5. Under the additional condition $p_+ < \infty$, Theorem 4.3 was proved in [11, 35] for the next special operators.

EXAMPLE 5.1. Considering the indices $j = i = n = m$ in the definition of $U_{\gamma,s}f$, only, we obtain

$$U_{\gamma,s}^{(1)}f(x) := \sup_{x \in I} \frac{1}{\lambda(I)} \left| \int_I f_n d\lambda \right| = Mf(x).$$

Note that $I^{n,n} := I$.

EXAMPLE 5.2. If $m = 0, \dots, n-1$, $j = i = m$, then

$$U_{\gamma,s}^{(2)}f(x) := \sup_{x \in I} \sum_{m=0}^{n-1} 2^{(m-n)\gamma} \frac{1}{\lambda(I^{m,m})} \left| \int_{I^{m,m}} f_n d\lambda \right|.$$

Here $I^{m,m} := I \dot{+} [0, 2^{-m}]$. It is easy to see that $U_{\gamma,s}^{(2)}f \leq Mf$.

EXAMPLE 5.3. If $m = n$, $j = 0, \dots, n-1$ and $i = n$, then

$$U_{\gamma,s}^{(3)}f(x) := \sup_{x \in I} \sum_{j=0}^{n-1} 2^{(j-n)(\gamma+s)} \frac{1}{\lambda(I^{j,n})} \left| \int_{I^{j,n}} f_n d\lambda \right|.$$

Note that $I^{j,n} := I \dot{+} 2^{-j-1}$.

EXAMPLE 5.4. If $m = n-1$, then

$$U_{\gamma,s}^{(4)}f(x) := \sup_{x \in I} \sum_{j=0}^{n-1} 2^{(j-n)\gamma} \sum_{i=j}^{n-1} 2^{(j-i)s} \frac{1}{\lambda(I^{j,i})} \left| \int_{I^{j,i}} f_n d\lambda \right|.$$

EXAMPLE 5.5. If $m = 0, \dots, n-1$, then

$$U_{\gamma,s}^{(5)}f(x) := \sup_{x \in I} \sum_{m=0}^{n-1} \sum_{j=0}^m 2^{(j-n)\gamma} \sum_{i=j}^m 2^{(j-i)s} \frac{1}{\lambda(I^{j,i})} \left| \int_{I^{j,i}} f_n d\lambda \right|.$$

6. The equivalence of maximal operators on $H_{p(\cdot)}$

In this section, we apply the atomic characterization to prove the boundedness of $U_{\gamma,s}$ from $H_{p(\cdot)}$ to $L_{p(\cdot)}$. We need the following result due to Jiao et al. [11].

THEOREM 6.1. Let $p(\cdot) \in C^{\log}$, $0 < p_- \leq p_+ < \infty$ and $0 < t < \underline{p}$. Suppose that the σ -sublinear operator $T : L_\infty \rightarrow L_\infty$ is bounded and

$$\left\| \sum_{k \in \mathbb{Z}} \mu_k^t T(a^k)^t \chi_{\{\tau_k = \infty\}} \right\|_{p(\cdot)/t} \lesssim \left\| \sum_{k \in \mathbb{Z}} 2^{kt} \chi_{\{\tau_k < \infty\}} \right\|_{p(\cdot)/t}, \quad (6.1)$$

where $\mu_k = 3 \cdot 2^k \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}$ and τ_k is the stopping time associated with the $p(\cdot)$ -atom a^k . Then we have

$$\|Tf\|_{p(\cdot)} \lesssim \|f\|_{H_{p(\cdot)}} \quad (f \in H_{p(\cdot)}).$$

THEOREM 6.2. *Let $1/p(\cdot) \in C^{\log}$, $0 < \gamma, s < \infty$ and $1 < p_- \leq p_+ \leq \infty$ or $0 < p_- \leq p_+ < \infty$. If (4.1) holds, then*

$$\|U_{\gamma,s}f\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{H_{p(\cdot)}} \quad (f \in H_{p(\cdot)}).$$

P r o o f. In case $1 < p_- \leq p_+ \leq \infty$, the result follows from Theorem 4.3. So we can suppose that $0 < p_- \leq p_+ < \infty$. Since $U_{\gamma,s}$ is bounded on L_∞ by Theorem 4.2, it is enough to prove (6.1) for $U_{\gamma,s}$. For a fixed $k \in \mathbb{Z}$, the sets $\{\tau_k = K\}$ are disjoint and there exist disjoint dyadic intervals $I_{k,K,\mu} \in \mathcal{F}_K$ such that

$$\{\tau_k = K\} = \bigcup_{\mu} I_{k,K,\mu} \quad (K \in \mathbb{N}),$$

where the union in μ is finite and $\lambda(I_{k,K,\mu}) = 2^{-K}$. Thus

$$\{\tau_k < \infty\} = \bigcup_{K \in \mathbb{N}} \bigcup_{\mu} I_{k,K,\mu},$$

where the dyadic intervals $I_{k,K,\mu}$ are disjoint for a fixed $k \in \mathbb{Z}$. Then

$$a^k = \sum_{K \in \mathbb{N}} \sum_{\mu} a^k \chi_{I_{k,K,\mu}}.$$

Since $\int_{I_{k,K,\mu}} a^k d\lambda = 0$,

$$\int_{I^{j,i}} a^k d\lambda = 0$$

if $i \leq K$. That is, we can suppose that $i > K$, thus $n \geq m > K$. If $x \notin I_{k,K,\mu}$, $x \in I$ and $j \geq K$, then $I^{j,i} \cap I_{k,K,\mu} = \emptyset$. Therefore we can suppose that $j < K$. Similarly, if $x \in I_{k,K,\mu} \cap [2^{-j-1}, 2^{-j}) \setminus (I_{k,K,\mu} \cap 2^{-j-1})$, then $I^{j,i} \cap I_{k,K,\mu} = \emptyset$, so we may assume that $x \in I_{k,K,\mu} \cap 2^{-j-1} = I_{k,K,\mu}^{j,K}$. Therefore, for $x \notin I_{k,K,\mu}$,

$$\begin{aligned} & U_{\gamma,s}(a^k \chi_{I_{k,K,\mu}})(x) \\ & \leq \sup_{n > K} \chi_I(x) \sum_{m=K+1}^n \sum_{j=0}^{K-1} 2^{(j-n)\gamma} \sum_{i=K+1}^m 2^{(j-i)s} \frac{1}{\lambda(I^{j,i})} \left| \int_{I^{j,i}} a d\lambda \right| \chi_{I_{k,K,\mu}^{j,K}}(x) \\ & \leq \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}^{-1} \sup_{n > K} \chi_I(x) \sum_{m=K+1}^n \sum_{j=0}^{K-1} 2^{(j-n)\gamma} \sum_{i=K+1}^m 2^{(j-i)s} \chi_{I_{k,K,\mu}^{j,K}}(x) \\ & \leq \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}^{-1} \sup_{n > K} \chi_I(x) \sum_{m=K+1}^n \sum_{j=0}^{K-1} 2^{(j-n)\gamma} 2^{(j-K)s} \chi_{I_{k,K,\mu}^{j,K}}(x) \\ & \leq \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}^{-1} \sup_{n > K} (n-K) 2^{(K-n)\gamma} \sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)} \chi_{I_{k,K,\mu}^{j,K}}(x). \end{aligned}$$

Since the function $x \mapsto x 2^{-\gamma x}$ is bounded, we obtain that

$$U_{\gamma,s}(a^k \chi_{I_{k,K,\mu}})(x) \leq \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}^{-1} \sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)} \chi_{I_{k,K,\mu}^{j,K}}(x).$$

Consequently, for $x \in \{\tau_k = \infty\}$,

$$U_{\gamma,s}(a^k)(x) \lesssim \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}^{-1} \sum_{K \in \mathbb{N}} \sum_{\mu} \sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)} \chi_{I_{k,K,\mu}^{j,K}}(x) \quad (6.2)$$

and

$$\begin{aligned} & \left\| \sum_{k \in \mathbb{Z}} \mu_k^t U_{\gamma,s}(a^k)^t \chi_{\{\tau_k = \infty\}} \right\|_{p(\cdot)/t} \\ & \lesssim \left\| \sum_{k \in \mathbb{Z}} 2^{kt} \sum_{K \in \mathbb{N}} \sum_{\mu} \sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)t} \chi_{I_{k,K,\mu}^{j,K}} \right\|_{p(\cdot)/t} =: Z, \end{aligned}$$

where $0 < t < \underline{p}$.

Let us choose $\max(1, p_+) < r < \infty$. By Lemma 2.2, there is $g \in L_{(\frac{p(\cdot)}{t})'}$ with norm less than 1 such that

$$\begin{aligned} Z & \lesssim \int_0^1 \sum_{k \in \mathbb{Z}} 2^{kt} \sum_{K \in \mathbb{N}} \sum_{\mu} \sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)t} \chi_{I_{k,K,\mu}^{j,K}} g \, d\lambda \\ & \leq \sum_{k \in \mathbb{Z}} 2^{kt} \sum_{K \in \mathbb{N}} \sum_{\mu} \sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)t} \|\chi_{I_{k,K,\mu}^{j,K}}\|_{\frac{r}{t}} \|\chi_{I_{k,K,\mu}^{j,K}} g\|_{(\frac{r}{t})'} \\ & \lesssim \sum_{k \in \mathbb{Z}} 2^{kt} \sum_{K \in \mathbb{N}} \sum_{\mu} \sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)t} \\ & \quad \times \int_0^1 \chi_{I_{k,K,\mu}} \left(\frac{1}{\lambda(I_{k,K,\mu}^{j,K})} \int_{I_{k,K,\mu}^{j,K}} |g|^{(\frac{r}{t})'} \, d\lambda \right)^{1/(\frac{r}{t})'} \, d\lambda, \end{aligned}$$

because $\lambda(I_{k,K,m}) = \lambda(I_{k,K,m}^{j,K}) = 2^{-K}$. Then

$$\begin{aligned} Z & \lesssim \int_0^1 \sum_{k \in \mathbb{Z}} 2^{kt} \sum_{K \in \mathbb{N}} \sum_{\mu} \chi_{I_{k,K,\mu}} \sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)t(1/(\frac{r}{t})+1/(\frac{r}{t})')} \\ & \quad \times \left(\frac{1}{\lambda(I_{k,K,\mu}^{j,K})} \int_{I_{k,K,\mu}^{j,K}} |g|^{(\frac{r}{t})'} \, d\lambda \right)^{1/(\frac{r}{t})'} \, d\lambda, \end{aligned}$$

and by Hölder's inequality,

$$\begin{aligned}
 Z &\lesssim \int_0^1 \sum_{k \in \mathbb{Z}} 2^{kt} \sum_{K \in \mathbb{N}} \sum_{\mu} \chi_{I_{k,K,\mu}} \left(\sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)t} \right)^{1/(\frac{r}{t})} \\
 &\quad \times \left(\sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)t} \frac{1}{\lambda(I_{k,K,\mu}^{j,K})} \int_{I_{k,K,\mu}^{j,K}} |g|^{(\frac{r}{t})'} \right)^{1/(\frac{r}{t})'} d\lambda \\
 &\lesssim \int_0^1 \sum_{k \in \mathbb{Z}} 2^{kt} \sum_{K \in \mathbb{N}} \sum_{\mu} \chi_{I_{k,K,\mu}} \\
 &\quad \times \left(\sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)t} \frac{1}{\lambda(I_{k,K,\mu}^{j,K})} \int_{I_{k,K,\mu}^{j,K}} |g|^{(\frac{r}{t})'} \right)^{1/(\frac{r}{t})'} d\lambda.
 \end{aligned}$$

Example 5.3 and Theorem 4.3 imply

$$\begin{aligned}
 Z &\leq \int_0^1 \sum_{k \in \mathbb{Z}} 2^{kt} \sum_{K \in \mathbb{N}} \sum_{\mu} \chi_{I_{k,K,\mu}} \left(U_{\gamma t, st}^{(3)}(|g|^{(\frac{r}{t})'}) \right)^{1/(\frac{r}{t})'} d\lambda \\
 &\leq \left\| \sum_{k \in \mathbb{Z}} 2^{kt} \sum_{K \in \mathbb{N}} \sum_{\mu} \chi_{I_{k,K,\mu}} \right\|_{p(\cdot)/t} \left\| \left(U_{\gamma t, st}^{(3)}(|g|^{(\frac{r}{t})'}) \right)^{1/(\frac{r}{t})'} \right\|_{(p(\cdot)/t)'} \\
 &\lesssim \left\| \sum_{k \in \mathbb{Z}} 2^{kt} \sum_{K \in \mathbb{N}} \sum_{\mu} \chi_{I_{k,K,\mu}} \right\|_{p(\cdot)/t} \|g\|_{L_{(p(\cdot)/t)'}} \\
 &\lesssim \left\| \sum_{k \in \mathbb{Z}} 2^{kt} \chi_{\{\tau_k < \infty\}} \right\|_{p(\cdot)/t},
 \end{aligned}$$

whenever

$$\begin{aligned}
 \frac{1}{((p(\cdot)/t)'/(r/t)')_-} - \frac{1}{((p(\cdot)/t)'/(r/t)')_+} &= \frac{r/(r-t)}{p_+/(p_+ - t)} - \frac{r/(r-t)}{p_-/(p_- - t)} \\
 &< (\gamma + s)t.
 \end{aligned}$$

Since r can be arbitrarily large, this means that

$$\frac{p_+ - t}{p_+} - \frac{p_- - t}{p_-} < (\gamma + s)t,$$

which is exactly (4.1). This completes the proof. $\square$

REMARK 6.1. Theorem 6.2 holds if $p_- > 1/(\gamma + s)$.

Now we generalize Corollary 4.1 and verify that $\|\cdot\|_{H_{p(\cdot)}} \sim \|U_{\gamma,s}(\cdot)\|_{p(\cdot)}$.

COROLLARY 6.1. *Let $1/p(\cdot) \in C^{\log}$, $0 < \gamma, s < \infty$ and $1 < p_- \leq p_+ \leq \infty$ or $0 < p_- \leq p_+ < \infty$. If (4.1) holds, then*

$$\|f\|_{H_{p(\cdot)}} \leq \|U_{\gamma,s}f\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{H_{p(\cdot)}} \quad (f \in H_{p(\cdot)}).$$

Theorem 6.2 does not hold if (4.1) is not satisfied. More exactly, we show

THEOREM 6.3. *Let $1/p(\cdot) \in C^{\log}$. If*

$$\frac{1}{p_-(I_{0,n-1})} - \frac{1}{p_+(I_{0,n}^{0,n})} > \gamma + s \quad (6.3)$$

for all $n \in \mathbb{N}$, then $U_{\gamma,s}$ is not bounded from $H_{p(\cdot)}$ to $L_{p(\cdot)}$.

P r o o f. Let

$$a_{n-1}(t) = 2^{(n-1)/p_-(I_{0,n-1})} (\chi_{I_{0,n}} - \chi_{I_{1,n}})$$

and $x \notin I_{0,n-1}$. By Lemma 2.4, a_{n-1} is an atom for all $n \geq 1$, and so $\|a_{n-1}\|_{H_{p(\cdot)}} \leq 1$. Choosing $m = n = N$ and $i = n$, we can see that

$$\begin{aligned} & U_{\gamma,s}a_{n-1} \\ &= \sup_{N \in \mathbb{N}} \sum_{k=0}^{2^N-1} \chi_{I_{k,N}} \sum_{m=0}^N \sum_{j=0}^m 2^{(j-N)\gamma} \sum_{i=j}^m 2^{(j-i)s} \frac{1}{\lambda(I_{k,N}^{j,i})} \left| \int_{I_{k,N}^{j,i} \cap I_{0,n-1}} a_{n-1} d\lambda \right| \\ &\geq \chi_J \sum_{j=0}^n 2^{(j-n)(\gamma+s)} \frac{1}{\lambda(J^{j,n})} \left| \int_{J^{j,n} \cap I_{0,n-1}} a_{n-1} d\lambda \right|, \end{aligned}$$

where $J = [1/2, 1/2 + 2^{-n}) = I_{0,n}^{0,n}$. The terms except $j = 0$ are all 0, so

$$\begin{aligned} U_{\gamma,s}a_{n-1} &\geq \chi_{I_{0,n}^{0,n}} 2^{-n(\gamma+s)} \frac{1}{\lambda(I_{0,n})} \left| \int_{I_{0,n} \cap I_{0,n-1}} a_{n-1} d\lambda \right| \\ &\geq \chi_{I_{0,n}^{0,n}}(x) 2^{-n(\gamma+s)} 2^{(n-1)/p_-(I_{0,n-1})}. \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 U_{\gamma,s}a_{n-1}(x)^{p(x)} dx &\geq \int_{I_{0,n}^{0,n}} 2^{-n(\gamma+s)p(x)} 2^{(n-1)p(x)/p_-(I_{0,n-1})} dx \\ &\geq C \int_{I_{0,n}^{0,n}} 2^{np_+(I_{0,n}^{0,n})(1/p_-(I_{0,n-1})-(\gamma+s))} dx \\ &= C 2^{np_+(I_{0,n}^{0,n})(1/p_-(I_{0,n-1})-(\gamma+s))} 2^{-n} \end{aligned}$$

which tends to infinity as $n \rightarrow \infty$ if (6.3) holds. $\square$

7. The equivalence of maximal operators on $H_{p(\cdot),q}$

In this section, we extend the main results in Section 6 to the variable Hardy-Lorentz space setting. The next result was proved in Jiao et al. [11].

THEOREM 7.1. *Let $p(\cdot) \in C^{\log}$, $0 < p_- \leq p_+ < \infty$ and $0 < q \leq \infty$. Suppose that the σ -sublinear operator $T : L_\infty \rightarrow L_\infty$ is bounded and*

$$\left\| |Ta|^\delta \chi_{\{\tau=\infty\}} \right\|_{p(\cdot)} \leq C \left\| \chi_{\{\tau<\infty\}} \right\|_{p(\cdot)}^{1-\delta}$$

for some $0 < \delta < 1$ and all $p(\cdot)$ -atoms a , where τ is the stopping time associated with a . Then we have

$$\|Tf\|_{L_{p(\cdot),q}} \lesssim \|f\|_{H_{p(\cdot),q}} \quad (f \in H_{p(\cdot),q}).$$

THEOREM 7.2. *Let $p(\cdot) \in C^{\log}$, $0 < p_- \leq p_+ < \infty$, $0 < q \leq \infty$ and $0 < \gamma, s < \infty$. If (4.1) holds, then*

$$\|U_{\gamma,s}f\|_{p(\cdot),q} \leq C_{p(\cdot),q} \|f\|_{H_{p(\cdot),q}} \quad (f \in H_{p(\cdot),q}).$$

P r o o f. Let us chose $0 < \delta < 1$ and $0 < \epsilon < \underline{p}$. Instead of (4.2), we will show that

$$\left\| |U_{\gamma,s}a|^{\delta\epsilon} \chi_{\{\tau=\infty\}} \right\|_{p(\cdot)/\epsilon} \leq C \left\| \chi_{\{\tau<\infty\}} \right\|_{p(\cdot)/\epsilon} \left\| \chi_{\{\tau<\infty\}} \right\|_{p(\cdot)}^{-\delta\epsilon}.$$

By (6.2),

$$U_{\gamma,s}(a)(x) \lesssim \left\| \chi_{\{\tau<\infty\}} \right\|_{p(\cdot)}^{-1} \sum_{K \in \mathbb{N}} \sum_{\mu} \sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)} \chi_{I_{K,\mu}^{j,K}}(x)$$

for $x \in \{\tau_k = \infty\}$ and

$$\begin{aligned} \left\| |U_{\gamma,s}a|^{\delta\epsilon} \chi_{\{\tau=\infty\}} \right\|_{p(\cdot)/\epsilon} &\leq \left\| \chi_{\{\tau<\infty\}} \right\|_{p(\cdot)}^{-\delta\epsilon} \left\| \sum_{K \in \mathbb{N}} \sum_{\mu} \sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)\delta\epsilon} \chi_{I_{K,\mu}^{j,K}} \right\|_{p(\cdot)/\epsilon} \\ &=: Z \left\| \chi_{\{\tau<\infty\}} \right\|_{p(\cdot)}^{-\delta\epsilon}, \end{aligned}$$

where

$$\{\tau < \infty\} = \bigcup_{K \in \mathbb{N}} \bigcup_{\mu} I_{K,\mu}.$$

Choose $\max(1, \delta p_+) < r < \infty$. By Lemma 2.2, there exists a function $g \in L_{(\frac{p(\cdot)}{\epsilon})'}$ with $\|g\|_{(\frac{p(\cdot)}{\epsilon})'} \leq 1$ such that

$$\begin{aligned}
Z &= \int_0^1 \sum_{K \in \mathbb{N}} \sum_{\mu} \sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)\delta\epsilon} \chi_{I_{K,\mu}^{j,K}} g \, d\lambda \\
&\leq \sum_{K \in \mathbb{N}} \sum_{\mu} \sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)\delta\epsilon} \|\chi_{I_{K,\mu}^{j,K}}\|_{\frac{r}{\delta\epsilon}} \|\chi_{I_{K,\mu}^{j,K}} g\|_{(\frac{r}{\delta\epsilon})'} \\
&\lesssim \sum_{K \in \mathbb{N}} \sum_{\mu} \sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)\delta\epsilon} \\
&\quad \times \int_0^1 \chi_{I_{K,\mu}} \left(\frac{1}{\lambda(I_{K,\mu}^{j,K})} \int_{I_{K,\mu}^{j,K}} |g|^{(\frac{r}{\delta\epsilon})'} \, d\lambda \right)^{1/(\frac{r}{\delta\epsilon})'} d\lambda.
\end{aligned}$$

Then

$$\begin{aligned}
Z &\lesssim \int_0^1 \sum_{K \in \mathbb{N}} \sum_{\mu} \chi_{I_{K,\mu}} \sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)\delta\epsilon(1/(\frac{r}{\delta\epsilon})+1/(\frac{r}{\delta\epsilon})')} \\
&\quad \times \left(\frac{1}{\lambda(I_{K,\mu}^{j,K})} \int_{I_{K,\mu}^{j,K}} |g|^{(\frac{r}{\delta\epsilon})'} \, d\lambda \right)^{1/(\frac{r}{\delta\epsilon})'} d\lambda,
\end{aligned}$$

and by Hölder's inequality,

$$\begin{aligned}
Z &\lesssim \int_0^1 \sum_{K \in \mathbb{N}} \sum_{\mu} \chi_{I_{K,\mu}} \\
&\quad \times \left(\sum_{j=0}^{K-1} 2^{(j-K)(\gamma+s)\delta\epsilon} \frac{1}{\lambda(I_{K,\mu}^{j,K})} \int_{I_{K,\mu}^{j,K}} |g|^{(\frac{r}{\delta\epsilon})'} \, d\lambda \right)^{1/(\frac{r}{\delta\epsilon})'} d\lambda \\
&\leq \int_0^1 \sum_{K \in \mathbb{N}} \sum_{\mu} \chi_{I_{K,\mu}} \left(U_{\gamma\delta\epsilon, s\delta\epsilon}^{(3)}(|g|^{(\frac{r}{\delta\epsilon})'}) \right)^{1/(\frac{r}{\delta\epsilon})'} d\lambda \\
&\leq \left\| \sum_{K \in \mathbb{N}} \sum_{\mu} \chi_{I_{K,\mu}} \right\|_{p(\cdot)/\epsilon} \left\| \left(U_{\gamma\delta\epsilon, s\delta\epsilon}^{(3)}(|g|^{(\frac{r}{\delta\epsilon})'}) \right)^{1/(\frac{r}{\delta\epsilon})'} \right\|_{(p(\cdot)/\epsilon)'}.
\end{aligned}$$

Since $(r/\delta\epsilon)' < (p(\cdot)/\epsilon)'$, Theorem 4.3 implies that

$$Z \lesssim \|\chi_{\{\tau < \infty\}}\|_{p(\cdot)/\epsilon},$$

whenever

$$\begin{aligned} & \frac{1}{((p(\cdot)/\epsilon)'/(r/\delta\epsilon)')_-} - \frac{1}{((p(\cdot)/\epsilon)'/(r/\delta\epsilon)')_+} \\ &= \frac{r/(r-\delta\epsilon)}{p_+/(p_+-\epsilon)} - \frac{r/(r-\delta\epsilon)}{p_-/(p_--\epsilon)} < (\gamma+s)\delta\epsilon. \end{aligned}$$

This means that

$$\frac{p_+-\epsilon}{p_+} - \frac{p_--\epsilon}{p_-} < (\gamma+s)\delta\epsilon,$$

in other words,

$$\frac{1}{p_-} - \frac{1}{p_+} < (\gamma+s)\delta.$$

Since δ can be arbitrarily near to 1, we obtain (4.1). The proof of the theorem is complete. $\square$

COROLLARY 7.1. *Let $p(\cdot) \in C^{\log}$, $0 < p_- \leq p_+ < \infty$, $0 < q \leq \infty$ and $0 < \gamma, s < \infty$. If (4.1) holds, then*

$$\|f\|_{H_{p(\cdot),q}} \leq \|U_{\gamma,s}f\|_{p(\cdot),q} \leq C_{p(\cdot)}\|f\|_{H_{p(\cdot),q}} \quad (f \in H_{p(\cdot),q}).$$

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