Corrigendum

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Metrical universality for groups

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Corrigendum to: M. Doucha, Metrical universality for groups, Forum Math. 29 (2017), no. 4, 847–872**Abstract:** The aim of this note is to correct the proof of Proposition 2.15 in the original article [1].**Keywords:** Metrically universal group, free groups, Fraïssé theory, Urysohn space, Graev metric, SIN group**MSC 2010:** 22A05, 54E50, 03C98**Communicated by:** Manfred Droste

In [1], we construct a Polish metric group G with a compatible bi-invariant metric bounded by 1 (or any other fixed constant) so that any separable metrizable group admitting a compatible bi-invariant metric bounded by 1 can be embedded into G by an isometric monomorphism. In particular, we show that there exists a universal Polish SIN group. We refer to the original article for any unexplained notion and for the motivation. An important step in the construction is Proposition 2.15 which roughly says that if we have two bi-invariant metrics d_1 , resp. d_2 on finitely generated free groups F_1 , resp. F_2 , such that the metrics are “close” in a certain metric defined in the article, then we can define a bi-invariant metric on the free product $F_1 * F_2$ which extends both d_1 and d_2 , and such that the free generators of F_1 are “close” to the free generators of F_2 .

The aim of this note is to correct the proof of Proposition 2.15 in the original article [1]. At the end of the proof, in between equations (2.3) and (2.4), it was estimated that $|h_2| + |h_4| + \dots + |h_l| = |a| - K$. That is not true in general, however we shall show that without loss of generality we can assume that.

We restate the proposition here, following the notation of the original article, and provide a complete proof of it.

Proposition 1. *Take any natural numbers $n \leq m$. Let p_1 be a finitely generated metric on F_n and let p_2 be a finitely generated metric on F_m such that $p_1 \geq p_2 \upharpoonright F_n$ and p_1 is δ -close to $p_2 \upharpoonright F_n$, for some $\delta > 0$. Then there exists a finitely generated metric p on $F'_n * F_m$, with free generators $\{f'_1, \dots, f'_n, f_1, \dots, f_n, \dots, f_m\}$ such that $p \upharpoonright F'_n \cong p_1 \upharpoonright F_n$, $p \upharpoonright F_m \cong p_2 \upharpoonright F_m$ and for every $i \leq n$ we have $p(f'_i, f_i) \leq \delta$.*

Remark 2. We note that in the original statement there was an equality $p(f'_i, f_i) = \delta$. While it is possible to prove the equality (under the assumption stated just before the statement of Proposition 2.15 in [1] that $\delta \leq p_1(f_i, 1) + p_2(f_i, 1)$), in order to keep the size of the corrigendum appropriate we note that the inequality $p(f'_i, f_i) \leq \delta$ is sufficient for Embedding Construction 2.3 in [1]. Indeed, the reader can readily verify that the construction employs a sequence $(\varepsilon_i)_i$ with $\sum_i \varepsilon_i < \infty$, and sequences of group generators $(f_i^{(n)})_n$ satisfying $d(f_i^{(n)}, f_i^{(n+1)}) = \varepsilon_n$ in order to be Cauchy. Obviously, if we ensure $d(f_i^{(n)}, f_i^{(n+1)}) \leq \varepsilon_n$, the sequence is Cauchy too.

For the new proof, we introduce the notion of a conjugation-invariant norm on a group, which corresponds to the notion of a bi-invariant metric, but it is easier to deal with in some situations.

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Definition 3. If G is a group, then a function $\lambda : G \rightarrow [0, \infty)$ is called a *conjugation-invariant norm* (or shortly a CI-norm) if

- $\lambda(g) = 0$ if and only if $g = 1_G$,
- $\lambda(g) = \lambda(g^{-1})$ for every $g \in G$,
- $\lambda(g \cdot h) \leq \lambda(g) + \lambda(h)$ for every $g, h \in G$,
- $\lambda(h^{-1} \cdot g \cdot h) = \lambda(g)$ for every $g, h \in G$.

It is clear that if d is a bi-invariant metric on G , then $\lambda_d(g) = d(g, 1)$ is a conjugation-invariant norm on G , and conversely, if λ is a conjugation-invariant norm on G , then $d_\lambda(g, h) = \lambda(g^{-1} \cdot h) (= \lambda(h \cdot g^{-1}))$ is a bi-invariant metric on G .

The notion of a finitely generated metric, or more generally of a bi-invariant metric generated by values on some subset, corresponds to the notion of a conjugation-invariant norm generated by values on some subset, and there is an obvious correspondence between these two notions.

Definition 4. Let $A \subseteq G$ be a symmetric generating subset of a group G . We say that a conjugation-invariant norm λ on G is *generated by values on $A \subseteq G$* if for every $g \in G$ we have

$$\lambda(g) = \inf \left\{ \sum_{i \leq n} \lambda(g_i) : (g_i)_{i \leq n} \subseteq A, g = g_1^{h_1} \cdots g_n^{h_n}, (h_i)_{i \leq n} \subseteq G \right\}, \quad (1)$$

where g^h , for $g, h \in G$, denotes the element $h^{-1} \cdot g \cdot h$.

If moreover A is finite, we say that λ is *finitely generated*, and the infimum in the above definition can be then replaced by the minimum.

Observation 5. It is straightforward to check that if d is a bi-invariant metric on G generated by values on some subset $A \subseteq G^2$, then the corresponding CI-norm λ_d is generated by values on $\{g^{-1}h, h^{-1}g : (g, h) \in A\}$. Conversely, if λ is a CI-norm generated by values on some $A \subseteq G$, then λ_d is generated by values on the set $A' = \{(a, 1_G), (1_G, a) : a \in A\}$.

Finally, we note that we can consider the notion of distance, originally defined for bi-invariant metrics on free groups ([1, Definition 1.11]), also for CI-norms, and in exactly the same way. The reader can notice that the distance between metrics d and p in [1, Definition 1.11] is actually defined between the corresponding CI-norms λ_d and λ_p .

From now on, we assume that G is a finitely generated free group over some finite set of generators and λ is a CI-norm on G , generated by values on some finite symmetric subset $A \subseteq G$, which we assume that, without loss of generality, contains the free generators of G (as was also done in the appropriate section in [1] – see [1, Definition 2.13]). We identify the elements of G with the reduced words over the alphabet consisting of these generators and their inverses.

We shall need the following definition of a *match*.

Definition 6 (Match). Let $w = w_1 \dots w_n$ be a word in the alphabet consisting of the free generators of G and their inverses. A *match* for w is a bijection ρ on $\{1, \dots, n\}$ satisfying the following properties. We note that in order to simplify the notation, we shall often freely identify ρ as a bijection on $\{1, \dots, n\}$ with a bijection on $\{w_1, \dots, w_n\}$, i.e. write $\rho(w_i) = w_j$ instead of $w_{\rho(i)} = w_j$.

- for every $i \leq n$, we have $\rho(\rho(i)) = i$, i.e. $\rho \circ \rho = \text{id}$,
- for no $i, j \leq n$, we have $i < j < \rho(i) < \rho(j)$,
- if for some $i \leq n$, $\rho(w_i) \neq w_i$, then $\rho(w_i) = w_i^{-1}$ (the letter $\rho(w_i)$ is the inverse of the letter w_i).

Remark 7. The interpretation of this definition of a match ρ is that ρ describes how the word w is built from its subwords using the operations of concatenation and conjugation. For example if $w = aba^{-1}c$ and a match ρ is defined as $\rho(a) = a^{-1}$, $\rho(b) = b$, $\rho(c) = c$, then ρ provides the information that w was built by conjugating the letter b by the letter a (or a^{-1}) and then concatenating with c .

Let $a \in G$. Let $w = w_1 \dots w_n$ be such that $w' = a$. Let ρ be a match on w (freely identified with a match on $\{1, \dots, n\}$). Set $I_\rho \subseteq \{1, \dots, n\}$ to be the set $\{i \leq n : \rho(i) \neq i\}$. Let $J_\rho = \{1, \dots, n\} \setminus I_\rho$. Then J_ρ can be further

divided (not uniquely) into intervals $J_1, \dots, J_M$, where each such interval is a (reduced) subword of w that corresponds to an element from A (this subdivision is always possible since A contains the generators). Let $\mathcal{J}_\rho$ be such a subdivision.

We then define

$$\lambda_{\mathcal{J}_\rho}^\rho(w) = \sum_{J \in \mathcal{J}_\rho} \lambda(J),$$

where we identified the subwords from $\mathcal{J}_\rho$ with the corresponding elements from A .

Now for a word w set

$$\lambda(w) = \min\{\lambda_{\mathcal{J}_\rho}^\rho(w) : \rho \text{ is a match on } w, \mathcal{J}_\rho \text{ a subdivision of } J_\rho\}.$$

Finally, for an element $g \in F$, we may define

$$\lambda(g) = \min\{\lambda(w) : w' = g\}. \quad (2)$$

We claim that this definition of the CI-norm λ coincides with that one given by formula (1) (for free groups). We show that. Denote by λ_1 the function given by formula (1) and by λ_2 the function given by formula (2). Fix $g \in G$.

First we show that $\lambda_1(g) \leq \lambda_2(g)$. Pick an arbitrary word w such that $w' = g$, match ρ on w and a subdivision $\mathcal{J}_\rho = \{J_1, \dots, J_M\}$ of J_ρ . For each $i \leq M$, let $a(i)_1, \dots, a(i)_{j_i}$ be the subletters a of w , in the order as they appear in w , such that a precedes the subword J_i , $\rho(a) > a$ and $\rho(a)$ appears after the subword J_i in w . We then have that

$$g = J_1^{a(1)_1^{-1} \dots a(1)_{j_1}^{-1}} \dots J_M^{a(M)_1^{-1} \dots a(M)_{j_M}^{-1}}.$$

Then by definition

$$\lambda_1(g) \leq \sum_{i=1}^M \lambda(J_i) = \lambda_{\mathcal{J}_\rho}^\rho(w),$$

showing that $\lambda_1(g)$ is smaller or equal to $\lambda_2(g)$.

Conversely, choose any elements $g_1, \dots, g_n \in A$ and $h_1, \dots, h_n \in G$ such that

$$g = g_1^{h_1} \dots g_n^{h_n} = h_1^{-1} \cdot g_1 \cdot h_1 \dots h_n^{-1} \cdot g_n \cdot h_n.$$

Let w be the word corresponding to writing g as $h_1^{-1} \cdot g_1 \cdot h_1 \dots h_n^{-1} \cdot g_n \cdot h_n$ and let ρ be a match on w which is the identity on letters of g_i , for $i \leq n$, and for each subletter of h_i , $i \leq n$, ρ sends it to the corresponding inverse letter in h_i^{-1} . Let $\mathcal{J}_\rho = \{g_1, \dots, g_n\}$. Then by definition

$$\lambda_2(g) \leq \lambda_{\mathcal{J}_\rho}^\rho(w) = \sum_{i=1}^n \lambda(g_i),$$

showing that $\lambda_2(g)$ is smaller than or equal to $\lambda_1(g)$. This finishes the proof of the claim.

Remark 8. Another way how to prove the claim above is to directly verify that both λ_1 and λ_2 are the maximal CI-norms that are bounded from above on the set A by the prescribed values there. Since such a maximal CI-norm, if it exists, is unique, we get the equality between λ_1 and λ_2 .

Consider now some $a \in G$ and let $w = w_1 \dots w_n$ be a word such that $w' = a$, let ρ be some match on w and $\mathcal{J}_\rho$ be some subdivision of J_ρ . We shall define *admissible operations* on the word w , match ρ and $\mathcal{J}_\rho$ that change w, ρ and $\mathcal{J}_\rho$ to a word $v = v_1 \dots v_m$, match v on v and subdivision $\mathcal{J}_v$ such that still $v' = a$ and $\lambda_{\mathcal{J}_v}^\rho(v) = \lambda_{\mathcal{J}_\rho}^\rho(w)$.

Operation 1. Suppose that there are $i < i' < n$ and some $k \geq 1$ such that

- $w_i \dots w_{i'} \in \mathcal{J}_\rho$,
- $w_{i'+1} \dots w_{i'+k}$ is a trivial subword,
- $i' + k + 1 \in I_\rho$, or $i' + k + 1 \in J_\rho$ in which case $w_{i'+k+1}$ is the left-most letter of some $J_m \in \mathcal{J}_\rho$.

Then we can “move” the trivial subword $w_{i'+1} \dots w_{i'+k}$ in front of the subword $w_i \dots w_{i'}$ and change the match accordingly. That is, we get a new word

$$v = v_1 \dots v_n (= w_1 \dots w_{i-1} w_{i'+1} \dots w_{i'+k} w_i \dots w_{i'} w_{i'+k+1} \dots w_n)$$

such that $v_j = w_j$ for $j < i$ and $j > i' + k$, $v_{i-1+l} = w_{i'+l}$, for $l \leq k$, and $v_{j+k} = w_j$, for $i \leq j \leq i'$. The match ρ on w is changed accordingly in an obvious way to a match ν on v . That is, $\nu(s) = t$, for letters s, t in v , if and only if $\rho(s) = t$, if we view the word v as a permutation of letters in w . Also $\mathcal{J}_\rho$ is changed to $\mathcal{J}_\nu$ accordingly. Clearly, $v' = w' = a$ and $\lambda_{\mathcal{J}_\nu}^\nu(v) = \lambda_{\mathcal{J}_\rho}^\rho(w)$.

There is also a symmetric version of Operation 1 in which we assume that there are $1 < i' < i$ and some $k \geq 1$ such that

- $w_{i'} \dots w_i \in \mathcal{J}_\rho$,
- $w_{i'-k} \dots w_{i'}$ is a trivial subword,
- $i' - k - 1 \in I_\rho$, or $i' - k - 1 \in J_\rho$ in which case $w_{i'-k-1}$ is the right-most letter of some $J_m \in \mathcal{J}_\rho$.

The operation that we do then is the exact symmetric copy of the one above.

Operation 2. Suppose that there are $i < i' < n$ such that

- $w_i \dots w_{i'} \in \mathcal{J}_\rho$,
- $w_{i'+1} = w_{i'}^{-1}$ and $i' + 1 \in I_\rho$.

Denote the subword $w_i \dots w_{i'-1}$ by V . Set $j = \rho(i' + 1)$, therefore $w_j = w_{i'}$. We have either $j > i' + 1$, or $j < i' + 1$ (in which case necessarily $j < i$). We suppose the former, i.e. $j > i' + 1$, the other case is analogous. We modify the word w to a word v by deleting the letters $w_{i'}$ and $w_{i'+1}$ from w and replacing the letter $w_j (= w_{\rho(i'+1)})$ by the subword $V^{-1}Vw_j$ (note that Vw_j is equal to the word $w_i \dots w_{i'}$). That is, the original word w was

$$w_1 \dots Vw_{i'}w_{i'}^{-1} \dots w_j \dots w_n$$

while the new word v is

$$w_1 \dots Vw_{i'}w_{i'}^{-1} \dots V^{-1}Vw_j \dots w_n.$$

Clearly, $v' = w'$. We must also change the match ρ into a match ν on v . We put $\{i, \dots, i' - 1\}$ into I_ν as well as the indices corresponding to the subword V^{-1} of the subword $V^{-1}Vw_j$, and we ‘connect them’ by the match ν ; that is, for $i \leq l \leq i' - 1$, $\nu(w_l) = w_{l'}$, where $w_{l'}$ is the corresponding inverse letter to w_l in the subword V^{-1} . On the other hand, the subword Vw_j of $V^{-1}Vw_j$ now belongs to $\mathcal{J}_\nu$ (previously, the equivalent subword $w_i \dots w_{i'}$ belonged to $\mathcal{J}_\rho$). On the rest, ν is defined the same way as ρ ; that is, all other letters s, t from the word v can be identified with some letters in w , and we have $\nu(s) = t$ if and only if $\rho(s) = t$. It is clear that $\lambda_{\mathcal{J}_\nu}^\nu(v) = \lambda_{\mathcal{J}_\rho}^\rho(w)$.

Like Operation 1, Operation 2 also has its symmetric copy which we do not describe here.

We can now proceed to the proof of Proposition 1.

Proof of Proposition 1. Using Observation 5, we suppose that p_1 and p_2 are CI-norms. That is, we suppose p_1 to be a finitely generated CI-norm on the free group on n generators $\{f'_1, \dots, f'_n\}$ denoted by F'_n , generated by the values on some finite $A' \subseteq F'_n$, and p_2 to be a finitely generated CI-norm on the free group on m generators $\{f_1, \dots, f_m\}$ denoted by F_m , $m \geq n$, generated by the values on some finite $A \subseteq F_n$. Recall from the statement of Proposition 1 that we also have that $p_1 \geq p_2 \upharpoonright F_n$ and that p_1 is δ -close to $p_2 \upharpoonright F_n$, for some $\delta > 0$. We define a CI-norm p on $F'_n * F_m$. Set $D = \{(f'_i)^\varepsilon \cdot f_i^{-\varepsilon} : i \leq n, \varepsilon \in \{-1, 1\}\}$. We let p to be generated by the values of p_1 on F'_n , by the values of p_2 on F_m and by the value δ on D . That is, for $a \in F'_n \cup F_m \cup D$ we set

$$p(a) = \begin{cases} p_1(a), & a \in F'_n, \\ p_2(a), & a \in F_m, \\ \delta, & a \in D. \end{cases}$$

The norm p then extends to $F'_n * F_m$ using formula (2). It is clear that p is in fact finitely generated as it is generated by the values on $A' \cup A \cup D$.

For any $i \leq n$, we have by definition $p(f_i^{-1} \cdot f'_i) \leq \delta$, so we must check that p agrees with p_1 on F'_n and with p_2 on F_m . The latter is easy and we show it first.

Take some $a \in F_m$ and suppose that $p(a) < p_2(a)$. Then there are elements $a_1, \dots, a_k \in A' \cup A \cup D$ such that $a = a_1 \dots a_k$ and $\sum_{i=1}^k p(a_i) < p_2(a)$. Now for each $i \leq k$, let $a'_i \in F_m$ be an element obtained from a_i by replacing each generator f'_j , resp. its inverse, $j \leq n$, occurring in the element a_i as a reduced word over the alphabet $\{f'_1, \dots, f'_n\}^\pm \cup \{f_1, \dots, f_m\}^\pm \cup \{1\}$, by the generator f_j , resp. its inverse. Notice in particular that if

$a_i \in D$, then $a'_i = 1$. Since $a \in F_m$, we have that $a = a'_1 \cdots a'_k$. Note that we have $p(a'_j) \leq p(a_j)$, for all $j \leq k$. Indeed, if $a_j \in A$, then $a'_j = a_j$; if $a_j \in D$, then $a'_j = 1$, and finally if $a_j \in A'$, then since $p_1 \geq p_2 \upharpoonright F_n$ we get that $p(a'_j) \leq p(a_j)$. It follows that

$$p_2(a) \leq \sum_{j=1}^k p_2(a'_j) < p_2(a),$$

which is a contradiction.

So we focus on the former, i.e. that p agrees with p_1 on F'_n .

Take some $a \in F'_n$ and suppose that $p(a) < p_1(a)$. Then there is some word $w = w_1 \dots w_k$ in the alphabet $\{f'_1, \dots, f'_n\}^\pm \cup \{f_1, \dots, f_m\}^\pm \cup \{1\}$ such that $w' = a$ and a match ρ on w , and a subdivision $\mathcal{J}$ of J_ρ such that $p(a) = p_{\mathcal{J}}^\rho$. Set $K = \{i \leq k : w_i \in F_m\}$. Let us define a graph structure on K . We declare $i, j \in K$ to be connected by an edge

- of type (1) if j is the least element such that $i < j$, $w_j = w_i^{-1}$, and $w_{i+1} \dots w_{j-1}$ is a trivial word; or symmetrically, i.e. with the roles of i and j reversed,
- of type (2) if $i, j \in I_\rho$ and $\rho(i) = j$, $\rho(j) = i$.

We remark that this graph is analogous to that one defined by Gao in [2, Theorem 2.6.5] when computing the Graev metric.

Remark 9. The interpretation of the graph is that it tracks, for each $i \in K$, how the letter w_i gets cancelled in the reduced word w' . Consider for example the word $w = afa^{-1}af^{-1}bff^{-1}c$, where the letters a, b, c are from F'_n and f is from F_m . Suppose a match ρ is defined on w so that $\rho(1) = 3$ (matching the first a with the inverse a^{-1}), $\rho(5) = \rho(7)$ (matching the f^{-1} on the fifth position with f on the seventh position), and everywhere else ρ is the identity. Then $K = \{2, 5, 7, 8\}$, 2 is connected with 5 by an edge of type (1), 5 is moreover connected with 7 by an edge of type (2), and 7 is moreover connected with 8 by an edge of type (1).

Observation 10. If $i \in K$ lies in J_ρ , then, since w_i gets cancelled in the reduced word w' , it is connected by an edge of type (1) to exactly one element of K (which cancels it). While if $i \in K$ lies in I_ρ , then it is connected to exactly two elements: by an edge of type (1) to an element from K that cancels it and via the match ρ to another element by an edge of type (2).

It follows, since the degree of each vertex is either 1 or 2 (note that no vertex $i \in K$ can be isolated since the letter w_i must get cancelled), that the connected components of K are either simple paths or cycles.

Claim 11. *Without loss of generality, K contains no cycles.*

Proof of Claim 11. Suppose there is a cycle C in K . By Observation 10, C consists only of edges of type (2), so the letters $\{w_i : i \in C\}$ are paired by the match ρ . It follows that C consists of an even number l of edges. Denote the vertices of C by $i_1, \dots, i_{l+1}$ so that for $j \leq l+1$ odd, i_j is connected to i_{j+1} by an edge of type (2). That is, $\{(i_j, i_{j+1}) : j \leq l+1 \text{ odd}\}$ is a subset of the match ρ , and w_{i_j} and $w_{i_{j+1}}$ get cancelled in w , for $j \leq l+1$ even (mod $l+1$). It follows we may remove the letters $\{w_{i_j} : j \leq l\}$ from w to obtain a word v , and $v' = w'$. Also, we modify the match ρ on w to a match ρ' on v by removing $\{i_j : j \leq l+1\}$ from its domain. The set $\mathcal{J}_{\rho'}$ contains the same subwords as $\mathcal{J}_\rho$. It is straightforward to check that $\lambda_{\mathcal{J}_\rho}^\rho(w) = \lambda_{\mathcal{J}_{\rho'}}^{\rho'}(v)$. So we decreased the length of the word w without changing its reduced form and the value of the norm on it. By repeating the same procedure if necessary we can get rid of all the cycles. $\square$

Next notice that if $i \in K \cap J_\rho$, then w_i is a letter of a subword, denoted by $V(i)$, from $\mathcal{J}_\rho$. Either $V(i) \in F_m$ or $V(i) \in D$. The paths in K that have exactly one end i with $V(i) \in D$ are called *good paths*, the others are called *bad paths*.

Claim 12. *Without loss of generality, K contains no bad paths.*

Remark 13. The assumption that there are no bad paths means that every $i \in K$ such that the letter w_i is a part of some subword $V(i) \in F_m$ is an end of a simple path from K whose other end is some $i' \in K$ such that $w_{i'}$ is a part of some subword $V(i') \in D$. That informally means that the letter w_i gets eventually cancelled in a subword $f_i^{-\varepsilon} \cdot f'_i$ from D .

This assumption is what was taken for granted in the original proof of Proposition 1.15 in [1]. Let us show how to conclude the proof provided that Claim 12 is proved. Let us replace the word $w = w_1 \dots w_k$ by a word $v = v_1 \dots v_k$, where, for $i \leq k$,

- $v_i = w_i$ if $w_i \in F'_n$,
- $v_i = f_j^{\varepsilon}$ if $w_i = f_j^{\varepsilon} \in F_m$, for $j \leq n$, $\varepsilon \in \{-1, 1\}$.

We note that indeed necessarily $j \leq n$, since i lies in a good path. The match on v is the same as on w , however we shall denote it by v for notational reasons.

Now we still have $v' = a$, and by definition, $p_1(a) \leq p_{\mathcal{J}_v}^v(v)$. Thus we must show that $p_{\mathcal{J}_v}^v(v) \leq p_{\mathcal{J}_\rho}^\rho(w)$, i.e. $\sum_{J \in \mathcal{J}_v} p(J) \leq \sum_{J \in \mathcal{J}_\rho} p(J)$. Let ϕ be the isomorphism from $F_n \leq F_m$ to F'_n which sends f_i to f'_i , for $i \leq j$. Each $J \in \mathcal{J}_\rho$ from D was replaced by a trivial word in $\mathcal{J}_v$. Let k be the number of such J 's. On the other hand, each $J \in \mathcal{J}_\rho$ that lies in $F_n \leq F_m$ was replaced by $\phi(J)$ in $\mathcal{J}_v$ that lies in F'_n . Let k' be the total length of such words. By our assumption that there are only good paths in K , we have $k \geq k'$. Indeed, for each $J \in \mathcal{J}_\rho$ that lies in F_n and each subletter w_i of J there is one good path with one end i and the other end being j with $w_j = w_i^{-1}$, and $V(j) \in D$.

Since p_1 is δ -close to p_2 , we thus get that

$$p_{\mathcal{J}_\rho}^\rho(w) - p_{\mathcal{J}_v}^v(v) = \sum_{J \in \mathcal{J}_\rho} p(J) - \sum_{J \in \mathcal{J}_v} p(J) \geq k \cdot \delta - k' \cdot \delta \geq 0$$

and we are done. It remains to prove the claim.

Proof of Claim 12. Let $i \in K$ be an end of a bad path in K . It is connected by an edge to some $j \in K$. Without loss of generality, suppose that $i < j$. Also, without loss of generality suppose that $V(i)$ has w_i as its right-most letter. Indeed, otherwise $V(i)$ would be of the form $Vw_iV' \in \mathcal{J}_\rho$, where V and V' are some words, $V' = v_1 \dots v_l$, and we would replace Vw_iV' by its conjugate $V(i)' = V'Vw_i$ and putting $v_1 \dots v_l = V'$ after this word and $v_l^{-1} \dots v_1^{-1} = V'^{-1}$ before this word, and matching them together; that is, $Vw_iV' \in \mathcal{J}_\rho$ is replaced by a word $V'^{-1}(V'Vw_i)V'$, where $V'Vw_i$ belong to $\mathcal{J}_\rho$, while the left V'^{-1} and right V' are matched using ρ .

In case that $w_j \in J_\rho$, we may analogously assume that w_j is the left-most letter of $V(j)$.

- (1) In case that $w_j \in J_\rho$, we can move the trivial subword $w_{i+1} \dots w_{j-1}$ right before $V(i)$ using Operation 1. Then we get that $V(i)$ and $V(j)$ are neighbors and their neighboring letters are inverses to each other. By triangle inequality we may get these neighboring letters cancelled. This reduces the length of the bad path.
- (2) Assume that $w_j \in I_\rho$. We can again move the trivial subword $w_{i+1} \dots w_{j-1}$ right before $V(i)$, using Operation 1, so that w_i and w_j are neighbors. Then j is moreover connected by an edge to some $l = \rho(j)$. Suppose that $V(i)$ is the subword $v_1 \dots v_r$, where $v_r = w_i$. Using Operation 2, we can delete the letters w_i and w_j and we can replace the letter w_l by the subword $V(i)$ preceded by the letters $v_{r-1}^{-1} \dots v_1^{-1}$ which are connected by the match to the sequence of letters $v_1 \dots v_{r-1}$ on the former place of $V(i)$. This again reduces the length of the bad path.

Repeating the two steps above one can eventually get rid of all the bad paths by reducing their lengths. $\square$

This finishes the proof of Proposition 1. $\square$

References

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