Research Article

Sun-Sig Byun, Wontae Kim* and Minkyu Lim

Higher integrability near the initial boundary for nonhomogeneous parabolic systems of p -Laplacian type

<https://doi.org/10.1515/forum-2020-0068>

Received March 16, 2020; revised June 29, 2020

Abstract: We establish a sharp higher integrability near the initial boundary for a weak solution to the following p -Laplacian type system:

$$\begin{cases} u_t - \operatorname{div} \mathcal{A}(x, t, \nabla u) = \operatorname{div} |F|^{p-2} F + f & \text{in } \Omega_T, \\ u = u_0 & \text{on } \Omega \times \{0\}, \end{cases}$$

by proving that, for given $\delta \in (0, 1)$, there exists $\varepsilon > 0$ depending on δ and the structural data such that

$$|\nabla u_0|^{p+\varepsilon} \in L^1_{\text{loc}}(\Omega) \quad \text{and} \quad |F|^{p+\varepsilon}, |f|^{(\frac{\delta p(n+2)}{n})'+\varepsilon} \in L^1(0, T; L^1_{\text{loc}}(\Omega)) \implies |\nabla u|^{p+\varepsilon} \in L^1(0, T; L^1_{\text{loc}}(\Omega)).$$

Our regularity results complement established higher regularity theories near the initial boundary for such a nonhomogeneous problem with $f \neq 0$ and we provide an optimal regularity theory in the literature.

Keywords: Higher integrability, initial boundary data, non-divergence data, parabolic p -Laplacian system

MSC 2010: 35K20, 35K92

Communicated by: Frank Duzaar

1 Introduction

In this paper, we are interested in finding a sharp higher integrability near the initial boundary to a weak solution to the parabolic system

$$\begin{cases} u_t - \operatorname{div} \mathcal{A}(x, t, \nabla u) = \operatorname{div} |F|^{p-2} F + f & \text{in } \Omega_T, \\ u = u_0 & \text{on } \Omega \times \{0\}. \end{cases}$$

Here, $\frac{2n}{n+2} < p < \infty$, Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 2$, $\mathcal{A}(x, t, \zeta)$ is modeled after the p -Laplacian operator, $u_0 \in W^{1,p}(\Omega, \mathbb{R}^N)$, $F \in L^p(\Omega_T; \mathbb{R}^{Nn})$ and $f \in L^{q'}(\Omega_T, \mathbb{R}^N)$ for some $N \geq 1$, where $q = \frac{p(n+2)}{n}$ is the parabolic Sobolev conjugate of p and q' is Hölder conjugate of q .

In the case $f \equiv 0$, interior higher integrability results were proved by Kinnunen and Lewis in [13, 14] by providing a suitable application of DiBenedetto's intrinsic geometry method from [9] to the setting of Gehring type estimates. Higher integrability results near the initial and lateral boundary were proved by Parviainen in [19, 20]. These regularity results were extended to higher-order systems by Bögelein and Parviainen in [5].

***Corresponding author: Wontae Kim**, Department of Mathematical Sciences, Seoul National University, Seoul 08826, Korea, e-mail: m20258@snu.ac.kr

Sun-Sig Byun, Department of Mathematical Sciences; and Research Institute of Mathematics, Seoul National University, Seoul 08826, Korea, e-mail: byun@snu.ac.kr

Minkyu Lim, Department of Mathematical Sciences, Seoul National University, Seoul 08826, Korea, e-mail: mk0314@snu.ac.kr

On the other hand, in the case $f \neq 0$, a nonlinear relation between the gradient of a weak solution and the non-divergence data f coming from parabolic embeddings naturally occurs. To be specific, there is an exponent $\alpha > 1$ such that $\|\nabla u\|_{L^p}^p$ is related to $\|f\|_{L^{q'}}^{q'\alpha}$. Indeed, there have been regularity estimates coming from such a nonlinear relation rather than Gehring type estimates. In particular, Kuusi and Mingione in [15] proved gradient L^∞ regularity and gradient continuity of a weak solution with an nonlinear exponent α based on the intrinsic geometry method. The recent paper [2] provides a new representation of the nonlinear relation. It replaces the exponent α by 1 from the intrinsic geometry method, obtaining interior higher integrability results for a system of $p(x, t)$ -Laplacian type with scaling invariant estimates and extending gradient continuity results in [15] to the $p(x, t)$ -Laplacian system. For the elliptic case, there is a divergence representation for the non-divergence data, and Gehring type estimates directly follow from [10, 11, 17].

As already mentioned, the main purpose of this paper is to establish higher integrability results near the initial boundary. In addition, we are considering a class of the nonlinearities whose structures are associated with the divergence data F . Regarding the non-divergence data, $f \in L^{(\frac{\delta p(n+2)}{n})'}$ with $\delta \in (0, 1)$ is sharp in obtaining the reverse Hölder inequality by the parabolic Sobolev embedding theorem. A noteworthy feature of the present paper is to find the optimality and sharpness of the initial data u_0 from the intrinsic geometry method. Due to a suitable application of the Poincaré inequality in a type of the Caccioppoli inequality, $\|\nabla u_0\|_{L^s(B_\rho)}^2$ appears for some $s > 1$. The optimal exponent s can be found in the process of the use of the Fubini theorem. In fact, $(\iint |\nabla u_0|^s dz)^{\frac{p}{s}}$ appears after using the reverse Hölder inequality, and the Minkowski integral inequality enforces the condition $s = p$. On the other hand, there is an alternative for the non-divergence data f to avoid applying the Minkowski integral inequality, thanks to the presence of radius ρ (see (4.3) below). Therefore, p is the optimal exponent for s , and we can find the sharp exponent $\beta = \min\{2, p\}$ in $\|\nabla u_0\|_{L^p}^\beta$ from the stopping time argument.

Our work can be applicable to different problems including obstacle problem and system of $p(x, t)$ -Laplacian type problem as in [4, 6] as well as Calderón–Zygmund type estimates as in [1, 3, 7, 8, 16] both when $f \neq 0$ and when $\nabla u_0 \neq 0$.

The paper is organized as follows. In Section 2, we introduce basic notation and definitions to state our main results. Section 3 is devoted to proving reverse Hölder inequality. Finally, in Section 4, we prove our main results.

2 Basic notations and results

2.1 Notations

We shall clarify all the notations that will be used in this paper.

- (i) We use ∇ to denote derivatives with respect the space variable x and ∂_t to denote the time derivative.
- (ii) In what follows, we always assume the bounds $\frac{2n}{n+2} < p < \infty$.
- (iii) Let $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ be a point, $\rho, s > 0$ two given parameters, and let $\lambda \in [1, \infty)$. We use the following notations:

$$\begin{aligned} I_s(t_0) &:= (t_0 - s^2, t_0 + s^2) \subset \mathbb{R}, & Q_{\rho,s}(z_0) &:= B_\rho(x_0) \times I_s(t_0) \subset \mathbb{R}^{n+1}, \\ I_s^\lambda(t_0) &:= (t_0 - \lambda^{2-p}s^2, t_0 + \lambda^{2-p}s^2) \subset \mathbb{R}, & Q_{\rho,s}^\lambda(z_0) &:= B_\rho(x_0) \times I_s^\lambda(t_0) \subset \mathbb{R}^{n+1}, \\ Q_\rho^\lambda(z_0) &:= Q_{\rho,\rho}^\lambda(z_0), & Q_\rho(z_0) &:= Q_\rho^1(z_0). \end{aligned}$$

- (iv) We use $\int$ to denote the integral with respect to either space variable or time variable and use $\iint$ to denote the integral with respect to both space and time variables simultaneously.

Analogously, we use $\oint$ and $\overline{\iint}$ to denote the integral averages as defined below: for any set $A \times B \subset \mathbb{R}^n \times \mathbb{R}$, we define

$$(f)_A := \frac{1}{|A|} \int_A f(x) dx, \quad (f)_{A \times B} := \frac{1}{|A \times B|} \iint_{A \times B} f(x, t) dx dt.$$

- (v) We use the notation $\leq_{(a,b,\dots)}$ to denote an inequality with a constant depending on $a, b, \dots$.

Definition 2.1. Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and $u_0 \in L^2(\Omega, \mathbb{R}^N)$, $f \in L^{(\frac{p(n+2)}{n})'}(\Omega_T; \mathbb{R}^N)$ for some $N \geq 1$. A weak solution $u \in C(0, T; L^2(\Omega, \mathbb{R}^N)) \cap L^p(0, T; W^{1,p}(\Omega, \mathbb{R}^N))$ to

$$\begin{cases} u_t - \operatorname{div} \mathcal{A}(x, t, \nabla u) = f & \text{in } \Omega_T, \\ u = u_0 & \text{on } \Omega \times \{0\} \end{cases} \quad (2.1)$$

is a distributional energy solution in the sense

$$\iint_{\Omega_T} -u \phi_t + \langle \mathcal{A}(x, t, \nabla u), \nabla \phi \rangle dz = \iint_{\Omega_T} f \phi dz \quad \text{for all } \phi \in C_0^\infty(\Omega_T, \mathbb{R}^N),$$

and

$$\lim_{h \rightarrow 0^+} \int_0^h \int_{\Omega} |u(x, t) - u_0(x)|^2 dx dt = 0. \quad (2.2)$$

2.2 Structures of the operator

We now describe the assumptions on the nonlinear structures in (2.1). Assume $\mathcal{A}(x, t, \nabla u)$ is a Carathéodory function, i.e., we have that $(x, t) \mapsto \mathcal{A}(x, t, \zeta)$ is measurable for every $\zeta \in \mathbb{R}^n$ and $\zeta \mapsto \mathcal{A}(x, t, \zeta)$ is continuous for almost every $(x, t) \in \Omega_T$.

We further assume that, for a.e. $(x, t) \in \Omega_T$ and for any $\zeta \in \mathbb{R}^n$, there exist two positive constants Λ_0 and Λ_1 such that the following bounds are satisfied by the nonlinear structures:

$$\langle \mathcal{A}(x, t, \zeta), \zeta \rangle \geq \Lambda_0 |\zeta|^p - |F(x, t)|^p \quad \text{and} \quad |\mathcal{A}(x, t, \zeta)| \leq \Lambda_1 |\zeta|^{p-1} + |F(x, t)|^{p-1}, \quad (2.3)$$

where $F \in L^p(\Omega_T, \mathbb{R}^{Nn})$.

2.3 Main results

Before stating our main theorem, we fix some constants which will be frequently used in this paper.

Definition 2.2. For fixed constants

$$\max \left\{ \frac{n}{n+2}, \frac{n+1}{(n+2)p} \right\} < \delta < 1 \quad \text{and} \quad d := \begin{cases} \frac{p}{2} & \text{if } p \geq 2, \\ \frac{2p}{p(n+2)-2n} & \text{if } p < 2, \end{cases}$$

we denote

$$\alpha := \frac{\delta p(n+2)}{n}, \quad \nu := \alpha' \left(\frac{\min\{2, p\}}{2} - \frac{p}{\alpha} \right) \quad \text{and} \quad \alpha := \left(1 - \frac{d}{p} \nu \right)^{-1}.$$

Remark 2.3. (i) $0 < \frac{p}{d\alpha}$ holds, and (ii) α is a parabolic Sobolev conjugate of δp .

To see Remark 2.3 (i), we split two cases.

- Case $p \geq 2$: There hold $\frac{p}{d\alpha} = \frac{p}{d} - \alpha' \left(1 - \frac{p}{\alpha} \right)$ and

$$\frac{p}{d} - \alpha' \left(1 - \frac{p}{\alpha} \right) = 2 - \alpha' \left(1 - \frac{p}{\alpha} \right) > 0 \iff \frac{2}{\alpha'} - \left(1 - \frac{p}{\alpha} \right) > 0 \iff 2 - \frac{2}{\alpha} > 1 - \frac{p}{\alpha} \iff 1 > \frac{1}{\alpha} (2 - p).$$

- Case $p < 2$: There hold $\frac{p}{d\alpha} = \frac{p}{d} - \alpha' \left(\frac{p}{2} - \frac{p}{\alpha} \right)$ and

$$0 < \frac{p}{d\alpha} = \frac{p(n+2)-2n}{2} - \alpha' \left(\frac{p}{2} - \frac{p}{\alpha} \right) \iff 0 < \frac{n(p-2)}{\alpha'} + p \iff 2n - pn - p < \frac{2n - pn}{\alpha}.$$

Recall $\alpha = \frac{\delta p(n+2)}{n}$. Since $\delta \in (0, 1)$ and

$$(n+2)p[2n - (n+1)p] \leq n^2(2-p) \iff ((n+2)p - 2n)((n+1)p - n) \geq 0,$$

the remark follows.

To apply the intrinsic geometry method developed in [13], let us define the following notations.

Definition 2.4. Let $u \in C(0, T; L^2(\Omega, \mathbb{R}^N)) \cap L^p(0, T; W^{1,p}(\Omega, \mathbb{R}^N))$ be a weak solution of (2.1) under the assumption of (2.2), (2.3) and $f \in L^{a'}(\Omega_T, \mathbb{R}^N)$. For any $(x_0, t_0) = z_0 \in \Omega_T$ and $B_{2r}(x_0) \subset \Omega$, we define

$$\lambda_0^{\frac{p}{a}} := \iint_{Q_{2r}(z_0) \cap \Omega_T} (|\nabla u| + |F| + 1)^p dz + \left(\iint_{Q_{2r}(z_0) \cap \Omega_T} (2r)^{a'} |f|^{a'} dz \right)^{\alpha} + \left(\int_{B_{2r}(x_0)} |\nabla u_0|^p dx \right)^{\min\{1, \frac{2}{p}\}}.$$

Now, we state the main theorem.

Theorem 2.5. Let $u \in C(0, T; L^2(\Omega, \mathbb{R}^N)) \cap L^p(0, T; W^{1,p}(\Omega, \mathbb{R}^N))$ be a weak solution of (2.1) under the assumption of (2.2), (2.3) and $f \in L^{a'}(\Omega_T, \mathbb{R}^N)$. Then there exists $\varepsilon_0(n, N, p, \Lambda_0, \Lambda_1, \delta)$ such that, for any $\varepsilon \in (0, \varepsilon_0)$ and for any $(x_0, t_0) = z_0 \in \Omega_T$ such that $B_{2r}(x_0) \subset \Omega$, there holds

$$\begin{aligned} \iint_{Q_r(z_0) \cap \Omega_T} |\nabla u|^{p+\varepsilon} dz &\lesssim_{(n, N, p, \Lambda_0, \Lambda_1, \delta)} \lambda_0^{\varepsilon} \iint_{Q_{2r}(z_0) \cap \Omega_T} |\nabla u|^p dz + \iint_{Q_{2r}(z_0) \cap \Omega_T} (|F| + 1)^{p+\varepsilon} dz \\ &\quad + \int_{B_{2r}(x_0)} |\nabla u_0|^{p+\varepsilon} dx + \lambda_0^{v(1+\frac{\varepsilon}{p})} \iint_{Q_{2r}(z_0) \cap \Omega_T} (2r|f|)^{a' + \frac{a'}{p}\varepsilon} dz. \end{aligned}$$

Here, λ_0 is defined in Definition 2.4.

Remark 2.6. As a consequence of Theorem 2.5, we can also obtain global estimates of the weak solution to

$$\begin{cases} u_t - \operatorname{div} \mathcal{A}(x, t, \nabla u) = f & \text{in } \Omega_T, \\ u = \phi & \text{on } \partial\Omega \times (0, T), \\ u = u_0 & \text{on } \Omega \times \{0\}, \end{cases}$$

where $\mathcal{A}$, f and u_0 are assumed as in (2.1) and $\phi \in L^{p'}(0, T; W^{1,p}(\Omega_T, \mathbb{R}^N))$ such that $\phi_t \in L^{(\frac{p(n+2)}{n})'}(\Omega_T, \mathbb{R}^N)$. Especially, $\nabla \phi$ behaves like divergence data F does while ϕ_t behaves exactly in the same way as non-divergence data f does.

Before ending this section, we provide some important lemmas which will be used later in the proof of the main theorem. Let us state Gagliardo–Nirenberg’s inequality (see [18]).

Lemma 2.7. Let $B_\rho(x_0) \subset \mathbb{R}^n$ with $0 < \rho \leq 1$, $\sigma, q, r \in [1, \infty)$ and $\vartheta \in (0, 1)$ such that $-\frac{n}{\sigma} \leq \vartheta(1 - \frac{n}{q}) - (1 - \vartheta)\frac{n}{r}$. Then, for any $u \in W^{1,q}(B_\rho(x_0))$, there holds

$$\int_{B_\rho(x_0)} \frac{|u|^\sigma}{\rho^\sigma} dx \lesssim_{(n, \sigma, q)} \left(\int_{B_\rho(x_0)} \frac{|u|^q}{\rho^q} + |\nabla u|^q dx \right)^{\frac{\vartheta\sigma}{q}} \left(\int_{B_\rho(x_0)} \frac{|u|^r}{\rho^r} dx \right)^{\frac{(1-\vartheta)\sigma}{r}}.$$

The following iteration lemma can be found in [12, Lemma 6.1].

Lemma 2.8. Let $0 < r < R < \infty$ be given, and let $h: [r, R] \rightarrow \mathbb{R}$ be a non-negative and bounded function. Furthermore, let $\theta \in (0, 1)$ and $A, B, \gamma_1, \gamma_2 \geq 0$ be fixed constants, and suppose that

$$h(\rho_1) \leq \theta h(\rho_2) + \frac{A}{(\rho_2 - \rho_1)^{\gamma_1}} + \frac{B}{(\rho_2 - \rho_1)^{\gamma_2}},$$

holds for all $r \leq \rho_1 < \rho_2 \leq R$. Then the following conclusion holds:

$$h(r) \lesssim_{(\theta, \gamma_1, \gamma_2)} \frac{A}{(R - r)^{\gamma_1}} + \frac{B}{(R - r)^{\gamma_2}}.$$

3 Estimates near the initial boundary

In this section, we assume that $B_{4\rho}(x_0) \subset \Omega$, $0 < t_0$ and $\lambda \geq 1$. Also, note that, in the case $0 \in I_{4\rho}^\lambda(t_0)$, for any $\rho \leq \rho_1 \leq \rho_2 \leq 4\rho$, there holds

$$|I_{\rho_1}^\lambda(t_0)| \leq |I_{\rho_2}^\lambda(t_0)| \leq 4^2 |I_{\rho_1}^\lambda(t_0)| \quad \text{and} \quad \frac{1}{2} |I_{\rho_1}^\lambda(t_0)| \leq |I_{\rho_1}^\lambda(t_0) \cap (0, T)| \leq |I_{\rho_1}^\lambda(t_0)|. \quad (3.1)$$

We will show a reverse Hölder inequality in intrinsic cylinders under the following assumptions.

Assumption 3.1. We assume

$$\iint_{Q_{4\rho}^\lambda(z_0)\cap\Omega_T} (|\nabla u| + |F| + 1)^p dz + \lambda^{a'(1-\frac{p}{a})} \iint_{Q_{4\rho}^\lambda(z_0)\cap\Omega_T} (4\rho)^{a'} |f|^{a'} dz + \int_{B_{4\rho}(x_0)} |\nabla u_0|^p dx \leq \lambda^p, \quad (3.2)$$

$$\iint_{Q_\rho^\lambda(z_0)\cap\Omega_T} (|\nabla u| + |F| + 1)^p dz + \lambda^{a'(1-\frac{p}{a})} \iint_{Q_\rho^\lambda(z_0)\cap\Omega_T} \rho^{a'} |f|^{a'} dz + \int_{B_\rho(x_0)} |\nabla u_0|^p dx \geq \lambda^p, \quad (3.3)$$

Note that, by a choice of $\delta \in (0, 1)$ in Definition 2.2, $0 < a'(1 - \frac{p}{a}) < p$ holds, and this exponent $a'(1 - \frac{p}{a})$ is chosen to preserve a nonlinear relation between the gradient of a weak solution and the non-divergence data. Indeed, using Young's inequality, (3.3) becomes

$$\iint_{Q_\rho^\lambda(z_0)\cap\Omega_T} (|\nabla u| + |F| + 1)^p dz + \left(\iint_{Q_\rho^\lambda(z_0)\cap\Omega_T} \rho^{a'} |f|^{a'} dz \right)^{(1-\frac{a'}{p}(1-\frac{p}{a}))^{-1}} + \int_{B_\rho(x_0)} |\nabla u_0|^p dx \geq \lambda^p.$$

3.1 Caccioppoli inequality and Poincaré type inequality

Let us first state a Caccioppoli type inequality.

Lemma 3.2. Let u be a weak solution of (2.1). Suppose $B_{4\rho}(x_0) \subset \Omega$ and $0 < t_0$. Then, for any $\rho \leq \rho_a < \rho_b \leq 4\rho$, there holds

$$\begin{aligned} & \frac{1}{|I_{\rho_a}^\lambda(t_0)|} \sup_{t \in I_{\rho_a}^\lambda(t_0)} \int_{B_{\rho_a}} |u - (u_0)_{B_{\rho_a}(x_0)}|^2 dz + \iint_{Q_{\rho_a}^\lambda(z_0)\cap\Omega_T} |\nabla u|^p dz \\ & \lesssim_{(n,N,p,\Lambda_0,\Lambda_1)} \iint_{Q_{\rho_b}^\lambda(z_0)\cap\Omega_T} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b - \rho_a} \right]^p dz + \lambda^{p-2} \iint_{Q_{\rho_b}^\lambda(z_0)\cap\Omega_T} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b - \rho_a} \right]^2 dz \\ & + \iint_{Q_{\rho_b}^\lambda(z_0)\cap\Omega_T} |f| |u - (u_0)_{B_{\rho_b}(x_0)}| dz + \lambda^{p-2} \left(\int_{B_{\rho_b}(x_0)} |\nabla u_0|^p dz \right)^{\frac{2}{p}} + \iint_{Q_{\rho_b}^\lambda(z_0)\cap\Omega_T} |F|^p dz. \end{aligned}$$

Proof. The proof basically follows from [5, Lemma 5.1]. For the sake of completeness, we present the details in our setting.

In the space direction, take a cut-off function η satisfying

$$0 \leq \eta = \eta(x) \in C_c^\infty(B_{\rho_b}(x_0)), \quad \eta \equiv 1 \text{ on } B_{\rho_a}(x_0), \quad |\nabla \eta| \leq \frac{c}{\rho_b - \rho_a}.$$

For the time direction, we divide two cases. In the case that $0 \notin I_{4\rho}^\lambda(t_0)$, consider a cut-off function ζ satisfying

$$0 \leq \zeta = \zeta(t) \in C_c^\infty(I_{\rho_b}^\lambda(t_0)), \quad \zeta \equiv 1 \text{ on } I_{\rho_a}^\lambda(t_0), \quad |\partial_t \zeta| \leq \frac{c}{\lambda^{2-p}(\rho_b - \rho_a)^2}.$$

On the other hand, in the case that $0 \in I_{4\rho}^\lambda(t_0)$, consider a cut-off function ζ satisfying

$$0 \leq \zeta = \zeta(t) \in C_c^\infty(I_{\rho_b}^\lambda(t_0)), \quad \zeta \equiv 1 \text{ on } 0 \leq t \leq t_0 + \lambda^{2-p}\rho_a^2, \quad |\partial_t \zeta| \leq \frac{c}{\lambda^{2-p}(\rho_b - \rho_a)^2}.$$

Taking $(u - (u_0)_{B_{\rho_b}(x_0)})\eta^p \zeta^2$ as a test function in (2.1), we obtain

$$\begin{aligned} \text{I} + \text{II} &:= \iint_{Q_{\rho_b}^\lambda(z_0)\cap\Omega_T} \partial_t u (u - (u_0)_{B_{\rho_b}(x_0)}) \eta^p \zeta^2 dz + \iint_{Q_{\rho_b}^\lambda(z_0)\cap\Omega_T} \langle \mathcal{A}(z, \nabla u), \nabla u \rangle \eta^p \zeta^2 dz \\ &\leq \iint_{Q_{\rho_b}^\lambda(z_0)\cap\Omega_T} |\mathcal{A}(z, \nabla u)| |\nabla \eta| |u - (u_0)_{B_{\rho_b}(x_0)}| \eta^{p-1} \zeta^2 dz + \iint_{Q_{\rho_b}^\lambda(z_0)\cap\Omega_T} |f| |u - (u_0)_{B_{\rho_b}(x_0)}| \eta^p \zeta^2 dz =: \text{III} + \text{IV}. \end{aligned}$$

Estimate of I: We use integration by parts and (2.2) to find that

$$\begin{aligned} I \geq & \frac{1}{2|I_{\rho_b}^\lambda(t_0)|} \sup_{t \in I_{\rho_b}^\lambda(t_0)} \int_{B_{\rho_b}(x_0)} |u - (u_0)_{B_{\rho_b}(x_0)}|^2 \eta^p \zeta^2 dx - \frac{1}{2|I_{\rho_b}^\lambda(t_0)|} \int_{B_{\rho_b}(x_0)} |u_0 - (u_0)_{B_{\rho_b}(x_0)}|^2 \eta^p \zeta^2 dx \\ & - \frac{1}{2} \iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} |u - (u_0)_{B_{\rho_b}(x_0)}|^2 \eta^p \zeta \partial_t \zeta dz =: I_1 - I_2 - I_3. \end{aligned}$$

Estimate of I_1 : Using the triangle inequality along with the fact that $\rho \leq \rho_a \leq \rho_b \leq 4\rho_b$ and (3.1), we obtain

$$\frac{1}{|I_{\rho_a}^\lambda(t_0)|} \sup_{t \in I_{\rho_a}^\lambda(t_0)} \int_{B_{\rho_a}(x_0)} |u - (u_0)_{B_{\rho_a}(x_0)}|^2 dx \lesssim I_1.$$

Estimate of I_2 : Applying Poincaré's inequality, we have

$$I_2 \leq \frac{1}{2|I_{\rho_b}^\lambda(t_0)|} \int_{B_{\rho_b}(x_0)} |u_0 - (u_0)_{B_{\rho_b}(x_0)}|^2 dx \lesssim \lambda^{p-2} \left(\int_{B_{\rho_b}(x_0)} |\nabla u_0|^p dx \right)^{\frac{2}{p}}.$$

Estimate of I_3 : We get

$$I_3 \leq \lambda^{p-2} \iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b - \rho_a} \right]^2 dz.$$

Therefore, we obtain

$$\begin{aligned} I \geq & \frac{1}{|I_{\rho_a}^\lambda(t_0)|} \sup_{t \in I_{\rho_a}^\lambda(t_0)} \int_{B_{\rho_a}(x_0)} |u - (u_0)_{B_{\rho_a}(x_0)}|^2 dz - \lambda^{p-2} \left(\int_{B_{\rho_b}(x_0)} |\nabla u_0|^p dz \right)^{\frac{2}{p}} \\ & - \lambda^{p-2} \iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b - \rho_a} \right]^2 dz. \end{aligned}$$

Estimate of II: Applying (2.3), we discover

$$II \geq \iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} |\nabla u|^p \eta^p \zeta^2 dz - \iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} |F|^p dz.$$

Estimate of III:

$$\begin{aligned} III & \stackrel{(2.3)}{\leq} \iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} (|\nabla u|^{p-1} + |F|^{p-1}) \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b - \rho_a} \right] \eta^{p-1} \zeta^2 dz \\ & \stackrel{(a)}{\leq} \gamma \iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} |\nabla u|^p \eta^p \zeta^2 dz + C(\gamma) \iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b - \rho_a} \right]^p + |F|^p dz. \end{aligned}$$

Here, to obtain (a), we used Young's inequality with $\gamma \in (0, 1)$.

Estimate of IV: Clearly, there holds

$$IV \leq \iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} |f| |u - (u_0)_{B_{\rho_b}(x_0)}| dz.$$

Combining all the above calculations and taking $\gamma = \gamma(n, N, p, \Lambda_0, \Lambda_1)$ small enough, the conclusion follows. $\square$

In our definition of a weak solution to (2.1), there is no differentiability assumption on u with respect to time. Therefore, we cannot apply Poincaré's inequality directly. Nevertheless, we shall use (2.1) to estimate continuity of u with respect to time.

Lemma 3.3. *Let u be a weak solution of (2.1). Suppose that $B_r(x_0) \subset \Omega$ and $0 < t_0$. Then, for all $\theta \in [1, p]$, there holds*

$$\begin{aligned} \iint_{Q_r^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_r(x_0)}|}{r} \right]^\theta dz &\lesssim_{(n, N, p, \Lambda_1)} \iint_{Q_r^\lambda(z_0) \cap \Omega_T} |\nabla u|^\theta dz + \left(\lambda^{2-p} \iint_{Q_r^\lambda(z_0) \cap \Omega_T} |\nabla u|^{p-1} + |F|^{p-1} dz \right)^\theta \\ &\quad + \lambda^\theta \left(\lambda^{1-p} r \iint_{Q_r^\lambda(z_0) \cap \Omega_T} |f| dz \right)^\theta + \left(\int_{B_r(x_0)} |\nabla u_0| dx \right)^\theta. \end{aligned}$$

Proof. Let $0 \leq \eta \in C_c^\infty(B_r(x_0))$ such that $\int_{B_r(x_0)} \eta dz = 1$ and $\|\eta\|_\infty + r\|\nabla \eta\|_\infty \leq C(n)$. For a.e. t , we denote weighted integral averages of u by

$$(u)_\eta(t) := \int_{B_r(x_0)} u(x, t) \eta(x) dx.$$

The triangle inequality gives

$$\begin{aligned} \iint_{Q_r^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_r(x_0)}|}{r} \right]^\theta dz &\lesssim_{(p)} r^{-\theta} \left| \int_{I_r^\lambda(t_0) \cap (0, T)} (u)_\eta(t) - (u)_{B_r(x_0)}(t) dt \right|^\theta + r^{-\theta} \iint_{Q_r^\lambda(z_0) \cap \Omega_T} |u - (u)_\eta(t)|^\theta dz \\ &\quad + r^{-\theta} \left| \int_{I_r^\lambda(t_0) \cap (0, T)} (u)_{B_r(x_0)}(t) - (u_0)_{B_r(x_0)} dt \right|^\theta =: \text{I} + \text{II} + \text{III}. \end{aligned}$$

Estimate of I: Applying Poincaré's inequality in space direction, we obtain

$$\begin{aligned} \text{I} &\leq r^{-\theta} \left| \int_{I_r^\lambda(t_0) \cap (0, T)} \int_{B_r(x_0)} |u - (u)_{B_r(x_0)}(t)| \eta dx dt \right|^\theta \\ &\leq r^{-\theta} \left| \int_{I_r^\lambda(t_0) \cap (0, T)} \int_{B_r(x_0)} |u - (u)_{B_r(x_0)}(t)| dx dt \right|^\theta \leq \iint_{Q_r^\lambda(z_0) \cap \Omega_T} |\nabla u|^\theta dz. \end{aligned}$$

Estimate of II: Similarly, we have

$$\text{II} \leq r^{-\theta} \iint_{Q_r^\lambda(z_0) \cap \Omega_T} |u - (u)_{B_r(x_0)}(t)|^\theta dz \lesssim \iint_{Q_r^\lambda(z_0) \cap \Omega_T} |\nabla u|^\theta dz.$$

Estimate of III: Again, the triangle inequality implies

$$\begin{aligned} \text{III} &\leq r^{-\theta} \left| \int_{I_r^\lambda(t_0) \cap (0, T)} (u)_{B_r(x_0)}(t) - (u)_\eta(t) dt \right|^\theta + r^{-\theta} \left| \int_{I_r^\lambda(t_0) \cap (0, T)} (u)_\eta(t) - (u_0)_\eta dt \right|^\theta + r^{-\theta} |(u_0)_{B_r(x_0)} - (u_0)_\eta|^\theta \\ &\leq \iint_{Q_r^\lambda(z_0) \cap \Omega_T} |\nabla u|^\theta dz + r^{-\theta} \left| \int_{I_r^\lambda(t_0) \cap (0, T)} (u)_\eta(t) - (u_0)_\eta dt \right|^\theta + \left(\int_{B_r(x_0)} |\nabla u_0| dx \right)^\theta. \end{aligned}$$

To estimate the second term, we test η to (2.1) in $B_r(x_0) \times (t_1, t_2) \subset Q_r^\lambda(z_0) \cap \Omega_T$. Then (2.3) implies that

$$\begin{aligned} |(u)_\eta(t_1) - (u)_\eta(t_2)| &\lesssim_{(\Lambda_1)} \int_{t_1}^{t_2} \int_{B_r(x_0)} |\nabla u|^{p-1} |\nabla \eta| + |F|^{p-1} |\nabla \eta| + |f| |\eta| dx dt \\ &\stackrel{(3.1)}{\leq} \lambda^{2-p} r \iint_{Q_r^\lambda(z_0) \cap \Omega_T} |\nabla u|^{p-1} + |F|^{p-1} dz + \lambda^{2-p} r^2 \iint_{Q_r^\lambda(z_0) \cap \Omega_T} |f| dz. \end{aligned}$$

Using (2.2), we obtain

$$\sup_{t \in I_r^\lambda(t_0)} |(u)_\eta(t) - (u_0)_\eta| \leq \lambda^{2-p} r \iint_{Q_r^\lambda(z_0) \cap \Omega_T} |\nabla u|^{p-1} + |F|^{p-1} dz + \lambda^{2-p} r^2 \iint_{Q_r^\lambda(z_0) \cap \Omega_T} |f| dz.$$

Therefore, we get

$$\begin{aligned} \text{III} \lesssim_{(n,N,p,\Lambda_1)} & \iint_{Q_r^\lambda(z_0) \cap \Omega_T} |\nabla u|^\theta dz + \left(\lambda^{2-p} \iint_{Q_r^\lambda(z_0) \cap \Omega_T} |\nabla u|^{p-1} + |F|^{p-1} dz \right)^\theta \\ & + \lambda^\theta \left(\lambda^{1-p} r \iint_{Q_r^\lambda(z_0) \cap \Omega_T} |f| dz \right)^\theta + \left(\int_{B_r(x_0)} |\nabla u_0| dx \right)^\theta. \end{aligned}$$

This completes the proof. $\square$

3.2 Some crucial estimates

The purpose of this subsection is to refine the estimate u in $C(I_r^\lambda(t_0) \cap (0, T); L^2(B_r(x_0), \mathbb{R}^N))$. To this end, we first estimate the right-hand side of the inequality in Lemma 3.2 using Lemma 3.3 and assumption (3.2), and then return to the left-hand side of the inequality.

Lemma 3.4. *Let u be a weak solution of (2.1). Suppose $B_{4\rho}(x_0) \subset \Omega$ and $0 < t_0$. Also, assume (3.2) for some $\lambda \geq 1$. Then, for all $\theta \in [1, p]$, there holds*

$$\iint_{Q_{4\rho}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{4\rho}(x_0)}|}{4\rho} \right]^\theta dz \lesssim_{(n,N,p,\Lambda_0,\Lambda_1)} \lambda^\theta.$$

Proof. Applying Lemma 3.3 and Hölder's inequality, we find

$$\begin{aligned} \iint_{Q_{4\rho}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{4\rho}(x_0)}|}{4\rho} \right]^\theta dz & \leq \left(\iint_{Q_{4\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^p dz \right)^{\frac{\theta}{p}} + \left[\lambda^{2-p} \left(\iint_{Q_{4\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^p + |F|^p dz \right)^{\frac{p-1}{p}} \right]^\theta \\ & + \lambda^\theta \left(\lambda^{a'(1-p)} \iint_{Q_{4\rho}^\lambda(z_0) \cap \Omega_T} (4\rho)^{a'} |f|^{a'} dz \right)^{\frac{\theta}{a'}} + \left(\int_{B_{4\rho}(x_0)} |\nabla u_0|^p dx \right)^{\frac{\theta}{p}}. \end{aligned}$$

Note that

$$\lambda^{a'(1-p)} \iint_{Q_{4\rho}^\lambda(z_0) \cap \Omega_T} (4\rho)^{a'} |f|^{a'} dz = \lambda^{a'(1-\frac{p}{a})-p} \iint_{Q_{4\rho}^\lambda(z_0) \cap \Omega_T} (4\rho)^{a'} |f|^{a'} dz \stackrel{(3.2)}{\leq} 1. \quad (3.4)$$

Therefore, making use of (3.2), this completes the proof. $\square$

Lemma 3.5. *Under the assumptions and the conclusion in Lemma 3.4, we further have*

$$\sup_{I_{2\rho}^\lambda(t_0) \cap (0, T)} \int_{B_{2\rho}(x_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^2 dx \lesssim_{(n,N,p,\Lambda_0,\Lambda_1)} \lambda^2.$$

Proof. Take $2\rho \leq \rho_a < \rho_b \leq 4\rho$ in Lemma 3.2 to get

$$\begin{aligned} & \lambda^{p-2} \sup_{I_{\rho_a}^\lambda(t_0) \cap (0, T)} \int_{B_{\rho_a}(x_0)} \left[\frac{|u - (u_0)_{B_{\rho_a}(x_0)}|}{\rho_a} \right]^2 dx \\ & \leq \iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b - \rho_a} \right]^p dz + \lambda^{p-2} \iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b - \rho_a} \right]^2 dz \\ & \quad + \iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} |f| |u - (u_0)_{B_{\rho_b}(x_0)}| dz + \lambda^{p-2} \left(\int_{B_{\rho_b}(x_0)} |\nabla u_0|^p dz \right)^{\frac{2}{p}} + \iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} |F|^p dz \\ & =: \text{I} + \text{II} + \text{III} + \text{IV} + \text{V}. \end{aligned}$$

Estimate of I: Note that, under the restriction $2\rho \leq \rho_a < \rho_b \leq 4\rho$, we see that

$$\begin{aligned} \left(\frac{\rho_b - \rho_a}{\rho_b}\right)^p \text{I} &\leq \iint_{Q_{\rho_b}^\lambda(z_0)} \left[\frac{|u - (u_0)_{B_{4\rho}(x_0)}|}{\rho_b}\right]^p dz + \left[\frac{|(u_0)_{B_{\rho_b}(x_0)} - (u_0)_{B_{4\rho}(x_0)}|}{\rho_b}\right]^p \\ &\leq \iint_{Q_{4\rho}^\lambda(z_0)} \left[\frac{|u - (u_0)_{B_{4\rho}(x_0)}|}{4\rho}\right]^p dz + \int_{B_{4\rho}} \left[\frac{|u_0 - (u_0)_{B_{4\rho}}|}{4\rho}\right]^p dx \stackrel{(a)}{\leq} \lambda^p. \end{aligned} \quad (3.5)$$

Here, to obtain (a), we used Lemma 3.4 and Poincaré's inequality with (3.2) for initial data.

Estimate of II: There holds that

$$\begin{aligned} \text{II} &= \lambda^{p-2} \left(\frac{\rho_b}{\rho_b - \rho_a}\right)^2 \int_{I_{\rho_b}^\lambda(t_0) \cap (0, T)} \left(\int_{B_{\rho_b}(x_0)} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b}\right]^2 dx \right)^{\frac{1}{2}} \left(\int_{B_{\rho_b}(x_0)} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b}\right]^2 dx \right)^{\frac{1}{2}} dt \\ &\stackrel{(a)}{\leq} \lambda^{\frac{p-2}{2}} \left(\frac{\rho_b}{\rho_b - \rho_a}\right)^2 \left(\iint_{Q_{\rho_b}^\lambda(z_0)} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b}\right]^p + |\nabla u|^p dz \right)^{\frac{1}{p}} \\ &\quad \times \left(\lambda^{p-2} \sup_{I_{\rho_b}^\lambda(t_0) \cap (0, T)} \int_{B_{\rho_b}(x_0)} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b}\right]^2 dx \right)^{\frac{1}{2}} \\ &\stackrel{(b)}{\leq} \lambda^{\frac{p}{2}} \left(\frac{\rho_b}{\rho_b - \rho_a}\right)^2 \left(\lambda^{p-2} \sup_{I_{\rho_b}^\lambda(t_0) \cap (0, T)} \int_{B_{\rho_b}(x_0)} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b}\right]^2 dx \right)^{\frac{1}{2}}, \end{aligned}$$

where, to obtain (a), we used Sobolev's inequality with respect to space direction and then Hölder's inequality with respect to the time integral, and to obtain (b), we used (3.5) and (3.2).

Applying Young's inequality, for any $\gamma \in (0, 1)$, there holds

$$\text{II} \leq \gamma \left(\lambda^{p-2} \sup_{I_{\rho_b}^\lambda(t_0) \cap (0, T)} \int_{B_{\rho_b}(x_0)} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b}\right]^2 dx \right) + C(\gamma) \left(\frac{\rho_b}{\rho_b - \rho_a}\right)^4 \lambda^p.$$

Estimate of III: After applying Hölder's inequality, we get

$$\begin{aligned} \text{III} &\leq \int_{I_{\rho_b}^\lambda(t_0) \cap (0, T)} \left(\int_{B_{\rho_b}(x_0)} \rho_b^{\alpha'} |f|^{\alpha'} dx \right)^{\frac{1}{\alpha'}} \left(\int_{B_{\rho_b}(x_0)} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b}\right]^\alpha dx \right)^{\frac{1}{\alpha}} dt \\ &\stackrel{(a)}{\leq} \int_{I_{\rho_b}^\lambda(t_0) \cap (0, T)} \left(\int_{B_{\rho_b}(x_0)} \rho_b^{\alpha'} |f|^{\alpha'} dx \right)^{\frac{1}{\alpha'}} \left(\int_{B_{\rho_b}(x_0)} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b}\right]^{\delta p} + |\nabla u|^{\delta p} dx \right)^{\frac{1}{\alpha}} \\ &\quad \times \left(\int_{B_{\rho_b}(x_0)} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b}\right]^2 dx \right)^{\frac{1}{n+2}} dt \\ &\stackrel{(b)}{\leq} \left(\iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} \rho_b^{\alpha'} |f|^{\alpha'} dz \right)^{\frac{1}{\alpha'}} \left(\iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b}\right]^{\delta p} + |\nabla u|^{\delta p} dz \right)^{\frac{1}{\alpha}} \\ &\quad \times \left(\sup_{I_{\rho_b}^\lambda(t_0) \cap (0, T)} \int_{B_{\rho_b}(x_0)} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b}\right]^2 dx \right)^{\frac{1}{n+2}}. \end{aligned} \quad (3.6)$$

Here, to obtain (a), we used Lemma 2.7 with $\sigma = \alpha$, $q = \delta p$, $r = 2$ and $\vartheta = \frac{n}{n+2}$. To obtain (b), we used Hölder's inequality with respect to the time integral.

Also, the restriction $\rho \leq \rho_b \leq 4\rho$ and (3.1) imply

$$\left(\iint_{Q_{\rho_b}^\lambda(z_0) \cap \Omega_T} \rho_b^{\alpha'} |f|^{\alpha'} dz \right)^{\frac{1}{\alpha'}} \leq \left(\iint_{Q_{4\rho}^\lambda(z_0) \cap \Omega_T} (4\rho)^{\alpha'} |f|^{\alpha'} dz \right)^{\frac{1}{\alpha'}} \stackrel{(3.2)}{\leq} \lambda^{\frac{p}{\alpha'} - p(\frac{1}{p} - \frac{1}{\alpha})} = \lambda^{p(\frac{1}{\alpha'} + \frac{1}{\alpha}) - 1} = \lambda^{p-1}.$$

Thus, along with (3.5) and (3.2), (3.6) becomes

$$\begin{aligned} \text{III} &\leq \lambda^{p-1+\frac{n}{n+2}} \left(\sup_{I_{\rho_b}^\lambda(t_0) \cap (0, T)} \int_{B_{\rho_b}^\lambda(x_0)} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b} \right]^2 dx \right)^{\frac{1}{n+2}} \\ &= \lambda^{p(1-\frac{1}{n+2})} \left(\lambda^{p-2} \sup_{I_{\rho_b}^\lambda(t_0) \cap (0, T)} \int_{B_{\rho_b}^\lambda(x_0)} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b} \right]^2 dx \right)^{\frac{1}{n+2}}, \end{aligned}$$

and Young's inequality gives that

$$\text{III} \leq \gamma \left(\lambda^{p-2} \sup_{I_{\rho_b}^\lambda(t_0) \cap (0, T)} \int_{B_{\rho_b}^\lambda(x_0)} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b} \right]^2 dx \right) + C(\gamma) \lambda^p,$$

where $\gamma \in (0, 1)$.

Estimate of IV and V:

$$\text{IV} + \text{V} \stackrel{(3.2)}{\leq} \lambda^p.$$

Combining estimates and taking $\gamma = \gamma(n, N, p, \Lambda_0, \Lambda_1) \in (0, 1)$ small enough, there holds

$$\begin{aligned} \sup_{I_{\rho_a}^\lambda(t_0) \cap (0, T)} \int_{B_{\rho_a}(x_0)} \left[\frac{|u - (u_0)_{B_{\rho_a}(x_0)}|}{\rho_a} \right]^2 dx &\leq \frac{1}{2} \sup_{I_{\rho_b}^\lambda(t_0) \cap (0, T)} \int_{B_{\rho_b}^\lambda(x_0)} \left[\frac{|u - (u_0)_{B_{\rho_b}(x_0)}|}{\rho_b} \right]^2 dx \\ &\quad + C \left(\frac{\rho_b}{\rho_b - \rho_a} \right)^4 \lambda^2 + C \left(\frac{\rho_b}{\rho_b - \rho_a} \right)^p \lambda^2, \end{aligned}$$

where $C = C(n, N, p, \Lambda_0, \Lambda_1)$. Hence, from Lemma 2.8, the conclusion follows. $\square$

We will not use following corollary in further estimates for our purpose, but it is worthwhile observing the estimate of the corollary.

Corollary 3.6. *Under the assumptions and the conclusion in Lemma 3.5, we derive*

$$\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^{\frac{p(n+2)}{n}} dx \lesssim_{(n, N, p, \Lambda_0, \Lambda_1)} \lambda^{\frac{p(n+2)}{n}}.$$

Proof. Constants $\sigma = \frac{p(n+2)}{n}$, $q = p$, $r = 2$ and $\vartheta = \frac{n}{n+2} \in (0, 1)$ satisfy the condition in Lemma 2.7. Therefore, we obtain

$$\begin{aligned} &\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^{\frac{p(n+2)}{n}} dz \\ &\leq \int_{I_{2\rho}^\lambda(t_0) \cap (0, T)} \left(\int_{B_{2\rho}(x_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^p + |\nabla u|^p dx \right) \left(\int_{B_{2\rho}(x_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^2 dx \right)^{\frac{p}{n}} dt \\ &\leq \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^p + |\nabla u|^p dz \right) \left(\sup_{I_{2\rho}^\lambda(t_0) \cap (0, T)} \int_{B_{2\rho}(x_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^2 dx \right)^{\frac{p}{n}} \stackrel{(a)}{\lesssim} \lambda^{\frac{p(n+2)}{n}}, \end{aligned}$$

where, to obtain (a), we used Lemma 3.4, (3.2), (3.5) and Lemma 3.5. $\square$

3.3 Reverse Hölder inequality in intrinsic cylinders

We are now ready to obtain a reverse Hölder inequality under assumptions (3.2) and (3.3). We shall again estimate the right-hand side of Lemma 3.2 with $\rho_a = \rho$ and $\rho_b = 2\rho$ using Lemma 3.5. Let us fix a constant

$$q := \max \left\{ \frac{2n}{n+2}, p-1, \frac{np}{n+2}, \delta p \right\}. \quad (3.7)$$

Lemma 3.7. Let u be a weak solution of (2.1). Suppose $B_{4\rho}(x_0) \subset \Omega$ and $0 < t_0$. Also, assume (3.2) for some $\lambda \geq 1$. Then, for any $\gamma \in (0, 1)$, there holds

$$\begin{aligned} \lambda^{p-2} \iint_{Q_{2\rho}^\lambda(z_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^2 dz &\lesssim_{(n, N, p, \Lambda_0, \Lambda_1, \delta)} \gamma \lambda^p + C(\gamma) \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q dz \right)^{\frac{p}{q}} + C(\gamma) \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |F|^p dz \\ &\quad + C(\gamma) \lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} (2\rho)^{\alpha'} |f|^{\alpha'} dz + C(\gamma) \int_{B_{2\rho}(x_0)} |\nabla u_0|^p dx. \end{aligned}$$

Proof. We apply Lemma 2.7 with $\sigma = 2$, q as defined in (3.7), $r = 2$, and any $\vartheta \in (0, 1)$ is admissible since

$$-\frac{n}{2} \leq \vartheta \left(1 - \frac{n}{q} \right) - (1 - \vartheta) \frac{n}{2} \iff \frac{2n}{n+2} \leq q.$$

Let us take $\vartheta \in (0, 1)$ such that

$$0 < \vartheta < \min \left\{ \frac{q}{2}, \frac{p}{2(p-1)}, \frac{1}{2} \right\}.$$

Then we get

$$\begin{aligned} \iint_{Q_{2\rho}^\lambda(z_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^2 dz &\leq \int_{I_{2\rho}^\lambda(t_0)} \left(\int_{B_{2\rho}(x_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^q + |\nabla u|^q dx \right)^{\frac{2\vartheta}{q}} \left(\int_{B_{2\rho}(x_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^2 dx \right)^{1-\vartheta} dt \\ &\stackrel{(a)}{\leq} \left(\iint_{Q_{2\rho}^\lambda(z_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^q + |\nabla u|^q dz \right)^{\frac{2\vartheta}{q}} \left(\sup_{I_{2\rho}^\lambda(t_0) \cap (0, T)} \int_{B_{2\rho}(x_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^2 dx \right)^{1-\vartheta} \\ &\stackrel{(b)}{\leq} \lambda^{2(1-\vartheta)} \left(\iint_{Q_{2\rho}^\lambda(z_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^q + |\nabla u|^q dz \right)^{\frac{2\vartheta}{q}}, \end{aligned}$$

where, to obtain (a), we used Hölder's inequality along with the fact that $\frac{2\vartheta}{q} \leq 1$, and to obtain (b), we used Lemma 3.5.

Thus, applying Lemma 3.3 with $\theta = q$ to estimate the second term, we obtain

$$\begin{aligned} \lambda^{p-2} \iint_{Q_{2\rho}^\lambda(z_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^2 dz &\leq \lambda^{p-2\vartheta} \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q dz \right)^{\frac{2\vartheta}{q}} + \lambda^{p-2\vartheta} \left(\lambda^{2-p} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^{p-1} + |F|^{p-1} dz \right)^{2\vartheta} \\ &\quad + \lambda^p \left(\lambda^{1-p} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} (2\rho) |f| dz \right)^{2\vartheta} + \lambda^{p-2\vartheta} \left(\int_{B_{2\rho}(x_0)} |\nabla u_0| dx \right)^{2\vartheta} =: \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned}$$

Estimate of I: Applying Young's inequality along with the fact $\frac{2\vartheta}{p} < \frac{2\vartheta}{q} \leq 1$, we get

$$\text{I} \leq \gamma \lambda^p + C(\gamma) \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q dz \right)^{\frac{p}{q}}.$$

Estimate of II: Similarly, since $\frac{2\vartheta(p-1)}{p} < 1$, Hölder's inequality and Young's inequality give

$$\text{II} \leq \lambda^{p-2\vartheta(p-1)} \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q + |F|^q dz \right)^{\frac{p}{q} \frac{2\vartheta(p-1)}{p}}$$

$$\begin{aligned}
&\leq \gamma\lambda^p + C(\gamma)\left(\iint_{Q_{2\rho}^\lambda(z_0)\cap\Omega_T} |\nabla u|^q dz\right)^{\frac{p}{q}} + C(\gamma)\left(\iint_{Q_{2\rho}^\lambda(z_0)\cap\Omega_T} |F|^q dz\right)^{\frac{p}{q}} \\
&\leq \gamma\lambda^p + C(\gamma)\left(\iint_{Q_{2\rho}^\lambda(z_0)\cap\Omega_T} |\nabla u|^q dz\right)^{\frac{p}{q}} + C(\gamma)\iint_{Q_{2\rho}^\lambda(z_0)\cap\Omega_T} |F|^p dz.
\end{aligned}$$

Estimate of III: Since $\frac{2\vartheta}{\alpha'} < 1$, Young's inequality gives

$$\begin{aligned}
\text{III} &\leq \lambda^p \left(\lambda^{\alpha'(1-p)} \iint_{Q_{2\rho}^\lambda(z_0)\cap\Omega_T} (2\rho)^{\alpha'} |f|^{\alpha'} dz \right)^{\frac{2\vartheta}{\alpha'}} \\
&= \lambda^{p(1-\frac{2\vartheta}{\alpha'})} \left(\lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2\rho}^\lambda(z_0)\cap\Omega_T} (2\rho)^{\alpha'} |f|^{\alpha'} dz \right)^{\frac{2\vartheta}{\alpha'}} \\
&\leq \gamma\lambda^p + C(\gamma)\lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2\rho}^\lambda(z_0)\cap\Omega_T} (2\rho)^{\alpha'} |f|^{\alpha'} dz.
\end{aligned}$$

Estimate of IV: Since $\frac{2\vartheta}{p} < 1$, Hölder's inequality and Young's inequality give

$$\text{IV} \leq \lambda^{p-2\vartheta} \left(\int_{B_{2\rho}(x_0)} |\nabla u_0|^p dx \right)^{\frac{2\vartheta}{p}} \leq \gamma\lambda^p + C(\gamma) \int_{B_{2\rho}(x_0)} |\nabla u_0|^p dx.$$

Therefore, combining all the estimates, the proof is completed. $\square$

Lemma 3.8. Under the assumptions and the conclusion in Lemma 3.7, we further have

$$\begin{aligned}
\iint_{Q_{2\rho}^\lambda(z_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^p dz &\lesssim_{(n,N,p,\Lambda_0,\Lambda_1,\delta)} \gamma\lambda^p + C(\gamma)\left(\iint_{Q_{2\rho}^\lambda(z_0)\cap\Omega_T} |\nabla u|^q dz\right)^{\frac{p}{q}} + C(\gamma)\iint_{Q_{2\rho}^\lambda(z_0)\cap\Omega_T} |F|^p dz \\
&\quad + C(\gamma)\lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2\rho}^\lambda(z_0)\cap\Omega_T} (2\rho)^{\alpha'} |f|^{\alpha'} dz + C(\gamma) \int_{B_{2\rho}(x_0)} |\nabla u_0|^p dx.
\end{aligned}$$

Proof. We apply Lemma 2.7 with $\sigma = p$, q as defined in (3.7), $r = 2$ and $\vartheta = \frac{q}{p}$. Note that (3.7) implies

$$\frac{np}{n+2} \leq q \iff -\frac{n}{p} \leq \frac{q}{p} \left(1 - \frac{n}{q}\right) - \left(1 - \frac{q}{p}\right) \frac{n}{2}.$$

Therefore, we have

$$\begin{aligned}
&\iint_{Q_{2\rho}^\lambda(z_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^p dz \\
&\leq \int_{I_{2\rho}^\lambda(t_0)} \left(\int_{B_{2\rho}(x_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^q + |\nabla u|^q dx \right)^{\frac{p\vartheta}{q}} \left(\int_{B_{2\rho}(x_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^2 dx \right)^{\frac{p(1-\vartheta)}{2}} dt \\
&\stackrel{(a)}{\leq} \iint_{Q_{2\rho}^\lambda(z_0)\cap\Omega_T} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^q + |\nabla u|^q dz \left(\sup_{I_{2\rho}^\lambda(t_0)\cap(0,T)} \int_{B_{2\rho}(x_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^2 dx \right)^{\frac{p(1-\vartheta)}{2}} \\
&\stackrel{(b)}{\leq} \lambda^{p(1-\vartheta)} \iint_{Q_{2\rho}^\lambda(z_0)\cap\Omega_T} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^q + |\nabla u|^q dz \\
&= \lambda^{p-q} \iint_{Q_{2\rho}^\lambda(z_0)\cap\Omega_T} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^q + |\nabla u|^q dz.
\end{aligned}$$

Here, to obtain (a), we used $\frac{p\theta}{q} = 1$, and to obtain (b), we used Lemma 3.5. Now, we apply Lemma 3.3 with $\theta = q$ to the first term on the right-hand side to get

$$\begin{aligned} \iint_{Q_{2\rho}^\lambda(z_0)} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^p dz &\leq \lambda^{p-q} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q dz + \lambda^{p-q} \left(\lambda^{2-p} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^{p-1} + |F|^{p-1} dz \right)^q \\ &\quad + \lambda^p \left(\lambda^{1-p} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} (2\rho)|f| dz \right)^q + \lambda^{p-q} \left(\int_{B_{2\rho}(x_0)} |\nabla u_0| dx \right)^q \\ &=: \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned}$$

Estimate of I: Applying Young's inequality, there holds

$$I \leq \gamma \lambda^p + C(\gamma) \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q dz \right)^{\frac{p}{q}}.$$

Estimate of II: Applying Hölder's inequality and Young's inequality, we get

$$\begin{aligned} \text{II} &\leq \lambda^{p+q-pq} \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q + |F|^q dz \right)^{\frac{p}{q} \frac{q(p-1)}{p}} \\ &\leq \lambda^{p+q-pq} \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q + |F|^q dz \right)^{\frac{p}{q} \frac{p-1}{p}} \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^p + |F|^p dz \right)^{\frac{(q-1)(p-1)}{p}} \\ &\stackrel{(3.2)}{\leq} \lambda \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q + |F|^q dz \right)^{\frac{p}{q} \frac{p-1}{p}} \\ &\leq \gamma \lambda^p + C(\gamma) \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q dz \right)^{\frac{p}{q}} + C(\gamma) \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |F|^p dz. \end{aligned}$$

Estimate of III: We observe

$$\begin{aligned} \text{III} &\leq \lambda^p \left(\lambda^{a'(1-p)} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} (2\rho)^{a'} |f|^{a'} dz \right)^{\frac{q}{a'}} \\ &= \lambda^p \left(\lambda^{a'(1-p)} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} (2\rho)^{a'} |f|^{a'} dz \right)^{\frac{1}{a'}} \left(\lambda^{a'(1-p)} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} (2\rho)^{a'} |f|^{a'} dz \right)^{\frac{q-1}{a'}} \\ &\stackrel{(3.4)}{\leq} \lambda^p \left(\lambda^{a'(1-p)} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} (2\rho)^{a'} |f|^{a'} dz \right)^{\frac{1}{a'}} \\ &= \lambda^{p(1-\frac{1}{a'})} \left(\lambda^{a'(1-\frac{p}{a})} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} (2\rho)^{a'} |f|^{a'} dz \right)^{\frac{1}{a'}}. \end{aligned}$$

Therefore, Young's inequality gives

$$\text{III} \leq \gamma \lambda^p + C(\gamma) \lambda^{a'(1-\frac{p}{a})} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} (2\rho)^{a'} |f|^{a'} dz.$$

Estimate of IV: Hölder's inequality and Young's inequality imply

$$\text{IV} \leq \lambda^{p-q} \left(\int_{B_{2\rho}(x_0)} |\nabla u_0|^p dx \right)^{\frac{q}{p}} \leq \gamma \lambda^p + C(\gamma) \int_{B_{2\rho}(x_0)} |\nabla u_0|^p dx.$$

We combine all the estimates to complete the proof. $\square$

Lemma 3.9. *Under the assumptions and the conclusion in Lemma 3.7, we further have*

$$\begin{aligned} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |f||u - (u_0)| \, dz &\lesssim_{(n,N,p,\Lambda_0,\Lambda_1,\delta)} \gamma \lambda^p + C(\gamma) \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q \, dz \right)^{\frac{p}{q}} + C(\gamma) \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |F|^p \, dz \\ &\quad + C(\gamma) \lambda^{a'(1-\frac{p}{a})} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} (2\rho)^{a'} |f|^{a'} \, dz + C(\gamma) \int_{B_{2\rho}(x_0)} |\nabla u_0|^p \, dx. \end{aligned}$$

Proof. Apply Lemma 2.7 as in (3.6) and Lemma 3.5 to get

$$\begin{aligned} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |f||u - (u_0)| \, dz &\lesssim \lambda^{\frac{2}{n+2}} \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} (2\rho)^{a'} |f|^{a'} \, dz \right)^{\frac{1}{a'}} \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^{\delta p} + |\nabla u|^{\delta p} \, dz \right)^{\frac{1}{a}} \\ &\stackrel{(3.4)}{\leq} \lambda^{p-\frac{n}{n+2}} \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^{\delta p} + |\nabla u|^{\delta p} \, dz \right)^{\frac{1}{a}} \\ &\leq \lambda^{p-\frac{n}{n+2}} \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^q + |\nabla u|^q \, dz \right)^{\frac{n}{q(n+2)}}. \end{aligned}$$

Using Lemma 3.3 with $\theta = q$ to the first term on the right-hand side, we have

$$\begin{aligned} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |f||u - (u_0)| \, dz &\lesssim \lambda^{p-\frac{n}{n+2}} \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q \, dz \right)^{\frac{p}{q} \frac{n}{p(n+2)}} + \lambda^{p-\frac{n}{n+2}} \left(\lambda^{2-p} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^{p-1} + |F|^{p-1} \, dz \right)^{\frac{p}{q} \frac{qn}{p(n+2)}} \\ &\quad + \lambda^p \left(\lambda^{1-p} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} (2\rho) |f| \, dz \right)^{\frac{n}{n+2}} + \lambda^{p-\frac{n}{n+2}} \left(\int_{B_{2\rho}(x_0)} |\nabla u_0| \, dx \right)^{\frac{n}{n+2}} =: \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned}$$

Estimate of I: Applying Young's inequality, we obtain

$$\text{I} \leq \gamma \lambda^p + C(\gamma) \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q \, dz \right)^{\frac{p}{q}}.$$

Estimate of II: Note that $\frac{(p-1)n}{p(n+2)} < 1$. Apply Hölder's inequality and Young's inequality to get

$$\begin{aligned} \text{II} &\leq \lambda^{p+\frac{n(1-p)}{n+2}} \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q + |F|^q \, dz \right)^{\frac{p}{q} \frac{(p-1)n}{p(n+2)}} \\ &\leq \gamma \lambda^p + C(\gamma) \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q \, dz \right)^{\frac{p}{q}} + C(\gamma) \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |F|^p \, dz. \end{aligned}$$

Estimate of III: Applying Hölder's inequality and Young's inequality, we have

$$\begin{aligned} \text{III} &\leq \lambda^{p(1-\frac{n}{a'(n+2)})} \left(\lambda^{a'(1-\frac{p}{a})} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} (2\rho)^{a'} |f|^{a'} \, dz \right)^{\frac{n}{a'(n+2)}} \\ &\leq \gamma \lambda^p + C(\gamma) \lambda^{a'(1-\frac{p}{a})} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} (2\rho)^{a'} |f|^{a'} \, dz. \end{aligned}$$

Estimate of IV: Again, Hölder's inequality and Young's inequality give

$$\text{IV} \leq \gamma \lambda^p + C(\gamma) \int_{B_{2\rho}(x_0)} |\nabla u_0|^p \, dx.$$

The proof follows. $\square$

Lemma 3.10. *Let u be a weak solution of (2.1). Suppose $B_{4\rho}(x_0) \subset \Omega$ and $0 < t_0$. Also, assume (3.2) and (3.3) for some $\lambda \geq 1$. Then there holds*

$$\begin{aligned} \iint_{Q_\rho^\lambda(z_0) \cap \Omega_T} |\nabla u|^p dz &\lesssim_{(n, N, p, \Lambda_0, \Lambda_1, \delta)} \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q dz \right)^{\frac{p}{q}} + \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |F|^p dz \\ &\quad + \int_{B_{2\rho}(x_0)} |\nabla u_0|^p dx + \lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} (2\rho)^{\alpha'} |f|^{\alpha'} dz. \end{aligned}$$

Proof. From Lemma 3.2 with $\rho_a = \rho$ and $\rho_b = 2\rho$, there holds

$$\begin{aligned} \iint_{Q_\rho^\lambda(z_0) \cap \Omega_T} |\nabla u|^p dz &\leq \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^p dz + \lambda^{p-2} \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{2\rho}(x_0)}|}{2\rho} \right]^2 dz \\ &\quad + \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |f| |u - (u_0)_{B_{2\rho}(x_0)}| dz + \lambda^{p-1} \left(\int_{B_{2\rho}(x_0)} |\nabla u_0|^p dz \right)^{\frac{1}{p}} \\ &\quad + \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |F|^p dz \\ &=: \text{I} + \text{II} + \text{III} + \text{IV} + \iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |F|^p dz. \end{aligned}$$

We apply Lemma 3.7, Lemma 3.8 and Lemma 3.9 to I, II and III. Use Young's inequality to estimate IV. Then there holds

$$\begin{aligned} \lambda^p &\stackrel{(3.3)}{\leq} \iint_{Q_\rho^\lambda(z_0) \cap \Omega_T} |\nabla u|^p dz + \iint_{Q_\rho^\lambda(z_0) \cap \Omega_T} |F|^p dz + \int_{B_\rho(x_0)} |\nabla u_0|^p dx + \lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_\rho^\lambda(z_0) \cap \Omega_T} (2\rho)^{\alpha'} |f|^{\alpha'} dz \\ &\leq \gamma \lambda^p + \left(\iint_{Q_{2\rho}^\lambda(z_0) \cap \Omega_T} |\nabla u|^q dz \right)^{\frac{p}{q}} + \iint_{Q_\rho^\lambda(z_0) \cap \Omega_T} |F|^p dz + \int_{B_\rho(x_0)} |\nabla u_0|^p dx + \lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_\rho^\lambda(z_0) \cap \Omega_T} (2\rho)^{\alpha'} |f|^{\alpha'} dz \end{aligned}$$

Taking $\gamma = \gamma(n, N, p, \Lambda_0, \Lambda_1, \delta)$ small enough, we finish the proof. $\square$

4 Proof of Theorem 2.5

In this section, we prove the main results. First of all, we shall find intrinsic cylinders such that (3.2) and (3.3) hold.

Definition 4.1. Let $z_0 \in \Omega_T$, $r \leq r_1 < r_2 \leq 2r$ and $\mathfrak{z} = (x, t) \in Q_{r_1}(z_0)$, and let d and α be defined in Definition 2.2.

$$\begin{aligned} E_\lambda &:= \{z \in Q_{r_1}(z_0) \cap \Omega_T : |\nabla u|^p > \lambda^p\}, \\ \mathbb{B} &:= \left[2 \left(\frac{20r}{r_2 - r_1} \right) \right]^{(n+2)\frac{d}{p}} + \left[2 \left(\frac{20r}{r_2 - r_1} \right) \right]^{(n+2)\frac{d\alpha}{p}} + \left(\frac{20r}{r_2 - r_1} \right)^{\frac{n}{p}} + \left(\frac{20r}{r_2 - r_1} \right)^{\frac{2n}{p(n+2)-2n}}, \\ G(Q_\rho^\lambda(\mathfrak{z})) &:= \iint_{Q_\rho^\lambda(\mathfrak{z}) \cap \Omega_T} |\nabla u|^p dz + \iint_{Q_\rho^\lambda(\mathfrak{z}) \cap \Omega_T} |F|^p dz + \int_{B_\rho(x)} |\nabla u_0|^p dx + \lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_\rho^\lambda(\mathfrak{z}) \cap \Omega_T} \rho^{\alpha'} |f|^{\alpha'} dz. \end{aligned}$$

Lemma 4.2. Let λ_0 be defined in Definition 2.4. Then, for $\lambda > 3\mathbb{B}\lambda_0$ and $\mathfrak{z} \in E_\lambda$, there exists $\rho_\mathfrak{z} \in (0, \frac{r_2-r_1}{10})$ such that

$$G(Q_{\rho_\mathfrak{z}}^\lambda(\mathfrak{z})) = \lambda^p \quad \text{and} \quad G(Q_\rho^\lambda(\mathfrak{z})) < \lambda^p \quad \text{for all } \rho \in (\rho_\mathfrak{z}, r_2 - r_1). \quad (4.1)$$

Proof. Due to intrinsic geometry, we split the proof into two cases.

Case $p \geq 2$: For any $\frac{r_2-r_1}{10} < \rho < r_2 - r_1$, there holds

$$G(Q_\rho^\lambda(\mathfrak{z})) \leq \frac{|Q_{2r}(z_0) \cap \Omega_T|}{|Q_\rho^\lambda(\mathfrak{z}) \cap \Omega_T|} \iint_{Q_{2r}(z_0) \cap \Omega_T} (|\nabla u| + |F| + 1)^p dz \\ + \frac{|Q_{2r}(z_0) \cap \Omega_T|}{|Q_\rho^\lambda(\mathfrak{z}) \cap \Omega_T|} \lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2r}(z_0) \cap \Omega_T} (2r)^{\alpha'} |f|^{\alpha'} dz + \frac{|B_{2r}(x_0)|}{|B_\rho(\mathfrak{z})|} \int_{B_{2r}(x_0)} |\nabla u_0|^p dx.$$

Note that (3.1) implies

$$\frac{|Q_{2r}(z_0) \cap \Omega_T|}{|Q_\rho^\lambda(\mathfrak{z}) \cap \Omega_T|} \leq \frac{|Q_{2r}(z_0)|}{\frac{1}{2}|Q_\rho^\lambda(\mathfrak{z})|} \leq 2 \left(\frac{2r}{\rho} \right)^{n+2} \lambda^{p-2} \leq 2 \left(\frac{20r}{r_2-r_1} \right)^{n+2} \lambda^{p-2}. \quad (4.2)$$

From Remark 2.3, we have $\frac{p}{d\alpha} = 2 - \alpha'(1 - \frac{p}{\alpha}) > 0$, $d = \frac{p}{2}$,

$$\iint_{Q_{2r}(z_0) \cap \Omega_T} (|\nabla u| + |F| + 1)^p dz \leq \lambda_0^2, \quad \int_{B_{2r}(x_0)} |\nabla u_0|^p dx \leq \lambda_0^p, \\ \lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2r}(z_0) \cap \Omega_T} (2r)^{\alpha'} |f|^{\alpha'} dz \leq \lambda^{\alpha'(1-\frac{p}{\alpha})} \lambda_0^{\frac{p}{d\alpha}} = \lambda^{\alpha'(1-\frac{p}{\alpha})} \lambda_0^{2-\alpha'(1-\frac{p}{\alpha})}.$$

It follows that

$$G(Q_\rho^\lambda(\mathfrak{z})) \leq 2 \left(\frac{20r}{r_2-r_1} \right)^{n+2} \lambda^{p-2} \left(\lambda_0^2 + \lambda^{\alpha'(1-\frac{p}{\alpha})} \lambda_0^{2-\alpha'(1-\frac{p}{\alpha})} \right) + \left(\frac{20r}{r_2-r_1} \right)^n \lambda_0^p < \lambda^p.$$

On the other hand, since $\mathfrak{z} \in E_\lambda$, there exists $\rho_\lambda \in (0, \frac{r_2-r_1}{10})$ such that (4.1) holds.

Case $p < 2$: Let us denote $\rho_\lambda := \lambda^{\frac{p-2}{2}} \rho$. Then, for any $\frac{r_2-r_1}{10} < \rho < r_2 - r_1$, there holds

$$G(Q_{\rho_\lambda}^\lambda(\mathfrak{z})) \leq \frac{|Q_{2r}(z_0) \cap \Omega_T|}{|Q_{\rho_\lambda}^\lambda(\mathfrak{z}) \cap \Omega_T|} \iint_{Q_{2r}(z_0) \cap \Omega_T} (|\nabla u| + |F| + 1)^p dz \\ + \frac{|Q_{2r}(z_0) \cap \Omega_T|}{|Q_{\rho_\lambda}^\lambda(\mathfrak{z}) \cap \Omega_T|} \lambda^{\alpha'(\frac{p}{2}-\frac{p}{\alpha})} \iint_{Q_{2r}(z_0) \cap \Omega_T} (2r)^{\alpha'} |f|^{\alpha'} dz + \frac{|B_{2r}(x_0)|}{|B_{\rho_\lambda}(\mathfrak{z})|} \int_{B_{2r}(x_0)} |\nabla u_0|^p dx.$$

Here, we used the fact that $\rho_\lambda \leq \lambda^{\frac{p-2}{2}} r$ for the second term. Note that

$$\frac{|Q_{2r}(z_0) \cap \Omega_T|}{|Q_{\rho_\lambda}^\lambda(\mathfrak{z}) \cap \Omega_T|} \leq \frac{|Q_{2r}(z_0)|}{\frac{1}{2}|Q_{\rho_\lambda}^\lambda(\mathfrak{z})|} \leq 2 \left(\frac{20r}{r_2-r_1} \right)^{n+2} \lambda^{\frac{(2-p)n}{2}}.$$

Again, from Remark 2.3, we have $\frac{p}{d\alpha} = \frac{p(n+2)-2n}{2} - \alpha'(\frac{p}{2} - \frac{p}{\alpha}) > 0$, $\frac{p}{d} = \frac{p(n+2)-2n}{2}$,

$$\iint_{Q_{2r}(z_0) \cap \Omega_T} (|\nabla u| + |F| + 1)^p dz + \int_{B_{2r}(x_0)} |\nabla u_0|^p dx \leq \lambda_0^{\frac{(p-2)n+2p}{2}}, \\ \lambda^{\alpha'(\frac{p}{2}-\frac{p}{\alpha})} \iint_{Q_{2r}(z_0) \cap \Omega_T} (2r)^{\alpha'} |f|^{\alpha'} dz \leq \lambda^{\alpha'(\frac{p}{2}-\frac{p}{\alpha})} \lambda_0^{\frac{p}{d\alpha}} = \lambda^{\alpha'(\frac{p}{2}-\frac{p}{\alpha})} \lambda_0^{\frac{(p-2)n+2p}{2}-\alpha'(\frac{p}{2}-\frac{p}{\alpha})}.$$

It follows that

$$G(Q_{\rho_\lambda}^\lambda(\mathfrak{z})) \leq 2 \left(\frac{20r}{r_2-r_1} \right)^{n+2} \lambda^{\frac{(2-p)n}{2}} \left(\lambda_0^{\frac{(p-2)n+2p}{2}} + \lambda^{\alpha'(\frac{p}{2}-\frac{p}{\alpha})} \lambda_0^{\frac{(p-2)n+2p}{2}-\alpha'(\frac{p}{2}-\frac{p}{\alpha})} \right) + \left(\frac{20r}{r_2-r_1} \right)^n \lambda^{\frac{(2-p)n}{2}} \lambda_0^{\frac{p(n+2)-2n}{2}} < \lambda^p.$$

On the other hand, since $\mathfrak{z} \in E_\lambda$, there exists $\rho_\lambda \in (0, \frac{r_2-r_1}{10})$ such that (4.1) holds. The lemma follows. $\square$

We now define upper level sets.

Definition 4.3. Let $\eta > 0$ and $\lambda > 3B\lambda_0$. We define the following:

- $\Phi_{\eta\lambda}^\rho := \{z \in Q_\rho(z_0) \cap \Omega_T : |\nabla u|^p(z) > \eta\lambda^p\}$,
- $\Psi_{\eta\lambda}^\rho := \{z \in Q_\rho(z_0) \cap \Omega_T : H^p(z) > \eta\lambda^p\}$ where $H(z) := |F(z)| + |\nabla u_0(x)| + 1$,
- $\Sigma_{\eta\lambda}^\rho := \{z \in Q_\rho(z_0) \cap \Omega_T : |\tilde{f}_\eta|(z) > \eta\lambda^p\}$, where $\tilde{f}_\eta := 2^{\alpha'} \eta^{-\alpha'} \lambda_0^\nu (2r|f|(z))^{\alpha'}$.

Our covering argument is divided into three steps.

Step 1: Let $\lambda > 3\mathbb{B}\lambda_0$ and $\mathfrak{z} \in E_\lambda$. Assumptions (3.2) in $Q_{4\rho_3}^\lambda(\mathfrak{z})$ and (3.3) in $Q_{\rho_3}^\lambda(\mathfrak{z})$ are satisfied by (4.1). Applying Lemma 3.10, there exists $q < p$ defined in (3.7) such that

$$\begin{aligned} G(Q_{\rho_3}^\lambda(\mathfrak{z})) &\leq \left(\iint_{Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T} |\nabla u|^q dz \right)^{\frac{p}{q}} + \iint_{Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T} (|F| + 1)^p dz \\ &\quad + \iint_{Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T} |\nabla u_0|^p dx + \lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T} (2\rho_3)^{\alpha'} |f|^{\alpha'} dz =: \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned}$$

Estimate of I + II + III: Let $\eta \in (0, 1)$ to be chosen later. There holds

$$\begin{aligned} \text{I} &\leq \eta \lambda^p + \left(\frac{1}{|Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T|} \iint_{\Phi_{\eta\lambda}^{r_2} \cap Q_{2\rho_3}^\lambda(\mathfrak{z})} |\nabla u|^q dz \right)^{\frac{p}{q}} \\ &\leq \eta \lambda^p + \left(\frac{1}{|Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T|} \iint_{\Phi_{\eta\lambda}^{r_2} \cap Q_{2\rho_3}^\lambda(\mathfrak{z})} |\nabla u|^q dz \right) \left(\iint_{Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T} |\nabla u|^q dz \right)^{\frac{p}{q}-1} \\ &\stackrel{(a)}{\leq} \eta \lambda^p + \frac{1}{|Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T|} \iint_{\Phi_{\eta\lambda}^{r_2} \cap Q_{2\rho_3}^\lambda(\mathfrak{z})} \lambda^{p-q} |\nabla u|^q dz. \end{aligned}$$

Here, to obtain (a), we used Hölder's inequality and (4.1). Therefore, we get

$$\text{I} + \text{II} + \text{III} \leq \eta \lambda^p + \frac{1}{|Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T|} \iint_{\Phi_{\eta\lambda}^{r_2} \cap Q_{2\rho_3}^\lambda(\mathfrak{z})} \lambda^{p-q} |\nabla u|^q dz + \frac{1}{|Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T|} \iint_{\Psi_{\eta\lambda}^{r_2} \cap Q_{2\rho_3}^\lambda(\mathfrak{z})} |H|^p dz.$$

Estimate of IV: Let us consider the alternative

$$\lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T} (2\rho_3)^{\alpha'} |f|^{\alpha'} dz \leq \eta \lambda^p \quad \text{or} \quad \lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T} (2\rho_3)^{\alpha'} |f|^{\alpha'} dz \geq \eta \lambda^p. \quad (4.3)$$

Leaving the first case of (4.3), suppose the second case holds.

Case $p \geq 2$: Applying (4.2), we have

$$\eta \lambda^p \leq 2\lambda^{\alpha'(1-\frac{p}{\alpha})+p-\frac{p}{d}} \left(\frac{r}{\rho_3} \right)^{n+2-\alpha'} \iint_{Q_{2r}(z_0) \cap \Omega_T} (2r|f|)^{\alpha'} dz \leq 2\lambda^{\alpha'(1-\frac{p}{\alpha})+p-\frac{p}{d}} \lambda_0^{\frac{p}{d\alpha}} \left(\frac{r}{\rho_3} \right)^{n+2-\alpha'}. \quad (4.4)$$

Since Definition 2.2 implies the inequality

$$n + 2 - \alpha' > 1 \iff n + 1 > \alpha' \iff \frac{n+1}{n} = (n+1)' < \alpha = \delta \frac{p(n+2)}{n} \iff \frac{n+1}{p(n+2)} < \delta, \quad (4.5)$$

we see that (4.4) becomes

$$\frac{\rho_3}{r} \leq 2\eta^{-1} \lambda^{\frac{\alpha'(1-\frac{p}{\alpha})-\frac{p}{d}}{n+2-\alpha'}} \lambda_0^{\frac{1}{n+2-\alpha'} \frac{p}{d\alpha}}.$$

Therefore, we get

$$\begin{aligned} \lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T} (2\rho_3)^{\alpha'} |f|^{\alpha'} dz &= \lambda^{\alpha'(1-\frac{p}{\alpha})} \left(\frac{\rho_3}{r} \right)^{\alpha'} \iint_{Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T} (2r)^{\alpha'} |f|^{\alpha'} dz \\ &\leq 2^{\alpha'} \eta^{-\alpha'} \lambda^{\alpha'(1-\frac{p}{\alpha}) + \frac{\alpha'}{n+2-\alpha'} (\alpha'(1-\frac{p}{\alpha}) - \frac{p}{d})} \lambda_0^{\frac{\alpha'}{n+2-\alpha'} \frac{p}{d\alpha}} \iint_{Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T} (2r)^{\alpha'} |f|^{\alpha'} dz \\ &\stackrel{(a)}{\leq} 2^{\alpha'} \eta^{-\alpha'} \lambda_0^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T} (2r)^{\alpha'} |f|^{\alpha'} dz. \end{aligned}$$

Here, to obtain (a), we used $\lambda \geq \lambda_0$ and

$$\begin{aligned} \frac{\alpha' n}{n+2-\alpha'} \left(1 - \frac{1}{\delta}\right) + \frac{\alpha'}{n+2-\alpha'} \frac{p}{d\alpha} &= \alpha' \left(1 - \frac{p}{\alpha}\right) + \frac{\alpha'}{n+2-\alpha'} \left(\alpha' \left(1 - \frac{p}{\alpha}\right) - \frac{p}{d}\right) + \frac{\alpha'}{n+2-\alpha'} \frac{p}{d\alpha} \\ &= \alpha' \left(1 - \frac{p}{\alpha}\right). \end{aligned}$$

It follows that both cases of (4.3) give

$$\begin{aligned} \lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2\rho_3}^\lambda(z_0) \cap \Omega_T} (2\rho_3)^{\alpha'} |f|^{\alpha'} dz &\leq \eta \lambda^p + 2^{\alpha'} \eta^{-\alpha'} \lambda_0^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2\rho_3}^\lambda(z_0) \cap \Omega_T} (2r)^{\alpha'} |f|^{\alpha'} dz \\ &\leq 2\eta \lambda^p + \frac{2^{\alpha'} \eta^{-\alpha'} \lambda_0^{\alpha'}}{|Q_{2\rho_3}^\lambda(z) \cap \Omega_T|} \iint_{\Sigma_{\eta\lambda}^{r_2} \cap Q_{2\rho_3}^\lambda(z)} (2r)^{\alpha'} |f|^{\alpha'} dz. \end{aligned}$$

Case $p < 2$: Since $\rho_3 = \lambda^{\frac{p-2}{2}} \tilde{\rho}_3$ for some $\tilde{\rho}_3 \in (0, \frac{r_2-r_1}{10})$, (4.4) becomes

$$\eta \lambda^p \leq 2\lambda^{\alpha'(1-\frac{p}{\alpha}) + \frac{n(2-p)}{2} + \alpha'(\frac{p-2}{2})} \lambda_0^{\frac{p}{d\alpha}} \left(\frac{r}{\tilde{\rho}_3}\right)^{n+2-\alpha'} = 2\lambda^{\alpha'(\frac{p}{2}-\frac{p}{\alpha}) + \frac{n(2-p)}{2}} \lambda_0^{\frac{p}{d\alpha}} \left(\frac{r}{\tilde{\rho}_3}\right)^{n+2-\alpha'},$$

and thus (4.5) gives

$$\left(\frac{\tilde{\rho}_3}{r}\right) \leq 2\eta^{-1} \lambda^{\frac{\alpha'(\frac{p}{2}-\frac{p}{\alpha})-\frac{p}{d}}{n+2-\alpha'}} \lambda_0^{\frac{1}{n+2-\alpha'} \frac{p}{d\alpha}}.$$

Analogously, the second case of (4.3) implies

$$\begin{aligned} \lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2\rho_3}^\lambda(z) \cap \Omega_T} (2\rho_3)^{\alpha'} |f|^{\alpha'} dz &= \lambda^{\alpha'(\frac{p}{2}-\frac{p}{\alpha})} \left(\frac{\tilde{\rho}_3}{r}\right)^{\alpha'} \iint_{Q_{2\rho_3}^\lambda(z) \cap \Omega_T} (2r)^{\alpha'} |f|^{\alpha'} dz \\ &\leq 2^{\alpha'} \eta^{-\alpha'} \lambda^{\alpha'(\frac{p}{2}-\frac{p}{\alpha}) + \frac{\alpha'}{n+2-\alpha'} (\alpha'(\frac{p}{2}-\frac{p}{\alpha}) - \frac{p}{d})} \lambda_0^{\frac{\alpha'}{n+2-\alpha'} \frac{p}{d\alpha}} \iint_{Q_{2\rho_3}^\lambda(z) \cap \Omega_T} (2r)^{\alpha'} |f|^{\alpha'} dz \\ &\stackrel{(a)}{\leq} 2^{\alpha'} \eta^{-\alpha'} \lambda_0^{\alpha'(\frac{p}{2}-\frac{p}{\alpha})} \iint_{Q_{2\rho_3}^\lambda(z) \cap \Omega_T} (2r)^{\alpha'} |f|^{\alpha'} dz. \end{aligned}$$

Here, to obtain (a), we used $\lambda \geq \lambda_0$ and

$$\begin{aligned} \frac{\alpha' n}{n+2-\alpha'} \left(1 - \frac{1}{\delta}\right) + \frac{\alpha'}{n+2-\alpha'} \frac{p}{d\alpha} &= \alpha' \left(\frac{p}{2} - \frac{p}{\alpha}\right) + \frac{\alpha'}{n+2-\alpha'} \left(\alpha' \left(\frac{p}{2} - \frac{p}{\alpha}\right) - \frac{p}{d}\right) + \frac{\alpha'}{n+2-\alpha'} \frac{p}{d\alpha} \\ &= \alpha' \left(\frac{p}{2} - \frac{p}{\alpha}\right). \end{aligned}$$

Therefore, we have

$$\lambda^{\alpha'(1-\frac{p}{\alpha})} \iint_{Q_{2\rho_3}^\lambda(z_0) \cap \Omega_T} (2\rho_3)^{\alpha'} |f|^{\alpha'} dz \leq 2\eta \lambda^p + \frac{2^{\alpha'} \eta^{-\alpha'} \lambda_0^{\alpha'}}{|Q_{2\rho_3}^\lambda(z) \cap \Omega_T|} \iint_{\Sigma_{\eta\lambda}^{r_2} \cap Q_{2\rho_3}^\lambda(z)} (2r)^{\alpha'} |f|^{\alpha'} dz.$$

Combining all the estimates, we get

$$\begin{aligned} G(Q_{\rho_3}^\lambda(z)) &\leq \eta \lambda^p + \frac{1}{|Q_{2\rho_3}^\lambda(z) \cap \Omega_T|} \iint_{\Phi_{\eta\lambda}^{r_2} \cap Q_{2\rho_3}^\lambda(z)} \lambda^{p-q} |\nabla u|^q dz \\ &\quad + \frac{1}{|Q_{2\rho_3}^\lambda(z) \cap \Omega_T|} \iint_{\Psi_{\eta\lambda}^{r_2} \cap Q_{2\rho_3}^\lambda(z)} |H|^p dz \\ &\quad + \frac{2^{\alpha'} \eta^{-\alpha'} \lambda_0^{\alpha'}}{|Q_{2\rho_3}^\lambda(z) \cap \Omega_T|} \iint_{\Sigma_{\eta\lambda}^{r_2} \cap Q_{2\rho_3}^\lambda(z)} (2r)^{\alpha'} |f|^{\alpha'} dz. \end{aligned}$$

Taking $\eta = \eta(n, N, p, \Lambda_0, \Lambda_1, \delta) \in (0, 1)$ small enough, we obtain

$$\begin{aligned} \iint_{Q_{10\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T} |\nabla u|^p dz &\leq \frac{1}{|Q_{2\rho_3}^\lambda(z_0) \cap \Omega_T|} \iint_{\Phi_{\eta^\lambda}^{r_2} \cap Q_{2\rho_3}^\lambda(\mathfrak{z})} \lambda^{p-q} |\nabla u|^q dz \\ &\quad + \frac{1}{|Q_{2\rho_3}^\lambda(z_0) \cap \Omega_T|} \iint_{\Psi_{\eta^\lambda}^{r_2} \cap Q_{2\rho_3}^\lambda(\mathfrak{z})} |H|^p dz + \frac{2^{a'} \eta^{-a'} \lambda_0^v}{|Q_{2\rho_3}^\lambda(z_0) \cap \Omega_T|} \iint_{\Sigma_{\eta^\lambda}^{r_2}} (2r)^{a'} |f|^{a'} dz. \end{aligned}$$

Therefore, for any $\lambda > 3B\lambda_0$ and $\mathfrak{z} \in E_\lambda$, there holds

$$\iint_{Q_{10\rho_3}^\lambda(\mathfrak{z}) \cap \Omega_T} |\nabla u|^p dz \leq \iint_{\Phi_{\eta^\lambda}^{r_2} \cap Q_{2\rho_3}^\lambda(\mathfrak{z})} \lambda^{p-q} |\nabla u|^q dz + \iint_{\Psi_{\eta^\lambda}^{r_2} \cap Q_{2\rho_3}^\lambda(\mathfrak{z})} |H|^p dz + \iint_{\Sigma_{\eta^\lambda}^{r_2} \cap Q_{2\rho_3}^\lambda(\mathfrak{z})} |\tilde{f}_\eta| dz. \quad (4.6)$$

Step 2: We apply Vitali's covering lemma to $\{Q_{2\rho_3}^\lambda(\mathfrak{z})\}_{\mathfrak{z} \in E_\lambda}$ and to obtain a disjoint countable subfamily $\{Q_{2\rho_{3i}}^\lambda(\mathfrak{z}_i)\}_{i \in \mathbb{N}}$ such that

$$E_\lambda \subset \bigcup_{\mathfrak{z} \in E_\lambda} Q_{2\rho_3}^\lambda(\mathfrak{z}) \subset \bigcup_{1 \leq i < \infty} Q_{10\rho_{3i}}^\lambda(\mathfrak{z}_i) \subset Q_{r_2}(z_0).$$

Therefore, we get

$$\iint_{E_\lambda} |\nabla u|^p dz \leq \sum_{1 \leq i < \infty} \iint_{Q_{10\rho_{3i}}^\lambda(\mathfrak{z}_i) \cap \Omega_T} |\nabla u|^p dz \stackrel{(a)}{\leq} \iint_{\Phi_{\eta^\lambda}^{r_2}} \lambda^{p-q} |\nabla u|^q dz + \iint_{\Psi_{\eta^\lambda}^{r_2}} |H|^p dz + \iint_{\Sigma_{\eta^\lambda}^{r_2}} |\tilde{f}_\eta| dz.$$

where, to obtain (a), we used (4.6) for each i and disjointness of $\{Q_{2\rho_{3i}}^\lambda(\mathfrak{z}_i)\}_{i \in \mathbb{N}}$ in $\mathbb{R}^{n+1}$.

Also, since there holds

$$\iint_{\Phi_{\eta^\lambda}^{r_1} \setminus E_\lambda} |\nabla u|^p dz \leq \iint_{\Phi_{\eta^\lambda}^{r_1} \setminus E_\lambda} \lambda^{p-q} |\nabla u|^q dz,$$

it follows

$$\iint_{\Phi_{\eta^\lambda}^{r_1}} |\nabla u|^p dz \leq \iint_{\Phi_{\eta^\lambda}^{r_2}} \lambda^{p-q} |\nabla u|^q dz + \iint_{\Psi_{\eta^\lambda}^{r_2}} |H|^p dz + \iint_{\Sigma_{\eta^\lambda}^{r_2}} |\tilde{f}_\eta| dz.$$

Letting $\lambda_1 := 3\eta^{\frac{1}{p}} B\lambda_0$, for any $\lambda > \lambda_1$, we have

$$\iint_{\Phi_\lambda^{r_1}} |\nabla u|^p dz \leq \iint_{\Phi_\lambda^{r_2}} \lambda^{p-q} |\nabla u|^q dz + \iint_{\Psi_\lambda^{r_2}} |H|^p dz + \iint_{\Sigma_\lambda^{r_2}} |\tilde{f}_\eta| dz. \quad (4.7)$$

Step 3: For $k > \lambda_1$, let us define

$$|\nabla u|_k := \min\{|\nabla u|, k\} \quad \text{and} \quad \Phi_{\lambda,k}^\rho := \{z \in Q_\rho(z_0) \cap \Omega_T : |\nabla u|_k^p > \lambda^p\}.$$

We see that if $\lambda > k$, then $\Phi_{\lambda,k}^\rho = \emptyset$, and if $\lambda \leq k$, then $\Phi_{\lambda,k}^\rho = \Phi_\lambda^\rho$. From (4.7), we deduce

$$\iint_{\Phi_{\lambda,k}^{r_1}} |\nabla u|_k^{p-q} |\nabla u|^q dz \leq \iint_{\Phi_{\lambda,k}^{r_2}} \lambda^{p-q} |\nabla u|^q dz + \iint_{\Psi_\lambda^{r_2}} |H|^p dz + \iint_{\Sigma_\lambda^{r_2}} |\tilde{f}_\eta| dz. \quad (4.8)$$

Let $\varepsilon > 0$ to be chosen later. Multiply (4.8) by $\lambda^{\varepsilon-1}$ and integrate over (λ_1, ∞) to get

$$\begin{aligned} \text{I} := \int_{\lambda_1}^{\infty} \lambda^{\varepsilon-1} \iint_{\Phi_{\lambda,k}^{r_1}} |\nabla u|_k^{p-q} |\nabla u|^q dz d\lambda &\leq \int_{\lambda_1}^{\infty} \lambda^{\varepsilon-1} \iint_{\Phi_{\lambda,k}^{r_2}} \lambda^{p-q} |\nabla u|^q dz d\lambda + \int_{\lambda_1}^{\infty} \lambda^{\varepsilon-1} \iint_{\Psi_\lambda^{r_2}} |H|^p dz d\lambda \\ &\quad + \int_{\lambda_1}^{\infty} \lambda^{\varepsilon-1} \iint_{\Sigma_\lambda^{r_2}} |\tilde{f}_\eta| dz d\lambda =: \text{II} + \text{III} + \text{IV}. \end{aligned} \quad (4.9)$$

Estimate of I: Applying Fubini's theorem, we get

$$I = \iint_{\Phi_{\lambda_1, k}^{r_1}} |\nabla u|_k^{p-q} |\nabla u|^q \int_{\lambda_1}^{|\nabla u|_k} \lambda^{\varepsilon-1} d\lambda dz = \frac{1}{\varepsilon} \iint_{\Phi_{\lambda_1, k}^{r_1}} |\nabla u|_k^{p-q+\varepsilon} |\nabla u|^q dz - \frac{1}{\varepsilon} \lambda_1^\varepsilon \iint_{\Phi_{\lambda_1, k}^{r_1}} |\nabla u|_k^{p-q} |\nabla u|^q dz.$$

Estimate of II: Again, using Fubini's theorem, we obtain

$$II = \iint_{\Phi_{\lambda_1, k}^{r_2}} |\nabla u|^q \int_{\lambda_1}^{|\nabla u|_k} \lambda^{p-q+\varepsilon-1} d\lambda dz \leq \frac{1}{p-q} \iint_{\Phi_{\lambda_1, k}^{r_2}} |\nabla u|_k^{p-q+\varepsilon} |\nabla u|^q dz.$$

Estimate of III: Again, by Fubini's theorem, we have

$$III = \iint_{\Psi_{\lambda_1}^{r_2}} |H|^p \int_{\lambda_1}^{|H|} \lambda^{\varepsilon-1} d\lambda dz \leq \frac{1}{\varepsilon} \iint_{\Psi_{\lambda_1}^{r_2}} |H|^{p+\varepsilon} dz.$$

Estimate of IV: Similarly, we have

$$IV = \iint_{\Sigma_{\lambda_1}^{r_2}} |\tilde{f}_\eta| \int_{\lambda_1}^{|\tilde{f}_\eta|^{\frac{1}{p}}} \lambda^{\varepsilon-1} d\lambda dz \leq \frac{1}{\varepsilon} \iint_{\Sigma_{\lambda_1}^{r_2}} |\tilde{f}_\eta|^{1+\frac{\varepsilon}{p}} dz.$$

It follows that (4.9) becomes

$$\begin{aligned} \iint_{\Phi_{\lambda_1, k}^{r_1}} |\nabla u|_k^{p-q+\varepsilon} |\nabla u|^q dz &\leq \mathbf{C} \frac{\varepsilon}{p-q} \iint_{\Phi_{\lambda_1, k}^{r_2}} |\nabla u|_k^{p-q+\varepsilon} |\nabla u|^q dz + \lambda_1^\varepsilon \iint_{\Phi_{\lambda_1, k}^{r_1}} |\nabla u|_k^{p-q} |\nabla u|^q dz \\ &\quad + \mathbf{C} \iint_{\Psi_{\lambda_1}^{r_2}} |H|^{p+\varepsilon} dz + \mathbf{C} \iint_{\Sigma_{\lambda_1}^{r_2}} |\tilde{f}_\eta|^{1+\frac{\varepsilon}{p}} dz, \end{aligned}$$

where $\mathbf{C} = \mathbf{C}(n, N, p, \Lambda_0, \Lambda_1, \delta)$.

Since there holds

$$\iint_{Q_{r_1}(z_0) \cap \Omega_T \setminus \Phi_{\lambda_1, k}^{r_1}} |\nabla u|_k^{p-q+\varepsilon} |\nabla u|^q dz \leq \lambda_1^\varepsilon \iint_{Q_{r_1}(z_0) \cap \Omega_T \setminus \Phi_{\lambda_1, k}^{r_1}} |\nabla u|_k^{p-q} |\nabla u|^q dz,$$

$\lambda_1 := 3\eta^{\frac{1}{p}} \mathbb{B} \lambda_0$ and $\mathbb{B} \geq 1$, we obtain

$$\begin{aligned} \iint_{Q_{r_1}(z_0) \cap \Omega_T} |\nabla u|_k^{p-q+\varepsilon} |\nabla u|^q dz &\leq \mathbf{C} \frac{\varepsilon}{p-q} \iint_{Q_{r_2}(z_0) \cap \Omega_T} |\nabla u|_k^{p-q+\varepsilon} |\nabla u|^q dz + \mathbf{C} \mathbb{B} \lambda_0^\varepsilon \iint_{Q_{2r}(z_0) \cap \Omega_T} |\nabla u|_k^{p-q} |\nabla u|^q dz \\ &\quad + \mathbf{C} \iint_{Q_{2r}(z_0) \cap \Omega_T} |H|^{p+\varepsilon} dz + \mathbf{C} \iint_{Q_{2r}(z_0) \cap \Omega_T} |\tilde{f}_\eta|^{1+\frac{\varepsilon}{p}} dz. \end{aligned}$$

Take $\varepsilon_0 = \varepsilon_0(n, N, p, \Lambda_0, \Lambda_1, \delta)$ so that $\mathbf{C} \frac{\varepsilon_0}{p-q} = \frac{1}{2}$. Then, for any $\varepsilon \in (0, \varepsilon_0)$, it follows that

$$\begin{aligned} \iint_{Q_{r_1}(z_0) \cap \Omega_T} |\nabla u|_k^{p-q+\varepsilon} |\nabla u|^q dz &\leq \frac{1}{2} \iint_{Q_{r_2}(z_0) \cap \Omega_T} |\nabla u|_k^{p-q+\varepsilon} |\nabla u|^q dz + \mathbf{C} \mathbb{B} \lambda_0^\varepsilon \iint_{Q_{2r}(z_0) \cap \Omega_T} |\nabla u|_k^{p-q} |\nabla u|^q dz \\ &\quad + \mathbf{C} \iint_{Q_{2r}(z_0) \cap \Omega_T} |H|^{p+\varepsilon} dz + \mathbf{C} \iint_{Q_{2r}(z_0) \cap \Omega_T} |\tilde{f}_\eta|^{1+\frac{\varepsilon}{p}} dz. \end{aligned}$$

Hence, applying Lemma 2.8, we get

$$\iint_{Q_r(z_0) \cap \Omega_T} |\nabla u|_k^{p-q+\varepsilon} |\nabla u|^q dz \leq \lambda_0^\varepsilon \iint_{Q_{2r}(z_0) \cap \Omega_T} |\nabla u|_k^{p-q} |\nabla u|^q dz + \iint_{Q_{2r}(z_0) \cap \Omega_T} |H|^{p+\varepsilon} dz + \iint_{Q_{2r}(z_0) \cap \Omega_T} |\tilde{f}_\eta|^{1+\frac{\varepsilon}{p}} dz.$$

Let $k \rightarrow \infty$ to derive the desired estimate. This completes the proof.

Acknowledgment: The authors are grateful to Karthik Adimurthi for valuable conversations and constant support throughout all this work. The authors wish to thank the referees for careful reading of the early version of this manuscript and providing many valuable suggestions and comments.

Funding: S. Byun was supported by the National Research Foundation of Korea (NRF-2017R1A2B2003877). W. Kim and M. Lim were supported by the National Research Foundation of Korea (NRF-2019R1C1C1003844).

References

- [1] E. Acerbi and G. Mingione, Gradient estimates for a class of parabolic systems, *Duke Math. J.* **136** (2007), no. 2, 285–320.
- [2] K. Adimurthi, S.-S. Byun and W. Kim, Higher integrability for quasilinear parabolic equations with nondivergence data – with application, preprint (2020).
- [3] P. Baroni and J. Habermann, New gradient estimates for parabolic equations, *Houston J. Math.* **38** (2012), no. 3, 855–914.
- [4] V. Bögelein and F. Duzaar, Higher integrability for parabolic systems with non-standard growth and degenerate diffusions, *Publ. Mat.* **55** (2011), no. 1, 201–250.
- [5] V. Bögelein and M. Parviainen, Self-improving property of nonlinear higher order parabolic systems near the boundary, *NoDEA Nonlinear Differential Equations Appl.* **17** (2010), no. 1, 21–54.
- [6] V. Bögelein and C. Scheven, Higher integrability in parabolic obstacle problems, *Forum Math.* **24** (2012), no. 5, 931–972.
- [7] S.-S. Byun and Y. Cho, Nonlinear gradient estimates for parabolic obstacle problems in non-smooth domains, *Manuscripta Math.* **146** (2015), no. 3–4, 539–558.
- [8] S.-S. Byun, J. Ok and S. Ryu, Global gradient estimates for general nonlinear parabolic equations in nonsmooth domains, *J. Differential Equations* **254** (2013), no. 11, 4290–4326.
- [9] E. DiBenedetto, *Degenerate Parabolic Equations*, Universitext, Springer, New York, 1993.
- [10] F. W. Gehring, The L^p -integrability of the partial derivatives of a quasiconformal mapping, *Acta Math.* **130** (1973), 265–277.
- [11] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Ann. of Math. Stud. 105, Princeton University, Princeton, 1983.
- [12] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific, River Edge, 2003.
- [13] J. Kinnunen and J. L. Lewis, Higher integrability for parabolic systems of p -Laplacian type, *Duke Math. J.* **102** (2000), no. 2, 253–271.
- [14] J. Kinnunen and J. L. Lewis, Very weak solutions of parabolic systems of p -Laplacian type, *Ark. Mat.* **40** (2002), no. 1, 105–132.
- [15] T. Kuusi and G. Mingione, A nonlinear Stein theorem, *Calc. Var. Partial Differential Equations* **51** (2014), no. 1–2, 45–86.
- [16] M. Lee and J. Ok, Nonlinear Calderón–Zygmund theory involving dual data, *Rev. Mat. Iberoam.* **35** (2019), no. 4, 1053–1078.
- [17] N. G. Meyers and A. Elcrat, Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions, *Duke Math. J.* **42** (1975), 121–136.
- [18] L. Nirenberg, On elliptic partial differential equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (3)* **13** (1959), 115–162.
- [19] M. Parviainen, Global gradient estimates for degenerate parabolic equations in nonsmooth domains, *Ann. Mat. Pura Appl. (4)* **188** (2009), no. 2, 333–358.
- [20] M. Parviainen, Reverse Hölder inequalities for singular parabolic equations near the boundary, *J. Differential Equations* **246** (2009), no. 2, 512–540.