Research Article

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Weighted value distributions of the Riemann zeta function on the critical line

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Abstract: We prove a central limit theorem for $\log|\zeta(\frac{1}{2} + it)|$ with respect to the measure $|\zeta^{(m)}(\frac{1}{2} + it)|^{2k} dt$ ($k, m \in \mathbb{N}$), assuming RH and the asymptotic formula for twisted and shifted integral moments of zeta. Under the same hypotheses, we also study a shifted case, looking at the measure $|\zeta(\frac{1}{2} + it + i\alpha)|^{2k} dt$, with $\alpha \in (-1, 1)$. Finally, we prove unconditionally the analogue result in the random matrix theory context.

Keywords: Riemann zeta function, central limit theorem

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1 Introduction and statement of the main results

In 1946, Selberg [18] proved a central limit theorem for the real part of the logarithm of the Riemann zeta function on the critical line, showing that the distribution of $\log|\zeta(\frac{1}{2} + it)|$ is approximately Gaussian with mean 0 and variance $\frac{1}{2} \log \log T$, i.e.

$$\frac{1}{T} \text{meas} \left\{ t \in [T, 2T] : \frac{\log|\zeta(\frac{1}{2} + it)|}{\sqrt{\frac{1}{2} \log \log T}} \geq V \right\} \sim \int_V^\infty e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \quad (1.1)$$

for any fixed $V \in \mathbb{R}$, as T goes to infinity. The author [9] studied the distribution of $\log|\zeta(\frac{1}{2} + it)|$ with respect to the weighted measure $|\zeta(\frac{1}{2} + it)|^2 dt$ and, assuming the Riemann hypothesis (RH), proved that it is asymptotically Gaussian with mean $\log \log T$ and variance $\frac{1}{2} \log \log T$. In this paper, we investigate the value distribution of $\log|\zeta(\frac{1}{2} + it)|$ with respect to the measure

$$|\zeta^{(m)}(\frac{1}{2} + it)|^{2k} dt \quad (1.2)$$

for any fixed non-negative integers m, k . The motivation is due to the study of the large values of the Riemann zeta function, i.e. the uniformity in V in the central limit theorem (1.1). In [19], Soundararajan speculates that an upper bound like

$$\frac{1}{T} \text{meas} \left\{ t \in [T, 2T] : \frac{\log|\zeta(\frac{1}{2} + it)|}{\sqrt{\frac{1}{2} \log \log T}} \geq V \right\} \ll \frac{1}{V} \exp\left(-\frac{V^2}{2}\right)$$

holds in a large range for V . In particular, we expect (see [17, Conjecture 2] and [11]) that for any fixed k ,

$$\frac{1}{T} \text{meas} \left\{ t \in [T, 2T] : \log \left| \zeta\left(\frac{1}{2} + it\right) \right| \geq k \log \log T \right\} \ll_k \frac{1}{\sqrt{\log \log T}} \frac{1}{(\log T)^{k^2}}. \quad (1.3)$$

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Expressing the characteristic function of positive reals as a Mellin transform, the left-hand side of (1.3) can be written as

$$\frac{1}{(\log T)^{k^2}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{T(\log T)^{k^2}} \int_T^{2T} e^{iu(\log|\zeta(\frac{1}{2}+it)|-k\log\log T)} \left| \zeta\left(\frac{1}{2}+it\right) \right|^{2k} dt \frac{du}{2k+iu}$$

and is then related to the distribution of $\log|\zeta(\frac{1}{2}+it)|$ with respect to the weighted measure (1.2), with $m = 0$.

In the case $k = 1, 2$, we can prove a central limit theorem for $\log|\zeta(\frac{1}{2}+it)|$ with respect to the measure $|\zeta^{(m)}(\frac{1}{2}+it)|^{2k} dt$, assuming RH only.

Corollary 1.1. *Assume the Riemann hypothesis, let m be a non-negative integer and let $k = 1$ or $k = 2$. As t varies in $T \leq t \leq 2T$, the distribution of $\log|\zeta(\frac{1}{2}+it)|$ is asymptotically Gaussian with mean $k\log\log T$ and variance $\frac{1}{2}\log\log T$, with respect to the weighted measure $|\zeta^{(m)}(\frac{1}{2}+it)|^{2k} dt$.*

In the case $k > 2$, since not even the moments of zeta are known, we cannot expect to prove unconditionally a central limit theorem. However, assuming the asymptotic formula [13, Conjecture 7.1] suggested by the recipe [6] for the twisted and shifted $(2k)$ -th moments of the Riemann zeta function

$$\int_T^{2T} \left(\frac{a}{b}\right)^{-it} \zeta\left(\frac{1}{2}+\alpha_1+it\right) \cdots \zeta\left(\frac{1}{2}+\alpha_k+it\right) \zeta\left(\frac{1}{2}+\beta_1-it\right) \cdots \zeta\left(\frac{1}{2}+\beta_k-it\right) dt,$$

one can deal with the general case too. More precisely, we use the strategy of [4, Theorem 1.2] in order to re-write the conjecture of Hughes and Young and we assume the following statement.

Conjecture 1.2 (Hughes–Young). Let T be a large parameter, let $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \ll (\log T)^{-1}$, let Φ_j be the set of subsets of $\{\alpha_1, \dots, \alpha_k\}$ of cardinality j , for $j = 0, \dots, k$, and similarly let Ψ_j be the set of subsets of $\{\beta_1, \dots, \beta_k\}$ of cardinality j . If $S \in \Phi_j$ and $\mathcal{T} \in \Psi_j$, then write $S = \{\alpha_{i_1}, \dots, \alpha_{i_j}\}$ and $\mathcal{T} = \{\beta_{l_1}, \dots, \beta_{l_j}\}$, where $i_1 < i_2 < \dots < i_j$ and $l_1 < l_2 < \dots < l_j$. Let $(\alpha_S; \beta_{\mathcal{T}})$ be the tuple obtained from $(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_k)$ by replacing α_{i_r} with $-\beta_{l_r}$ and replacing β_{l_r} with $-\alpha_{i_r}$ for $1 \leq r \leq j$. Consider

$$A(s) = \sum_{a \leq T^\theta} \frac{f(a)}{a^s} \quad \text{and} \quad B(s) = \sum_{b \leq T^\theta} \frac{g(b)}{b^s},$$

where $f(a) \ll a^\varepsilon$ and $g(b) \ll b^\varepsilon$ for any $\varepsilon > 0$. We then conjecture that there exists a $\delta > 0$, depending on k , such that if $\theta < \delta$, then

$$\int_T^{2T} A\left(\frac{1}{2}+it\right) \overline{B\left(\frac{1}{2}+it\right)} \zeta\left(\frac{1}{2}+it+\alpha_1\right) \cdots \zeta\left(\frac{1}{2}+it+\alpha_k\right) \zeta\left(\frac{1}{2}-it+\beta_1\right) \cdots \zeta\left(\frac{1}{2}-it+\beta_k\right) dt$$

equals

$$\sum_{a, b \leq T^\theta} f(a) \overline{g(b)} \int_T^{2T} \sum_{0 \leq j \leq k} \sum_{\substack{S \in \Phi_j \\ \mathcal{T} \in \Psi_j}} Z_{\alpha_S; \beta_{\mathcal{T}}, a, b}(t) \left(\frac{t}{2\pi}\right)^{-S-\mathcal{T}} dt + O(T^{1-\eta})$$

for some $\eta > 0$, where we have written $(t/2\pi)^{-S-\mathcal{T}}$ for $(t/2\pi)^{-\sum_{x \in S} x - \sum_{y \in \mathcal{T}} y}$ and

$$Z_{\alpha; \beta, a, b}(t) := \sum_{a m_1 \cdots m_k = b n_1 \cdots n_k} \frac{1}{\sqrt{ab} m_1^{1/2+\alpha_1} \cdots m_k^{1/2+\alpha_k} n_1^{1/2+\beta_1} \cdots n_k^{1/2+\beta_k}} V\left(\frac{m_1 \cdots m_k n_1 \cdots n_k}{t^k}\right)$$

with

$$V(x) := \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} (2\pi)^{-ks} x^{-s} ds,$$

where $G(s)$ is an even entire function of rapid decay in any fixed strip $|\Re(s)| \leq C$ satisfying $G(0) = 1$.

With this assumption we can prove the main theorem.

Theorem 1.3. Let $k, m \in \mathbb{N}$ and assume the Riemann hypothesis and Conjecture 1.2 for k . As t varies in $T \leq t \leq 2T$, the distribution of $\log|\zeta(\frac{1}{2} + it)|$ is asymptotically Gaussian with mean $k \log \log T$ and variance $\frac{1}{2} \log \log T$, with respect to the weighted measure $|\zeta^{(m)}(\frac{1}{2} + it)|^{2k} dt$.

In particular, since Conjecture 1.2 is known in the cases $k = 1$ (see [2, 5] and, e.g., [3] for the easy modifications needed to account for the shifts) and $k = 2$ (see [4, 13]), we notice that Corollary 1.1 trivially follows from Theorem 1.3.

We notice that Theorem 1.3 shows that the m -th derivative has no effect in the weighted distribution of $\log|\zeta(\frac{1}{2} + it)|$. This is consistent with the conjecture (see [12, Conjecture 6.1] and [8])

$$\int_T^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k-2h} \left| \zeta'\left(\frac{1}{2} + it\right) \right|^{2h} dt \sim c(h, k) T (\log T)^{k^2+2h},$$

which indicates that $|\zeta'(\frac{1}{2} + it)|^{2h}$ amplifies the contribution coming from the large values in the same way as $|\zeta(\frac{1}{2} + it)|^{2h}$ would do (up to a normalization of $(\log T)^{2h}$). More generally, as far as moments are concerned, the m -th derivative of zeta should behave like zeta itself (see, e.g., [7]), up to a normalization of $\log^m T$, in accordance with Theorem 1.3. We also note that, while in Selberg's classical case the mean is 0 because the contribution of the small values and that of the large values of zeta balance out, by tilting with $|\zeta(\frac{1}{2} + it)|^{2k}$, the mean of $\log|\zeta(\frac{1}{2} + it)|$ moves to the right as k grows, and this reflects the fact that the measure $|\zeta(\frac{1}{2} + it)|^{2k} dt$ gives more and more weight to the large values of the Riemann zeta function.

Moreover, we look at the shifted weighted measure $|\zeta(\frac{1}{2} + it + i\alpha)|^{2k} dt$ with a real number α such that $|\alpha| < 1$. As we will see, the distribution of $\log|\zeta(\frac{1}{2} + it)|$ is quite sensitive to the parameter α . Indeed, in computing the integral

$$\int_T^{2T} \log \left| \zeta\left(\frac{1}{2} + it\right) \right| \left| \zeta\left(\frac{1}{2} + it + i\alpha\right) \right|^{2k} dt, \quad (1.4)$$

one expects the same magnitude as in the unshifted case if $|\alpha|$ is smaller than $(\log T)^{-1}$, which is the typical scale for the Riemann zeta function. On the other hand, if $|\alpha|$ is larger, then the two factors in the integral (1.4) start decorrelating, and thus the size of the integral decreases. This phenomenon is shown in the following result, in which we use the notation

$$\bar{\mu}_\alpha = \begin{cases} \log \log T + O(1) & \text{if } |\alpha| \log T \leq 1, \\ -\log |\alpha| + O(1) & \text{if } |\alpha| \log T > 1, \end{cases}$$

for any $\alpha \in (-1, 1)$.

Theorem 1.4. Let $k \in \mathbb{N}$ and assume the Riemann hypothesis and Conjecture 1.2 for k . As t varies in $T \leq t \leq 2T$, for any fixed and real α such that $|\alpha| < 1$, the distribution of $\log|\zeta(\frac{1}{2} + it)|$ is asymptotically Gaussian with mean $k\bar{\mu}_\alpha$ and variance $\frac{1}{2} \log \log T$, with respect to the measure $|\zeta(\frac{1}{2} + it + i\alpha)|^{2k} dt$.

This theorem shows that the shift has no effect if it is smaller than $(\log T)^{-1}$. On the contrary, for larger values of the shift the mean gets smaller; for instance, if

$$\alpha = \frac{(\log T)^\delta}{\log T}$$

with $\delta \in (0, 1)$, then the mean is $\sim (1 - \delta)k \log \log T$. In this shifted case too, if $k \leq 2$, then Theorem 1.4 holds assuming RH only.

Lastly, we show that in the random matrix theory setting, an analogous weighted central limit theorem can be proved unconditionally. We consider the characteristic polynomials

$$Z = Z(U, \theta) = \det(I - Ue^{-i\theta})$$

of $N \times N$ unitary matrices U and we investigate their distribution of values with respect to the circular unitary ensemble (CUE). It has been conjectured that the limiting distribution of the non-trivial zeros of the Riemann

zeta function, on the scale of their mean spacing, is the same as that of the eigenphases θ_n of matrices in the CUE in the limit as $N \rightarrow \infty$ (see, e.g., [14]). Then we consider a tilted version of the Haar measure and we have the following theorem.

Theorem 1.5. *As $N \rightarrow \infty$, the value distribution of $\log|Z|$ is asymptotically Gaussian with mean $k \log N$ and variance $\frac{1}{2} \log N$ with respect to the measure $|Z|^{2k} d_{\text{Haar}}$.*

As usual, the correspondence with Theorem 1.3 holds if we identify the mean density of the eigenangles θ_n , $N/2\pi$, with the mean density of the Riemann zeros at a height T up the critical line $\frac{1}{2\pi} \log \frac{T}{2\pi}$, i.e. if

$$N = \log \frac{T}{2\pi}.$$

2 Proof of Theorems 1.3 and 1.4

To prove both the theorems, we introduce a set of shifts $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k$ and we denote for the sake of brevity

$$\zeta_{\alpha, \beta}(t) := \zeta\left(\frac{1}{2} + \alpha_1 + it\right) \cdots \zeta\left(\frac{1}{2} + \alpha_k + it\right) \zeta\left(\frac{1}{2} + \beta_1 - it\right) \cdots \zeta\left(\frac{1}{2} + \beta_k - it\right).$$

The general strategy of the proof is similar to the one of [9, Theorem 1], but here we avoid the detailed combinatorial analysis we performed in [9], by working with Euler products instead of Dirichlet series, inspired by [1, Proposition 5.1]. The first step is then approximating the logarithm of the Riemann zeta function with a suitable Dirichlet polynomial. Let us set

$$\tilde{P}(t) := \sum_{p \leq x} \frac{1}{p^{1/2+it}},$$

where $x := T^\varepsilon$, with $\varepsilon := (\log \log \log T)^{-1}$. Now we show that $\log|\zeta(\frac{1}{2} + it)|$ has the same distribution as $\Re \tilde{P}(t)$ with respect to the measure $\zeta_{\alpha, \beta}(t) dt$. This is achieved in the following proposition by bounding the second moment of the difference.

Proposition 2.1. *For a non-negative integer k , assume Conjecture 1.2 and RH. Let T be a large parameter and let $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \mathbb{C}$ such that $|\alpha_i|, |\beta_j| < 1$, $|\Re(\alpha_i)|, |\Re(\beta_j)| \leq (\log T)^{-1}$ and $|\alpha_i - \beta_j| \ll (\log T)^{-1}$ for all $1 \leq i, j \leq k$. Then we have*

$$\int_T^{2T} \left| \log \left| \zeta\left(\frac{1}{2} + it\right) \right| - \Re \tilde{P}(t) \right|^2 \zeta_{\alpha, \beta}(t) dt \ll_k T (\log T)^{k^2} (\log \log \log T)^{5/2}.$$

Proof. From Tsang's work [20, equation (5.15)], we know that

$$\log \zeta\left(\frac{1}{2} + it\right) - \tilde{P}(t) = S_1 + S_2 + S_3 + O(R) - L(t)$$

with

$$\begin{aligned} S_1 &:= \sum_{p \leq x} (p^{-1/2-4/\log x} - p^{-1/2}) p^{-it}, \\ S_2 &:= \sum_{\substack{p' \leq x \\ r \geq 2}} \frac{p^{-r(1/2+4/\log x+it)}}{r}, \\ S_3 &:= \sum_{x < n \leq x^3} \frac{\Lambda(n)}{\log n} n^{-1/2-4/\log x-it}, \\ R &:= \frac{5}{\log x} \left(\left| \sum_{n \leq x^3} \frac{\Lambda(n)}{n^{1/2+4/\log x+it}} \right| + \log T \right), \\ L(t) &:= \sum_p \int_{1/2}^{1/2+4/\log x} \left(\frac{1}{2} + \frac{4}{\log x} - u \right) \frac{1}{u+it-\rho} \frac{1}{\frac{1}{2} + \frac{4}{\log x} - \rho} du, \end{aligned}$$

where the sum in the definition of $L(t)$ is over all non-trivial zeros of ζ . Then we have to bound the second moments of the terms on the right-hand side with respect to the weighted measure $\zeta_{\alpha,\beta}(t) dt$ by using Conjecture 1.2 (note that we are allowed to apply the conjecture since the shifts are small up to a change of variable, being $|\alpha_i - \beta_j| \ll (\log T)^{-1}$).

Let us start with

$$\begin{aligned} \int_T^{2T} |S_1|^2 \zeta_{\alpha,\beta}(t) dt &\ll \left| \int_T^{2T} \sum_{0 \leq j \leq k} \sum_{\substack{\mathcal{S} \in \Phi_j \\ \mathcal{T} \in \Psi_j}} \left(\frac{t}{2\pi} \right)^{-\mathcal{S}-\mathcal{T}} \sum_{p,q \leq x} (p^{-4/\log x} - 1)(q^{-4/\log x} - 1) Z_{\alpha,\beta,\mathcal{T},p,q}(t) dt \right| \\ &\ll \int_T^{2T} \sum_{0 \leq j \leq k} \sum_{\substack{\mathcal{S} \in \Phi_j \\ \mathcal{T} \in \Psi_j}} \left| \sum_{a,b \leq x} \mathbb{1}_x(a) \mathbb{1}_x(b) (a^{-\frac{4}{\log x}} - 1)(b^{-\frac{4}{\log x}} - 1) Z_{\alpha,\beta,\mathcal{T},a,b}(t) \right| dt, \end{aligned}$$

where $\mathbb{1}_x(\cdot)$ is the indicator function of primes up to x . If we denote by $\Omega(n)$ the function which counts the number of prime factors of n with multiplicity, then we have

$$\sum_{n \leq x} \mathbb{1}_x(n) f(n) = \sum_{\substack{n: \\ p|n \Rightarrow p \leq x}} \partial_z [z^{\Omega(n)} f(n)]_{z=0}.$$

Hence we get

$$\int_T^{2T} |S_1|^2 \zeta_{\alpha,\beta}(t) dt \ll \int_T^{2T} \sum_{0 \leq j \leq k} \sum_{\substack{\mathcal{S} \in \Phi_j \\ \mathcal{T} \in \Psi_j}} \left| \partial_z \partial_w \left[\frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} \left(\frac{t}{2\pi} \right)^{ks} I_{\alpha,\beta}^{j,\mathcal{S},\mathcal{T}}(z, w; s) ds \right]_{z=0} \right|_{w=0} dt, \quad (2.1)$$

where

$$I_{\alpha,\beta}^{j,\mathcal{S},\mathcal{T}}(z, w; s) := \sum_{\substack{a,b: \\ p|ab \Rightarrow p \leq x}} z^{\Omega(a)} w^{\Omega(b)} g_x(a) g_x(b) \bar{Z}_{\alpha,\beta,\mathcal{T},a,b}(s) \quad (2.2)$$

with g_x being the multiplicative function defined by $g_x(p^a) = p^{-4a/\log x} - 1$ and

$$\bar{Z}_{\alpha,\beta,a,b}(s) := \sum_{am_1 \cdots m_k = bn_1 \cdots n_k} \frac{1}{\sqrt{ab} m_1^{1/2+\alpha_1+s} \cdots m_k^{1/2+\alpha_k+s} n_1^{1/2+\beta_1+s} \cdots n_k^{1/2+\beta_k+s}}. \quad (2.3)$$

We now analyze the first term $I_{\alpha,\beta}^0(z, w; s)$ and then we will see how to apply the method to deal with all others. Since the sum in the definition (2.2) is multiplicative, we have

$$\begin{aligned} I_{\alpha,\beta}^0(z, w; s) &= \sum_{\substack{am_1 \cdots m_k = bn_1 \cdots n_k: \\ p|ab \Rightarrow p \leq x}} \frac{z^{\Omega(a)} w^{\Omega(b)} g_x(a) g_x(b)}{\sqrt{ab} m_1^{\frac{1}{2}+\alpha_1+s} \cdots m_k^{\frac{1}{2}+\alpha_k+s} n_1^{\frac{1}{2}+\beta_1+s} \cdots n_k^{\frac{1}{2}+\beta_k+s}} \\ &= \prod_{p \leq x} \sum_{\substack{a+m_1+\cdots+m_k \\ =b+n_1+\cdots+n_k}} \frac{z^{\Omega(p^a)} w^{\Omega(p^b)} g_x(p^a) g_x(p^b)}{p^{\frac{a}{2}+\frac{b}{2}+m_1(\frac{1}{2}+\alpha_1+s)+\cdots+m_k(\frac{1}{2}+\alpha_k+s)+n_1(\frac{1}{2}+\beta_1+s)+\cdots+n_k(\frac{1}{2}+\beta_k+s)}} \\ &\quad \cdot \prod_{\substack{p > x \\ m_1+\cdots+m_k \\ =n_1+\cdots+n_k}} \frac{1}{p^{m_1(\frac{1}{2}+\alpha_1+s)+\cdots+m_k(\frac{1}{2}+\alpha_k+s)+n_1(\frac{1}{2}+\beta_1+s)+\cdots+n_k(\frac{1}{2}+\beta_k+s)}} \quad (2.4) \end{aligned}$$

By putting in evidence the first terms in the Euler products, this equals

$$\begin{aligned} A_{\alpha,\beta}(z, w; s) &\prod_{p \leq x} \left(1 + \frac{z w g_x(p)^2}{p} + \frac{z g_x(p)}{p^{1+\beta_1+s}} + \cdots + \frac{w g_x(p)}{p^{1+\alpha_k+s}} \right) \\ &\quad \cdot \prod_{p \leq x} \left(1 + \frac{1}{p^{1+\alpha_1+\beta_1+2s}} + \cdots + \frac{1}{p^{1+\alpha_k+\beta_k+2s}} \right) \prod_{p > x} \left(1 + \frac{1}{p^{1+\alpha_1+\beta_1+2s}} + \cdots + \frac{1}{p^{1+\alpha_k+\beta_k+2s}} \right) \\ &= A_{\alpha,\beta}^*(z, w; s) \prod_{i,j=1}^k \zeta(1 + \alpha_i + \beta_j + 2s) \cdot \exp \left(\sum_{p \leq x} \left\{ \frac{z w g_x(p)^2}{p} + \frac{z g_x(p)}{p^{1+\beta_1+s}} + \cdots + \frac{w g_x(p)}{p^{1+\alpha_k+s}} \right\} \right), \quad (2.5) \end{aligned}$$

where $A_{\alpha,\beta}(z, w; s)$ and $A_{\alpha,\beta}^*(z, w; s)$ are arithmetical factors (Euler products) converging absolutely in a half-plane $\Re(s) > -\delta$ for some $\delta > 0$, uniformly for $|z|, |w| \leq 1$, such that their derivatives with respect to z and w at 0 also converge in the same half-plane. We now have extracted the polar part, and hence we are ready to shift the integral over s in (2.1) to the left of zero. To do so, it is convenient to prescribe the same conditions as in [4, remarks after Lemma 2.1], assuming that $G(s)$ vanishes at $s = -\frac{\alpha_i + \beta_j}{2}$ for all i, j , so that the only pole we pick in the contour shift is at $s = 0$. Moreover, we assume that the shifts are such that $|\alpha_i + \beta_j| \gg (\log T)^{-1}$ for every $1 \leq i, j \leq k$, so that

$$\prod_{i,j=1}^k |\zeta(1 + \alpha_i + \beta_j)| \ll_k (\log T)^{k^2}. \quad (2.6)$$

Hence by (2.4)–(2.6), we get

$$\begin{aligned} \partial_z \partial_w \left[\frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} \left(\frac{t}{2\pi} \right)^{ks} I_{\alpha,\beta}^0(z, w; s) ds \right]_{\substack{z=0 \\ w=0}} \\ \ll_k (\log T)^{k^2} \left| \partial_z \partial_w \left[\exp \left(\sum_{p \leq x} \left\{ \frac{zw g_x(p)^2}{p} + \frac{z g_x(p)}{p^{1+\beta_1}} + \dots + \frac{w g_x(p)}{p^{1+\alpha_k}} \right\} \right) \right]_{\substack{z=0 \\ w=0}} \right| \\ \ll_k (\log T)^{k^2} \left(\sum_{p \leq x} \frac{|g_x(p)|}{p} \right)^2 \\ \ll_k (\log T)^{k^2} (\log \log \log T)^2, \end{aligned}$$

where (see, e.g., [9, p. 5])

$$|\Re(\alpha_i)|, |\Re(\beta_j)| \leq (\log T)^{-1} \quad \text{and} \quad \sum_{p \leq x} \frac{|p^{-4/\log x} - 1|}{p} \ll \log \log \log T.$$

This last bound can be obtained by using the Taylor approximation $e^{-z} = 1 + O(z)$ for $z \ll 1$, which yields

$$\sum_{p \leq x} \frac{|p^{-4/\log x} - 1|}{p} = \sum_{p \leq x} \frac{|p^{-4 \log p / \log x} - 1|}{p} \ll \frac{1}{\log x} \sum_{p \leq x} \frac{\log p}{p} \ll 1$$

by Merten's first theorem. All the other terms $I_{\alpha,\beta}^{j,S,\mathcal{T}}(z, w)$ can be treated exactly in the same way as $I_{\alpha,\beta}^0(z, w)$ by assuming that $|\alpha_i \pm \beta_j| \gg (\log T)^{-1}$ for every i, j , since they only differ from the first case by permutations and changes of signs of the shifts. Therefore, we get

$$\partial_z \partial_w \left[\frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} \left(\frac{t}{2\pi} \right)^{ks} I_{\alpha,\beta}^{j,S,\mathcal{T}}(z, w; s) ds \right]_{\substack{z=0 \\ w=0}} \ll_k (\log T)^{k^2} (\log \log \log T)^2$$

provided that

$$|\alpha_i \pm \beta_j| \gg (\log T)^{-1} \quad \text{for every } 1 \leq i, j \leq k. \quad (2.7)$$

Plugging this into (2.1), we prove that

$$\int_T^{2T} |S_1|^2 \zeta_{\alpha,\beta}(t) dt \ll_k T (\log T)^{k^2} (\log \log \log T)^2. \quad (2.8)$$

Moreover, since the left-hand side in (2.8) is holomorphic in terms of the shifts, although we have proved the above for α_i, β_j such that (2.7) holds, the maximum modulus principle can be applied to obtain the bound to the enlarged domain we need.

We will treat the other pieces similarly. As regards the second one, we have that

$$\int_T^{2T} |S_2|^2 \zeta_{\alpha,\beta}(t) dt \ll \int_T^{2T} \sum_{0 \leq j \leq k} \sum_{\substack{S \in \Phi_j \\ \mathcal{T} \in \Psi_j}} \left| \sum_{\substack{p_1^{r_1}, p_2^{r_2} \leq x \\ r_1, r_2 \geq 2}} \frac{p_1^{-\frac{4r_1}{\log x}}}{r_1} \frac{p_2^{-\frac{4r_2}{\log x}}}{r_2} Z_{\alpha_S, \beta_{\mathcal{T}}, p_1^{r_1}, p_2^{r_2}}(t) \right| dt.$$

As before, we analyze the first term only since all others are completely analogous. The term for $j = 0$ is

$$\frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} \left(\frac{t}{2\pi}\right)^{ks} \sum_{\substack{p_1^{r_1} m_1 \dots m_k = p_2^{r_2} n_1 \dots n_k: \\ p_1^{r_1}, p_2^{r_2} \leq x, r_1, r_2 \geq 2}} \frac{p_1^{-\frac{4r_1}{\log x}} p_2^{-\frac{4r_2}{\log x}}}{r_1 r_2} \frac{1}{\sqrt{p_1^{r_1} p_2^{r_2} m_1^{\frac{1}{2} + \alpha_1 + s} \dots n_k^{\frac{1}{2} + \beta_k + s}}} ds.$$

This time, because of the condition $r_1, r_2 \geq 2$, when we estimate the sum via the first terms of its Euler product, we just get that the above is

$$\ll_k \prod_{i,j=1}^k |\zeta(1 + \alpha_i + \beta_j)|$$

Applying the same machinery as before, this yields

$$\int_T^{2T} |S_2|^2 \zeta_{\alpha,\beta}(t) dt \ll_k T(\log T)^{k^2}.$$

We use the same approach in order to bound the second moment of S_3 as well, which is

$$\begin{aligned} & \ll_k \int_T^{2T} \left| \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} \left(\frac{t}{2\pi}\right)^{ks} \sum_{\substack{am_1 \dots m_k = bn_1 \dots n_k \\ x < a, b < x^3}} \frac{\Lambda(a)\Lambda(b)}{\log a \log b} \frac{a^{-4/\log x} b^{-4/\log x}}{\sqrt{abm_1^{\frac{1}{2} + \alpha_1 + s} \dots n_k^{\frac{1}{2} + \beta_k + s}}} \right| dt \\ & \ll_k T \left(\sum_{x < p \leq x^3} \frac{1}{p} \right)^2 \prod_{i,j=1}^k |\zeta(1 + \alpha_i + \beta_j)| \\ & \ll_k T(\log T)^{k^2} \end{aligned}$$

since the sum $\sum_{x < p \leq x^3} p^{-1}$ is bounded.

We deal with R in the same way and we get that

$$\int_T^{2T} |R|^2 \zeta_{\alpha,\beta}(t) dt \ll_k T(\log T)^{k^2} (\log \log \log T)^2,$$

where the extra factor with the triple log comes from the second term in the definition of R , while the first one can be treated analogously to S_3 .

Finally, we have to bound the second moment of $\Re L(t)$. To do so, in view of equations [9, (2.8) and (2.9)], it suffices to study

$$\frac{1}{(\log x)^2} \int_T^{2T} \left(\log^+ \frac{1}{\eta_t \log x} \right)^2 \left(\left| \sum_{n \leq x^3} \frac{\Lambda(n) n^{-4/\log x}}{n^{1/2+it}} \right| + \log T \right)^2 \zeta_{\alpha,\beta}(t) dt,$$

where $\eta_t := \min_\rho |t - \gamma|$ and $\log^+ t := \max(\log t, 0)$, with the aim of proving that this is

$$\ll_k T(\log T)^{k^2} (\log \log \log T)^{5/2}.$$

By applying the Cauchy–Schwarz inequality, the above is

$$\begin{aligned} & \leq \left(\int_T^{2T} \left(\frac{1}{\log x} \left| \sum_{n \leq x^3} \frac{\Lambda(n) n^{-4/\log x}}{n^{1/2+it}} \right| + \frac{1}{\varepsilon} \right)^4 \left| \zeta\left(\frac{1}{2} + \beta_1 - it\right) \right|^2 \dots \left| \zeta\left(\frac{1}{2} + \beta_k - it\right) \right|^2 dt \right)^{\frac{1}{2}} \\ & \quad \cdot \left(\int_T^{2T} \left(\log^+ \frac{1}{\eta_t \log x} \right)^4 \left| \zeta\left(\frac{1}{2} + \alpha_1 + it\right) \right|^2 \dots \left| \zeta\left(\frac{1}{2} + \alpha_k + it\right) \right|^2 dt \right)^{\frac{1}{2}} \end{aligned} \quad (2.9)$$

and the first term can be treated as R above and it is

$$\ll_k \sqrt{T(\log T)^{k^2} (\log \log \log T)^4}.$$

We now conclude the proof, bounding the second term in (2.9). If we denote

$$\mathcal{T} := \left[T - \frac{1}{\log x}, 2T + \frac{1}{\log x} \right],$$

then we have

$$\begin{aligned} & \int_T^{2T} \left(\log^+ \frac{1}{\eta_t \log x} \right)^4 \left| \zeta \left(\frac{1}{2} + \alpha_1 + it \right) \right|^2 \cdots \left| \zeta \left(\frac{1}{2} + \alpha_k + it \right) \right|^2 dt \\ & \leq \sum_{y \in \mathcal{T}} \int_{-1/\log x}^{1/\log x} \left(\log^+ \frac{1}{|w| \log x} \right)^4 \prod_{j=1}^k \left| \zeta \left(\frac{1}{2} + \alpha_j + i(w + y) \right) \right|^2 dw \\ & = \sum_{y \in \mathcal{T}} \int_{-1}^1 \left(\log^+ \frac{1}{|t|} \right)^4 \prod_{j=1}^k \left| \zeta \left(\frac{1}{2} + \alpha_j + i \left(y + \frac{t}{\log x} \right) \right) \right|^2 \frac{dt}{\log x} \\ & \ll \frac{1}{\log x} \int_{-1}^1 (\log |t|)^4 \prod_{j=1}^k \left(\sum_{y \in \mathcal{T}} \left| \zeta \left(\frac{1}{2} + iy + \alpha_j + i \frac{t}{\log x} \right) \right|^{2k} \right)^{1/k} dt \end{aligned} \quad (2.10)$$

by the Hölder inequality. The remaining sum can be bounded under RH in view of Kirila's [15, Theorem 1.2] and Milinovich's [16] works, which generalize the well-known result due to Gonek about the sum over the non-trivial zeros of zeta [10, Corollary 1]. Indeed, since the shifts $\alpha_j + i \frac{t}{\log x}$ in the sum over zeros in (2.10) have modulus ≤ 1 and real part $\leq (\log T)^{-1}$ in absolute value, then we have

$$\sum_{y \in \mathcal{T}} \left| \zeta \left(\frac{1}{2} + iy + \alpha_j + i \frac{t}{\log x} \right) \right|^{2k} \ll_k T \log T (\log T)^{k^2}$$

for every $j = 1, \dots, k$. Putting this into (2.10), we get

$$\int_T^{2T} \left(\log^+ \frac{1}{\eta_t \log x} \right)^4 \left| \zeta \left(\frac{1}{2} + \alpha_1 + it \right) \right|^2 \cdots \left| \zeta \left(\frac{1}{2} + \alpha_k + it \right) \right|^2 dt \ll_k \frac{T \log T}{\log x} (\log T)^{k^2}. \quad (2.11)$$

Plugging (2.11) into (2.9), we prove that

$$\begin{aligned} & \int_T^{2T} |\mathbb{R}L(t)|^2 \zeta_{\alpha, \beta}(t) dt \ll_k \sqrt{T(\log T)^{k^2} (\log \log \log T)^4} \sqrt{T(\log T)^{k^2} \log \log \log T} \\ & \ll_k T(\log T)^{k^2} (\log \log \log T)^{5/2}, \end{aligned}$$

concluding the proof of the proposition. $\square$

The second step is getting rid of the small primes, showing that their contribution does not affect the distribution asymptotically. This simple fact will simplify the third and last step of the proof, as we will see in the following. Let us define

$$P(t) := \sum_{p \in X} \frac{1}{p^{1/2+it}},$$

where $X := (\log T, x]$ (we recall that $x = T^\varepsilon$ and $\varepsilon = (\log \log \log T)^{-1}$).

Proposition 2.2. *For a non-negative integer k , assume Conjecture 1.2 and RH. Let T be a large parameter and let $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \mathbb{C}$ such that $|\alpha_i|, |\beta_j| < 1$, $|\Re(\alpha_i)|, |\Re(\beta_j)| \leq (\log T)^{-1}$ and $|\alpha_i - \beta_j| \ll (\log T)^{-1}$ for all $1 \leq i, j \leq k$. Then we have*

$$\int_T^{2T} |\mathbb{R}\tilde{P}(t) - \mathbb{R}P(t)|^2 \zeta_{\alpha, \beta}(t) dt \ll_k T(\log T)^{k^2} (\log \log \log T)^2.$$

Proof. This can be proved with the same method used in Proposition 2.1. We recall that $\mathbb{1}_{\log T}(\cdot)$ denotes the indicator function of primes up to $\log T$. We start by studying

$$\frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} \left(\frac{t}{2\pi}\right)^{ks} \sum_{\substack{am_1 \dots m_k \\ =bm_1 \dots m_k}} \frac{\mathbb{1}_{\log T}(a) \mathbb{1}_{\log T}(b)}{\sqrt{ab} m_1^{\frac{1}{2} + \alpha_1 + s} \dots n_k^{\frac{1}{2} + \beta_k + s}} ds \quad (2.12)$$

and, as usual, we estimate the sum with the first terms of its Euler product, extract the polar part, shift the integral over s to the left, getting that (2.12) is

$$\ll_k \prod_{i,j=1}^k |\zeta(1 + \alpha_i + \beta_j)| \cdot \left| \partial_z \partial_w \left[\exp \left(\sum_{p \leq \log T} \left\{ \frac{zw}{p} + \frac{z}{p^{1+\beta_1}} + \dots + \frac{w}{p^{1+\alpha_k}} \right\} \right) \right] \right|_{\substack{z=0 \\ w=0}}.$$

By the same argument as in the proof of Proposition 2.1 the above is

$$\ll_k (\log T)^{k^2} \left(\sum_{p \leq \log T} \frac{1}{p} \right)^2 \ll (\log T)^{k^2} (\log \log \log T)^2,$$

and this concludes the proof. $\square$

Finally, we investigate the distribution of the polynomial $\mathbb{R}P(t)$, which, thanks to Propositions 2.1 and 2.2, has the same distribution as $\log |\zeta(\frac{1}{2} + it)|$. The most natural method to do so is studying the moments, and this is achieved in the following result.

Proposition 2.3. *For a non-negative integer k , assume Conjecture 1.2 and RH. Let T be a large parameter and let $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \mathbb{C}$ such that $|\alpha_i|, |\beta_j| < 1$, $|\mathbb{R}(\alpha_i)|, |\mathbb{R}(\beta_j)| \leq (\log T)^{-1}$ and $|\alpha_i - \beta_j| \ll (\log T)^{-1}$ for all $1 \leq i, j \leq k$. Set $\mathcal{L} := \sum_{p \in X} \frac{1}{p} \sim \log \log T$ and $\mu \in \mathbb{R}$ such that $\mu \ll \log \log T$. Then for every fixed integer n we have*

$$\begin{aligned} & \int_T^{2T} (\mathbb{R}P(t) - k\mu)^n \zeta_{\alpha, \beta}(t) dt \\ &= \int_T^{2T} \sum_{0 \leq j \leq k} \sum_{\substack{\mathcal{S} \in \Phi_j \\ \mathcal{T} \in \Psi_j}} \left(\frac{t}{2\pi}\right)^{-\mathcal{S} - \mathcal{T}} M_{\alpha_{\mathcal{S}}, \beta_{\mathcal{T}}}(0) \partial_z^n \left[e^{\frac{z^2}{4} \mathcal{L} - kz\mu} \exp \left(\frac{z}{2} \sum_{p \in X} \frac{g_p(\mathcal{S}, \mathcal{T})}{p} \right) \right]_{z=0} dt + O_{k,n}(T(\log T)^{k^2-1+\varepsilon}) \end{aligned}$$

where

$$g_p(\mathcal{S}, \mathcal{T}) := \sum_{x_1 \notin \mathcal{S}} p^{-x_1} + \sum_{x_2 \notin \mathcal{T}} p^{-x_2} + \sum_{x_3 \in \mathcal{S}} p^{x_3} + \sum_{x_4 \in \mathcal{T}} p^{x_4}$$

and

$$M_{\alpha, \beta}(s) := \sum_{m_1 \dots m_k = n_1 \dots n_k} \frac{1}{m_1^{\frac{1}{2} + \alpha_1 + s} \dots m_k^{\frac{1}{2} + \alpha_k + s} n_1^{\frac{1}{2} + \beta_1 + s} \dots n_k^{\frac{1}{2} + \beta_k + s}},$$

so that $M_{\alpha, \beta}(0)$ is the first term of the moment of $\zeta_{\alpha, \beta}$, predicted by the recipe [6] (the sum which defines $M_{\alpha, \beta}(s)$ does not converge for $s = 0$, so we appeal to [6, Theorem 2.4.1] for the analytic continuation).

Proof. Expanding out the powers, since $2\mathbb{R}(z) = z + \bar{z}$, we get

$$\int_T^{2T} (\mathbb{R}P(t) - k\mu)^n \zeta_{\alpha, \beta}(t) dt = \sum_{j+h=n} \binom{n}{h} (-k\mu)^j 2^{-h} \sum_{r+s=h} \binom{h}{r} \int_T^{2T} P(t)^r \overline{P(t)}^s \zeta_{\alpha, \beta}(t) dt. \quad (2.13)$$

By using Conjecture 1.2 and ignoring the error term which is negligible in this context, the inner integral in (2.13) is

$$\int_T^{2T} \sum_{0 \leq j \leq k} \sum_{\substack{\mathcal{S} \in \Phi_j \\ \mathcal{T} \in \Psi_j}} \left(\frac{t}{2\pi}\right)^{-\mathcal{S} - \mathcal{T}} \sum_{a, b} \mathbb{I}_X^{*r}(a) \mathbb{I}_X^{*s}(b) Z_{\alpha_{\mathcal{S}}, \beta_{\mathcal{T}}, a, b}(t) dt, \quad (2.14)$$

where $\mathbb{1}_X^{*r}(a)$ denotes the indicator function of primes in the interval X , self-convoluted r times. If we define temporarily the multiplicative function g given by $g(p^n) = 1/n!$, then the inner sum over a, b in (2.14) equals

$$\partial_z^r \partial_w^s \left[\sum_{\substack{a,b: \\ p|ab \Rightarrow p \in X}} z^{\Omega(a)} w^{\Omega(b)} g(a)g(b) Z_{\alpha_s, \beta_{\mathcal{T}}, a, b}(t) \right]_{\substack{z=0 \\ w=0}}. \quad (2.15)$$

Then, plugging (2.14) and (2.15) into (2.13) and recollecting together the powers we expanded before, we get

$$\int_T^{2T} \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} \left(\frac{t}{2\pi} \right)^{ks} \sum_{0 \leq j \leq k} \sum_{\substack{s \in \Phi_j \\ \mathcal{T} \in \Psi_j}} \left(\frac{t}{2\pi} \right)^{-s-\mathcal{T}} \partial_z^n [e^{-k\mu z} I_{\alpha, \beta}^{j, s, \mathcal{T}}(z; s)]_{z=0} ds dt, \quad (2.16)$$

with

$$I_{\alpha, \beta}^{j, s, \mathcal{T}}(z; s) := \sum_{\substack{a,b: \\ p|ab \Rightarrow p \in X}} \left(\frac{z}{2} \right)^{\Omega(a)+\Omega(b)} g(a)g(b) \tilde{Z}_{\alpha_s, \beta_{\mathcal{T}}, a, b}(s);$$

see (2.3) for the definition of $\tilde{Z}_{\alpha, \beta, a, b}(s)$. Now study the first term in (2.16), i.e. $j = 0$, since all other terms can be understood from the first one with a slight modification. Then we look at

$$\begin{aligned} I_{\alpha, \beta}^0(z; s) &= \sum_{\substack{a m_1 \dots m_k = b n_1 \dots n_k \\ p|ab \Rightarrow p \in X}} \frac{(z/2)^{\Omega(a)+\Omega(b)} g(a)g(b)}{\sqrt{ab} m_1^{\frac{1}{2}+\alpha_1+s} \dots m_k^{\frac{1}{2}+\alpha_k+s} n_1^{\frac{1}{2}+\beta_1+s} \dots n_k^{\frac{1}{2}+\beta_k+s}} \\ &= \prod_{\substack{p \in X \\ a+m_1+\dots+m_k \\ =b+n_1+\dots+n_k}} \frac{(z/2)^{a+b} g(p^a)g(p^b)}{p^{\frac{a}{2}+\frac{b}{2}+m_1(\frac{1}{2}+\alpha_1+s)+\dots+n_k(\frac{1}{2}+\beta_k+s)}} \prod_{\substack{p \notin X \\ m_1+\dots+m_k \\ =n_1+\dots+n_k}} \frac{1}{p^{m_1(\frac{1}{2}+\alpha_1+s)+\dots+n_k(\frac{1}{2}+\beta_k+s)}} \\ &= \exp \left(\sum_{p \in X} \left\{ \frac{(z/2)^2}{p} + \frac{z/2}{p^{1+\alpha_1+s}} + \dots + \frac{z/2}{p^{1+\alpha_k+s}} + \frac{z/2}{p^{1+\beta_1+s}} + \frac{z/2}{p^{1+\beta_k+s}} \right\} \right) F_{X, \alpha, \beta}(z; s) M_{\alpha, \beta}(s), \end{aligned}$$

where $F_{X, \alpha, \beta}(z; s)$ is an arithmetical factor (Euler product) converging absolutely in a product of half-planes containing the origin, such that $F_{X, \alpha, \beta}(z; 0)$ is holomorphic at $z = 0$, $F_{X, \alpha, \beta}(0, 0) = 1$ and all its derivatives at $z = 0$ are small, i.e.

$$\partial_z^c [F_{X, \alpha, \beta}(z, 0)]_{z=0} \ll (\log T)^{-1}$$

for any positive integer c . Now we want to shift the integral over s in (2.16) to the left of zero, picking the contribution of the (only) pole at $s = 0$. To do so, we appeal to the meromorphic continuation of the function $M_{\alpha, \beta}(s)$; see [6, Theorem 2.4.1]. Thus we can shift the path of integration to the vertical line (say) $\Re(s) = -\frac{1}{10}$, where the integral is trivially bounded by $\ll T^{1-1/10+\varepsilon}$ for any positive ε . Moreover, the contribution from the pole at $s = 0$ gives

$$T \partial_z^n \left[e^{-k\mu z} e^{\frac{z^2}{4}\mathcal{L}} \exp \left(\frac{z}{2} \sum_{p \in X} \frac{p^{-\alpha_1} + \dots + p^{-\beta_k}}{p} \right) F_{X, \alpha, \beta}(z; 0) \right]_{z=0} M_{\alpha, \beta}(0).$$

Thanks to the bounds for $\mu, \mathcal{L}$ and the derivatives of $F_{X, \alpha, \beta}$ being $F_{X, \alpha, \beta}(0; 0) = 1$ and $M_{\alpha, \beta}(0) \ll (\log T)^{k^2}$ (again this is due to an argument similar to the one in the proof of Proposition 2.1, when we assume extra conditions on the shifts and then we appeal to the maximum modulus principle), we get that the term for $j = 0$ in (2.16) equals

$$T \partial_z^n \left[e^{-k\mu z} e^{\frac{z^2}{4}\mathcal{L}} \exp \left(\frac{z}{2} \sum_{p \in X} \frac{p^{-\alpha_1} + \dots + p^{-\beta_k}}{p} \right) \right]_{z=0} M_{\alpha, \beta}(0) + O_{k, n}(T(\log T)^{k^2-1+\varepsilon}).$$

Analogously, the general term turns out to be

$$\int_T^{2T} \left(\frac{t}{2\pi} \right)^{-s-\mathcal{T}} \partial_z^n \left[e^{\frac{z^2}{4}\mathcal{L}-k\mu z} \exp \left(\frac{z}{2} \sum_{p \in X} \frac{g_p(s, \mathcal{T})}{p} \right) \right]_{z=0} M_{\alpha_s, \beta_{\mathcal{T}}}(0) dt + O_{k, n}(T(\log T)^{k^2-1+\varepsilon}),$$

and putting this into (2.14), i.e. summing over $j, s, \mathcal{T}$, we get the thesis. $\square$

Proof of Theorem 1.3. This proof follows easily from the three propositions we have proved above. If we take $\mu = \mathcal{L}$ and all shifts small enough, i.e.

$$\alpha_i, \beta_j \ll (\log T)^{-1} \quad \text{for any } i, j,$$

then the exponent on the right-hand side of the thesis in Proposition 2.3 becomes

$$\begin{aligned} \frac{z^2}{4} \mathcal{L} + \frac{z}{2} \sum_{p \in X} \frac{g_p(\mathcal{S}, \mathcal{T})}{p} - zk\mathcal{L} &= \frac{z^2}{4} \mathcal{L} + O\left(z \sum_{p \in X} \frac{|p^{-\alpha_1} - 1| + \dots + |p^{-\beta_k} - 1|}{p}\right) \\ &= \frac{z^2}{4} \mathcal{L} + O_k\left(z \sum_{p \in X} \frac{|p^{-\alpha_1} - 1|}{p}\right) \\ &= \frac{z^2}{4} \mathcal{L} + O_k(z) \end{aligned}$$

and does not depend on $\mathcal{S}$ and $\mathcal{T}$ asymptotically, in fact. Hence we can bring that factor outside and reconstruct the moment of $\zeta_{\alpha, \beta}$, as follows:

$$\begin{aligned} &\int_T^{2T} (\Re P(t) - k\mathcal{L})^n \zeta_{\alpha, \beta}(t) dt \\ &= \partial_z^n \left[e^{\frac{z^2}{4} \mathcal{L} + O_k(z)} \right]_{z=0} \int_T^{2T} \sum_{0 \leq j \leq k} \sum_{\substack{\mathcal{S} \in \Phi_j \\ \mathcal{T} \in \Psi_j}} \left(\frac{t}{2\pi} \right)^{-\mathcal{S} - \mathcal{T}} M_{\alpha, \beta, \mathcal{T}}(0) dt + O_{k,n}(T(\log T)^{k^2-1+\varepsilon}) \\ &= \left(\partial_z^n \left[e^{\frac{z^2}{4} \mathcal{L}} \right]_{z=0} + O_{k,n}((\log \log T)^{\frac{n-1}{2}}) \right) \int_T^{2T} \zeta_{\alpha, \beta}(t) dt + O_{k,n}(T(\log T)^{k^2-1+\varepsilon}). \end{aligned}$$

The thesis follows by analyzing

$$\partial_z^n \left[e^{\frac{z^2}{4} \mathcal{L}} \right]_{z=0} = \left[\sum_{m_1+2m_2=n} \frac{n!}{m_1!m_2!} \left(\frac{2z\mathcal{L}}{4} \right)^{m_1} \left(\frac{1}{2!} \frac{\mathcal{L}}{2} \right)^{m_2} \right]_{z=0}.$$

If n is odd then the coefficient

$$\partial_z^n \left[e^{\frac{z^2}{4} \mathcal{L}} \right]_{z=0}$$

vanishes, while if n is even, then only the term for $m_1 = 0$ and $m_2 = \frac{n}{2}$ survives and gives

$$\frac{n!}{(n/2)!} \left(\frac{\mathcal{L}}{4} \right)^{n/2} = (n-1)!! \left(\frac{\mathcal{L}}{2} \right)^{n/2},$$

i.e.

$$\frac{1}{\int_T^{2T} \zeta_{\alpha, \beta}(t) dt} \int_T^{2T} (\Re P(t) - k\mathcal{L})^n \zeta_{\alpha, \beta}(t) dt = \begin{cases} (1 + o_{k,n}(1))(n-1)!! \left(\frac{\mathcal{L}}{2} \right)^{n/2} & \text{if } n \text{ is even,} \\ o_{k,n}((\log \log T)^{n/2}) & \text{if } n \text{ is odd.} \end{cases}$$

This then matched with the Gaussian coefficient proves that, in the limit $T \rightarrow \infty$, $\Re P(t)$ has Gaussian distribution, with mean $k \log \log T$ and variance $\frac{1}{2} \log \log T$, and then so does $\log |\zeta(\frac{1}{2} + it)|$, in view of Propositions 2.1 and 2.2. Theorem 1.3 follows by taking the derivatives with respect to the shifts. $\square$

Proof of Theorem 1.4. To derive Theorem 1.4, in Proposition 2.3 we set

$$\alpha_1 = \dots = \alpha_k = i\alpha \quad \text{and} \quad \beta_1 = \dots = \beta_k = -i\alpha,$$

with $\alpha \in \mathbb{R}$, $|\alpha| < 1$, and we take μ as

$$\mu_\alpha := \sum_{p \in X} \frac{\cos(\alpha \log p)}{p} = \tilde{\mu}_\alpha + O(1) = \begin{cases} \log \log T + O(1) & \text{if } |\alpha| \log T \leq 1, \\ -\log |\alpha| + O(1) & \text{if } |\alpha| \log T > 1, \end{cases}$$

by partial summation. Then we get

$$\int_T^{2T} (\Re P(t) - k\mu_\alpha)^n \left| \zeta\left(\frac{1}{2} + i\alpha + it\right) \right|^{2k} dt = (1 + o_{k,n}(1)) \partial_z^n [e^{\frac{z^2}{4}\mathcal{L}}]_{z=0} \int_T^{2T} \left| \zeta\left(\frac{1}{2} + i\alpha + it\right) \right|^{2k} dt$$

since the quantity

$$\frac{z}{2} \sum_{p \in X} \frac{g_p(\mathcal{S}, \mathcal{T})}{p} - zk\mu_\alpha$$

vanishes for all $\mathcal{S}, \mathcal{T}$ for this choice of the shifts. The thesis follows as in the proof of Theorem 1.3. $\square$

3 Proof of Theorem 1.5

In the usual notations we set in the introduction, let us define the moment generating function

$$M_N(s) = \langle |Z|^s \rangle = \sum_{j=0}^{\infty} \frac{\langle (\log |Z|)^j \rangle}{j!} s^j, \quad (3.1)$$

where the mean has to be considered over the group $U(N)$ with respect to the Haar measure, and the cumulants $Q_j = Q_j(N)$ by

$$\log M_N(s) = \sum_{j=1}^{\infty} \frac{Q_j}{j!} s^j.$$

In [14], among other things, Keating and Snaith studied the cumulants showing that

$$Q_n = \begin{cases} 0 & \text{if } n = 1, \\ \frac{1}{2} \log N + O(1) & \text{if } n = 2, \\ O(1) & \text{if } n \geq 3, \end{cases}$$

and deduced a central limit theorem proving that the limiting distribution of $\log |Z|$ is Gaussian with mean 0 and variance $\frac{1}{2} \log N$. Here, for any $k \in \mathbb{N}$, we study the distribution of the random variable $\log |Z|$ with respect to the tilted measure $|Z|^{2k} d_{\text{Haar}}$. Before starting with our analysis, we recall that the moments of $|Z|$ are known also for a non-negative integer k (see [14, (6) and (16)]):

$$M_N(2k) = \langle |Z|^{2k} \rangle = \exp\left(\sum_{j=1}^{\infty} \frac{(2k)^j}{j!} Q_j\right) = \prod_{j=1}^N \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2} \sim N^{k^2} \frac{G^2(1+k)}{G(1+2k)}, \quad (3.2)$$

where $k \in \mathbb{R}$ and G denotes the Barnes G -function. We set

$$\mathcal{M}_{2k} := \prod_{j=1}^N \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2}.$$

Now we are ready to consider the first moment

$$\langle |Z|^{2k} \log |Z| \rangle = \frac{d}{dx} [\langle |Z|^{2k+x} \rangle]_{x=0} = \frac{d}{dx} \left[\prod_{j=1}^N \frac{\Gamma(j)\Gamma(j+2k+x)}{\Gamma(j+k+\frac{x}{2})^2} \right]_{x=0}$$

by (3.1) and (3.2). We compute the derivative by Leibniz's rule, writing

$$\prod_{j=1}^N \frac{\Gamma(j)\Gamma(j+2k+x)}{\Gamma(j+k+\frac{x}{2})^2} = \exp\left(\sum_{j=1}^N \left\{ \log \Gamma(j) + \log \Gamma(j+2k+x) - 2 \log \Gamma\left(j+k+\frac{x}{2}\right) \right\}\right),$$

and we get

$$\langle |Z|^{2k} \log |Z| \rangle = \prod_{j=1}^N \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2} \sum_{j=1}^N \left\{ \frac{\Gamma'}{\Gamma}(j+2k) - \frac{\Gamma'}{\Gamma}\left(j+k\right) \right\}. \quad (3.3)$$

Moreover, an application of Stirling's formula yields

$$\frac{\Gamma'}{\Gamma}(j+2k) - \frac{\Gamma'}{\Gamma}(j+k) = \frac{k}{j} + O_k\left(\frac{1}{j^2}\right). \quad (3.4)$$

Hence, by (3.3) and (3.4), the weighted mean of the random variable $\log|Z|$ is

$$\mu_{2k} := \frac{1}{\mathcal{M}_{2k}} \langle |Z|^{2k} \log|Z| \rangle = k \log N + O_k(1).$$

Then we study the weighted n -th moment of the random variable $\log|Z|$:

$$\begin{aligned} \langle |Z|^{2k} (\log|Z| - \mu_{2k})^n \rangle &= \sum_{h+j=n} \binom{n}{h} (-\mu_{2k})^j \langle |Z|^{2k} (\log|Z|)^h \rangle \\ &= \sum_{h+j=n} \binom{n}{h} \frac{d^j}{dx^j} [e^{-x\mu_{2k}}]_{x=0} \frac{d^h}{dx^h} \left[\prod_{j=1}^N \frac{\Gamma(j)\Gamma(j+2k+x)}{\Gamma(j+k+\frac{x}{2})^2} \right]_{x=0} \\ &= \frac{d^n}{dx^n} \left[\exp\left(-x\mu_{2k} + \sum_{j=1}^N \log\left(\frac{\Gamma(j)\Gamma(j+2k+x)}{\Gamma(j+k+\frac{x}{2})^2}\right)\right) \right]_{x=0}. \end{aligned} \quad (3.5)$$

If we set

$$f_j(x) = f_{N,k,j}(x) := \log \Gamma(j) + \log \Gamma(j+2k+x) - 2 \log \Gamma\left(j+k+\frac{x}{2}\right),$$

then we can carry on the computation in (3.5) by computing the derivative, getting

$$\begin{aligned} &\frac{d^n}{dx^n} \left[\exp\left(-x\mu_{2k} + \sum_{j=1}^N \log\left(\frac{\Gamma(j)\Gamma(j+2k+x)}{\Gamma(j+k+\frac{x}{2})^2}\right)\right) \right]_{x=0} \\ &= \sum_{m_1, \dots, m_n} \frac{n!}{m_1! \dots m_n!} e^{\sum_{j=1}^N f_j(0)} \prod_{i=1}^n \left(\frac{1}{i!} \frac{d^i}{dx^i} \left[-x\mu_{2k} + \sum_{j=1}^N f_j(x) \right]_{x=0} \right)^{m_i} \\ &= \mathcal{M}_{2k} \sum_{m_1, \dots, m_n} \frac{n!}{m_1! \dots m_n!} \left(-\mu_{2k} + \sum_{j=1}^N f_j'(0) \right)^{m_1} \left(\frac{1}{2} \sum_{j=1}^N f_j''(0) \right)^{m_2} \prod_{i=3}^n \left(\frac{1}{i!} \sum_{j=1}^N f_j^{(i)}(0) \right)^{m_i}, \end{aligned} \quad (3.6)$$

where the sums in (3.6) are over the n -tuple $(m_1, \dots, m_n)$ such that

$$m_1 + 2m_2 + \dots + nm_n = n.$$

Using Stirling's approximation formula, one can easily estimate the derivatives of $f_j(x)$ and prove

$$\begin{aligned} \sum_{j=1}^N f_j'(0) &= \sum_{j=1}^N \left\{ \frac{k}{j} + O_k(j^{-2}) \right\} = k \log N + O_k(1), \\ \sum_{j=1}^N f_j''(0) &= \sum_{j=1}^N \left\{ \frac{1}{j+2k} - \frac{1/2}{j+k} + O(j^{-2}) \right\} = \frac{1}{2} \log N + O_k(1), \\ \sum_{j=1}^N f_j^{(i)}(0) &= \sum_{j=1}^N O(j^{-2}) = O(1) \quad \text{for all } i \geq 3. \end{aligned} \quad (3.7)$$

Putting together (3.5), (3.6) and (3.7), one has

$$\langle |Z|^{2k} (\log|Z| - \mu_{2k})^n \rangle = \mathcal{M}_{2k} \sum_{m_1+2m_2+\dots+nm_n=n} \frac{n!}{m_1! \dots m_n!} \left(\frac{1}{4} \log N + O_k(1) \right)^{m_2} (O_k(1))^{m_1+m_3+\dots+m_n}.$$

Then, if n is even, the asymptotic is given by $m_2 = \frac{n}{2}$ and $m_i = 0$ for $i \neq 2$, giving

$$\langle |Z|^{2k} (\log|Z| - \mu_{2k})^n \rangle \sim_{k,n} \mathcal{M}_{2k} \frac{n!}{(n/2)!} \left(\frac{1}{4} \log N \right)^{n/2} = \mathcal{M}_{2k} (n-1)! \left(\frac{1}{2} \log N \right)^{n/2},$$

while if n is odd, then the n -th moment is surely $o_{k,n}(\mathcal{M}_{2k}(\log N)^{n/2})$.

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References

- [1] L.-P. Arguin, D. Belius, P. Bourgade, M. Radziwiłł and K. Soundararajan, Maximum of the Riemann zeta function on a short interval of the critical line, *Comm. Pure Appl. Math.* **72** (2019), no. 3, 500–535.
- [2] R. Balasubramanian, J. B. Conrey and D. R. Heath-Brown, Asymptotic mean square of the product of the Riemann zeta-function and a Dirichlet polynomial, *J. Reine Angew. Math.* **357** (1985), 161–181.
- [3] S. Bettin, The second moment of the Riemann zeta function with unbounded shifts, *Int. J. Number Theory* **6** (2010), no. 8, 1933–1944.
- [4] S. Bettin, H. M. Bui, X. Li and M. Radziwiłł, A quadratic divisor problem and moments of the Riemann zeta-function, *J. Eur. Math. Soc. (JEMS)* **22** (2020), no. 12, 3953–3980.
- [5] S. Bettin, V. Chandee and M. Radziwiłł, The mean square of the product of the Riemann zeta-function with Dirichlet polynomials, *J. Reine Angew. Math.* **729** (2017), 51–79.
- [6] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith, Integral moments of L -functions, *Proc. Lond. Math. Soc. (3)* **91** (2005), no. 1, 33–104.
- [7] J. B. Conrey and A. Ghosh, Zeros of derivatives of the Riemann zeta-function near the critical line, in: *Analytic Number Theory* (Allerton Park 1989), Progr. Math. 85, Birkhäuser, Boston (1990), 95–110.
- [8] J. B. Conrey, M. O. Rubinstein and N. C. Snaith, Moments of the derivative of characteristic polynomials with an application to the Riemann zeta function, *Comm. Math. Phys.* **267** (2006), no. 3, 611–629.
- [9] A. Fazzari, A weighted central limit theorem for $\log|\zeta(1/2 + it)|$, *Mathematika* **67** (2021), no. 2, 324–341.
- [10] S. M. Gonek, Mean values of the Riemann zeta function and its derivatives, *Invent. Math.* **75** (1984), no. 1, 123–141.
- [11] A. Harper, Sharp conditional bounds for moments of the Riemann zeta functions, preprint (2013), <https://arxiv.org/abs/1305.4618v1>.
- [12] C. P. Hughes, *On the characteristic polynomial of a random unitary matrix and the Riemann zeta function*, PhD thesis, University of Bristol, 2001.
- [13] C. P. Hughes and M. P. Young, The twisted fourth moment of the Riemann zeta function, *J. Reine Angew. Math.* **641** (2010), 203–236.
- [14] J. P. Keating and N. C. Snaith, Random matrix theory and $\zeta(1/2 + it)$, *Comm. Math. Phys.* **214** (2000), no. 1, 57–89.
- [15] S. Kirila, An upper bound for discrete moments of the derivative of the Riemann zeta-function, *Mathematika* **66** (2020), no. 2, 475–497.
- [16] M. B. Milinovich, Upper bounds for moments of $\zeta'(\rho)$, *Bull. Lond. Math. Soc.* **42** (2010), no. 1, 28–44.
- [17] M. Radziwiłł, Large deviations in Selberg’s central limit theorem, preprint (2011), <https://arxiv.org/abs/1108.5092>.
- [18] A. Selberg, Contributions to the theory of the Riemann zeta-function, *Arch. Math. Naturvid.* **48** (1946), no. 5, 89–155.
- [19] K. Soundararajan, Moments of the Riemann zeta function, *Ann. of Math. (2)* **170** (2009), no. 2, 981–993.
- [20] K.-M. Tsang, *The Distribution of the Values of the Riemann Zeta-Function*, ProQuest LLC, Ann Arbor, 1984; Thesis (Ph.D.)—Princeton University.