Research Article

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Fiberwise linear differential operators

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Abstract: We define a new notion of *fiberwise linear differential operator* on the total space of a vector bundle E . Our main result is that fiberwise linear differential operators on E are equivalent to (polynomial) derivations of an appropriate line bundle over E^* . We believe this might represent a first step towards a definition of multiplicative (resp. infinitesimally multiplicative) differential operators on a Lie groupoid (resp. a Lie algebroid). We also discuss the *linearization* of a differential operator around a submanifold.

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1 Introduction

Given a vector bundle $E \rightarrow M$, it is often interesting to look at geometric structures (functions, vector fields, differential forms etc.) on the total space E that satisfy appropriate compatibility conditions with the vector bundle structure. Such compatibility is often referred to as *linearity* in the literature. Accordingly, one speaks about *linear functions*, *linear vector fields*, *linear differential forms* etc., on the total space of a vector bundle. In the present paper, we will rather use the terminology “fiberwise linearity”, to avoid confusion with other types of linearities. Now, the vector bundle structure is completely determined by (the smooth structure on E and) the action $h : \mathbb{R} \times E \rightarrow E$ of the monoid $(\mathbb{R}, \cdot)$ of multiplicative reals by fiberwise scalar multiplication: $h(t, e) = te$ for all $t \in \mathbb{R}$ and all $e \in E$ (see, e.g., [4]). It follows that the fiberwise linearity of a geometric structure can usually be expressed purely in terms of h . For instance, a function f on E is fiberwise linear if and only if $h_t^* f = tf$ for all t . Similarly, a vector field X (resp. a differential form ω) on E is fiberwise linear if and only if $h_t^* X = X$ for all $t \neq 0$ (resp. $h_t^* \omega = t\omega$ for all t). A fiberwise linear function is equivalent to a section of the dual vector bundle E^* , a fiberwise linear vector field is equivalent to a section of the *gauge algebroid* of E (see, e.g., [9]), and a fiberwise linear differential 1-form is equivalent to a section of the first jet bundle $J^1 E \rightarrow M$. The latter examples already show that fiberwise linear structures on E can encode interesting geometric structures on (vector bundles over) M . There are even more interesting examples. A fiberwise linear symplectic structure ω on E is equivalent to a vector bundle isomorphism $E \cong T^*M$. To see this, notice that ω , together with the standard direct sum decomposition $TE|_M \cong E \oplus TM$ of the tangent bundle of E restricted to the zero section, gives rise to a bilinear map $E \times_M TM \rightarrow \mathbb{R}$. As ω is non-degenerate, the induced map $I_\omega : E \rightarrow T^*M$ is a vector bundle isomorphism. From closedness and fiberwise linearity, $\omega = I_\omega^* \omega_M$, the pull-back along I_ω of the canonical 2-form ω_M on T^*M . It is clear that I_ω is the unique vector bundle isomorphism with this property (see, e.g., [4], see also [11]).

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Another interesting example is the following: a fiberwise linear metric is equivalent to an isomorphism $E \cong T^*M$ together with a torsion free connection in TM [12].

As a final remarkable example, we recall that a fiberwise linear Poisson structure on E is the same as a Lie algebroid structure on E^* [9].

In this paper, we propose the following definition of *fiberwise linear scalar differential operator*. An $\mathbb{R}$ -linear differential operator $\Delta : C^\infty(E) \rightarrow C^\infty(E)$ of order q is *fiberwise linear* if $h_t^* \Delta = t^{1-q} \Delta$ for all $t \neq 0$. This definition might seem weird at a first glance. However, it is supported by several different facts. For instance, according to our definition, a function and a vector field are fiberwise linear if and only if they are fiberwise linear when regarded as a 0-th order and a first order scalar differential operator, respectively. Moreover, the principal symbol of a fiberwise linear differential operator is a fiberwise linear symmetric multivector field. Another supporting remark is that the Laplacian (acting on functions) of a fiberwise linear metric is a fiberwise linear differential operator. Finally, a scalar differential operator Δ can be *linearized* around a submanifold producing a fiberwise linear differential operator representing a first order approximation to Δ in the transverse direction with respect to the submanifold. All these facts suggest that our definition might indeed be the “correct one”. Our main result is a description of fiberwise linear differential operators in terms of somehow simpler data. More precisely, we prove the following theorem (see Theorem 6.7 for a more precise statement).

Theorem. *Let $E \rightarrow M$ be a vector bundle. Then there is a degree inverting $C^\infty(M)$ -linear bijection between fiberwise linear scalar differential operators $\Delta : C^\infty(E) \rightarrow C^\infty(E)$ and polynomial derivations of the line bundle $E^* \times_M \wedge^{\text{top}} E \rightarrow E^*$.*

This theorem is a little surprising because it describes objects of higher order in derivatives (fiberwise linear differential operators) in terms of objects of order 1 in derivatives (derivations of an appropriate vector bundle). We hope that this result might be the starting point of a more thorough investigation of *multiplicative differential operators* on Lie groupoids and, at the infinitesimal level, *infinitesimally multiplicative differential operators* on Lie algebroids. Multiplicative (resp. infinitesimally multiplicative) structures are geometric structures on a Lie groupoid (resp. Lie algebroid) which are additionally compatible with the groupoid (resp. algebroid) structure. In the last thirty years, starting from the pioneering works of Weinstein on symplectic groupoids [13], multiplicative structures captured the interest of a large community of people working in Poisson geometry and related fields, and today we have a precise description of several different multiplicative structures and their infinitesimal counterparts: infinitesimally multiplicative structures (see [7] for a survey). However, all examples investigated so far are of order 1 in derivatives and it would be interesting to investigate the compatibility of a Lie groupoid/algebroid with structures of higher order in derivatives, e.g., higher order differential operators. This is a natural issue that might conjecturally lead to new important developments. As infinitesimally multiplicative structures are, in particular, fiberwise linear structures, this paper might be also considered as a first step in this direction.

The paper is organized as follows. In Section 2, we recall what it means for a vector field on the total space of a vector bundle $E \rightarrow M$ to be *fiberwise linear*, i.e. compatible with the vector bundle structure. In Section 3, we discuss *fiberwise linear symmetric multivector fields* and we describe them in terms of simpler data. This material is well known to experts (although it is scattered in the literature and it is hard to find a universal reference) and Sections 2 and 3 are mainly intended to fix our notation. In Section 4, we recall what a derivation of a vector bundle E is and introduce what we call E -multivectors, a “derivation analogue” of plain multivector fields. We also discuss fiberwise linear E -multivectors. These objects are not exactly of our primary interest, but they play a very useful role in the proofs of our main theorems (Theorems 6.7 and 7.5). To the best of our knowledge, the material in Section 4 is mostly new. Section 5 is an extremely compact introduction to linear differential operators on vector bundles, and, in particular, scalar differential operators. Section 6 contains our main constructions and results: we define and study fiberwise linear (scalar) differential operators on the total space of a vector bundle E . Somehow surprisingly, fiberwise linear differential operators on E form a transitive Lie–Rinehart algebra over fiberwise polynomial functions on E^* , with abelian isotropies (Theorem 6.4). The reason is ultimately explained by our main result, the theorem above (see also Theorem 6.7 below). As already announced, E -multivectors play a prominent role in the proof. In Section 7, we discuss

the *linearization* of a scalar differential operator Δ around a submanifold M in a larger manifold. The linearization of Δ is a fiberwise linear differential operator on the total space of the normal bundle to M , and can be seen as a first order approximation to Δ in the direction transverse to M . The existence of a *linearization construction* strongly supports our definition of fiberwise linear differential operators.

We will adopt systematically the Einstein convention on the sum of repeated upper-lower indexes. We will not adopt the Einstein convention for sums over (repeated) multi-indexes.

2 Core and linear vector fields on a vector bundle

As we mentioned in Section 1, the main aim of this paper is to explain what it means for a differential operator on the total space E of a vector bundle $E \rightarrow M$ to be compatible with the vector bundle structure. We will reach our definition (Definition 6.1) by stages. We first need to recall what it means for a function, a vector field and, more generally, a multivector field on E , to be compatible with the vector bundle structure. We do this in the present and the next section. We adopt the general philosophy of [4, 5] where it is shown that a vector bundle structure is encoded in the fiberwise scalar multiplication, and compatibility with the vector bundle structure is expressed in terms of such multiplication.

So, let $\pi : E \rightarrow M$ be a vector bundle. The fiberwise scalar multiplication by a real number

$$h : \mathbb{R} \times E \rightarrow E$$

is an action of the multiplicative monoid of reals $(\mathbb{R}, \cdot)$. The algebra $C_{\text{poly}}^{\infty}(E)$ of fiberwise polynomial functions on the total space E is non-negatively graded:

$$C_{\text{poly}}^{\infty}(E) = \bigoplus_{k=0}^{\infty} C^{\infty}(E)_k,$$

and its k -th homogeneous piece $C^{\infty}(E)_k$ consists of homogeneous polynomial functions of degree k , i.e. functions $f \in C^{\infty}(E)$ such that

$$h_t^*(f) = t^k f$$

for all $t \in \mathbb{R}$. Functions in $C^{\infty}(E)_0$ are just (pull-backs via the projection $\pi : E \rightarrow M$ of) functions on M . We call them *core functions* and also denote them by $C_{\text{core}}^{\infty}(E)$. They form a subalgebra in $C_{\text{poly}}^{\infty}(E)$. Functions in $C^{\infty}(E)_1$ are fiberwise linear (FWL for short in what follows) functions, and identify naturally with sections of the dual vector bundle E^* . We denote them by $C_{\text{lin}}^{\infty}(E)$. They form a $C_{\text{core}}^{\infty}(E)$ -submodule in $C_{\text{poly}}^{\infty}(E)$. We denote by ℓ_{φ} the linear function corresponding to the section $\varphi \in \Gamma(E^*)$. The terminology “core function” (and similarly “core vector field” etc., see below) is motivated by the theory of double vector bundles, where “core sections” are sections with an appropriate degree with respect to certain actions of $(\mathbb{R}, \cdot)$ (see, e.g., [9]).

A vector field $X \in \mathfrak{X}(E)$ on E is *fiberwise polynomial*, or simply *polynomial*, if it maps (fiberwise) polynomial functions to polynomial functions. Polynomial vector fields $\mathfrak{X}_{\text{poly}}(E)$ form a (graded) *Lie–Rinehart algebra* over polynomial functions $C_{\text{poly}}^{\infty}(E)$:

$$\mathfrak{X}_{\text{poly}}(E) = \bigoplus_{k=-1}^{\infty} \mathfrak{X}(E)_k.$$

We recall for later purposes that a Lie–Rinehart algebra over a commutative algebra A is a vector space L , which is both an A -module and a Lie algebra acting on A by derivations with the following two compatibilities:

- The Lie algebra action map $\rho : L \rightarrow \text{Der } A$ is A -linear (it is often called the *anchor*).
- The Lie bracket $[-, -] : L \times L \rightarrow L$ is a bi-derivation, i.e. it satisfies the following Leibniz rule:

$$[\lambda, a\mu] = \rho(\lambda)(a)\mu + a[\lambda, \mu], \quad \lambda, \mu \in L, a \in A.$$

Lie–Rinehart algebras are purely algebraic counterparts of Lie algebroids. For more on Lie–Rinehart algebras, see, e.g., [6] and the references therein.

Coming back to polynomial vector fields, the k -th homogeneous piece $\mathfrak{X}(E)_k$ of $\mathfrak{X}_{\text{poly}}(E)$ consists of homogeneous polynomial vector fields of degree k , i.e. vector fields $X \in \mathfrak{X}(E)$ such that

$$h_t^*(X) = t^k X$$

for all $t \neq 0$. Vector fields in $\mathfrak{X}(E)_{-1}$ are vertical lifts of sections of E . We call them *core vector fields* and also denote them by $\mathfrak{X}_{\text{core}}(E)$. They form an abelian Lie–Rinehart subalgebra (note: over the subalgebra $C_{\text{core}}^\infty(E) = C^\infty(M)$) in $\mathfrak{X}_{\text{poly}}(E)$. We denote by

$$e^\uparrow \in \mathfrak{X}(E)$$

the vertical lift of a section $e \in \Gamma(E)$.

Vector fields in $\mathfrak{X}(E)_0$ are, by definition, fiberwise linear (FWL) vector fields. They can be equivalently characterized as vector fields preserving linear functions and they satisfy the following property:

$$[X, Y] \in \mathfrak{X}_{\text{core}}(E) \quad \text{for all } Y \in \mathfrak{X}_{\text{core}}(E).$$

We denote FWL vector fields by $\mathfrak{X}_{\text{lin}}(E)$. They form a Lie–Rinehart subalgebra (over $C_{\text{core}}^\infty(E)$) in $\mathfrak{X}_{\text{poly}}(E)$.

If (x^i, u^α) are vector bundle coordinates, then a function $f \in C^\infty(E)$ is a core function if and only if, locally, $f = f(x)$ and it is a linear function if and only if, locally, $f = f_\alpha(x)u^\alpha$. Similarly, a vector field $X \in \mathfrak{X}(E)$ is a core vector field if and only if, locally,

$$X = X^\alpha(x) \frac{\partial}{\partial u^\alpha},$$

and it is a linear vector field if and only if, locally,

$$X = X^i(x) \frac{\partial}{\partial x^i} + X_\beta^\alpha(x) u^\beta \frac{\partial}{\partial u^\alpha}.$$

There is also a useful notion of *FWL tensor* on E . Let $\mathcal{T} \in \Gamma(T^{\otimes r}E \otimes T^{*\otimes s}E)$ be a tensor field of type (r, s) .

Definition 2.1. Then $\mathcal{T}$ is a *FWL tensor* if

$$h_t^* \mathcal{T} = t^{1-r} \mathcal{T}$$

for all $t \neq 0$.

Notice that FWL tensor fields are called *linear tensor fields* in [1], where they are characterized in terms of the fiberwise addition in E (rather than via the fiberwise multiplication h as we do).

Example 2.2. As an instance, consider a metric $g \in \Gamma(S^2 T^*E)$. It is linear if $h_t^* g = tg$ for all t , and it is easy to see that this is in turn equivalent to g being locally of the form

$$g = g_{\alpha i}(x) du^\alpha \odot dx^i + g_{\alpha |ij}(x) u^\alpha dx^i \odot dx^j.$$

Notice that the non-degeneracy condition then implies that the x -dependent matrix $(g_{\alpha i})$ is invertible. In particular, the dimension of M and the rank of E must agree. Even more, by denoting by $T^\pi E = \ker d\pi$ the π -vertical bundle, the composition

$$E \xrightarrow{\cong} T^\pi E|_M \hookrightarrow TE|_M \xrightarrow{\flat} T^*E|_M \rightarrow T^*M$$

is a vector bundle isomorphism. Here $\flat : TE \rightarrow T^*E$ is the musical isomorphism, the second and the fourth arrow are those induced by the canonical direct sum decomposition, $TE|_M = TM \oplus T^\pi E|_M$, and the first arrow is the canonical isomorphism. In other words, a non-degenerate symmetric covariant 2-tensor g can only exist on the total space of (a vector bundle isomorphic to) the cotangent bundle. Finally, in standard coordinates (x^i, p_i) on T^*M , g looks like

$$g = dp_i \odot dx^i - \Gamma_{ij}^k(x) p_k dx^i \odot dx^j$$

for some appropriate local functions $\Gamma_{ij}^k(x)$. In particular, g is necessarily of split signature. For more on FWL metrics, see [12].

3 More on FWL multivector fields

The material in this section is well known to experts, and it is partly folklore, partly scattered in the literature. For this reason, it is hard to give precise references (the reader may consult, e.g., [8, Appendix A] and the references therein, although that reference does not cover the same exact material as the present one). In any case, most of the proofs are straightforward and we omit them.

We will need to consider FWL symmetric multivector fields. According to Definition 2.1, a k -multivector field P on the total space E of a vector bundle $E \rightarrow M$ is *FWL* if

$$h_t^*(P) = t^{1-k}P$$

for all $t \neq 0$. We denote by $\mathfrak{X}_{\text{sym,lin}}^*(E)$ FWL symmetric multivector fields.

There is a useful characterization of FWL k -multivector fields. Namely, a k -multivector field P on E is FWL if and only if

$$\begin{aligned} P(f_1, \dots, f_k) &\in C_{\text{lin}}^\infty(E), \\ P(f_1, \dots, f_{k-1}, h_1) &\in C_{\text{core}}^\infty(E), \\ P(f_1, \dots, f_{k-2}, h_1, h_2) &= 0 \end{aligned}$$

for all $f_i \in C_{\text{lin}}^\infty(E)$ and all $h_j \in C_{\text{core}}^\infty(E)$. In particular, an FWL symmetric k -multivector field P determines a pair of maps (D_P, l_P) :

$$D_P : \underbrace{\Gamma(E^*) \times \dots \times \Gamma(E^*)}_{k \text{ times}} \rightarrow \Gamma(E^*)$$

and

$$l_P : \underbrace{\Gamma(E^*) \times \dots \times \Gamma(E^*)}_{k-1 \text{ times}} \times C^\infty(M) \rightarrow C^\infty(M)$$

via

$$\begin{aligned} \ell_{D_P(\varphi_1, \dots, \varphi_k)} &= P(\ell_{\varphi_1}, \dots, \ell_{\varphi_k}), \\ l_P(\varphi_1, \dots, \varphi_{k-1}, f) &= P(\ell_{\varphi_1}, \dots, \ell_{\varphi_{k-1}}, f) \end{aligned}$$

for all $\varphi_i \in \Gamma(E^*)$ and all $f \in C^\infty(M)$. The maps D_P, l_P satisfy the following properties:

- (i) D_P is $\mathbb{R}$ -multilinear and symmetric.
- (ii) l_P is $C^\infty(M)$ -multilinear and symmetric in the first $(k-1)$ -arguments.
- (iii) One has

$$D_P(\varphi_1, \dots, \varphi_{k-1}, f\varphi_k) = fD_P(\varphi_1, \dots, \varphi_k) + l_P(\varphi_1, \dots, \varphi_{k-1}, f)\varphi_k \quad \text{for } \varphi_i \in \Gamma(E^*) \text{ and } f \in C^\infty(M).$$

- (iv) l_P is a derivation in its last argument.

In particular, l_P can be seen as a vector bundle map $l_P : S^{k-1}E^* \rightarrow TM$, and we will often write

$$l_P(\varphi_1, \dots, \varphi_{k-1})(f)$$

instead of $l_P(\varphi_1, \dots, \varphi_{k-1}, f)$. The assignment $P \mapsto (D_P, l_P)$ establishes a $C^\infty(M)$ -linear bijection between FWL symmetric k -multivector fields on E and k -multiderivations of E^* , i.e. pairs (D, l) consisting of a map

$$D : \Gamma(E^*) \times \dots \times \Gamma(E^*) \rightarrow \Gamma(E^*)$$

and a vector bundle map $l : S^{k-1}E^* \rightarrow TM$ (equivalently, a section of $S^{k-1}E \otimes TM$) satisfying

$$D(\varphi_1, \dots, \varphi_{k-1}, f\varphi_k) = fD(\varphi_1, \dots, \varphi_k) + l(\varphi_1, \dots, \varphi_{k-1})(f)\varphi_k.$$

The map l is sometimes called the *symbol* of D and it is completely determined by D . For this reason, we will often refer to D itself as a k -multiderivation (see, e.g., [2, 3] for a skew-symmetric version of multiderivations).

Recall that there is a natural Poisson bracket $\{-, -\}$ on symmetric multivector fields given by the following (Gerstenhaber-type) formula:

$$\begin{aligned} \{P_1, P_2\}(f_1, \dots, f_{k_1+k_2+1}) = & \sum_{\sigma \in S_{k_2+1, k_1}} P_1(P_2(f_{\sigma(1)}, \dots, f_{\sigma(k_2+1)}), f_{\sigma(k_2+2)}, \dots, f_{\sigma(k_1+k_2+1)}) \\ & - \sum_{\sigma \in S_{k_1+1, k_2}} P_2(P_1(f_{\sigma(1)}, \dots, f_{\sigma(k_1+1)}), f_{\sigma(k_1+2)}, \dots, f_{\sigma(k_1+k_2+1)}) \end{aligned} \quad (3.1)$$

for all $(k_1 + 1)$ -multivector fields P_1 , all $(k_2 + 1)$ -multivector fields P_2 , and all functions f_i , where $S_{k,h}$ denotes (k, h) -unshuffles. The Poisson bracket (3.1) preserves FWL symmetric multivector fields, and the Poisson bracket $\{P_1, P_2\}$ of FWL Poisson multivector fields $P_1, P_2 \in \mathfrak{X}_{\text{sym}, \text{lin}}^*(E)$ identifies with the obvious Gerstenhaber-like bracket $\{D_1, D_2\}$ of the associated multiderivations D_1, D_2 .

FWL symmetric multivector fields on E do also identify with polynomial vector fields on E^* . To see this, it is useful to talk about *core multivector fields* first. A k -multivector field P on E is *core* if

$$h_t^*(P) = t^{-k}P$$

for all $t \neq 0$. Core k -multivector fields can be characterized as those multivector fields P such that

$$\begin{aligned} P(f_1, \dots, f_k) & \in C_{\text{core}}^\infty(E) = C^\infty(M), \\ P(f_1, \dots, f_k, h) & = 0 \end{aligned}$$

for all $f_i \in C_{\text{lin}}^\infty(E)$ and $h \in C_{\text{core}}^\infty(E)$, and they form a subalgebra $\mathfrak{X}_{\text{sym}, \text{core}}^*(E)$ in the associative, commutative algebra $\mathfrak{X}_{\text{sym}}^*(E)$ (with the symmetric product). More precisely, $\mathfrak{X}_{\text{sym}, \text{core}}^*(E)$ is the subalgebra spanned by core functions and core vector fields. In particular, $\mathfrak{X}_{\text{sym}, \text{core}}^*(E)$ identifies with sections $\Gamma(S^*E)$ of the symmetric algebra of E via

$$(e_1)^\uparrow \odot \dots \odot (e_k)^\uparrow \mapsto e_1 \odot \dots \odot e_k,$$

with $e_i \in \Gamma(E)$ (recall that $e^\uparrow \in \mathfrak{X}_{\text{core}}(E)$ denotes the vertical lift of a section $e \in \Gamma(E)$). In its turn, $\Gamma(S^*E)$ identifies with polynomial functions on E^* via the (degree preserving) algebra isomorphism

$$\Gamma(S^*E) \rightarrow C_{\text{poly}}^\infty(E^*), \quad e_1 \odot \dots \odot e_q \mapsto \ell_{e_1} \cdots \ell_{e_q},$$

and, in what follows, we will often understand the latter identifications. Notice that the resulting isomorphism

$$\mathfrak{X}_{\text{sym}, \text{core}}^*(E) \rightarrow C_{\text{poly}}^\infty(E^*), \quad P \mapsto F_P,$$

is given by

$$F_P(\varphi_x) = \frac{1}{k!} P(\ell_\varphi, \dots, \ell_\varphi)(x), \quad (3.2)$$

where $P \in \mathfrak{X}_{\text{sym}, \text{core}}^k(E)$, $\varphi \in \Gamma(E^*)$ and $x \in M$.

We can now go back to FWL symmetric multivector fields. The symmetric product of a core symmetric multivector field and an FWL one is an FWL multivector field, and this turns $\mathfrak{X}_{\text{sym}, \text{lin}}^*(E)$ into a $\Gamma(S^*E)$ -module. Now, let $Q \in \mathfrak{X}_{\text{sym}, \text{lin}}^*(E)$. It is easy to see that the Poisson bracket $H_Q = \{Q, -\}$ preserves core multivector fields. Hence, it is a derivation of the commutative algebra $\mathfrak{X}_{\text{sym}, \text{core}}^*(E) \cong C_{\text{poly}}^\infty(E^*)$. In its turn, H_Q extends uniquely to a polynomial vector field, also denoted by H_Q , on E^* . The assignment

$$\mathfrak{X}_{\text{sym}, \text{lin}}^*(E) \rightarrow \mathfrak{X}_{\text{poly}}(E^*), \quad Q \mapsto H_Q = \{Q, -\},$$

establishes a degree inverting isomorphism of Lie algebras, between the Lie algebra of linear symmetric multivector fields on E (with the Poisson bracket) and polynomial vector fields on E^* (with the commutator). When we equip $\mathfrak{X}_{\text{sym}, \text{lin}}^*(E)$ with the symmetric product by a core multivector field, the latter isomorphism becomes an isomorphism of Lie–Rinehart algebras. We summarize and complement the above discussion on core and FWL multivector fields with the following proposition.

Proposition 3.1. *Let $E \rightarrow M$ be a vector bundle. The map*

$$\mathfrak{X}_{\text{sym,core}}^{\bullet}(E) \rightarrow C_{\text{poly}}^{\infty}(E^*), \quad P \mapsto F_P,$$

given by (3.2) is a degree inverting isomorphism of (graded) commutative algebras, and the map

$$\mathfrak{X}_{\text{sym,lin}}^{\bullet}(E) \rightarrow \mathfrak{X}_{\text{poly}}(E^*), \quad Q \mapsto H_Q = \{Q, -\},$$

is a degree inverting isomorphism of (graded) Lie–Rinehart algebras over $\mathfrak{X}_{\text{sym,core}}^{\bullet}(E) \cong C_{\text{poly}}^{\infty}(E^*)$.

Let (x^i, u^{α}) be vector bundle coordinates on E and let (x^i, u_{α}) be the corresponding dual vector bundle coordinates on E^* . Then every core symmetric q -multivector field $P \in \mathfrak{X}_{\text{sym,core}}^{\bullet}(E)$ is locally of the form

$$P = P^{\alpha_1 \dots \alpha_q}(x) \frac{\partial}{\partial u^{\alpha_1}} \odot \dots \odot \frac{\partial}{\partial u^{\alpha_q}}$$

and we have

$$F_P = P^{\alpha_1 \dots \alpha_q}(x) u_{\alpha_1} \dots u_{\alpha_q}.$$

Additionally, every FWL symmetric q -multivector field $Q \in \mathfrak{X}_{\text{sym,lin}}^{\bullet}(E)$ is locally of the form

$$Q = Q^{i|\alpha_1 \dots \alpha_{q-1}}(x) \frac{\partial}{\partial x^i} \odot \frac{\partial}{\partial u^{\alpha_1}} \odot \dots \odot \frac{\partial}{\partial u^{\alpha_{q-1}}} + Q_{\alpha}^{\beta_1 \dots \beta_q}(x) u^{\alpha} \frac{\partial}{\partial u^{\beta_1}} \odot \dots \odot \frac{\partial}{\partial u^{\beta_q}}$$

and we have

$$H_Q = Q^{i|\alpha_1 \dots \alpha_{q-1}}(x) u_{\alpha_1} \dots u_{\alpha_{q-1}} \frac{\partial}{\partial x^i} - Q_{\alpha}^{\beta_1 \dots \beta_q}(x) u_{\beta_1} \dots u_{\beta_q} \frac{\partial}{\partial u^{\alpha}}.$$

Proof. The proof of those parts of the statement that have not been already discussed is straightforward, and we leave the easy details to the reader. $\square$

Finally, we remark that linear symmetric multivector fields fit in the following short exact sequence:

$$0 \rightarrow \Gamma(S^*E \otimes E^*) \rightarrow \mathfrak{X}_{\text{sym,lin}}^{\bullet}(E) \xrightarrow{l} \Gamma(S^{*-1}E \otimes TM) \rightarrow 0, \quad (3.3)$$

where the second arrow identifies the section $e_1 \odot \dots \odot e_k \otimes \varphi$ of $S^k E \otimes E^*$ with the FWL k -multivector field $\ell_{\varphi} e_1^{\uparrow} \odot \dots \odot e_k^{\uparrow}$.

4 More on derivations of a vector bundle

In this section, for a vector bundle $V \rightarrow M$, we introduce a notion of (symmetric) V -multivector (Definition 4.1). To the best of our knowledge, this notion is new. It will play a significant role in the description of fiberwise linear differential operators provided in Section 6. Symmetric V -multivectors are in many respects similar to plain symmetric multivector fields, so the proofs of most of the statements in this section parallel the proofs of the analogous statements for multivector fields, and we usually omit them.

We begin with a vector bundle $E \rightarrow M$ and remark that the space $\mathfrak{X}_{\text{lin}}^1(E) = \mathfrak{X}_{\text{lin}}(E)$ of linear vector fields is of particular interest. The assignment $X \mapsto D_X$ establishes an isomorphism of Lie–Rinehart algebras (over $C^{\infty}(M)$) between linear vector fields on E and *derivations* of E^* , i.e. 1-multiderivations. We stress that

$$X(\ell_{\varphi}) = \ell_{D_X \varphi}, \quad \varphi \in \Gamma(E^*).$$

In the following, we denote by $\mathfrak{D}(V)$ the Lie–Rinehart algebra of derivations of a vector bundle V . It is the Lie–Rinehart algebra of sections of a Lie algebroid $DV \rightarrow M$ whose Lie bracket is the commutator of derivations and whose anchor is the symbol map $D \mapsto l_D$.

Notice also that the assignment $X \mapsto H_X$ (see Proposition 3.1) does also establish an isomorphism of Lie–Rinehart algebras (over $C^{\infty}(M)$) between linear vector fields on E and linear vector fields on E^* . Accordingly, we have a canonical Lie algebroid isomorphism

$$DE \rightarrow DE^*, \quad D \mapsto D^*,$$

which is explicitly given by

$$\langle D^* \varphi, e \rangle = l_D(\langle \varphi, e \rangle) - \langle \varphi, De \rangle$$

for every $\varphi \in \Gamma(E^*)$ and $e \in \Gamma(E)$, where $\langle -, - \rangle : E^* \otimes E \rightarrow \mathbb{R}_M := M \times \mathbb{R}$ is the duality pairing. In the following, we will simply denote by D the derivation of E^* induced by a derivation of E (and vice-versa). It is easy to see that

$$[X, e^\dagger] = (D_X e)^\dagger, \quad e \in \Gamma(E).$$

More generally, a derivation D of a vector bundle V induces a derivation, also denoted by D , in each component of the whole (symmetric, resp. alternating) tensor algebra of $V \oplus V^*$. The latter derivation is defined imposing the obvious Leibniz rule with respect to the tensor product and the contraction by an element in the dual.

We are now ready to define the algebra of V -multivectors, which is a “derivation analogue” of the Poisson algebra of symmetric multivector fields. So, let $V \rightarrow M$ be a vector bundle, and consider the graded space $\tilde{\mathfrak{D}}^\bullet := \mathfrak{X}_{\text{sym}}^{\bullet-1}(M) \otimes \mathfrak{D}(V)$, where the tensor product is over functions on M . Notice that $\tilde{\mathfrak{D}}^\bullet$ is concentrated in positive degrees. Consider the graded subspace $\tilde{\mathfrak{D}}_{\text{sym}}^\bullet(V) \subset \tilde{\mathfrak{D}}^\bullet$ consisting of elements projecting on symmetric multivector fields $\mathfrak{X}_{\text{sym}}^\bullet(M) \hookrightarrow \mathfrak{X}_{\text{sym}}^{\bullet-1}(M) \otimes \mathfrak{X}(M)$ via

$$\text{id} \otimes l : \tilde{\mathfrak{D}}^\bullet \rightarrow \mathfrak{X}_{\text{sym}}^\bullet(M) \otimes \mathfrak{X}(M).$$

Finally, we define the graded space $\mathfrak{D}_{\text{sym}}^\bullet(V)$ by putting

$$\mathfrak{D}_{\text{sym}}^0(V) = C^\infty(M) \quad \text{and} \quad \mathfrak{D}_{\text{sym}}^k(V) = \tilde{\mathfrak{D}}_{\text{sym}}^k(V) \quad \text{for } k > 0.$$

We denote by

$$L : \mathfrak{D}_{\text{sym}}^\bullet(V) \rightarrow \mathfrak{X}_{\text{sym}}^\bullet(M)$$

the projection (given simply by the identity $\text{id} : C^\infty(M) \rightarrow C^\infty(M)$ in degree 0). Clearly, $\mathfrak{D}_{\text{sym}}^\bullet(V)$ fits in an exact sequence:

$$0 \rightarrow \mathfrak{X}_{\text{sym}}^{\bullet-1}(M) \otimes \Gamma(\text{End } V) \rightarrow \mathfrak{D}_{\text{sym}}^\bullet(V) \xrightarrow{L} \mathfrak{X}_{\text{sym}}^\bullet(M) \rightarrow 0.$$

Definition 4.1. Elements in $\mathfrak{D}_{\text{sym}}^k(V)$ are *symmetric k - V -multivectors*.

For $k > 0$, a symmetric k - V -multivector $D \in \mathfrak{D}_{\text{sym}}^k(V)$ will often be interpreted as an operator

$$D : C^\infty(M) \times \cdots \times C^\infty(M) \times \Gamma(V) \rightarrow \Gamma(V), \quad (f_1, \dots, f_{k-1}, v) \mapsto D(f_1, \dots, f_{k-1}|v).$$

As such it is symmetric and a derivation in the first $k-1$ arguments. Additionally, there exists a (necessarily unique) symmetric multivector field L_D such that

$$D(f_1, \dots, f_{k-1}|fv) = L_D(f_1, \dots, f_{k-1}, f)v + fD(f_1, \dots, f_{k-1}|v)$$

for all $f_1, \dots, f_{k-1}, f \in C^\infty(M)$ and $v \in \Gamma(V)$.

Lemma 4.2. The space $\mathfrak{D}_{\text{sym}}^\bullet(V)$ of symmetric V -multivectors is a Poisson algebra.

(i) The associative product is the usual $C^\infty(M)$ -module product when one of the two factors has 0 degree. It is given by

$$\begin{aligned} D_1 \cdot D_2(f_1, \dots, f_{k_1+k_2+1}|v) &= \sum_{\sigma \in S_{k_1+1, k_2}} L_{D_1}(f_{\sigma(1)}, \dots, f_{\sigma(k_1+1)}) D_2(f_{\sigma(k_1+2)}, \dots, f_{\sigma(k_1+k_2+1)}|v) \\ &\quad + \sum_{\sigma \in S_{k_2+1, k_1}} L_{D_2}(f_{\sigma(1)}, \dots, f_{\sigma(k_2+1)}) D_1(f_{\sigma(k_2+2)}, \dots, f_{\sigma(k_1+k_2+1)}|v) \end{aligned}$$

for all $f_i \in C^\infty(M)$ and $v \in \Gamma(V)$, if $D_1 \in \mathfrak{D}_{\text{sym}}^{k_1+1}(V)$, $D_2 \in \mathfrak{D}_{\text{sym}}^{k_2+1}(V)$, in which case

$$D_1 \cdot D_2 \in \mathfrak{D}_{\text{sym}}^{k_1+k_2+2}(V).$$

(ii) The Lie bracket $\{-, -\}$ is given by

$$\{D, f\} = -\{f, D\} = D(f, -, \dots, -|-)$$

when $f \in \mathfrak{D}_{\text{sym}}^0(V) = C^\infty(M)$. It is given by

$$\{D_1, D_2\} = D_1 \bullet D_2 - D_2 \bullet D_1, \quad (4.1)$$

where

$$D_1 \bullet D_2 : C^\infty(M) \times \dots \times C^\infty(M) \times \Gamma(V) \rightarrow \Gamma(V)$$

is the operator given by

$$\begin{aligned} D_1 \bullet D_2(f_1, \dots, f_{k_1+k_2}|v) &= \sum_{\sigma \in S_{k_1, k_2}} D_1(f_{\sigma(1)}, \dots, f_{\sigma(k_1)}) D_2(f_{\sigma(k_1+1)}, \dots, f_{\sigma(k_1+k_2)}|v) \\ &+ \sum_{\sigma \in S_{k_1-1, k_2+1}} D_1(f_{\sigma(1)}, \dots, f_{\sigma(k_1-1)}, L_{D_2}(f_{\sigma(k_1)}, \dots, f_{\sigma(k_1+k_2)}|v) \end{aligned}$$

for all $f_i \in C^\infty(M)$ and $v \in \Gamma(V)$, if $D_1 \in \mathfrak{D}_{\text{sym}}^{k_1+1}(V)$, $D_2 \in \mathfrak{D}_{\text{sym}}^{k_2+1}(V)$, in which case

$$\{D_1, D_2\} \in \mathfrak{D}_{\text{sym}}^{k_1+k_2+1}(V).$$

The map

$$L : \mathfrak{D}_{\text{sym}}^\bullet(V) \rightarrow \mathfrak{X}_{\text{sym}}^\bullet(M)$$

is a surjective Poisson algebra map.

Proof. The proof is straightforward, but cumbersome. We have to prove associativity (and commutativity) of the product $\cdot$, and Jacobi identity and Leibniz rule for the bracket $\{-, -\}$. The easiest case is when one of the involved V -multivectors has degree 0 and the rest have degree 1. We skip this case. The next to the easiest case is when all involved V -multivectors have degree 1. We only treat in some (but not all) details this case. The general case is similar and it is left to the reader. We begin by noticing that 1- V -multivectors are just derivations of V . Moreover, when $D \in \mathfrak{D}_{\text{sym}}^1(V) = \mathfrak{D}(V)$, then L_D is just the symbol of D , i.e. $L_D = l_D$. Now let $D_1, D_2 \in \mathfrak{D}_{\text{sym}}^1(V)$. Then

$$D_1 \cdot D_2 : C^\infty(M) \times \Gamma(V) \rightarrow \Gamma(V)$$

satisfies

$$\begin{aligned} D_1 \cdot D_2(f|gv) &= l_{D_1}(f)D_2(gv) + l_{D_2}(f)D_1(gv) \\ &= l_{D_1}(f)l_{D_2}(g)v + gl_{D_1}(f)D_2(v) + l_{D_2}(f)l_{D_1}(g)v + gl_{D_2}(f)D_1(v) \\ &= l_{D_1} \odot l_{D_2}(f, g)v + gD_1 \cdot D_2(f|v) \end{aligned}$$

for all $f, g \in C^\infty(M)$ and $v \in \Gamma(V)$. This shows that $D_1 \cdot D_2$ is a 2- V -multivector with

$$L_{D_1 \cdot D_2} = l_{D_1} \odot l_{D_2}. \quad (4.2)$$

This holds similarly in higher degrees.

Next let $D_1, D_2, D_3 \in \mathfrak{D}_{\text{sym}}^1(V)$. Then $D_1 \cdot (D_2 \cdot D_3)$ is the 3- V -multivector given by

$$\begin{aligned} D_1 \cdot (D_2 \cdot D_3)(f_1, f_2|v) &= l_{D_1}(f_1)D_2 \cdot D_3(f_2|v) + l_{D_1}(f_2)D_2 \cdot D_3(f_1|v) + L_{D_2 \cdot D_3}(f_1, f_2)D_1(v) \\ &= l_{D_1}(f_1)l_{D_2}(f_2)D_3(v) + l_{D_1}(f_1)l_{D_3}(f_2)D_2(v) + l_{D_1}(f_2)l_{D_2}(f_1)D_3(v) \\ &\quad + l_{D_1}(f_2)l_{D_3}(f_1)D_2(v) + l_{D_2} \odot l_{D_3}(f_1, f_2)D_1(v) \\ &= l_{D_1} \odot l_{D_2}(f_1, f_2)D_3(v) + l_{D_1} \odot l_{D_3}(f_1, f_2)D_2(v) + l_{D_2} \odot l_{D_3}(f_1, f_2)D_1(v) \end{aligned}$$

for all $f_1, f_2 \in C^\infty(M)$ and $v \in \Gamma(V)$, where we also used (4.2). This clearly agrees with $(D_1 \cdot D_2) \cdot D_3(f_1, f_2|v)$, showing associativity (in low degree). This holds similarly in higher degrees. Commutativity is obvious. The unit is the constant function 1.

We now pass to the bracket $\{-, -\}$. When the entries have degree 1, the latter is just the commutator of derivations and the Jacobi identity is obvious in this case. For the Leibniz rule let $D_1, D_2, D_3 \in \mathfrak{D}_{\text{sym}}^1(V)$, $f \in C^\infty(M)$ and $v \in \Gamma(V)$, and compute

$$\begin{aligned} \{D_1, D_2 \cdot D_3\}(fv) &= D_1(D_2 \cdot D_3(fv)) - D_2 \cdot D_3(fD_1(v)) - D_2 \cdot D_3(l_{D_1}(f)v) \\ &= D_1(l_{D_2}(f)D_3(v) + l_{D_3}(f)D_2(v)) - l_{D_2}(f)D_3(D_1(v)) - l_{D_3}(f)D_2(D_1(v)) \\ &\quad - l_{D_2}(l_{D_1}(f))D_3(v) - l_{D_3}(l_{D_1}(f))D_2(v) \\ &= l_{D_1}(l_{D_2}(f))D_3(v) + l_{D_2}(f)D_1(D_3(v)) + l_{D_1}(l_{D_3}(f))D_2(v) + l_{D_3}(f)D_1(D_2(v)) \\ &\quad - l_{D_2}(f)D_3(D_1(v)) - l_{D_3}(f)D_2(D_1(v)) - l_{D_2}(l_{D_1}(f))D_3(v) - l_{D_3}(l_{D_1}(f))D_2(v) \\ &= l_{\{D_1, D_2\}}(f)D_3(v) + l_{D_3}(f)\{D_1, D_2\}(v) + l_{\{D_1, D_3\}}(f)D_2(v) + l_{D_2}(f)\{D_1, D_3\}(v) \\ &= (\{D_1, D_2\} \cdot D_3 + \{D_1, D_3\} \cdot D_2)(fv). \end{aligned}$$

Hence,

$$\{D_1, D_2 \cdot D_3\} = \{D_1, D_2\} \cdot D_3 + \{D_1, D_3\} \cdot D_2.$$

This concludes the proof. $\square$

Remark 4.3. When V is a line bundle, the map $\mathfrak{X}_{\text{sym}}^{\bullet-1}(M) \otimes \Gamma(\text{End } V) \rightarrow \mathfrak{D}_{\text{sym}}^\bullet(V)$ embeds

$$\mathfrak{X}_{\text{sym}}^{\bullet-1}(M) = \mathfrak{X}_{\text{sym}}^{\bullet-1}(M) \otimes C^\infty(M) = \mathfrak{X}_{\text{sym}}^{\bullet-1}(M) \otimes \Gamma(\text{End } V)$$

into $\mathfrak{D}_{\text{sym}}^\bullet(V)$ as an abelian Poisson subalgebra and an ideal.

There is also a notion of FWL V -multivector. In order to discuss it, it is useful to discuss derivations of pull-back vector bundles first. So, let V be a vector bundle, and consider its pull-back $V_{\mathcal{P}} := \pi^*V$ along a surjective submersion $\pi : \mathcal{P} \rightarrow M$. Clearly, a derivation D of $V_{\mathcal{P}}$ is completely determined by its symbol and its action on pull-back sections. The restriction $D_M := D|_{\Gamma(V)}$ of D to pull-back sections is a *derivation along π* , i.e. it is an $\mathbb{R}$ -linear map $D_M : \Gamma(V) \rightarrow \Gamma(V_{\mathcal{P}})$ and there exists a, necessarily unique, vector field along π , denoted by $l_{D_M} \in \Gamma(\pi^*TM)$, fitting in the Leibniz rule

$$D_M(fv) = \pi^*(f)D_M(v) + l_{D_M}(f)v, \quad f \in C^\infty(M), v \in \Gamma(V).$$

The correspondence $D \mapsto (l_D, D_M)$ establishes a $C^\infty(M)$ -linear bijection between derivations D of $V_{\mathcal{P}}$ and pairs (X, D_M) consisting of a vector field $X \in \mathfrak{X}(\mathcal{P})$ and a derivation along π satisfying the following additional compatibility: $d\pi \circ X = l_{D_M}$. When $\mathcal{P} = E \rightarrow M$ is a vector bundle, it makes sense to talk about *polynomial sections of V_E* . Namely, $\Gamma(V_E) = C^\infty(E) \otimes \Gamma(V)$, where the tensor product is over $C^\infty(M)$, and multiplicative reals act on $\Gamma(V_E)$ via their action on the first factor. As for functions, we denote by h^* this action. A section v of $\Gamma(V_E)$ is *homogeneous* (polynomial) of degree k if $h_t^*(v) = t^k v$ for all $t \in \mathbb{R}$; in other words, $v \in C_{\text{poly}}^\infty(E) \otimes \Gamma(V)$. Denote by $\Gamma(V_E)_k$ the space of homogeneous sections of degree k , and by

$$\Gamma_{\text{poly}}(V_E) := \bigoplus_{k=0}^{\infty} \Gamma(V_E)_k$$

the space of all polynomial sections. *FWL sections* are degree 1 sections and they identify with sections of $E^* \otimes V$. *Core sections* are degree 0 sections and they identify simply with sections of V . Similarly to vector fields, a derivation D of V_E is *homogeneous* (polynomial) of degree k if it maps homogeneous sections of degree h to homogeneous sections of degree $k + h$; in other words,

$$h_t^* D = t^k D$$

for all $t \neq 0$. The space of all polynomial derivations of V_E will be denoted by $\mathfrak{D}_{\text{poly}}(V_E)$, and they correspond to pairs (X, D_M) where $X \in \mathfrak{X}_{\text{poly}}(E)$ and D_M takes values in polynomial sections.

We can also consider polynomial symmetric V_E -multivectors. We will only need core and FWL ones. A symmetric k - V_E -multivector D is *FWL* (resp. *core*) if it is homogeneous of degree $1 - k$ (resp. $-k$), i.e.

$$h_t^* D = t^{1-k} D \quad (\text{resp. } h_t^* D = t^{-k} D)$$

for all $t \neq 0$. We denote by $\mathfrak{D}_{\text{sym}, \text{lin}}^*(V_E)$ (resp. $\mathfrak{D}_{\text{sym}, \text{core}}^*(V_E)$) the space of FWL (resp. core) symmetric V_E -multivectors. The Lie bracket $\{-, -\}$ on V -multivectors preserves FWL ones. Additionally, the projection $L : \mathfrak{D}_{\text{sym}}^*(V_E) \rightarrow \mathfrak{X}_{\text{sym}}^*(E)$ maps FWL V -multivectors to FWL multivector fields, and we get a short exact sequence of Lie algebras:

$$0 \rightarrow \mathfrak{X}_{\text{sym}, \text{lin}}^{*-1}(E) \otimes \Gamma(\text{End}(V)) \rightarrow \mathfrak{D}_{\text{sym}, \text{lin}}^*(V_E) \xrightarrow{L} \mathfrak{X}_{\text{sym}, \text{lin}}^*(E) \rightarrow 0. \quad (4.3)$$

Proposition 4.4. *A k - V_E -multivector D is core if and only if*

$$\begin{aligned} D(f_1, \dots, f_{k-1}|v) &\in \Gamma_{\text{core}}(V_E), \\ D(f_1, \dots, f_{k-1}|w) &= 0, \\ D(f_1, \dots, f_{k-2}, h|v) &= 0 \end{aligned}$$

for all $f_i \in C_{\text{lin}}^\infty(E)$, $h \in C_{\text{core}}^\infty(E)$, $v \in \Gamma_{\text{lin}}(V_E) := \Gamma(V_E)_1$, and all $w \in \Gamma_{\text{core}}(V_E) := \Gamma(V_E)_0 = \Gamma(V)$.

Proof. The proof is straightforward. $\square$

It easily follows from the above proposition that a core k - V_E -multivector is completely determined by its symbol. More precisely, the symbol map $D \mapsto L_D$ establishes a one-to-one correspondence between core k - V_E -multivectors and core multivector fields. Let us discuss this in detail in the case $k = 1$. The general case is similar. So, let D be a core 1- V_E -multivector. In other words, D is a degree -1 derivation of V_E . From Proposition 4.4 it maps linear sections to core sections, and core sections to 0. Now let f be an FWL function on E and let $v \in \Gamma(V)$. Then v and $f \otimes v$ can be seen as a core and an FWL section of V_E , respectively. Hence, we have

$$D(f \otimes v) = l_D(f)v,$$

with $l_D(f) \in C^\infty(M)$. As the action of D on linear sections determines D completely, we see that D is necessarily of the form

$$X \otimes \text{id} : \Gamma(V_E) \cong C^\infty(E) \otimes \Gamma(V) \rightarrow \Gamma(V_E) \cong C^\infty(E) \otimes \Gamma(V)$$

for some core vector field $X = l_D$ on E . In particular, $X = e^\uparrow$, the vertical lift of some section $e \in \Gamma(E)$. Conversely, every derivation of V_E of the type $e^\uparrow \otimes \text{id}$, with $e \in \Gamma(E)$, is a core derivation. We conclude that core 1- V_E -multivectors identify with core vector fields on E via the symbol map: $\mathfrak{D}_{\text{sym}, \text{core}}^1(V_E) \cong \mathfrak{X}_{\text{core}}^1(E)$. Similarly,

$$\mathfrak{D}_{\text{sym}, \text{core}}^*(V_E) \cong \mathfrak{X}_{\text{sym}, \text{core}}^*(E) \cong C_{\text{poly}}^\infty(E^*).$$

In particular, $\mathfrak{D}_{\text{sym}, \text{core}}^*(V_E)$ does not really depend on V (but only on E).

Proposition 4.5. *A symmetric k - V_E -multivector D is FWL if and only if*

$$\begin{aligned} D(f_1, \dots, f_{k-1}|v) &\in \Gamma_{\text{lin}}(V_E), \\ D(f_1, \dots, f_{k-1}|w) &\in \Gamma_{\text{core}}(V_E), \\ D(f_1, \dots, f_{k-2}, h_1|v) &\in \Gamma_{\text{core}}(V_E), \\ D(f_1, \dots, f_{k-2}, h_1|w) &= 0, \\ D(f_1, \dots, f_{k-3}, h_1, h_2|v) &= 0 \end{aligned}$$

for all $f_i \in C_{\text{lin}}^\infty(E)$, $h_j \in C_{\text{core}}^\infty(E)$, $v \in \Gamma_{\text{lin}}(V_E) := \Gamma(V_E)_1$, and all $w \in \Gamma_{\text{core}}(V_E) := \Gamma(V_E)_0 = \Gamma(V)$.

Proof. The proof is straightforward. $\square$

In particular, an FWL symmetric k - V_E -multivector D determines a map

$$\Phi_D : \underbrace{\Gamma(E^*) \times \dots \times \Gamma(E^*)}_{k-1 \text{ times}} \rightarrow \mathfrak{D}(V)$$

via

$$\Phi_D(\varphi_1, \dots, \varphi_{k-1})(w) = D(\ell_{\varphi_1}, \dots, \ell_{\varphi_{k-1}}|w)$$

for all $\varphi_i \in \Gamma(E^*)$ and all $w \in \Gamma(V)$. The map Φ_D is $C^\infty(M)$ -multilinear and symmetric. Hence, it can be seen as a vector bundle map $\Phi_D : S^{k-1}E^* \rightarrow DV$ or, equivalently, as a section of $S^{k-1}E \otimes DV$.

Proposition 4.6. *The assignment $D \mapsto (L_D, \Phi_D)$ establishes a $C^\infty(M)$ -linear bijection between FWL symmetric k - V_E -multivectors $D \in \mathfrak{D}_{\text{sym}, \text{lin}}^k(V_E)$ and pairs (P, Φ) consisting of an FWL symmetric multivector field $P \in \mathfrak{X}_{\text{sym}, \text{lin}}^k(E)$ and a vector bundle map $\Phi : S^{k-1}E^* \rightarrow DV$ such that $l_P = l \circ \Phi$.*

Proof. The proof is easy and left to the reader. $\square$

According to Proposition 4.6 we will sometimes call the pair (L_D, Φ_D) itself an FWL symmetric V_E -multivector.

Now, we can combine the exact sequences (3.3) and (4.3) in one exact commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \Gamma(S^{*-1}E \otimes \text{End } V) & \longrightarrow & \Gamma(S^{*-1}E \otimes DV) & \xrightarrow{\text{id} \otimes l} & \Gamma(S^{*-1}E \otimes TM) \longrightarrow 0 \\
 & & \parallel & & \uparrow \Phi & & \uparrow l \\
 0 & \longrightarrow & \Gamma(S^{*-1}E \otimes \text{End } V) & \longrightarrow & \mathfrak{D}_{\text{sym}, \text{lin}}^*(V_E) & \xrightarrow{L} & \mathfrak{X}_{\text{sym}, \text{lin}}^*(E) \longrightarrow 0 \\
 & & \uparrow & & \uparrow I & & \uparrow \\
 & & 0 & \longrightarrow & \Gamma(S^*E \otimes E^*) & \xlongequal{\quad} & \Gamma(S^*E \otimes E^*) \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & & 0.
 \end{array}$$

We only need to explain the map I . To do that, we first remark that π -vertical vector fields act naturally on sections of V_E , via

$$X(f \otimes v) = X(f) \otimes v$$

for all $X \in \Gamma(T^\pi E)$, $f \in C^\infty(E)$ and $v \in \Gamma(V)$. Now, take $e_1, \dots, e_k \in \Gamma(E)$ and $\varphi \in \Gamma(E^*)$. Then

$$I(e_1 \odot \dots \odot e_k \otimes \varphi)(f_1, \dots, f_{k-1}|v) = \frac{1}{k!} \cdot \ell_\varphi \sum_{\sigma \in S_k} e_{\sigma(1)}^\uparrow(f_1) \cdots e_{\sigma(k-1)}^\uparrow(f_{k-1}) e_{\sigma(k)}^\uparrow(v).$$

Equivalently, we can interpret $e_1 \odot \dots \odot e_k$ as a core V_E -multivector, via the isomorphism

$$\mathfrak{D}_{\text{sym}, \text{core}}^*(V_E) \cong \Gamma(S^*E),$$

and then multiply by the FWL function ℓ_φ to get an FWL V_E -multivector.

We conclude this section showing that FWL symmetric V_E -multivectors do also identify with polynomial derivations of V_{E^*} . This is an easy consequence (among other things) of Proposition 4.6. Indeed, take $D \in \mathfrak{D}_{\text{sym}, \text{lin}}^*(V_E)$ and let (L_D, Φ_D) be the corresponding pair. Denote by $\pi : E^* \rightarrow M$ the projection. We claim that Φ_D can be seen as a derivation along π . Indeed, Φ_D is a section of $S^{*-1}E \otimes DL$ and, by acting on a section $v \in \Gamma(V)$ with the DL -factor, we get a section $\Phi_D(v)$ of $S^{*-1}E \otimes V$, i.e. a polynomial section of $\Gamma(V_{E^*})$. In the following, we use this construction to interpret Φ_D as a derivation along π . If we do so, the pair (H_{L_D}, Φ_D) consists of a vector field on E^* , and a derivation Φ_D along π , with the additional property that $d\pi \circ H_{L_D} = l \circ \Phi_D$. Hence, it corresponds to a (polynomial) derivation D^* of the pull-back vector bundle V_{E^*} (recall from Section 3 that H_{L_D} is the vector field on E^* completely determined by the property of acting as $\{L_D, -\}$ on polynomial functions on E^* , or, equivalently, core symmetric multivector fields on E , where $\{-, -\}$ is the Poisson bracket of symmetric multivector fields). Finally, a tedious but straightforward computation shows that the bijection $D \mapsto D^*$ between linear V_E -multivectors and polynomial derivations of V_{E^*} obtained in this way also preserves the Lie algebra structures. When we equip $\mathfrak{D}_{\text{sym}, \text{lin}}^*(V_E)$ with the product by a core V_E -multivector, the latter bijection becomes an isomorphism of Lie–Rinehart algebras. We have thus proved the following main result in this section.

Theorem 4.7. *Let $E \rightarrow M$ and $V \rightarrow M$ be vector bundles. The assignment $D \mapsto D^*$ establishes a degree inverting isomorphism of Lie–Rinehart algebras over $\mathfrak{D}_{\text{sym}, \text{core}}^*(V_E) \cong C_{\text{poly}}^\infty(E^*)$ between linear V_E -multivectors and polynomial derivations of V_{E^*} .*

5 Differential operators and their symbols

We finally come to the object of our primary interest: differential operators. This section is a super-short review of the subject.

Let $V, W \rightarrow M$ be vector bundles. A (linear) *differential operator* (DO in the following) *of order q from V to W* is an $\mathbb{R}$ -linear map

$$\Delta : \Gamma(V) \rightarrow \Gamma(W)$$

such that

$$[\cdots [[\Delta, f_0], f_1], \dots, f_q] = 0$$

for all $f_i \in C^\infty(M)$. In particular, DOs of order zero are just vector bundle maps $V \rightarrow W$. We denote by $\text{DO}_q(V, W)$ the space of order q DOs from V to W . Clearly, a DO of order q is also a DO of order $q + 1$, and we get the filtration

$$\text{DO}_0(V, W) = \Gamma(\text{Hom}(V, W)) \subseteq \text{DO}_1(V, W) \subseteq \cdots \subseteq \text{DO}_q(V, W) \subseteq \cdots.$$

The union of all $\text{DO}_q(V, W)$ will be denoted simply by $\text{DO}(V, W)$. A *scalar DO* on M is a DO acting on functions over M , i.e. a DO from the trivial line bundle $\mathbb{R}_M := M \times \mathbb{R}$ to itself. We use the symbol $\text{DO}_q(\mathbb{R}_M)$ (instead of $\text{DO}_q(\mathbb{R}_M, \mathbb{R}_M)$) for scalar DOs.

The composition of an order q and an order r DO is an order $q + r$ DO. In particular, for all q , $\text{DO}_q(V, W)$ is a $C^\infty(M)$ -module in two different ways: via composition on the left and composition on the right with a function on M (seen as an order 0 DO). We will consider the first module structure unless otherwise stated. The space $\text{DO}(\mathbb{R}_M)$ is a filtered non-commutative algebra with the composition. It is actually the universal enveloping algebra of the tangent Lie algebroid $TM \rightarrow M$. Being an associative algebra, $\text{DO}(\mathbb{R}_M)$ is also a Lie algebra with the commutator. Notice that the commutator of an order q and an order r scalar DO is an order $q + r - 1$ scalar DO.

Given an order q DO $\Delta : \Gamma(V) \rightarrow \Gamma(W)$ from V to W , and functions $f_1, \dots, f_q$, the nested commutator

$$[\cdots [\Delta, f_1], \dots, f_q]$$

is an order 0 DO. Additionally, it is a derivation in each of the arguments f_i and it is symmetric in those arguments. In this way, we get a map

$$\sigma : \text{DO}_q(V, W) \rightarrow \Gamma(S^q TM \otimes \text{Hom}(V, W)), \quad \Delta \mapsto \sigma(\Delta),$$

with

$$\sigma(\Delta)(f_1, \dots, f_q) = [\cdots [\Delta, f_1], \dots, f_q].$$

The map σ is called the *symbol* and it fits in a short exact sequence of $C^\infty(M)$ -modules

$$0 \rightarrow \text{DO}_{q-1}(V, W) \rightarrow \text{DO}_q(V, W) \xrightarrow{\sigma} \Gamma(S^q TM \otimes \text{Hom}(V, W)) \rightarrow 0, \quad (5.1)$$

where the second arrow is the inclusion.

Example 5.1. Vector fields are first order scalar DOs. Derivations of the vector bundle V are first order DOs D from V to itself such that $\sigma(D)$ belongs to $\mathfrak{X}(M) \subseteq \Gamma(TM \otimes \text{End } V)$. Additionally, we have $\sigma(D) = l_D$.

For scalar DOs, the short exact sequence (5.1) becomes

$$0 \rightarrow \text{DO}_{q-1}(\mathbb{R}_M) \rightarrow \text{DO}_q(\mathbb{R}_M) \xrightarrow{\sigma} \mathfrak{X}_{\text{sym}}^q(M) \rightarrow 0.$$

The symbol of scalar DOs intertwines the commutator with the Poisson bracket (of symmetric multivector fields) in the sense that

$$\sigma([\Delta, \Delta']) = \{\sigma(\Delta), \sigma(\Delta')\}$$

whenever $\Delta \in \text{DO}_q(\mathbb{R}_M)$ and $\Delta' \in \text{DO}_{q'}(\mathbb{R}_M)$, in which case we take $[\Delta, \Delta'] \in \text{DO}_{q+q'-1}(\mathbb{R}_M)$.

We conclude this short review section by commenting briefly on the coordinate description of (scalar) DOs. To do this, we first fix our conventions on the multi-index notation for multiple partial derivatives. Let (x^i) , $i = 1, \dots, n$, be variables. A length k multi-index I is a word $I = i_1 \cdots i_k$, with $i_j = 1, \dots, n$, where words are considered modulo permutations of their letters. The length k of a multi-index $I = i_1 \cdots i_k$ is also denoted by $|I|$. Words can be composed by concatenation and we also consider the empty multi-index $\emptyset$. If we do so, then multi-indexes are elements in the free abelian monoid generated by $1, \dots, n$. The length is then a monoid homomorphism. A length k multi-index $I = i_1 \cdots i_k$, determines an order k DO

$$\frac{\partial^{|I|}}{\partial x^I} := \frac{\partial^k}{\partial x^{i_1} \cdots \partial x^{i_k}}.$$

Now, we go back to manifolds M (and vector bundles over them). Actually, $\text{DO}_q(\mathbb{R}_M)$ (likewise $\text{DO}_q(V, W)$) is the $C^\infty(M)$ -module of sections of a vector bundle over M . If (x^i) are coordinates on M , then $\text{DO}_q(\mathbb{R}_M)$ is spanned locally (in the corresponding coordinate neighborhood) by

$$\frac{\partial^{|I|}}{\partial x^I}, \quad |I| = 0, 1, \dots, q.$$

More precisely, locally, every DO $\Delta \in \text{DO}_q(\mathbb{R}_M)$ can be uniquely written in the form

$$\Delta = \sum_{|I| \leq q} \Delta^I(x) \frac{\partial^{|I|}}{\partial x^I}, \quad (5.2)$$

where the $\Delta^I(x)$ are local functions on M . The $\Delta^I(x)$ can be recovered via formulas

$$\Delta^{i_1 \cdots i_k}(x) = \frac{1}{(i_1 \cdots i_k)!} [\cdots [\Delta, x^{i_1}], \dots, x^{i_k}](1), \quad k = 0, 1, \dots, q,$$

where, for a multi-index I , we denote by $I!$ the product $I[1]! \cdots I[n]!$ where $I[i]$ is the number of times the letter i occurs in I .

Finally, if Δ is an order q scalar DO locally given by (5.2), then its symbol $\sigma(\Delta)$ is locally given by

$$\sigma(\Delta) = \frac{1}{q!} \Delta^{i_1 \cdots i_q} \frac{\partial}{\partial x^{i_1}} \odot \cdots \odot \frac{\partial}{\partial x^{i_q}}.$$

6 Core and fiberwise linear differential operators

This is the main section of the paper. We propose a notion of *FWL* (scalar) *DO* on the total space of a vector bundle. Our definition is partly motivated by the fact that the symbol of an FWL DO is an FWL multivector field. It is also motivated by the *linearization construction* discussed in the next section. Yet another motivating little fact is that the Laplacian of an FWL metric is an FWL DO (Example 6.2).

Let $E \rightarrow M$ be a vector bundle. We have learnt from Sections 2–4 that, given a type $\mathfrak{T}$ of geometric structures on manifolds (functions, vector fields, tensors etc.) appropriate notions of *core* and *FWL* structures of the type $\mathfrak{T}$ on E exist, and these notions can be identified by means of the following recipe: (1) notice that the space $\mathfrak{T}(E)$ of structures of type $\mathfrak{T}$ on E possesses a subspace $\mathfrak{T}_{\text{poly}}(E)$ which is naturally graded (via the action of multiplicative reals on E by fiberwise scalar multiplication); (2) identify the smallest degree k for which the degree k homogeneous component $\mathfrak{T}(E)_k$ of $\mathfrak{T}_{\text{poly}}(E)$ is non-trivial; (3) put $\mathfrak{T}_{\text{core}}(E) = \mathfrak{T}(E)_k$ and $\mathfrak{T}_{\text{lin}}(E) = \mathfrak{T}(E)_{k+1}$. A quick check shows that this recipe cooks up the required definitions in all cases considered so far. Notice that we could make this recipe much more rigorous using for the rather vague “geometric structure of type $\mathfrak{T}$ ” the very precise notion of *natural vector bundle* $\mathfrak{T}$, but we will not need this level of abstraction.

We adopt the strategy described above to define core and FWL DOs on E . Consider the non-commutative algebra $\text{DO}(\mathbb{R}_E)$ of scalar DOs $\Delta : C^\infty(E) \rightarrow C^\infty(E)$. We begin noticing that, for each q , the space $\text{DO}_q(\mathbb{R}_E)$ of DOs $\Delta : C^\infty(E) \rightarrow C^\infty(E)$ of order q possesses a subspace $\text{DO}_{q,\text{poly}}(\mathbb{R}_E)$ which is naturally graded:

$$\text{DO}_{q,\text{poly}}(\mathbb{R}_E) = \bigoplus_{k=-q}^{\infty} \text{DO}_q(\mathbb{R}_E)_k,$$

where $\mathrm{DO}_q(\mathbb{R}_E)_k$ consists of degree k DOs (of order q), i.e. DOs Δ such that

$$h_t^*(\Delta) = t^k \Delta$$

for all $t \neq 0$. The smallest degree k for which $\mathrm{DO}_q(\mathbb{R}_E)_k$ is non-trivial is $k = -q$. So, following our recipe, we put

$$\mathrm{DO}_{q,\mathrm{core}}(E) := \mathrm{DO}_q(\mathbb{R}_E)_{-q},$$

and call them *core DOs*. We also put

$$\mathrm{DO}_{\mathrm{core}}(E) := \bigoplus_q \mathrm{DO}_{q,\mathrm{core}}(E). \quad (6.1)$$

Let (x^i, u^α) be vector bundle coordinates on E , and let (x^i, u_α) be dual coordinates on E^* . A DO $F \in \mathrm{DO}_q(\mathbb{R}_E)$ is a core DO if and only if, locally,

$$F = \sum_{|A|=q} F^A(x) \frac{\partial^{|A|}}{\partial u^A}, \quad (6.2)$$

where $A = \alpha_1 \cdots \alpha_q$ is a length q multi-index.

It follows from (6.2) that $\mathrm{DO}_{\mathrm{core}}(E) \subseteq \mathrm{DO}(\mathbb{R}_E)$ is the subalgebra generated by core functions $C_{\mathrm{core}}^\infty(M)$ and core vector fields $\mathfrak{X}_{\mathrm{core}}(E)$. Equivalently, it is the universal enveloping algebra of the abelian Lie algebroid $E \Rightarrow M$. Because of the latter description, there is an algebra isomorphism

$$\Gamma(S^*E) \rightarrow \mathrm{DO}_{\mathrm{core}}(E),$$

mapping a monomial

$$e_1 \odot \cdots \odot e_q, \quad e_i \in \Gamma(E),$$

to the DO

$$e_1^\dagger \circ \cdots \circ e_q^\dagger.$$

In its turn, as already mentioned, $\Gamma(S^*E)$ identifies with polynomial functions on E^* . In the following, we will often identify $\mathrm{DO}_{\mathrm{core}}(E)$ with both $\Gamma(S^*E)$ and $C_{\mathrm{poly}}^\infty(E^*)$ via the latter isomorphisms. If $F \in \mathrm{DO}_{q,\mathrm{core}}(E)$ is locally given by (6.1), then it identifies with

$$\frac{1}{q!} F^{\alpha_1 \cdots \alpha_q}(x) u_{\alpha_1} \odot \cdots \odot u_{\alpha_q} \in \Gamma(S^q E)$$

and

$$\sum_{|A|=q} F^A(x) u_A = \frac{1}{q!} F^{\alpha_1 \cdots \alpha_q}(x) u_{\alpha_1} \cdots u_{\alpha_q} \in C^\infty(E^*)_q,$$

where, for $A = \alpha_1 \cdots \alpha_q$, we denote by u_A the monomial $u_{\alpha_1} \cdots u_{\alpha_q}$.

We will always consider $\mathrm{DO}(\mathbb{R}_E)$ as a $\mathrm{DO}_{\mathrm{core}}(E)$ -module with the scalar multiplication given by the *left* composition.

We now pass to *FWL DOs*. Following our recipe again, for each q we put

$$\mathrm{DO}_{q,\mathrm{lin}}(E) := \mathrm{DO}_q(\mathbb{R}_E)_{-q+1}.$$

Definition 6.1. DOs in $\mathrm{DO}_{q,\mathrm{lin}}(E)$ are called *fiberwise linear differential operators* (FWL DOs) of order q .

For instance,

$$\mathrm{DO}_{0,\mathrm{lin}}(E) = C_{\mathrm{lin}}^\infty(E) \quad \text{and} \quad \mathrm{DO}_{1,\mathrm{lin}}(E) = \mathfrak{X}_{\mathrm{lin}}(E) \oplus C^\infty(M).$$

It is also clear that $\mathrm{DO}_{q,\mathrm{lin}}(E) \supseteq \mathrm{DO}_{q-1,\mathrm{core}}$ for all q . More precisely,

$$\mathrm{DO}_{q-1,\mathrm{core}}(E) = \mathrm{DO}_{q,\mathrm{lin}}(E) \cap \mathrm{DO}_{q-1}(\mathbb{R}_E).$$

We put

$$\mathrm{DO}_{\mathrm{lin}}(E) := \bigoplus_q \mathrm{DO}_q(\mathbb{R}_E)_{-q+1}.$$

Clearly, $\mathrm{DO}_{\mathrm{core}}(E) \subseteq \mathrm{DO}_{\mathrm{lin}}(E) \subseteq \mathrm{DO}(\mathbb{R}_E)$.

A DO $\Delta \in \text{DO}_q(\mathbb{R}_E)$ is FWL if and only if, in vector bundle coordinates, it looks like

$$\Delta = \sum_{|A|=q-1} \Delta^{i|A}(x) \frac{\partial^{|A|+1}}{\partial x^i \partial u^A} + \sum_{|B|=q} \Delta_\alpha^B(x) u^\alpha \frac{\partial^{|B|}}{\partial u^B} + \sum_{|C|=q-1} \Delta^C(x) \frac{\partial^{|C|}}{\partial u^C}. \quad (6.3)$$

It is easy to see from this formula that $\text{DO}_{\text{lin}}(E) \subseteq \text{DO}(\mathbb{R}_E)$ is the $\text{DO}_{\text{core}}(E)$ -submodule spanned by 1, $C_{\text{lin}}^\infty(E)$ and $\mathfrak{X}_{\text{lin}}(E)$.

Example 6.2. Let g be a metric on E , and assume it is FWL. Then the associated Laplacian operator

$$\Delta_g : C^\infty(E) \rightarrow C^\infty(E)$$

is an FWL DO operator (of order 2). One can see this working in vector bundle coordinates. But there is also a (basically) coordinate free proof that we now illustrate. First of all, from g being FWL, it immediately follows that the inverse tensor g^{-1} is FWL as well. Now, the covariant derivative $\nabla \vartheta$ of a 1-form ϑ along the Levi-Civita connection ∇ is the covariant 2-tensor given by the formula

$$\nabla \vartheta = \frac{1}{2}(d\vartheta + \mathcal{L}_{\sharp(\vartheta)}g), \quad (6.4)$$

where $\sharp : T^*E \rightarrow TE$ is the musical isomorphism. Equivalently, the covariant derivative $\nabla_X Y$ of a vector field Y along another vector field X is the vector field $\nabla_X Y$ that acts on functions $f \in C^\infty(M)$ as follows:

$$\nabla_X Y(f) = X(Y(f)) - \frac{1}{2}(\mathcal{L}_{\text{grad} f}g)(X, Y), \quad (6.5)$$

where $\text{grad} f = \sharp(df)$ is the *gradient* of f . By using (6.4) (or (6.5)) and the naturality of both the de Rham differential and the Lie derivative, it is easy to see that the covariant derivative of arbitrary tensor fields commutes with the pull-back along h_t for all $t \neq 0$. As the Laplacian $\Delta_g f$ of a function f is obtained by contracting the covariant derivative of df with g^{-1} , Δ_g decreases by one the degree of a homogeneous (fiberwise polynomial) function. So it is a second order DO of degree $1 - 2 = -1$, i.e. an FWL DO of order 2, as claimed. It might also be interesting to remark that the Levi-Civita connection of an FWL metric is an FWL connection according to a definition introduced in [10].

Lemma 6.3. *The space $\text{DO}_{\text{lin}}(E)$ of FWL DOs is the stabilizer of $\text{DO}_{\text{core}}(E)$ in the Lie algebra $\text{DO}(\mathbb{R}_E)$, i.e. a DO $\Delta \in \text{DO}(\mathbb{R}_E)$ is in $\text{DO}_{\text{lin}}(E)$ if and only if $[\Delta, F] \in \text{DO}_{\text{core}}(E)$ for all $F \in \text{DO}_{\text{core}}(E)$.*

Proof. The “only if part” of the statement immediately follows from an obvious order/degree argument. For the “if part”, consider a DO Δ of order r . Locally,

$$\Delta = \sum_{|I|+|A| \leq r} \Delta^{I|A}(x, u) \frac{\partial^{|I|+|A|}}{\partial x^I \partial u^A}.$$

Assume that Δ is in the stabilizer of $\text{DO}_{\text{core}}(E)$. We want to show that Δ is the sum of operators of the form (6.3) (with possibly varying $q \leq r$). As x^i is a core function for all i , the commutator $[\Delta, x^i]$ is a core DO. But

$$[\Delta, x^i] = \sum_{|I|+|A| \leq r-1} (I[i] + 1) \Delta^{I|A}(x, u) \frac{\partial^{|I|+|A|-1}}{\partial x^I \partial u^A},$$

so it can only be a core DO for all i if $\Delta^{I|A}(x, u) = 0$ for $|I| > 1$, and $\Delta^{i|A}(x, u) = \Delta^{i|A}(x)$. In other words, Δ is necessarily of the form

$$\Delta = \sum_{|A| \leq r-1} \Delta^{i|A}(x) \frac{\partial^{|A|+1}}{\partial x^i \partial u^A} + \sum_{|B| \leq k} \Delta^B(x, u) \frac{\partial^{|B|}}{\partial u^B}. \quad (6.6)$$

To conclude, it is enough to prove that $\Delta^B(x, u)$ is of the form $\Delta^B(x, u) = \Delta_\alpha^B(x) u^\alpha + \Delta^B(x)$. To do this, recall that $\frac{\partial}{\partial u^\alpha}$ is a core vector field for all α , and hence $[\Delta, \frac{\partial}{\partial u^\alpha}]$ is a core DO. But, from (6.6),

$$\left[\Delta, \frac{\partial}{\partial u^\alpha} \right] = - \frac{\partial \Delta^B(x, u)}{\partial u^\alpha} \frac{\partial^{|B|}}{\partial u^B},$$

which is a core DO for all α if and only if

$$\frac{\partial^2 \Delta^B(x, u)}{\partial u^\alpha \partial u^\beta} = 0$$

for all α, β , i.e. $\Delta^B(x, u)$ is a (non-necessarily homogeneous) first order polynomial in the variables u , as desired. $\square$

It follows from Lemma 6.3 that $\text{DO}_{\text{lin}}(E)$ is a Lie subalgebra in $\text{DO}(\mathbb{R}_E)$. As already mentioned, it is also a $\text{DO}_{\text{core}}(E)$ -submodule. Actually, it is a Lie–Rinehart algebra over $\text{DO}_{\text{core}}(E)$, the anchor being the adjoint operator $\text{ad} : \Delta \mapsto \text{ad}(\Delta) := [\Delta, -]$. To see this, first notice that $\text{ad}(\Delta)$ is indeed a well-defined derivation of $\text{DO}_{\text{core}}(E)$ for all $\Delta \in \text{DO}_{\text{lin}}(E)$. Now, take $\Delta, \Delta' \in \text{DO}_{\text{lin}}(E)$ and $F, F' \in \text{DO}_{\text{core}}(E)$, and compute

$$\begin{aligned} [\Delta, F \circ \Delta'] &= \text{ad}(\Delta)(F) \circ \Delta' + F \circ [\Delta, \Delta'], \\ \text{ad}(F \circ \Delta)(F') &= [F \circ \Delta, F'] = F \circ [\Delta, F'] = F \circ \text{ad}(\Delta)(F'). \end{aligned}$$

Finally, from $\text{DO}_{\text{core}}(E) \cong C_{\text{poly}}^\infty(E^*)$, we see that, for every $\Delta \in \text{DO}_{\text{lin}}(E)$, the derivation $\text{ad}(\Delta)$ determines a polynomial vector field (of the same degree) on E^* , also denoted by $\text{ad}(\Delta)$.

Theorem 6.4. *The sequence of Lie–Rinehart algebras*

$$0 \rightarrow \text{DO}_{\text{core}}(E) \rightarrow \text{DO}_{\text{lin}}(E) \xrightarrow{\text{ad}} \mathfrak{X}_{\text{poly}}(E^*) \rightarrow 0 \quad (6.7)$$

is exact.

Proof. First of all, as already remarked, $\text{DO}_{\text{core}}(E)$ is in $\text{DO}_{\text{lin}}(E)$. Even more, as it is an abelian subalgebra in $\text{DO}(\mathbb{R}_E)$, it is actually in the kernel of $\text{ad} : \text{DO}_{\text{lin}}(E) \rightarrow \mathfrak{X}_{\text{poly}}(E^*)$. To see that core DOs exhaust the kernel of ad (i.e. $\text{DO}_{\text{core}}(E)$ is its own centralizer), assume that $[\Delta, F] = 0$ for all $F \in \text{DO}_{\text{core}}(E)$. Then exactly the same computation as in the proof of Lemma 6.3 shows that Δ is locally of the form (6.3) with $\Delta^{iA}(x) = \Delta_\alpha^B(x) = 0$, i.e. $\Delta \in \text{DO}_{\text{core}}(E)$. For the exactness of the sequence (6.7) it remains to show that the map $\text{ad} : \text{DO}_{\text{lin}}(E) \rightarrow \mathfrak{X}_{\text{poly}}(E^*)$ is surjective. To do that, we work in local coordinates again. So, let (x^i, u^α) be vector bundle coordinates on E , and let (x^i, u_α) be dual coordinates on E^* . It is not hard to see that, if Δ is locally given by (6.3), then the vector field $\text{ad}(\Delta)$ is locally given by

$$\text{ad}(\Delta) = \sum_{|A|=q-1} \Delta^{iA}(x) u_A \frac{\partial}{\partial x^i} - \sum_{|B|=q} \Delta_\alpha^B(x) u_B \frac{\partial}{\partial u_\alpha}, \quad (6.8)$$

where, for a multi-index $A = \alpha_1 \cdots \alpha_s$, we denote by u_A the monomial $u_{\alpha_1} \cdots u_{\alpha_s}$ ($s = q, q-1$). As (6.8) is the local expression of a generic homogeneous polynomial vector field of degree $q-1$, we are done. $\square$

Our next aim is to prove that the Lie–Rinehart algebra $\text{DO}_{\text{lin}}(E)$ is canonically isomorphic to the Lie–Rinehart algebra of polynomial derivations of an appropriate line bundle on E^* . We begin with a simple proposition.

Proposition 6.5. *The symbol $\sigma(\Delta)$ of an FWL DO $\Delta \in \text{DO}_{q,\text{lin}}(E)$ is an FWL symmetric q -multivector field on E . Every FWL symmetric q -multivector field is the symbol of an order q FWL DO.*

Proof. The statement of this proposition immediately follows from (6.3) and the coordinate description of symmetric q -multivector fields (see Proposition 3.1). $\square$

Now let $\Delta \in \text{DO}_{q,\text{lin}}(E)$. Notice that the adjoint operator $\text{ad}(\Delta)$, seen as a polynomial vector field on E^* , corresponds exactly to the symbol $\sigma(\Delta)$ via the isomorphism $\mathfrak{X}_{\text{sym},\text{lin}}^q(E) \cong \mathfrak{X}(E^*)_{q-1}$. It is also clear that, in view of its coordinate form (6.3), Δ is completely determined by $\sigma(\Delta)$ or, equivalently, $\text{ad}(\Delta)$, together with the map

$$\Psi_\Delta : \underbrace{\Gamma(E^*) \times \cdots \times \Gamma(E^*)}_{q-1 \text{ times}} \rightarrow C^\infty(M), \quad (\varphi_1, \dots, \varphi_{q-1}) \mapsto [\cdots [\Delta, \ell_{\varphi_1}], \dots, \ell_{\varphi_{q-1}}](1).$$

The map Ψ_Δ is clearly well-defined. Additionally, it enjoys the following properties:

- (i) Ψ_Δ is symmetric.
- (ii) Ψ_Δ is a first order DO in each entry.

More precisely, we have the following lemma.

Lemma 6.6. *The map Ψ_Δ satisfies*

$$\Psi_\Delta(\varphi_1, \dots, \varphi_{q-2}, f\varphi_{q-1}) = f\Psi_\Delta(\varphi_1, \dots, \varphi_{q-2}, \varphi_{q-1}) + l_{\sigma(\Delta)}(\varphi_1, \dots, \varphi_{q-1})(f)$$

for all $\varphi_i \in \Gamma(E^*)$ and $f \in C^\infty(M)$.

Proof. Let φ_i and f be as in the statement, and compute

$$\begin{aligned} \Psi_\Delta(\varphi_1, \dots, \varphi_{q-2}, f\varphi_{q-1}) &= [[\cdots [\Delta, \ell_{\varphi_1}], \dots, \ell_{\varphi_{q-2}}], f\ell_{\varphi_{q-1}}](1) \\ &= f[[\cdots [\Delta, \ell_{\varphi_1}], \dots, \ell_{\varphi_{q-2}}], \ell_{\varphi_{q-1}}](1) + [[\cdots [\Delta, \ell_{\varphi_1}], \dots, \ell_{\varphi_{q-2}}], f](\ell_{\varphi_{q-1}}) \\ &= f\Psi_\Delta(\varphi_1, \dots, \varphi_{q-2}, \varphi_{q-1}) + [[\cdots [\Delta, \ell_{\varphi_1}], \dots, \ell_{\varphi_{q-2}}], f](\ell_{\varphi_{q-1}}). \end{aligned}$$

It remains to compute the last summand. So,

$$\begin{aligned} &[[\cdots [\Delta, \ell_{\varphi_1}], \dots, \ell_{\varphi_{q-2}}], f](\ell_{\varphi_{q-1}}) \\ &= [[[\cdots [\Delta, \ell_{\varphi_1}], \dots, \ell_{\varphi_{q-2}}], f], \ell_{\varphi_{q-1}}] + \ell_{\varphi_{q-1}}[[\cdots [\Delta, \ell_{\varphi_1}], \dots, \ell_{\varphi_{q-2}}], f](1) \\ &= l_{\sigma(\Delta)}(\varphi_1, \dots, \varphi_{q-1})(f), \end{aligned}$$

where we used that, from (6.3) again, the last summand in the second line is necessarily zero. This concludes the proof. $\square$

The data $(\sigma(\Delta), \Psi_\Delta)$ (determines Δ completely and) can be repackaged in a very useful way. Namely, consider the line bundle

$$L = \wedge^{\text{top}} E.$$

Then the pair $(\sigma(\Delta), \Psi_\Delta)$ determines an FWL q - L_E -multivector in the following way. Recall that an FWL q - L_E -multivector can be equivalently presented as a pair (P, Φ) consisting of an FWL symmetric multivector field $P \in \mathfrak{X}_{\text{sym}, \text{lin}}^q(E)$ and a vector bundle map $\Phi : S^{q-1}E^* \rightarrow DL$ such that $l_P = l \circ \Phi$. We claim that we can construct such a pair from the pair $(\sigma(\Delta), \Psi_\Delta)$. Namely, we put

$$P = \sigma(\Delta)$$

and define $\Phi = \Phi_\Delta$ by putting

$$\Phi_\Delta(\varphi_1, \dots, \varphi_{q-1})(U) := \sigma(\Delta)(\varphi_1, \dots, \varphi_{q-1}, -)U + \Psi_\Delta(\varphi_1, \dots, \varphi_{q-1})U \quad (6.9)$$

for all $\varphi_i \in \Gamma(E^*)$ and $U \in \Gamma(L)$. Equation (6.9) needs some explanations. In the first summand of the right-hand side, we interpret $\sigma(\Delta)$ as a q -multiderivation of E^* , so, when contracting it with the $q-1$ sections $\varphi_1, \dots, \varphi_{q-1}$, we get a plain derivation $\sigma(\Delta)(\varphi_1, \dots, \varphi_{q-1}, -) \in \mathfrak{D}(E^*) \cong \mathfrak{D}(E)$. As already remarked, derivations of E act on the exterior algebra of E . In our case we have

$$D(e_1 \wedge \cdots \wedge e_{\text{top}}) = \sum_i e_1 \wedge \cdots \wedge D e_i \wedge \cdots \wedge e_{\text{top}}$$

for all $D \in \mathfrak{D}(E)$ and all $e_i \in \Gamma(E)$.

The next theorem is the main result of the paper.

Theorem 6.7. *The assignment $\Delta \mapsto (\sigma(\Delta), \Phi_\Delta)$ establishes a degree inverting isomorphism of Lie–Rinehart algebras $A : \mathcal{DO}_{\text{lin}}(E) \rightarrow \mathfrak{D}_{\text{sym}, \text{lin}}(L_E)$.*

Proof. First of all, we have to show that Φ_Δ is well-defined, i.e. it is symmetric and $C^\infty(M)$ -linear in all its arguments. The symmetry is obvious. For the linearity, let $f \in C^\infty(M)$, and compute

$$\Phi_\Delta(\varphi_1, \dots, \varphi_{q-2}, f\varphi_{q-1})(U) = \sigma(\Delta)(\varphi_1, \dots, \varphi_{q-2}, f\varphi_{q-1}, -)U + \Psi_\Delta(\varphi_1, \dots, \varphi_{q-2}, f\varphi_{q-1})U.$$

Let us compute the two summands separately. First of all, for every $\varphi \in \Gamma(E^*)$,

$$\sigma(\Delta)(\varphi_1, \dots, \varphi_{q-2}, f\varphi_{q-1}, \varphi) = l_{\sigma(\Delta)}(\varphi_1, \dots, \varphi_{q-2}, \varphi)(f)\varphi_{q-1} + f\sigma(\Delta)(\varphi_1, \dots, \varphi_{q-1}, \varphi)$$

showing that

$$\sigma(\Delta)(\varphi_1, \dots, \varphi_{q-2}, f\varphi_{q-1}, -) = \langle -, e \rangle \cdot \varphi_{q-1} + f\sigma(\Delta)(\varphi_1, \dots, \varphi_{q-1}, -),$$

where $e \in \Gamma(E)$ is the section implicitly defined by

$$\langle \varphi, e \rangle = l_{\sigma(\Delta)}(\varphi_1, \dots, \varphi_{q-2}, \varphi)(f)$$

for all $\varphi \in \Gamma(E^*)$. We remark for future use that, in particular,

$$\langle \varphi_{q-1}, e \rangle = l_{\sigma(\Delta)}(\varphi_1, \dots, \varphi_{q-1})(f). \quad (6.10)$$

Now

$$\sigma(\Delta)(\varphi_1, \dots, \varphi_{q-2}, f\varphi_{q-1}, -)U = (\langle -, e \rangle \cdot \varphi_{q-1})U + f\sigma(\Delta)(\varphi_1, \dots, \varphi_{q-1}, -)U. \quad (6.11)$$

The endomorphism $\langle -, e \rangle \cdot \varphi_{q-1} : E^* \rightarrow E^*$ acts on E via its dual, which is minus its transpose, and hence

$$(\langle -, e \rangle \cdot \varphi_{q-1})U = -\text{trace}(\langle -, e \rangle \cdot \varphi_{q-1})U = -\langle \varphi_{q-1}, e \rangle U = -l_{\sigma(\Delta)}(\varphi_1, \dots, \varphi_{q-1})(f)U, \quad (6.12)$$

where we used (6.10). Substituting (6.12) into (6.11), we find

$$\sigma(\Delta)(\varphi_1, \dots, \varphi_{q-2}, f\varphi_{q-1}, -)U = f\sigma(\Delta)(\varphi_1, \dots, \varphi_{q-1}, -)U - l_{\sigma(\Delta)}(\varphi_1, \dots, \varphi_{q-1})(f)U.$$

We also have

$$\Psi_{\Delta}(\varphi_1, \dots, \varphi_{q-2}, f\varphi_{q-1})U = f\Psi_{\Delta}(\varphi_1, \dots, \varphi_{q-1})U + l_{\sigma(\Delta)}(\varphi_1, \dots, \varphi_{q-1})(f)U.$$

Putting everything together, we find

$$\begin{aligned} \Phi_{\Delta}(\varphi_1, \dots, \varphi_{q-2}, f\varphi_{q-1})U &= f\sigma(\Delta)(\varphi_1, \dots, \varphi_{q-1}, -)U + f\Psi_{\Delta}(\varphi_1, \dots, \varphi_{q-1})U \\ &= f\Phi_{\Delta}(\varphi_1, \dots, \varphi_{q-1})U. \end{aligned}$$

We conclude that Φ_{Δ} is a vector bundle map $S^{q-1}E^* \rightarrow DL$ as desired. Additionally, the composition $l \circ \Phi_{\Delta}$ does clearly agree with $\sigma(\Delta)$, so that $(\sigma(\Delta), \Phi_{\Delta})$ is indeed a q - L_E -multivector. It is also clear that Ψ_{Δ} can be reconstructed from $(\sigma(\Delta), \Phi_{\Delta})$ showing that the correspondence $\Delta \mapsto (\sigma(\Delta), \Phi_{\Delta})$ is injective. Next we prove the $\text{DO}_{\text{core}}(E)$ -linearity. So, take

$$F \in \text{DO}_{p, \text{core}}(E) = \Gamma(S^p E) = C^{\infty}(E^*)_p \quad \text{and} \quad \Delta \in \text{DO}_{q, \text{lin}}(E),$$

so that $F \circ \Delta \in \text{DO}_{p+q, \text{lin}}(E)$. We want to show that $(\sigma(F \circ \Delta), \Phi_{F \circ \Delta}) = F \cdot (\sigma(\Delta), \Phi_{\Delta})$. To do this, we begin by noticing that the product $F \cdot (\sigma(\Delta), \Phi_{\Delta})$ is the pair (D', Φ') where $D' \in \mathfrak{X}_{\text{sym}, \text{lin}}^{p+q}(E)$ is the symmetric multivector field that, when interpreted as a multiderivation $D' : \Gamma(E^*) \times \dots \times \Gamma(E^*) \rightarrow \Gamma(E^*)$, is given by

$$\begin{aligned} D'(\varphi_1, \dots, \varphi_{p+q}) &= F \cdot \sigma(\Delta)(\varphi_1, \dots, \varphi_{p+q}) \\ &= \sum_{\sigma \in S_{p,q}} \langle F, \varphi_{\sigma(1)} \odot \dots \odot \varphi_{\sigma(p)} \rangle \sigma(\Delta)(\varphi_{\sigma(p+1)}, \dots, \varphi_{\sigma(p+q)}), \end{aligned}$$

and, similarly, $\Phi' : S^{p+q-1}E^* \rightarrow DL$ is the bundle map given by

$$\begin{aligned} \Phi'(\varphi_1, \dots, \varphi_{p+q-1}) &= F \cdot \Phi_{\Delta}(\varphi_1, \dots, \varphi_{p+q-1}) \\ &= \sum_{\sigma \in S_{p,q-1}} \langle F, \varphi_{\sigma(1)} \odot \dots \odot \varphi_{\sigma(p)} \rangle \Phi_{\Delta}(\varphi_{\sigma(p+1)}, \dots, \varphi_{\sigma(p+q-1)}) \end{aligned}$$

for all $\varphi_i \in \Gamma(E^*)$. From the properties of the symbol map, we have $\sigma(F \circ \Delta) = F \cdot \sigma(\Delta)$, and it remains to take care of $\Phi_{F \circ \Delta}$. So choose $\varphi_i \in \Gamma(E^*)$, and compute $\Phi_{F \circ \Delta}(\varphi_1, \dots, \varphi_{p+q-1})$. From symmetry, it is enough to choose $\varphi_i = \varphi$ for all i and some φ . First of all, we have

$$\begin{aligned} \Psi_{F \circ \Delta}(\underbrace{\varphi, \dots, \varphi}_{p+q-1 \text{ times}}) &= [\dots [F \circ \Delta, \underbrace{\varphi, \dots, \varphi}_{p+q-1 \text{ times}}](1) \\ &= \sum_{l+m=p+q-1} \frac{1}{l!m!} [\dots [F, \underbrace{\varphi, \dots, \varphi}_{l \text{ times}}] \circ [\dots [\Delta, \underbrace{\varphi, \dots, \varphi}_{m \text{ times}}](1). \end{aligned}$$

Only the terms with $l = p, p - 1$ (and hence $m = q - 1, q$, respectively) survive, and we get

$$\begin{aligned}\Psi_{F \circ \Delta}(\underbrace{\varphi, \dots, \varphi}_{p+q-1 \text{ times}}) &= \frac{1}{p!(q-1)!} [\cdots \underbrace{[F, \varphi], \dots, \varphi}_{p \text{ times}} \circ \cdots \underbrace{[\Delta, \varphi], \dots, \varphi]_{q-1 \text{ times}}}(1) \\ &\quad + \frac{1}{(p-1)!q!} [\cdots \underbrace{[F, \varphi], \dots, \varphi}_{p-1 \text{ times}} \circ \cdots \underbrace{[\Delta, \varphi], \dots, \varphi]_q}(1) \\ &= \frac{1}{p!(q-1)!} \langle F, \underbrace{\varphi \odot \cdots \odot \varphi}_{p \text{ times}} \rangle \Psi_{\Delta}(\underbrace{\varphi, \dots, \varphi}_{q-1 \text{ times}}) \\ &\quad + \frac{1}{(p-1)!q!} \langle F, \underbrace{\varphi \odot \cdots \odot \varphi}_{p-1 \text{ times}} \odot \sigma(\Delta)(\underbrace{\varphi, \dots, \varphi}_{q \text{ times}}) \rangle.\end{aligned}\quad (6.13)$$

Now, for all $U \in \Gamma(L)$,

$$\Phi_{F \circ \Delta}(\underbrace{\varphi, \dots, \varphi}_{p+q-1 \text{ times}})U = \sigma(F \circ \Delta)(\underbrace{\varphi, \dots, \varphi}_{p+q-1 \text{ times}}, -)U + \Psi_{F \circ \Delta}(\underbrace{\varphi, \dots, \varphi}_{p+q-1 \text{ times}})U.$$

We already computed the second summand, while the first summand is

$$\begin{aligned}\sigma(F \circ \Delta)(\underbrace{\varphi, \dots, \varphi}_{p+q-1 \text{ times}}, -)U &= \frac{1}{p!(q-1)!} \langle F, \underbrace{\varphi \odot \cdots \odot \varphi}_{p \text{ times}} \rangle \sigma(\Delta)(\underbrace{\varphi, \dots, \varphi}_{q-1 \text{ times}}, -)U \\ &\quad + \frac{1}{(p-1)!q!} \langle F, \underbrace{\varphi \odot \cdots \odot \varphi}_{p-1 \text{ times}} \odot - \rangle \sigma(\Delta)(\underbrace{\varphi, \dots, \varphi}_{q \text{ times}}, -)U \\ &= \frac{1}{p!(q-1)!} \langle F, \underbrace{\varphi \odot \cdots \odot \varphi}_{p \text{ times}} \rangle \sigma(\Delta)(\underbrace{\varphi, \dots, \varphi}_{q-1 \text{ times}}, -)U \\ &\quad - \frac{1}{(p-1)!q!} \langle F, \underbrace{\varphi \odot \cdots \odot \varphi}_{p-1 \text{ times}} \odot \sigma(\Delta)(\underbrace{\varphi, \dots, \varphi}_{q \text{ times}}, -) \rangle U.\end{aligned}\quad (6.14)$$

From (6.13) and (6.14) it easily follows that $\Phi_{F \circ \Delta} = F \cdot \Phi_{\Delta}$, as claimed.

The surjectivity of the map $\Delta \mapsto (\sigma(\Delta), \Phi_{\Delta})$ now follows from (local) dimension counting.

It remains to check that the isomorphism $A : \text{DO}_{\text{lin}}(E) \rightarrow \mathfrak{D}_{\text{sym, lin}}(L_E)$ defined in this way is both anchor and bracket preserving. For the anchor, the anchor of $(\sigma(\Delta), \Phi_{\Delta})$ is the derivation of

$$\text{DO}_{\text{core}}(E) = \Gamma(S^*E) = C_{\text{poly}}^{\infty}(E)$$

corresponding to the FWL multivector field $\sigma(\Delta)$, which is exactly $\text{ad}(\Delta)$.

For the bracket, as we already discussed $C_{\text{poly}}^{\infty}(E)$ -linearity and compatibility with the anchor, it is enough to discuss the brackets of generators. As already remarked, $\text{DO}_{\text{lin}}(E)$ is generated (over $\text{DO}_{\text{core}}(E)$) by 1, $C_{\text{lin}}^{\infty}(E)$ and $\mathfrak{X}_{\text{lin}}(E)$. A direct check shows that

$$A(1) = (0, 1), \quad A(\ell_{\varphi}) = (-\varphi^{\dagger}, 0), \quad A(X) = (X^*, D_X) \quad (6.15)$$

for all $\varphi \in \Gamma(E^*)$ and all $X \in \mathfrak{X}_{\text{lin}}(E)$. Here X^* and D_X are, respectively, the linear vector field on E^* and the derivation of $L = \wedge^{\text{top}} E$ corresponding to X . It is now easy to check that the brackets are preserved on these generators, and this concludes the proof. $\square$

Composing with the isomorphism $\mathfrak{D}_{\text{sym, lin}}^*(L_E) \cong \mathfrak{D}(L_{E^*})$ from Theorem 4.7 (in the case $V = L$), we get a (degree inverting) Lie–Rinehart algebra isomorphism

$$\text{DO}_{\text{lin}}(E) \xrightarrow{\cong} \mathfrak{D}(L_{E^*})$$

that we denote by A again.

Remark 6.8. Let (x^i, u^{α}) be vector bundle coordinates on E , and let (x^i, u_{α}) be dual coordinates on E^* . Denote by $\text{Vol}_u = u_1 \wedge \cdots \wedge u_{\text{top}}$ the local coordinate generator of $\Gamma(L)$. It is easy to check using, e.g., (6.8), (6.15) and the $C_{\text{poly}}^{\infty}(E^*)$ -linearity, that, if the operator $\Delta \in \text{DO}_{\text{lin}}(E)$ is locally given by (6.3), then the corresponding derivation $A(\Delta) \in \mathfrak{D}(L_{E^*})$ maps a local section $\lambda = f(x, u) \text{Vol}_u$ of $\Gamma(L_{E^*})$ to

$$A(\Delta)(\lambda) = \left(\sum_{|A|=q-1} \Delta^{iA}(x) u_A \frac{\partial f}{\partial x^i}(x, u) - \sum_{|B|=q} \Delta_{\alpha}^B(x) u_B \frac{\partial f}{\partial u_{\alpha}}(x, u) + \sum_{|C|=q-1} \Delta^C(x) u_C f(x, u) \right) \text{Vol}_u.$$

7 Linearization of differential operators

Let $\mathcal{E}$ be a manifold, let $M \subseteq \mathcal{E}$ be a submanifold and let $\Delta \in \text{DO}(\mathbb{R}_{\mathcal{E}})$ be a scalar DO. Denote by $E \rightarrow M$ the normal bundle to M , i.e. $E = T\mathcal{E}|_M/TM$. In this section, we show that, under appropriate *linearizability conditions*, the DO Δ can be *linearized* around M yielding an FWL differential operator $\Delta_{\text{lin}} \in \text{DO}_{\text{lin}}(E)$. The DO Δ_{lin} represents the first order approximation of Δ around M in the direction transverse to M . This *linearization construction* is a further motivation supporting our definition of FWL DOs.

So, let $M \subseteq \mathcal{E}$ be a submanifold and let $E \rightarrow M$ be its normal bundle. We will often consider *adapted coordinates* on $\mathcal{E}$ around points of M , i.e. coordinates (X^i, U^α) such that $M : \{U^\alpha = 0\}$. In particular, the restrictions $(x^i = X^i|_M)$ are coordinates on M . From $U^\alpha|_M = 0$ we see that $u^\alpha := dU^\alpha|_M$ are conormal 1-forms and (x^i, u^α) are vector bundle coordinates on E .

We want to explain what it means to *linearize* an order q DO operator $\Delta \in \text{DO}_q(\mathbb{R}_{\mathcal{E}})$ around M . We proceed as follows: first, we recall the linearization of a function, second we discuss the linearization of a symmetric multivector field, and finally we define the linearization of a generic DO. So, let $F \in C^\infty(\mathcal{E})$. We say that F is *linearizable* (around M) if $F|_M = 0$. In this case, $dF|_M$ is a conormal 1-form to M , i.e. a section of the conormal bundle $E^* \hookrightarrow T^*\mathcal{E}|_M$. Hence, it corresponds to an FWL function on E . We put $F_{\text{lin}} = \ell_{dF|_M}$ and call it the *linearization* of F . For instance, if (X^i, U^α) are adapted coordinates on $\mathcal{E}$, then the (U^α) are linearizable and the linear fiber coordinates (u^α) on E are their linearizations. If F is any linearizable function on $\mathcal{E}$, then locally, around a point of M , $F(X, U) = F_\alpha(X)U^\alpha + \mathcal{O}(U^2)$ for some functions $F_\alpha(X)$ of the (X^i) (given by $F_\alpha(X) = \frac{\partial F}{\partial U^\alpha}(X, 0)$), and, in this case, $F_{\text{lin}} = F_\alpha(x)u^\alpha$. Notice that every linear function $\varphi \in C^\infty_{\text{lin}}(E)$ is the linearization of a (non-unique) linearizable function $F \in C^\infty(\mathcal{E})$: $\varphi = F_{\text{lin}}$.

We now pass to symmetric multivector fields. So, let P be a symmetric q -multivector field on $\mathcal{E}$. We say that P is *linearizable* if it belongs to the ideal $\mathcal{J}_M$ in $\mathfrak{X}^*_{\text{sym}}(\mathcal{E})$ generated by vector fields that are tangent to M . In other words, M is a *coisotropic submanifold* of $\mathcal{E}$ with respect to P .

Proposition 7.1. *Let $P \in \mathfrak{X}^q_{\text{sym}}(\mathcal{E})$ be a linearizable symmetric multivector field on $\mathcal{E}$. Then there exists a unique FWL symmetric q -multivector field P_{lin} on E such that*

$$P_{\text{lin}}((F_1)_{\text{lin}}, \dots, (F_q)_{\text{lin}}) = P(F_1, \dots, F_q)_{\text{lin}} \quad (7.1)$$

for all linearizable functions $F_i \in C^\infty(\mathcal{E})$. The linearization $P \mapsto P_{\text{lin}}$ preserves the Poisson bracket of symmetric multivector fields.

Proof. We begin by remarking that, as $P \in \mathcal{J}_M$, the function $P(F_1, \dots, F_q)$ is clearly linearizable for any choice of linearizable functions F_i . Now we want to show that the right-hand side of (7.1) does only depend on $(F_i)_{\text{lin}}$. We do this in coordinates. So, let (X^i, U^α) be adapted coordinates on $\mathcal{E}$, and let (x^i, u^α) be the associated vector bundle coordinates on E . Locally

$$P = \sum_{l+m=q} P^{i_1 \dots i_l, \alpha_1 \dots \alpha_m}(X, U) \frac{\partial}{\partial X^{i_1}} \odot \dots \odot \frac{\partial}{\partial X^{i_l}} \odot \frac{\partial}{\partial U^{\alpha_1}} \odot \dots \odot \frac{\partial}{\partial U^{\alpha_m}}. \quad (7.2)$$

Hence,

$$P(F_1, \dots, F_q) = \sum_{l+m=q} \sum_{\sigma \in S_{l,m}} P^{i_1 \dots i_l, \alpha_1 \dots \alpha_m}(X, U) \frac{\partial F_{\sigma(1)}}{\partial X^{i_1}} \dots \frac{\partial F_{\sigma(l)}}{\partial X^{i_l}} \frac{\partial F_{\sigma(l+1)}}{\partial U^{\alpha_1}} \dots \frac{\partial F_{\sigma(l+m)}}{\partial U^{\alpha_m}}.$$

It follows from $P \in \mathcal{J}_M$ that $P^{\alpha_1, \dots, \alpha_q}(X, 0) = 0$. Now compute

$$P(F_1, \dots, F_q)_{\text{lin}} = \left(\frac{\partial}{\partial U^\alpha} \Big|_{(x,0)} P(F_1, \dots, F_q) \right) u^\alpha.$$

Using that $F_i|_M = 0$ and that $P^{\alpha_1, \dots, \alpha_q}(X, 0) = 0$, we find

$$\begin{aligned} & \frac{\partial}{\partial U^\alpha} \Big|_{(x,0)} P(F_1, \dots, F_q) \\ &= \frac{\partial P^{\alpha_1, \dots, \alpha_q}}{\partial U^\alpha}(x, 0) (F_1)_{\alpha_1}(x) \dots (F_q)_{\alpha_q}(x) + \sum_{\sigma \in S_{1, q-1}} P^{i, \alpha_1 \dots \alpha_{q-1}}(x, 0) \frac{\partial (F_{\sigma(1)})_\alpha}{\partial X^i}(x, 0) (F_{\sigma(2)})_{\alpha_1} \dots (F_{\sigma(q)})_{\alpha_{q-1}}, \end{aligned}$$

which does only depend on the $(F_i)_{\text{lin}}$. As every FWL function is a linearization, this also shows that P_{lin} is well-defined on linear functions. Finally, P_{lin} can be uniquely extended to all functions on E , as a symmetric q -multivector field, also denoted by P_{lin} , and locally given by

$$P_{\text{lin}} = P_{\text{lin}}^{\alpha_1 \dots \alpha_q}(x, u) \frac{\partial}{\partial u^{\alpha_1}} \odot \dots \odot \frac{\partial}{\partial u^{\alpha_q}} + P^{i, \alpha_1 \dots \alpha_{q-1}}(x, 0) \frac{\partial}{\partial x^i} \odot \frac{\partial}{\partial u^{\alpha_1}} \odot \dots \odot \frac{\partial}{\partial u^{\alpha_{q-1}}}.$$

In particular, P_{lin} is an FWL multivector field.

For the last part of the statement, first notice that the ideal $\mathcal{I}_M$ is preserved by the Poisson bracket. So, if $P, Q \in \mathcal{I}_M$, then it makes sense to linearize $\{P, Q\}$. The rest follows easily from equation (7.1). $\square$

Remark 7.2. Proposition 7.1 is a “symmetric multivector analogue” of the following well-known fact. Let $(\mathcal{P}, \pi)$ be a Poisson manifold, let $(T^*\mathcal{P})_\pi$ be its cotangent algebroid and let $M \subseteq \mathcal{P}$ be a coisotropic submanifold. Then the conormal bundle $N^*M \subseteq T^*\mathcal{P}$ is a subalgebroid $(T^*\mathcal{P})_\pi$, and hence the normal bundle NM is equipped with a linear Poisson structure π_{lin} (see, e.g., [14, Section 3]).

By definition, the multivector field P_{lin} in Proposition 7.1 is the *linearization* of P .

We finally come to a generic differential operator $\Delta \in \text{DO}_q(\mathbb{R}_{\mathcal{E}})$.

Definition 7.3. An order q DO $\Delta \in \text{DO}(\mathbb{R}_{\mathcal{E}})$ is *order q linearizable* (around M) if its symbol $\sigma(\Delta) \in \mathfrak{X}_{\text{sym}}^q(\mathcal{E})$ is linearizable.

Remark 7.4.

- An order $q - 1$ DO $\Delta \in \text{DO}(\mathbb{R}_{\mathcal{E}})$ is always order q linearizable, but it might not be order $q - 1$ linearizable.
- A function $F \in C^\infty(\mathcal{E}) = \text{DO}_0(\mathbb{R}_{\mathcal{E}})$ is order 0 linearizable if and only if $F|_M = 0$, i.e. F is a linearizable function.
- A vector field $X \in \mathfrak{X}(\mathcal{E}) \subset \text{DO}_1(\mathbb{R}_{\mathcal{E}})$ is order 1 linearizable if and only if it is tangent to M , i.e. it is a linearizable vector field.

Now, let $\Delta \in \text{DO}_q(\mathbb{R}_{\mathcal{E}})$ be an order q linearizable DO. In order to *linearize* it, we use Theorem 6.7. In other words, from Δ we cook up a q - L_E -multivector (P_{lin}, Φ) of $L_E^* = E^* \times_M L$, with $L = \wedge^{\text{top}} E$ as in Section 6. The construction is inspired by the proof of Theorem 6.7 itself. We let $P_{\text{lin}} \in \mathfrak{X}_{\text{sym}, \text{lin}}^q(E)$ be the linearization of the symbol $P = \sigma(\Delta)$ of Δ . It remains to define the vector bundle map

$$\Phi : S^{q-1}E^* \rightarrow DL.$$

For $\varphi_1, \dots, \varphi_{q-1} \in \Gamma(E^*)$, we define $\Phi(\varphi_1, \dots, \varphi_{q-1})$ as follows. Pick linearizable functions $F_i \in C^\infty(\mathcal{E})$ such that $(F_i)_{\text{lin}} = \ell_{\varphi_i}$ and consider the following function on M :

$$f_{F_1, \dots, F_{q-1}} := [[\dots [\Delta, F_1], \dots], F_{q-1}](1)|_M.$$

We put

$$\Phi(\varphi_1, \dots, \varphi_{q-1})(U) = P_{\text{lin}}(\varphi_1, \dots, \varphi_{q-1}, -)U + f_{F_1, \dots, F_{q-1}}U$$

for all $U \in \Gamma(L)$. On the right-hand side, we interpret P_{lin} as a multiderivation of E^* . We have to show that Φ is well defined, i.e. $f_{F_1, \dots, F_{q-1}}$ does only depend on the φ_i , and Φ is indeed a vector bundle map $\Phi : S^{q-1}E^* \rightarrow DL$ as desired, i.e. it is (symmetric and) $C^\infty(M)$ -linear in its arguments. To see that $f_{F_1, \dots, F_{q-1}}$ does not depend on the choice of the F_i , assume that $(F_{q-1})_{\text{lin}} = 0$. Hence, we have $F_{q-1}|_M = 0$ and $dF_{q-1}|_M = 0$. Additionally,

$$\begin{aligned} f_{F_1, \dots, F_{q-1}} &= [[\dots [\Delta, F_1], \dots], F_{q-1}](1)|_M \\ &= (F_{q-1}[[\dots [\Delta, F_1], \dots], F_{q-2}](1) - [[\dots [\Delta, F_1], \dots], F_{q-2}](F_{q-1}))|_M \\ &= -[[\dots [\Delta, F_1], \dots], F_{q-2}](F_{q-1})|_M. \end{aligned}$$

Now, $[[\dots [\Delta, F_1], \dots], F_{q-2}]$ is a second order DO. As the first jet of F_{q-1} vanishes on M , the expression $[[\dots [\Delta, F_1], \dots], F_{q-2}](F_{q-1})|_M$ does only depend on the symbol of $[[\dots [\Delta, F_1], \dots], F_{q-2}]$ (and the Hessian of F_{q-1}). But

$$\sigma([[\dots [\Delta, F_1], \dots], F_{q-2}]) = \sigma(\Delta)(F_1, \dots, F_{q-2}, -, -).$$

So, if $P = \sigma(\Delta)$ is locally given by (7.2), using $P^{\alpha_1 \cdots \alpha_q}(X, 0) = 0$, we find that, locally,

$$f_{F_1, \dots, F_{q-1}} \propto P^{i, \alpha_1 \cdots \alpha_{q-2} \alpha_{q-1}}(x, 0)(F_1)_{\alpha_1}(x) \cdots (F_{q-2})_{\alpha_{q-2}}(x) \frac{\partial}{\partial x^i} (F_{q-1})_{\alpha_{q-1}}(x) = 0.$$

We conclude that $\Phi(\varphi_1, \dots, \varphi_{q-1})$ is well-defined. It remains to see that Φ is $C^\infty(M)$ -linear in one, and hence in all, of its arguments. So, take $f \in C^\infty(M)$. The simple formula

$$(FG)_{\text{lin}} = F|_M G_{\text{lin}} \quad \text{for all } F \in C^\infty(\mathcal{E}) \text{ and all linearizable } G \in C^\infty(\mathcal{E}) \quad (7.3)$$

shows that, when we replace φ_{q-1} with $f\varphi_{q-1}$ in $\Phi(\varphi_1, \dots, \varphi_{q-1})$, we can replace F_{q-1} with FF_{q-1} , where $F \in C^\infty(\mathcal{F})$ is any function such that $F|_M = f$. So, we have

$$\Phi(\varphi_1, \dots, \varphi_{q-2}, f\varphi_{q-1})U = P_{\text{lin}}(\varphi_1, \dots, \varphi_{q-2}, f\varphi_{q-1}, -)U + f_{F_1, \dots, F_{q-2}, FF_{q-1}}U.$$

For the first summand, recall from the proof of Theorem 6.7 (equations (6.11) and (6.12)) that

$$P_{\text{lin}}(\varphi_1, \dots, \varphi_{q-2}, f\varphi_{q-1}, -)U = fP_{\text{lin}}(\varphi_1, \dots, \varphi_{q-2}, \varphi_{q-1}, -)U - l_{P_{\text{lin}}}(\varphi_1, \dots, \varphi_{q-1})(f)U$$

For the second summand, compute

$$\begin{aligned} f_{F_1, \dots, F_{q-2}, FF_{q-1}} &= [[\cdots [\Delta, F_1], \dots, F_{q-2}], FF_{q-1}](1)|_M \\ &= F[[\cdots [\Delta, F_1], \dots, F_{q-2}], F_{q-1}](1)|_M + [[\cdots [\Delta, F_1], \dots, F_{q-2}], F](F_{q-1})|_M \\ &= ff_{F_1, \dots, F_{q-1}} + [[[\cdots [\Delta, F_1], \dots, F_{q-2}], F], F_{q-1}](1)|_M \\ &= Ff_{F_1, \dots, F_{q-1}} + \sigma(\Delta)(F_1, \dots, F_{q-1}, F)|_M. \end{aligned}$$

Now, to conclude the proof of the $C^\infty(M)$ -multilinearity, it is enough to show that

$$\sigma(\Delta)(F_1, \dots, F_{q-1}, F)|_M = l_{P_{\text{lin}}}(\varphi_1, \dots, \varphi_{q-1})(f). \quad (7.4)$$

But this follows easily from the fact that P_{lin} is the linearization of $\sigma(\Delta)$ and formula (7.3) again.

Finally, by construction, $l \circ \Phi = l_{P_{\text{lin}}}$, so the pair (P_{lin}, Φ) corresponds to a q - L_E -multivector. We call the linear DO $A^{-1}(P_{\text{lin}}, \Phi)$ the *linearization* of Δ and denote it Δ_{lin} .

The above discussion leads to the following theorem.

Theorem 7.5. *The assignment $\Delta \mapsto \Delta_{\text{lin}}$ is a well-defined (linearization) map from order q linearizable DOs on $\mathcal{E}$ and order q FWL DOs on E . The linearization preserves the commutator of DOs in the following sense: let $\Delta \in \text{DO}_{q+1}(\mathcal{E})$ and $\Delta' \in \text{DO}_{q'+1}(\mathcal{E})$ be an order $q+1$ linearizable and an order $q'+1$ linearizable DO, respectively. Then $[\Delta, \Delta']$ is order $q+q'+1$ linearizable, and its linearization is $[\Delta_{\text{lin}}, \Delta'_{\text{lin}}]$.*

Proof. We only need to prove the last part of the statement. To do that, recall that the commutator $[\Delta, \Delta']$ is a DO of order $q+q'+1$ and its symbol is the Poisson bracket $\{\sigma(\Delta), \sigma(\Delta')\}$. Let $\Delta_{\text{lin}} = A^{-1}(P_{\text{lin}}, \Phi)$ and $\Delta'_{\text{lin}} = A^{-1}(P'_{\text{lin}}, \Phi')$ be their linearizations. As the ideal $\mathcal{J}_M$ in $\mathfrak{X}_{\text{sym}}^*(\mathcal{E})$ is preserved by the Poisson bracket, $[\Delta, \Delta']$ is order $q+q'+1$ linearizable as desired. Let $[\Delta, \Delta']_{\text{lin}} = A^{-1}(P''_{\text{lin}}, \Phi'')$ be its linearization. As the linearization of multivector fields preserves the Poisson bracket, $P''_{\text{lin}} = \{P_{\text{lin}}, P'_{\text{lin}}\}$. Finally, we have to take care of Φ'' . It is a straightforward computation, which we sketch to point out the possible subtleties related to the properties of the various objects involved. Recall from the discussion preceding the statement of the theorem that

$$\Phi(\varphi_1, \dots, \varphi_q)U = P_{\text{lin}}(\varphi_1, \dots, \varphi_q)U + f_{F_1, \dots, F_q}U,$$

where

$$f_{F_1, \dots, F_q} = [\cdots [\Delta, F_1], \dots, F_q](1)|_M,$$

and likewise for Φ', Φ'' . Here F_i is any linearizable function on $\mathcal{E}$ such that $(F_i)_{\text{lin}} = \varphi_i$. In the following, to stress that, actually, the function $f_{F_1, \dots, F_q}$ does only depend on the φ_i , we write $f_{\varphi_1, \dots, \varphi_q}$ (instead of $f_{F_1, \dots, F_q}$). Additionally, we call f (resp. f', f'') the f -component of the FWL DO Δ_{lin} (resp. Δ'_{lin} , $[\Delta_{\text{lin}}, \Delta'_{\text{lin}}]$). We have to show that the f -component $f^{[\Delta, \Delta']}$ of $[\Delta, \Delta']_{\text{lin}}$ agrees with f'' . To do this, we compute $f^{[\Delta, \Delta']}$ explicitly. By sym-

metry and $\mathbb{R}$ -multilinearity, it is enough to evaluate it on $q + q'$ equal sections $\varphi_1 = \dots = \varphi_{q+q'} = \varphi \in \Gamma(E^*)$. So, let $F \in C^\infty(\mathcal{E})$ be a linearizable function such that $\varphi = F_{\text{lin}}$. We have

$$\begin{aligned} \underbrace{f_{\varphi, \dots, \varphi}^{[\Delta, \Delta']}}_{q+q' \text{ times}} &= [\dots [\underbrace{[\Delta, \Delta'], F, \dots, F}_{q+q' \text{ times}}](1)]_M \\ &= \sum_{i+j=q+q'} \frac{1}{i!j!} [\underbrace{[\dots [\Delta, F, \dots, F]]}_{i \text{ times}}, \underbrace{[\dots [\Delta', F, \dots, F]]}_{j \text{ times}}](1)]_M. \end{aligned}$$

The only terms that survive are those with $i = q - 1, q, q + 1$ (and hence $j = q' + 1, q', q' - 1$, respectively). We call them T_{q-1}, T_q, T_{q+1} , respectively. The first one is given by

$$\begin{aligned} T_{q-1} &= \frac{1}{(q-1)!(q'+1)!} [\underbrace{[\dots [\Delta, F, \dots, F]]}_{q-1 \text{ times}}, \underbrace{[\dots [\Delta', F, \dots, F]]}_{q'+1 \text{ times}}](1)]_M \\ &= \frac{1}{(q-1)!(q'+1)!} \underbrace{f_{\varphi, \dots, \varphi}}_{q-1 \text{ times}}, \underbrace{P'_{\text{lin}}(\varphi, \dots, \varphi)}_{q'+1 \text{ times}}. \end{aligned}$$

Similarly, the third one is

$$T_{q+1} = -\frac{1}{(q+1)!(q'-1)!} \underbrace{f'_{\varphi, \dots, \varphi}}_{q'-1 \text{ times}}, \underbrace{P_{\text{lin}}(\varphi, \dots, \varphi)}_{q+1 \text{ times}}.$$

To compute T_q , we use a simple trick: for every two scalar DOs $\square, \square'$ we have

$$[\square, \square'](1) = [\square, \square'(1)](1) - [\square', \square(1)](1).$$

After using this formula, we get

$$\begin{aligned} T_q &= \frac{1}{q!q'!} [\underbrace{[\dots [\Delta, F, \dots, F]]}_{q \text{ times}}, \underbrace{[\dots [\Delta', F, \dots, F]]}_{q' \text{ times}}](1)]_M \\ &= \frac{1}{q!q'!} \sigma(\Delta)(\underbrace{F, \dots, F}_{q \text{ times}}, \underbrace{[\dots [\Delta', F, \dots, F]](1)}_{q' \text{ times}})]_M - \frac{1}{q!q'!} \sigma(\Delta')(\underbrace{F, \dots, F}_{q' \text{ times}}, \underbrace{[\dots [\Delta, F, \dots, F]](1)}_{q \text{ times}})]_M. \end{aligned}$$

Now we use that both Δ and F are linearizable to replace $[\dots [\Delta', F], \dots, F](1)$ with its restriction to M in the first summand (likewise for Δ in the second summand). We get

$$\begin{aligned} T_q &= \frac{1}{q!q'!} \sigma(\Delta)(\underbrace{F, \dots, F}_{q \text{ times}}, \underbrace{f'_{\varphi, \dots, \varphi}}_{q' \text{ times}})]_M - \frac{1}{q!q'!} \sigma(\Delta')(\underbrace{F, \dots, F}_{q' \text{ times}}, \underbrace{f_{\varphi, \dots, \varphi}}_{q \text{ times}})]_M \\ &= \frac{1}{q!q'!} \underbrace{I_{P_{\text{lin}}}(\varphi, \dots, \varphi)}_{q \text{ times}}(\underbrace{f'_{\varphi, \dots, \varphi}}_{q' \text{ times}}) - \frac{1}{q!q'!} \underbrace{I_{P'_{\text{lin}}}(\varphi, \dots, \varphi)}_{q' \text{ times}}(\underbrace{f_{\varphi, \dots, \varphi}}_{q \text{ times}}), \end{aligned}$$

where we also used (7.4). At this point, it can be checked directly, exploiting the explicit formula (4.1) for the Poisson bracket of L_E -multivectors, that $T_{q-1} + T_q + T_{q+1}$ agrees with the f -component f'' of $[\Delta_{\text{lin}}, \Delta'_{\text{lin}}]$. We conclude that $[\Delta_{\text{lin}}, \Delta'_{\text{lin}}] = [\Delta, \Delta']_{\text{lin}}$, as desired. $\square$

Remark 7.6. Let $\mathcal{E}$ be the total space of a vector bundle $E \rightarrow M$ and interpret M as a submanifold in $\mathcal{E} = E$ via the zero section. In this situation, the normal bundle NM to M identifies canonically with E itself. Clearly, an order q FWL DO $\Delta \in \text{DO}_{q, \text{lin}}(E)$ is automatically order q linearizable, and it easily follows from the proof of Theorem 7.5 that the vector bundle isomorphism $E \cong NM$ identifies Δ with its own linearization Δ_{lin} .

Remark 7.7. As mentioned in Section 1, the results in the present paper should be considered as a first step towards a theory of multiplicative/infinitesimally multiplicative DOs on Lie groupoids/Lie algebroids and their Lie theory. The reader may consult, e.g., [7] for a recent review on multiplicative structures on Lie groupoids and their infinitesimal counterparts: infinitesimally multiplicative structures on Lie algebroids. We speculate that the ultimate definition of a multiplicative DO on a Lie groupoid (which we do not know yet) will fit into a theorem of the following form: Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid $A \rightrightarrows M$ and let

$\Delta \in \text{DO}(\mathbb{R}_{\mathcal{G}})$ be a *multiplicative DO* on $\mathcal{G}$. Then Δ is linearizable around M and its linearization $\Delta_{\text{lin}} \in \text{DO}_{\text{lin}}(A)$ is an infinitesimally multiplicative DO on A ; if $\mathcal{G}$ is source simply connected, then this correspondence between multiplicative and infinitesimally multiplicative DO can be inverted. Finally, we expect that non-trivial higher order examples will come from the theory of *multiplicative metrics* on Lie groupoids. Namely, we speculate that the Laplacian operator (on functions) determined by a multiplicative metric on a Lie groupoid $\mathcal{G}$ is a multiplicative DO, and hence its linearization is an infinitesimally multiplicative DO on the Lie algebroid of A . Exploring this class of examples requires a lot of work, including finding the appropriate definition of multiplicative/infinitesimally multiplicative DO and this goes far beyond the scopes of the present paper.

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