Research Article

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Finite rigid sets of the non-separating curve complex

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Abstract: We prove that the non-separating curve complex of every surface of finite type and genus at least three admits an exhaustion by finite rigid sets.

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1 Introduction

Let $S$ be a connected, orientable, finite-type surface. The curve complex is the simplicial complex $\mathcal{C}(S)$ whose $k$-simplices are sets of $k + 1$ distinct isotopy classes of essential simple curves on $S$ that are pairwise disjoint.

The extended mapping class group, denoted by $\text{Mod}^\pm(S)$, acts naturally on the set of curves up to isotopy on $S$. This action preserves disjointness of curves, and therefore extends to an action on the complex $\mathcal{C}(S)$. Via this action, the curve complex works as a combinatorial model to study properties of $\text{Mod}^\pm(S)$. For instance, a celebrated theorem of Ivanov in [12] asserts that, for sufficiently complex surfaces, the group $\text{Mod}^\pm(S)$ is isomorphic to the group of simplicial automorphisms of the curve complex, a result commonly known as simplicial rigidity. In turn, this result is a key ingredient in establishing the isomorphism $\text{Aut}(\text{Mod}^\pm(S)) \cong \text{Mod}^\pm(S)$.

The curve complex, and its applications to the mapping class group, has motivated the study of similar complexes associated to surfaces. For example, simplicial rigidity has been established for the arc complex [11], the non-separating curve complex [8], the separating curve complex [3, 13], the Hatcher–Thurston complex [10], and the pants graph [14] (see [16] for a survey on complexes associated to surfaces).

Another notion of rigidity which has been of recent interest is that of finite rigidity: the simplicial complex $\mathcal{C}(S)$ is said to be finitely rigid if there exists a finite subcomplex $X$ such that any locally injective simplicial map

$$\phi : X \to \mathcal{C}(S)$$

is induced by a unique mapping class, that is, there exists a unique $h \in \text{Mod}^\pm(S)$ such that the simplicial action $h : \mathcal{C}(S) \to \mathcal{C}(S)$ satisfies $h|_X = \phi$. Such $X$ is called a finite rigid set of $\mathcal{C}(S)$ with trivial pointwise stabilizer.

The finite rigidity of the curve complex was proven by Aramayona and Leininger in [1], thus answering a question by Lars Louder. Furthermore, they constructed in [2] an exhaustion of $\mathcal{C}(S)$ by finite rigid sets with trivial pointwise stabilizers, thus recovering Ivanov’s result [12] on the simplicial rigidity of $\mathcal{C}(S)$.

Following the result of Aramayona and Leininger, finite rigidity has been proven for other complexes: Shinkle proved it for the arc complex [17] and the flip graph [18], Hernández, Leininger and Maungchang proved a slightly different notion for the pants graph [5, 15], and Huang and Tshishiku proved a weaker notion for the separating curve complex [6].

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The main goal of this article is to prove the finite rigidity of the non-separating curve complex $\mathcal{N}(S)$, which is the subcomplex of $\mathcal{C}(S)$ spanned by the non-separating curves. To prove the finite rigidity of $\mathcal{N}(S)$, one would like to restrict the finite rigid set of $\mathcal{C}(S)$ in [1] to $\mathcal{N}(S)$; however, it is not clear why this restriction would yield a finite rigid set in $\mathcal{N}(S)$ as their proof uses separating curves in a fundamental way. Below, we construct a different subcomplex of $\mathcal{N}(S)$ and prove its rigidity.

Our main result is compiled in the next theorem.

**Theorem 1.1.** Let $S$ be a connected, orientable, finite-type surface of genus $g \geq 3$. There exists a finite simplicial complex $X \subset \mathcal{N}(S)$ such that any locally injective simplicial map

$$\phi : X \to \mathcal{N}(S)$$

is induced by a unique $h \in \text{Mod}^+ (S)$.

Our second result produces an exhaustion of the non-separating curve complex by finite rigid sets.

**Theorem 1.2.** Let $S$ be an orientable finite-type surface of genus $g \geq 3$. There exist subcomplexes

$$X_1 \subset X_2 \subset \cdots \subset \mathcal{N}(S)$$

such that

$$\bigcup_{i=1}^{\infty} X_i = \mathcal{N}(S)$$

and each $X_i$ is a finite rigid set with trivial pointwise stabilizer.

From Theorem 1.2 we can recover the simplicial rigidity of $\mathcal{N}(S)$ (see [9, Theorem 1.1]).

**Corollary 1.3.** Let $S$ be a connected, orientable, finite-type surface of genus $g \geq 3$. Any locally injective simplicial map $\phi : \mathcal{N}(S) \to \mathcal{N}(S)$ is induced by a unique $h \in \text{Mod}^+ (S)$. In particular, this yields an isomorphism between $\text{Mod}^+ (S)$ and the group of simplicial automorphisms of $\mathcal{N}(S)$.

**Plan of the paper.** In Section 2, we introduce some basic definitions that will be required. In Section 3, we introduce the notion of finite rigid sets. Sections 4 and 5 deal with the proofs of Theorems 1.1 and 1.2 for closed surfaces. Lastly, Sections 6 and 7 present the proofs of Theorems 1.1 and 1.2 for punctured surfaces.

## 2 Preliminaries

Let $S$ be a connected, orientable surface without boundary. We will further assume that $S$ has finite type, i.e., $\pi_1(S)$ is finitely generated. As such, $S$ is homeomorphic to $S_{g,n}$, the result of removing $n$ points from a genus $g$ surface. We refer to the removed points as punctures. If $S$ has no punctures, we will say that $S$ is closed. Otherwise, we will refer to $S$ as a punctured surface.

Before fervently jumping into the proofs, we warn the reader that the classification of surfaces, the change of coordinates principle and the Alexander method will be frequently used in proofs, sometimes without mention. For these and other fundamental results on mapping class groups, we refer the reader to [4].

### 2.1 Curves

By a curve $c$ in $S$ we will mean the isotopy class of an unoriented simple closed curve that does not bound a disk or a punctured disk. Throughout the article, we will make no distinction between curves and their representatives. We will say that $c$ is non-separating if any representative $\gamma$ of $c$ has connected complement in $S$.

The intersection number $i(a, b)$ between two curves $a$ and $b$ is the minimum intersection number between representatives of $a$ and $b$. If $i(a, b) = 0$, we will say that the curves $a$ and $b$ are disjoint. Given two representatives $a \in a$ and $\beta \in b$, we say that they are in minimal position if $i(a, b) = |a \cap \beta|$. A fact that will be often used is
that for any set of curves we may pick a single representative for each curve such that the representatives are pairwise in minimal position (see [4, Chapter 1.2]).

Given a set of curves \( \{c_1, \ldots, c_k\} \), consider representatives \( y_i \in c_i \) pairwise in minimal position. We will denote a regular neighborhood of \( \bigcup y_i \) by \( N(\bigcup y_i) \). The set of curves in the boundary of \( N(\bigcup y_i) \) will be denoted by

\[ \partial(c_1, \ldots, c_k). \]

We emphasize that implicit in the definition of \( b \in \partial(c_1, \ldots, c_k) \) is that \( b \) is an isotopy class of a simple closed curve which does not bound a disk or a punctured disk.

### 2.2 Non-separating curve complex

The **non-separating curve complex** \( \mathcal{N}(S) \) is the simplicial complex whose \( k \)-simplices are sets of \( k + 1 \) isotopy classes of pairwise disjoint curves.

Note that we can endow \( \mathcal{N}(S) \) with a metric by declaring each \( k \)-simplex to have the standard euclidean metric and considering the resulting path metric on \( \mathcal{N}(S) \).

#### 2.2.1 Pants decompositions

The dimension of \( \mathcal{N}(S_{g,n}) \) is \( 3g - 3 + n \), and the vertex set of a top-dimensional simplex in \( \mathcal{N}(S_{g,n}) \) is called a **non-separating pants decomposition** of \( S_{g,n} \). If \( P = \{c_1, \ldots, c_k\} \) is a pants decomposition of \( S \), then \( S \setminus \bigcup c_i \) is a union of pairs of pants (i.e., a union of subsurfaces homeomorphic to \( S_{0,3} \)).

Let \( P = \{c_1, \ldots, c_k\} \) be a pants decomposition of the surface \( S \). Two curves \( c_i, c_j \in P \) are said to be adjacent rel to \( P \) if there exists a curve \( c_k \in P \) such that \( c_i, c_j, c_k \) bound a pair of pants in \( S \); see Figure 1a for an example.

We record the following observation for future use.

**Remark 2.1.** Let \( P \) be a non-separating pants decomposition of \( S_{g,n} \), where \( g \geq 3 \).

- If \( n \leq 1 \), then every curve in \( P \) is adjacent rel to \( P \) to at least three other curves.
- If \( n > 1 \), then every curve is adjacent to at least two other curves.

Consider \( A \subset P \), where \( P \) is a pants decomposition of \( S \). We say that a set of curves \( A \) substitutes \( A \) in \( P \) if

\[ (\hat{A} \cup P) \setminus A \]

is a pants decomposition. In words, we say that \( \hat{A} \) substitutes \( A \) in \( P \) if both sets have no curves in common and we can replace the curves in \( A \) by the curves in \( \hat{A} \) and still get a pants decomposition.

### 3 Finite rigid sets

For a simplicial subcomplex \( X \subset \mathcal{N}(S) \), a map \( \phi : X \to \mathcal{N}(S) \) is said to be a **locally injective simplicial map** if \( \phi \) is simplicial and injective on the star of each vertex. A first elementary observation is the following lemma.

**Lemma 3.1.** Let \( \phi : X \to \mathcal{N}(S) \) be a locally injective simplicial map. If \( P \subset X \) is a pants decomposition, then \( \phi(P) \) is a pants decomposition.

**Proof.** Take a vertex \( p \in P \). Since \( \phi \) is injective in the star of \( p \) and \( P \) is a simplex, \( \phi \) is injective on \( P \). Thus, \( \phi(P) \) is a maximal-dimensional simplex, i.e., \( \phi(P) \) is a pants decomposition.

As mentioned in Section 1, the main goal of this article is to construct a finite subcomplex \( X \subset \mathcal{N}(S) \) with the following properties.
**Definition 3.2 (Finite rigid set).** A **finite rigid set** \(X\) of \(\mathcal{N}(S)\) is a finite subcomplex such that any locally injective simplicial map \(\phi : X \to \mathcal{N}(S)\) is induced by a mapping class, i.e., there exists \(h \in \text{Mod}^+\) with \(h|_X = \phi\).

In addition, if \(h\) is unique, we say that \(X\) has **trivial pointwise stabilizer**.

Observe that a subcomplex \(X \subset \mathcal{N}(S)\) has trivial pointwise stabilizer if and only if the inclusion \(X \hookrightarrow \mathcal{N}(S)\) is induced uniquely by the identity \(1 \in \text{Mod}^+(S)\), hence the name.

**Remark 3.3.** By the change of coordinates principle (see [4, Chapter 1.3]), every vertex \(\{v\} \subset \mathcal{N}(S)\) is a finite rigid set. However, the stabilizer of \(\{v\}\) is not trivial.

**Remark 3.4.** If \(X\) is a finite rigid set and \(X \subset Y\), then \(Y\) may not be a finite rigid set.

For example, consider a disjoint curves \(v_1, v_2\) such that \(S \setminus \bigcup v_i\) is connected, and two disjoint curves \(v'_1, v'_2\) such that \(S \setminus \bigcup v'_i\) is disconnected. Now, take \(X = \{v_1\}, Y = \{v_1, v_2\}\) and the locally injective simplicial map \(\phi(v_1) = v'_1\). Clearly, \(\phi\) is not induced by a mapping class, and so \(Y\) is not a finite rigid set of \(\mathcal{N}(S)\). Note that \(X\) is a finite rigid set of \(\mathcal{N}(S)\) by the remark above.

Following Aramayona and Leininger in [1], we will say that a subcomplex \(X \subset \mathcal{N}(S)\) **detects the intersection** of two curves \(a, b \in X\) if every locally injective simplicial map \(\phi : X \to \mathcal{N}(S)\) satisfies

\[i(a, b) \neq 0\quad \text{if and only if} \quad i(\phi(a), \phi(b)) \neq 0.\]

### 4 Finite rigid sets for closed surfaces

In this section, we construct finite rigid sets for closed surfaces and prove their rigidity. This will establish Theorem 1.1 for closed surfaces.

#### 4.1 Constructing the finite rigid set

Let \(S\) be a closed surface of genus \(g \geq 3\). We will start by defining the curves in the finite rigid set. The reader should keep Figures 1(a)–(e) in mind throughout the section.

Fix a set \(\{p_1, c_1, \ldots, p_g, c_g, p_{g+1}\}\) of non-separating curves such that \(i(c_i, p_i) = i(c_i, p_{i+1}) = 1\) and the rest of the curves are pairwise disjoint (see Figures 1(a) and 1(b)). Such a set of curves is unique up to homeomorphism. Let \(c_{g+1}\) be a curve such that \(i(p_1, c_{g+1}) = i(p_{g+1}, c_{g+1}) = 1\) and it is disjoint from every other curve in the set above. We define

\[C = \{c_1, \ldots, c_{g+1}\}.\]

Notice that \(S \setminus p_i\) has two connected components \((S \setminus p_i)^+\) and \((S \setminus p_i)^-\); we will call \((S \setminus p_i)^+\) the **top component** and \((S \setminus p_i)^-\) the **bottom component**. In the same fashion, \(S \setminus c_i\) has two connected components \((S \setminus c_i)^+\) and \((S \setminus c_i)^-\); we will call \((S \setminus c_i)^+\) the **front component** and \((S \setminus c_i)^-\) the **back component**.

For each \(k = 2, \ldots, g-1\), the set \(\partial(p_1, c_1, \ldots, p_k, c_k)\) consists of two curves: one of them in \((S \setminus p_{k})^+\) and the other one in \((S \setminus p_{k})^-\). We will call \(p_k^+\) the curve of \(\partial(p_1, c_1, \ldots, p_k, c_k)\) contained in \((S \setminus p_{k})^+\), and we will call \(p_k^-\) the curve of \(\partial(p_1, c_1, \ldots, p_k, c_k)\) in \((S \setminus p_{k})^-\). We set

\[P = \{p_1, \ldots, p_{g+1}\} \cup \{p_2^+, p_2^-, \ldots, p_{g-1}^+, p_{g-1}^-, p_{g+1}^-, p_{g+1}^+\}.\]

Notice that \(P\) is a pants decomposition (see Figure 1(a)).

For each \(k = 2, \ldots, g-1\), the set \(\partial(p_{k-1}, c_k, p_k)\) has two curves, one in \((S \setminus p_{k})^+\) and the other one in \((S \setminus p_{k})^-\). We will denote by \(u_k\) the curve in \((S \setminus p_{k})^+\) and by \(d_k\) the curve in \((S \setminus p_{k})^-\) (see Figure 1(c)). We set

\[U = \{u_2, \ldots, u_{g-1}\}\]

and

\[D = \{d_2, \ldots, d_{g-1}\}.\]
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(a) Pants decomposition $P$ of surface $S$. As an example, note that $p_1$ and $p_2$ are adjacent rel to $P$, while $p_1$ and $p_3$ are not adjacent rel to $P$.

(b) Circular curves $C$.

(c) Up and down curves $u_k$, $d_k$.

(d) Left and right curves $l_k$, $r_k$.

(e) Non-symmetrical down curve $nd$.

(f) Non-symmetrical left and right curves $nl$, $nr$.

Figure 1: Curves in $F_R$ for a closed surface.

Given $k \in \{2, \ldots, g - 1\}$, the set $\partial(p_k, c_k, p_k^+)$ contains two curves, and only one of them is also a curve in $P$. We will denote by $l_k$ the curve in $\partial(p_k, c_k, p_k^+)$ not already in $P$ (see Figure 1(d)). We set

$$L = \{l_2, \ldots, l_{g-2}\}.$$

Analogously, let $R = \{r_2, \ldots, r_{g-2}\}$ be the set of curves where $r_k$ is the unique curve in $\partial(p_{k+1}, c_{k+1}, p_{k+2})$ that is not in $P$ (see Figure 1(d)).

The set $\partial(p_2, c_2, \ldots, p_{g-1}, c_{g-1}, p_g)$ has two curves, one curve in each component of $S \setminus \bigcup p_i$. Let $b$ be the curve contained in the bottom component $(S \setminus \bigcup c_i)^+$. Then the set $\partial(c_1, b, c_g)$ has exactly two curves, one curve contained in $(S \setminus \bigcup c_i)^+$ and the other in $(S \setminus \bigcup c_i)^-$. Denote by $nd$ the curve in $(S \setminus \bigcup c_i)^+$ (see Figure 1(e)).

Lastly, consider the torus $T_1$ that contains $p_1$ and is bounded by the curves $p_1^+, p_1^-$. Let $nl$ be the unique curve contained in $T_1 \setminus (nd \cup c_1)$ distinct from $c_1$ and $p_1^+$. In the same way, $p_{g-1}^+, p_{g-1}^-$ bound a torus $T_g$ such that $p_g \subset T_g$, let $nr$ be the unique curve in $T_g \setminus (nd \cup c_g)$ distinct from $c_g$ and $p_{g-1}^+$ (see Figure 1(f)). We set

$$N = \{nl, nr, nd\}.$$

We set $F_R$ to be the subcomplex of $N(S)$ spanned by the vertices in

$$P \cup C \cup U \cup D \cup L \cup R \cup N.$$

Remark 4.1. Note that the subcomplex $F_R$ has diameter two. Therefore, any locally injective simplicial map $\phi : F_R \to N(S)$ is injective.

4.2 Proving the rigidity of $F_R$

The first step is to check that the locally injective simplicial map $\phi : F_R \to N(S)$ preserves the non-adjacency rel to $P$. To do so, we require the following technical lemma.

Lemma 4.2. Let $S$ be a finite-type surface, let $F_R \subset N(S)$ be a subcomplex, let $P \subset F_R$ be a pants decomposition, and let $a, b \in P$ be two curves. Suppose that there exist subsets $A, B \subset P$ and $\tilde{A}, \tilde{B} \subset F_R$ satisfying the following
This concludes the proof.

If $\phi$ and the image, that is, the curves $\phi$ conditions, the lemma ensures that a Lemma 4.3.

Let $\phi$ Using the previous result, we prove that $\phi$ Proof. We will proceed by contradiction. Suppose that the curves $a, b$ are adjacent in a pair of pants $Q$ bound by $a, b, c$. Since $A \cap B = \emptyset$, either $c \notin A$ or $c \notin B$. Without loss of generality, suppose $c \notin A$.

If $A$ is a substitution of $A$ and $A \cap B = \emptyset$, then there is a curve $\tilde{a} \in A$ such that $i(\tilde{a}, a) \neq 0$, $i(\tilde{a}, b) = 0$ and $i(\tilde{b}, a) = 0$. Now, note that, since $\tilde{B}$ and $A \cup \tilde{B}$ are substitutions, there exists $\tilde{b} \in \tilde{B}$ with $i(\tilde{b}, b) \neq 0$, $i(\tilde{b}, a) = 0$ and $i(\tilde{b}, \tilde{a}) = 0$. However, it is impossible to have arcs $\tilde{a} \cap Q$ and $\tilde{b} \cap Q$ satisfying the intersections above (see Figure 2).

Now, we prove that $\phi$ preserves non-adjacency rel to $P$.

**Lemma 4.3.** Let $S$ be a closed surface of genus $g \geq 3$ and let $\phi : F_R \to \mathcal{N}(S)$ be a locally injective simplicial map. If $a, b \in P$ are not adjacent rel to $P$, then $\phi(a), \phi(b)$ are not adjacent rel to $\phi(P)$.

**Proof.** Assume that for two curves $a, b \in P$ we have subsets $A, B \subset P$ and $\tilde{A}, \tilde{B} \subset F_R$ as in Lemma 4.2. Under these conditions, the lemma ensures that $a$ and $b$ are not adjacent rel to $P$. Moreover, these properties are carried to the image, that is, the curves $\phi(a), \phi(b) \in \phi(P)$ satisfy the conditions of Lemma 4.2 for the sets $\phi(A), \phi(B) \subset \phi(P)$ and $\phi(A), \phi(B)$. As a consequence, we deduce that $\phi(a)$ and $\phi(b)$ are not adjacent rel to $\phi(P)$.

By means of the method above, we are only left to find appropriate subsets $A, \tilde{A}, B, \tilde{B}$ for any non-adjacent curves $a, b \in P$. We will find such subsets for certain $a, b \in P$, as the rest of the cases are similar.

If $a = p_k$ and $b = p_{k+1}$ for $k \in \{3, \ldots, g-3\}$, we can consider

$$A = \{p_k, p_k\}, \quad \tilde{A} = \{k_{-1}, d_{k-1}\},$$

$$B = \{p_{k+1}, p_{k+1}\}, \quad \tilde{B} = \{r_k, d_{k+1}\}.$$ 

It is straightforward to check that these subsets satisfy the conditions of Lemma 4.2, and so $\phi(a), \phi(b)$ are not adjacent rel to $\phi(P)$.

If $a = p_1$, consider $A = \{p_1, p_2\}$ and $\tilde{A} = \{nl, c_1\}$.

- If $b \in \{p_{g-1}, p_{g+1}\}$, take $B = \{p_{g-1}, p_{g+1}\}$ and $\tilde{B} = \{nr, c_g\}$.

- If $b \in \{p_k, p_1\}$ for $k \in \{3, \ldots, g-1\}$, consider $B = \{p_k, p_{k-1}\}$ and $\tilde{B} = \{c_k, r_{k-1}\}$.

- If $b = p_k$ for $k \in \{3, \ldots, g-1\}$, consider $B = \{p_{k+1}, p_k\}$ and $\tilde{B} = \{u_k\}$.

This concludes the proof.

Using the previous result, we prove that $\phi$ preserves adjacency rel to $P$.

**Lemma 4.4.** Let $S$ be a closed surface of genus $g \geq 3$ and let $\phi : F_R \to \mathcal{N}(S)$ be a locally injective simplicial map. If $a, b \in P$ are adjacent rel to $P$, then $\phi(a), \phi(b)$ are adjacent rel to $\phi(P)$.
Proof. Take $\phi(p_1) \in \phi(P)$. From the non-adjacency rel to $\phi(P)$, it follows that $\phi(p_1)$ has at most three adjacent curves. On the other hand, Remark 2.1 implies that $\phi(p_1)$ has at least three adjacent curves. Thus, we conclude that $\phi(p_1)$ has exactly three adjacent curves, namely $\phi(p_2)$, $\phi(p_3)$, $\phi(p_4)$. The same argument applies to $\phi(p_2)$, so it is adjacent rel to $\phi(P)$ to exactly three curves, namely $\phi(p_1)$, $\phi(p_3)$, $\phi(p_4)$.

We now determine the curves adjacent to $\phi(p_1)$ and $\phi(p_2)$. First, note that the adjacency rel to $\phi(P)$ of $\phi(p_1)$ and $\phi(p_2)$ implies that $\phi(p_1)$ and $\phi(p_2)$ bound a subsurface homeomorphic to $S_{1,2}$. Since both curves $\phi(p_1)$ and $\phi(p_2)$ are non-separating, it follows that both have four adjacent curves rel to $\phi(P)$. Finally, considering the non-adjacency rel to $\phi(P)$, it follows that the curves adjacent to $\phi(p_3)$ are

$$\{\phi(p_1), \phi(p_2), \phi(p_3), \phi(p_4)\}$$

and the curves adjacent to $\phi(p_4)$ are

$$\{\phi(p_1), \phi(p_2), \phi(p_3), \phi(p_4)\}.$$

In the same style, we can argue inductively to determine the adjacency of each curve in $\phi(P)$. □

As a corollary, we obtain that $\phi$ preserves the topological type of $P$.

**Corollary 4.5.** Let $S$ be a finite-type surface and let $\phi : F_R \to \pi_1(S)$ be a map that preserves adjacency rel to $P$. There exists $h \in \Mod^1(S)$ such that $h|_P = \phi|_P$.

**Proof.** We construct a homeomorphism $h$ inductively by gluing abstract pairs of pants.

Consider the pairs of pants $P_0 \subseteq S$ bounded by curves in $P$ and that $P_0$ is a disjoint union of pants $P_1, \ldots, P_k$. Each $P_i$ is homeomorphic to $F_R$. Any two pairs of pants are homeomorphic, and thus we can consider homeomorphisms $h_i : P_i \to P_i$. Since $\phi$ preserves the adjacency rel to $P$, it follows that $\phi_i$ preserves the adjacency rel to $P_i$, where $h_i$ is the homeomorphism of $P_i$ to $P_i$.

The next three lemmas prove that $F_R$ detects intersection among certain curves. Recall that $F_R$ detects the intersection between $a$ and $b$ if for any locally injective simplicial map $\phi : F_R \to \pi_1(S)$ we have that

$$i(a, b) \neq 0 \quad \text{if and only if} \quad i(\phi(a), \phi(b)) \neq 0.$$

**Lemma 4.6.** The subcomplex $F_R \subseteq \pi_1(S)$ detects the following intersections for every $k = 2, \ldots, g - 1$:

(i) $u_k$ with $p_k$

(ii) $d_k$ with $p_k$

**Proof.** Let $\phi : F_R \to \pi_1(S)$ be a locally injective simplicial map. We need to check that $i(\phi(u_k), \phi(p_k)) \neq 0$ and $i(\phi(d_k), \phi(p_k)) \neq 0$.

Seeking a contradiction to case (i), we assume $i(\phi(u_k), \phi(p_k)) = 0$. Since $\phi$ is locally injective, it sends disjoint curves to disjoint curves. Thus, $\phi(u_k)$ is disjoint from every curve in the pants decomposition $\phi(P)$, which implies $\phi(u_k) \in \phi(P)$. However, this contradicts the injectivity of $\phi$ (see Remark 4.1).

To prove case (ii), the same argument works. □

**Remark 4.7.** Notice that from the previous lemma we actually know that $F_R$ detects the intersection of $u_k$ with every curve in $P$. Indeed, $u_k$ is disjoint from any curve in $P \setminus \{p_k\}$ and $\phi$ preserves disjointness. In the same way, $F_R$ detects the intersection of $d_k$ with every curve in $P$.

**Lemma 4.8.** The subcomplex $F_R \subseteq \pi_1(S)$ detects the intersection of $c_k$ with $p_k$ and $p_k$ for every $k = 2, \ldots, g - 1$.

**Proof.** Let $\phi : F_R \to \pi_1(S)$ be a locally injective simplicial map. By Corollary 4.5, there exists $h \in \Mod^1(S)$ such that $h \circ \phi$ fixes the pants decomposition $P$. Observe that detecting intersection is equivalent for $\phi$ and for $h \circ \phi$, since we have

$$i(\phi(a), \phi(b)) \neq 0 \quad \text{if and only if} \quad i(h \circ \phi(a), h \circ \phi(b)) \neq 0.$$

So, we can rename $h \circ \phi$ to $\phi$, and prove the statement assuming that $\phi$ fixes every $p \in P$. □
With the previous simplification, the proof boils down to check that \( \phi(c_k) \) intersects \( p^*_k \) and \( p^-_k \). In this direction, consider the torus \( T_k \) bounded by the curves \( p^*_{k-1}, p^-_{k-1}, p^*_k, \) and \( p^-_{k+1} \) (see Figure 3). Further, define the top of the torus by \( T_k^+ = T_k \cap (S \setminus \cup p_i)^+ \), and the bottom of the torus by \( T_k^- = T_k \cap (S \setminus \cup p_i)^- \).

By Lemma 4.6, \( \phi(u_k) \) is a curve in \( T_k^- \) intersecting \( p^*_k \), and \( \phi(d_k) \) is a curve in \( T_k^+ \) intersecting \( p^-_k \) (see Figure 3). Notice that \( \phi(c_k) \) is a curve in \( T_k \) distinct and disjoint from \( \phi(d_k) \) and \( \phi(u_k) \). It follows that \( \phi(c_k) \) intersects both \( T_k^+ \) and \( T_k^- \).

To finish, note that \( \phi(c_k) \cap T_k^+ \) is disjoint from \( \phi(u_k) \), so \( \phi(c_k) \) must intersect \( p^*_k \). Indeed, an arc disjoint from a curve in a sphere with four boundary components must intersect every other curve in the sphere. Similarly, \( \phi(c_k) \cap T_k^- \) is disjoint from \( \phi(d_k) \), so \( \phi(c_k) \) must intersect \( p^-_k \).

\[ \square \]

**Lemma 4.9.** The subcomplex \( F_R \subset \mathcal{N}(S) \) detects the intersection of \( c_k \in C \) with every curve in \( P \).

**Proof.** Let \( \phi : F_R \to \mathcal{N}(S) \) be a locally injective simplicial map. As in the previous proof, we may assume that \( \phi \) fixes every curve in \( P \).

Now, we start by proving the cases \( 2 \leq k \leq g-1 \). Note that, with the simplification above and Lemma 4.8, we only need to check that \( \phi(c_k) \) intersects \( p_k \) and \( p_{k+1} \).

To prove that \( \phi(c_k) \) intersects \( p_k \), consider the pair of pants \( Q \) bounded by the curves \( p^*_{k-1}, p_k, \) and \( p^*_k \). By Lemma 4.8, there are disjoint arcs \( Q \cap \phi(c_{k-1}) \) and \( Q \cap \phi(c_k) \) with at least one endpoint in \( p^*_{k-1} \) and \( p^*_k \), respectively (see Figure 4). Using that \( \phi(c_{k-1}) \cap p^*_{k-1} \) and \( \phi(c_k) \cap p^*_k \) are also disjoint, we conclude that any such arc configuration requires \( \phi(c_k) \) to intersect \( p_k \). The same argument yields that \( \phi(c_k) \) intersects \( p_{k+1} \).

It is left to prove the cases \( k \in \{1, g, g+1\} \). First, we prove that \( \phi(c_1) \) intersects \( p_2 \). Consider the torus \( T_1 \) bounded by the curves \( p^*_2 \) and \( p^-_2 \), and denote by \( T_1^+ \) the pair of pants bounded by \( p_1, p_2, \) and \( p^*_2 \) (see Figure 5). Noting that \( \phi(c_1) \) is a curve in \( T_1^+ \) distinct from \( p_1, p_2, \) and \( p^*_2 \), it follows that \( \phi(c_1) \) intersects \( T_1^+ \). Thus, we have disjoint arcs \( \phi(c_1) \cap T_1^+ \) and \( \phi(c_2) \cap T_1^+ \), the latter one having an endpoint in \( p^*_2 \) (see Figure 5). Since \( \phi(c_1), p^*_2 \) are disjoint and \( \phi(c_2), p_1 \) are disjoint, we conclude that \( \phi(c_1) \) must intersect \( p_2 \). Again, the same argument with slight changes yields that \( \phi(c_g) \) intersects \( p_g \).
Before proving that \( \phi(c_1) \) intersects \( p_1 \) and \( \phi(c_g) \) intersects \( p_{g+1} \), we need to check that \( \phi(c_{g+1}) \) intersects both \( p_1, p_g \) and every \( p_k^+, p_k^- \). Surely, \( \phi(c_{g+1}) \) intersects at least one of these curves, since otherwise \( \phi(c_{g+1}) \) would be disjoint and distinct from every curve in the pants decomposition \( \phi(P) \). Suppose that \( \phi(c_{g+1}) \) intersects \( p_k^+ \), and consider the pair of pants \( Q \) bound by \( p_k^- \), \( p_k^+ \) and \( p_{k-1}^- \). Using the intersections above, we deduce that there are disjoint arcs \( \phi(c_{k-1}) \cap Q \) and \( \phi(c_{g+1}) \cap Q \), so that \( \phi(c_{g+1}) \) must also intersect \( p_{k-1}^- \). Repeating the argument iteratively, we can detect the intersection of \( \phi(c_{g+1}) \) with every curve in \( P \).

To finish the proof, we check that \( \phi(c_1) \) intersects \( p_1 \). Consider the disjoint arcs \( a_2 = \phi(c_2) \cap T_1^+ \) and \( a_{g+1} = \phi(c_{g+1}) \cap T_1^+ \), where \( a_2 \) has an endpoint in \( p_2^+ \) and \( a_{g+1} \) has an endpoint in \( T_1^- \). These two arcs exist by the above intersections. Moreover, \( a_2 \) is disjoint from \( p_1 \), and \( a_{g+1} \) is disjoint from \( p_2 \). Recall that \( \phi(c_1) \) intersects \( T_1^- \) and, since it is disjoint from both \( a_1, a_2 \), it follows that \( \phi(c_1) \) intersects \( p_1 \). A similar argument yields that \( \phi(c_g) \) intersects \( p_{g+1} \).

So far we have seen that the map \( \phi : F_R \to \mathcal{N}(S) \) can be taken to agree with a homeomorphism on \( P \subset F_R \), and that it detects some intersections. In the next lemma, we extend it so that \( \phi \) agrees with a homeomorphism on \( F_R \setminus N \).

**Lemma 4.10.** Let \( S \) be a closed surface of genus \( g \geq 3 \) and let \( \phi : F_R \to \mathcal{N}(S) \) be a locally injective simplicial map. There exists \( h \in \text{Mod}^+(S) \) such that \( h|_{F_R \setminus N} = \phi|_{F_R \setminus N} \).

**Proof.** By Corollary 4.5, there exists \( h \in \text{Mod}^+(S) \) such that \( h \circ \phi \) fixes every \( p \in P \); we rename \( h \circ \phi \) to \( \phi \). Recall that by Remark 4.1 we know that \( \phi \) is injective.

First, we find a homeomorphism that agrees with \( \phi \) on \( c_1 \in C \). Observe that \( \phi(c_1) \) is contained in the torus \( T_1 \) bounded by \( p_2^+ \) and \( p_2^- \) (see Figure 6). Moreover, Lemma 4.9 implies that there exist disjoint arcs \( a \in \phi(c_2) \cap T_1 \) and \( \hat{a} \in \phi(c_{g+1}) \cap T_1 \). Notice that both arcs have endpoints in \( p_2^+ \) and \( p_2^- \), as otherwise \( \phi(c_1) \) would not intersect \( p_1 \), contradicting Lemma 4.9. It follows that \( \phi(c_1) \) is the curve contained in the annulus \( T_1 \setminus (a \cup \hat{a}) \). Even more, there exists a twist \( h' \in \text{Mod}^+(S) \) along \( p_1 \) and \( p_2 \) such that \( h' \circ \phi(c_1) = c_1 \). We rename \( h' \circ \phi \) to \( \phi \).

The same argument with minor changes yield homeomorphisms that agree with \( \phi \) on every \( c_k \in C \setminus \{c_{g+1}\} \). Thus, we may assume that \( \phi \) fixes every curve in \( P \cup C \setminus \{c_{g+1}\} \subset F_R \).
To finish the proof, we are going to check that \( \phi \) is fixing \( c_{g+1} \) and every curve in \( U \cup D \cup L \cup R \):

- Notice that \( \phi(c_{g+1}) \) is contained in the torus with boundary \( T' = S \setminus \bigcup_{i=2}^{k} p_i \). In fact, \( \phi(c_{g+1}) \) is the unique curve contained in the annulus \( S' \setminus \bigcup_{i=1}^{k} c_k \), i.e., \( \phi(c_{g+1}) = c_{g+1} \).

- To prove that \( \phi \) fixes \( U \), consider \( u_k \in U \). Observe that \( \phi(u_k) \) is contained in the sphere \( S_k^+ \) bounded by the curves \( p_{k-1}^-, p_k, p_{k+1} \) and \( p_{k+1}^- \). Moreover, \( \phi(u_k) \) must be the only curve in the annulus \( S_k^- \setminus \{ c_k, c_{g+1} \} \), i.e., \( \phi(u_k) = u_k \).

- To prove that \( \phi \) fixes \( D \), consider \( d_k \in U \). Notice that \( \phi(d_k) \) is contained in the sphere \( S_k^- \) bounded by the curves \( p_{k-1}^-, p_k, p_{k+1} \) and \( p_{k+1}^- \). Now, \( \phi(d_k) \) must be the only curve in the annulus \( S_k^- \setminus \{ c_k, c_{g+1} \} \), i.e., \( \phi(d_k) = d_k \).

- To prove that \( \phi \) fixes \( L \), consider the curve \( l_k \in L \). The image \( \phi(l_k) \) is a curve in the subsurface \( S_1 \) bounded by \( p_{k-1}^-, p_k, p_{k+1}^-, p_{k-1}^- \). Since \( \phi(l_k) \) is disjoint from curves in \( C \), we know that \( \phi(l_k) \) is contained in the pair of pants \( Q = S_1 \setminus \{ c_k, c_{g+1} \} \). Note that \( Q \) has only one boundary component not already in \( P \), and so we conclude \( \phi(l_k) = l_k \). Naturally, a similar argument yields that \( \phi \) fixes \( r_k \in R \).

Summarizing, we have proven that there exists a mapping class \( f \in \text{Mod}^+(S) \) such that \( f \circ \phi \) fixes \( F_R \setminus N \). In other words,

\[
\left. f^{-1} \right|_{F_R \setminus N} = \phi|_{F_R \setminus N}.
\]

This concludes the proof.

To finish the proof of Theorem 1.1, we need to check that we can take \( \phi \) to agree with a homeomorphism on \( N \subset F_R \), and that such homeomorphism is unique. Before doing so, we require one more definition: The back-front (orientation reversing) involution is the unique non-trivial mapping class \( \iota \in \text{Mod}^+(S) \) fixing every curve \( c \in P \cup C \).

**Proof of Theorem 1.1 for closed surfaces.** Let \( \phi : F_R \to N(S) \) be a locally injective simplicial map. By Lemma 4.10, there exists \( h \in \text{Mod}^+(S) \) such that \( h \circ \phi \) fixes \( F_R \setminus N \); we rename \( h \circ \phi \) to \( \phi \).

Consider \( nd \in N \) and notice that \( \phi(nd) \) is a curve in the genus two surface \( T \) bounded by the curves \( p_{k-1}^-, p_3, p_4, \ldots, p_{k-1}^+, p_{k+1}^- \). Since \( \phi(nd) \) is disjoint from \( C \), we have that \( \phi(nd) \) is a curve in \( Q = T \setminus \bigcup_{c \in C} c \). Note that \( Q \) is the disjoint union of two pairs of pants and it has only two boundary components not contained in \( C \), namely \( nd \) and \( \iota(nd) \), where \( \iota \) is the back-front involution. It follows that \( \phi(nd) \in \{ nd, \iota(nd) \} \). Thus, by precomposing by \( \iota \) if necessary, we may assume that \( \phi \) fixes \( F_R \setminus \{ nl, nr \} \).

We continue by proving that \( \phi \) fixes \( nl \). Note that \( \phi(nl) \) is a curve in the torus \( T_1 \) bounded by \( p_{k-1}^- \) and \( p_{k+1}^+ \). Even more, \( \phi(nl) \) is contained in \( Q = T_1 \setminus (c_1 \cup nd) \), which is the union of an annulus and a pair of pants. Note that only one curve in \( Q \) is not a curve in \( P \cup C \), and so we deduce \( \phi(nl) = nl \). An analogous argument leads to the conclusion \( \phi(nr) = nr \).

So far, we have found a composition of mapping classes \( f \in \text{Mod}^+(S) \) such that \( f \circ \phi \) is the identity in \( F_R \). Therefore, \( \phi \) is induced by the mapping class \( f^{-1} \).

To prove the uniqueness of the inducing mapping class, suppose that \( f, \bar{f} \in \text{Mod}^+(S) \) both induce the same \( \phi : F_R \to N(S) \). Since \( g = \bar{f} \circ f^{-1} \) fixes the curves \( P \cup C \subset F_R \), the Alexander method implies that \( g \) is either the identity or the back-front orientation reversing involution. However, since \( g \) also fixes the curve \( nd \), we have that \( g \) is the identity and \( f = \bar{f} \).

5 Finite rigid exhaustion for closed surfaces

In this section, we prove that, for any closed surface \( S \) of genus \( g \geq 3 \), there exist subcomplexes

\[
F_1 \subset F_2 \subset \cdots \subset N(S)
\]

such that each \( F_i \) is a finite rigid set with trivial pointwise stabilizer and

\[
\bigcup_{i=1}^{\infty} F_i = N(S).
\]
The strategy to produce an exhaustion is to first extend \( F_R \) to a larger finite rigid set \( F_1 \) with desirable properties. Then the set \( F_1 \) will work as base case for an induction that enlarges \( F_i \) into \( F_i+1 \). This method heavily resembles the proof given by Aramayona and Leininger in \cite{2} to produce an exhaustion of the curve complex.

The plan of the proof is summarized in the following lemma (cf. \cite[Lemma 3.13]{2}).

**Lemma 5.1.** Let \( S \) be a finite-type surface, let \( F_1 \subset \mathcal{N}(S) \) be a finite rigid set with trivial pointwise stabilizer, and let \( \{h_1, \ldots, h_k\} \) be a set of generators of \( \text{Mod}^\pm(S) \). If \( F_1 \cup h_i(F_1) \) is a finite rigid set with trivial pointwise stabilizer for every \( i \), then the sets

\[
F_{i+1} = F_i \cup \bigcup_{j=1}^k (h_j(F_i) \cup h_j^{-1}(F_i))
\]

satisfy that \( F_1 \subset F_2 \subset \cdots \subset \mathcal{N}(S) \) and every \( F_i \) is finite rigid with trivial pointwise stabilizer.

**Proof.** First, notice that if \( F_1 \cup h_i(F_1) \) is a finite rigid set with trivial stabilizer, then the same holds for \( F_1 \cup h_i^{-1}(F_1) \).

We will now check that \( F_2 \) is finite rigid with trivial stabilizer: Take \( \phi : F_2 \to \mathcal{N}(S) \) and observe that by hypothesis there exists \( f_j \in \text{Mod}^\pm(S) \) such that

\[
\phi|_{F_1 \cup h_j(F_1)} = f_j|_{F_1 \cup h_j(F_1)}
\]

for every \( j \in \{1, \ldots, k\} \). Since \( F_1 \) has trivial pointwise stabilizer and

\[
\phi|_{F_1} = f_k|_{F_1},
\]

we deduce \( f_j = f_k \) for all \( j, k \). Clearly, the same argument holds when considering \( F_1 \cup h_j^{-1}(F_1) \). It follows that \( \phi = f \) for \( f \in \text{Mod}^\pm(S) \) and \( F_2 \) is a finite rigid set with trivial pointwise stabilizer.

For \( F_{i+1} \), we proceed by induction. Consider \( \phi : F_{i+1} \to \mathcal{N}(S) \). By induction, we have \( \phi|_{F_i} = f|_{F_i} \in \text{Mod}^\pm(S) \) and

\[
\phi|_{h_j(F_i)} = f_j|_{h_j(F_i)}
\]

for some \( f_j \in \text{Mod}^\pm(S) \). Observe that \( F_1 \subset F_i \cap h_j(F_i) \) and \( F_1 \) is trivially stabilized. Thus \( f = f_j \) for every \( j \), and so \( F_{i+1} \) is finite rigid with trivial stabilizer.

To produce the finite rigid exhaustion of \( \mathcal{N}(S) \), we are going to use the previous lemma. In this direction, we enlarge \( F_R \) into a set \( F_1 \) and provide a set of generators for \( \text{Mod}^\pm(S) \).

### 5.1 Enlarging the finite rigid set

Let \( t \in \text{Mod}^\pm(S) \) be the back-front orientation reversing involution. We define the set of curves \( A := A_1 \cup A_2 \), where

\[
A_1 := \bigcup_{k=1}^{g-1} \partial(c_k, p_{k+1}, c_{k+1}).
\]

Note that the set \( \partial(c_k, p_{k+1}, c_{k+1}) \) has two curves, one curve in the front component and one curve in the back component of the surface. We will call \( c_{k,k+1} \) the curve in the front component, so

\[
\partial(c_k, p_{k+1}, c_{k+1}) = \{c_{k,k+1}, t(c_{k,k+1})\}.
\]

We proceed to define \( A_2 \). Consider the torus \( T \) bounded by \( p_{k-1}, p_{k+1}^-, \) and \( u_k \). Now, there is only one curve in the pair of pants \( T \setminus (c_{k-1} \cup c_k \cup t(c_{k-1}, k)) \) that is not already in \( P \cup C \); we denote this curve by \( n_k \) (see Figure 2 (a)). Analogously, let \( nr_k \) be the unique curve in \( T \setminus (c_k \cup c_{k+1} \cup t(c_{k,k+1})) \) that is not already a curve in \( P \cup C \) (see Figure 2 (b)). We set

\[
A_2 = \{n_k, nr_k \mid k \in \{2, \ldots, g-1\}\}.
\]

**Lemma 5.2.** Let \( S \) be a closed surface of genus \( g \geq 3 \). The subcomplex spanned by \( F_R \cup A \subset \mathcal{N}(S) \) is a finite rigid set with trivial pointwise stabilizer.
Remark 5.3. In the argument below, we sometimes abuse notation using \( p^+_1 = p^+_1 = p_1 \) and \( p^+_k = p^+_k = p_{g+1} \).

**Proof.** Take a locally injective simplicial map \( \phi : F_R \cup A \to \mathcal{N}(S) \). By Theorem 1.1 in the closed case, there is an \( h \in \Mod^+(S) \) such that \( h \circ \phi \) fixes \( F_R \); we rename \( h \circ \phi \) to \( \phi \). Furthermore, the subcomplex \( F_R \cup A \) has diameter two, so \( \phi \) is injective.

First, we check that \( \phi \) fixes \( A_1 \). Note that \( \phi((c_{1,2})) \) is contained in the torus \( T \) bounded by the curves \( p_2, p^x_3 \) and \( p^+_3 \). Since \( \phi((c_{1,2})) \) is disjoint from the curves in \( C \), we have that \( \phi((c_{1,2})) \) is contained in the pair of pants \( Q = T \setminus (c_1 \cup c_2 \cup c_3) \). Notice that only one curve in \( Q \) is non-separating and disjoint from \( \eta_1 \), namely \( \iota(c_{1,2}) \), and so \( \phi((c_{1,2})) = \iota(c_{1,2}) \). Now, \( \phi(c_{1,2}) \) is the unique non-separating curve in \( Q \) distinct from \( \iota(c_{1,2}) \), that is, \( \phi(c_{1,2}) = c_{1,2} \).

By an obvious modification of the above argument, we check that
\[
\phi(c_{k,k+1}) = c_{k,k+1} \quad \text{and} \quad \phi((c_{k,k+1})) = \iota(c_{k,k+1})
\]
for \( k \leq g - 1 \).

It is left to check that \( A_2 \) is also fixed by \( \phi \). Take \( n \iota_k \in A_2 \); we have that \( \phi(n \iota_k) \) is contained in the torus \( T \) bounded by the curves \( p_{k-1}, p^x_{k+1} \) and \( u_k \). Also, \( \phi(n \iota_k) \) is disjoint from \( c_k, c_{k+1} \) and \( \iota(c_{k,k+1}) \), and thus \( \phi(n \iota_k) \) is a curve in the pair of pants
\[
Q = T \setminus (c_k \cup c_{k+1} \cup \iota(c_{k,k+1})).
\]
Notice that the only curve in \( Q \) distinct from curves in \( P \cup C \) is the curve \( n \iota_k \), and thus we have \( \phi(n \iota_k) = \iota(n \iota_k) \).

For \( n \iota_k \in A_2 \), the argument is analogous.

We have proven that if \( \phi \) fixes \( F_R \), then it fixes \( F_R \cup A \). The statement follows immediately.

We define \( F_1 := F_R \cup A \) and proceed to prove that it satisfies the hypotheses of Lemma 5.1.

### 5.2 Constructing the exhaustion for closed surfaces

Let \( i \) be the back-front orientation reversing involution and let \( \delta_a \) be the right Dehn twist along the curve \( a \). The set of Dehn twists
\[
H := \{ \delta_{p_1}, \delta_{p_2}, \ldots, \delta_{p_{g+1}}, \delta_{p^x_k} \cup \{ \delta_{c_1}, \ldots, \delta_{c_k} \}
\]
are known as the Humphries generators, which are known to generate the group of orientation preserving mapping classes \( \Mod(S) \) (see [7]). Hence, \( H \cup \{ i \} \) generates \( \Mod^+(S) \).

**Lemma 5.4.** Let \( h \in H \cup \{ i \} \). The set \( F_1 \cup h(F_1) \) is a finite rigid set with trivial pointwise stabilizer.

**Proof.** First, we prove it for \( h = \delta_{p_i} \). Let \( \phi : F_1 \cup h(F_1) \to \mathcal{N}(S) \) be any locally injective simplicial map.

Since \( F_1 \) is a finite rigid set, we may assume that \( \phi \) fixes \( F_1 \) by precomposing with a mapping class. Moreover, since \( h(F_1) \) is also finite rigid, there exists a mapping class \( f \in \Mod^+(S) \) such that
\[
\phi|_{h(F_1)} = \phi h(F_1).
\]

Proving that \( F_1 \cup h(F_1) \) is a finite rigid set with trivial pointwise stabilizer boils down to proving that \( f = 1 \in \Mod^+(S) \).

Recall that \( p_1 \) is the associated curve to the Dehn twist \( h = \delta_{p_i} \). Now, let \( S' = S \setminus p_1 \) and consider the well-known cutting homomorphism (see [4, Proposition 3.20])
\[
1 \to \langle \delta_{p_1} \rangle \to \Mod(S, p_1) \to \Mod(S'),
\]
where \( \text{Mod}(S') \subset \text{Mod}^+(S') \) is the subgroup of orientation preserving classes, and \( \text{Mod}(S, p_1) \subset \text{Mod}^+(S) \) is the subgroup of orientation preserving classes that fix \( p_1 \). Notice that \( f \) fixes all curves in \( F_1 \cap h(F_1) \) which fill \( S' \). In particular, \( f \) is orientation preserving and fixes \( p_1 \). Furthermore, by means of the Alexander method, the image of \( f \) by the cutting homomorphism is trivial and the above sequence implies that \( f = \delta_{p_1}^k \) for some \( k \in \mathbb{Z} \). It is left to see that \( k = 0 \).

Note that \( i(nl, \delta_{p_1}(c_{g+1})) = 0 \). As \( \phi \) is locally injective, we know that

\[
i((\phi(nl), \phi(\delta_{p_1}(c_{g+1}))) = 0, \tag{5.1}
\]

which is equivalent to

\[
i(nl, \delta_{p_1}^{k+1}(c_{g+1})) = 0,
\]

since \( \phi|_{F_1} = \text{id} \) and \( \phi|_{h(F_1)} = \delta_{p_1}^k \).

It can be directly checked that equation (5.1) is satisfied if and only if \( k = 0 \), thus proving the statement of the lemma for \( h = \delta_{p_1} \).

The rest of the cases \( h \in H \) are proved in exactly the same way, but changing the curves in equation (5.1). We list all cases for completeness:

- If \( h = \delta_{p_{g+1}} \), change \( (nl, c_{g+1}) \) in (5.1) by \( (c_{g+1}, nr) \).
- If \( h = \delta_{p_i} \) for \( i \in \{2, \ldots, g - 2\} \), change \( (nl, c_{g+1}) \) by \( (nl_i, c_{g+1}) \).
- If \( h = \delta_{p_{g+1}} \), change \( (nl, c_{g+1}) \) by \( (c_{g-2}, c_{g+1}, nr_{g-1}) \).
- If \( h = \delta_{p_{g+1}} \), change \( (nl, c_{g+1}) \) by \( (c_{g+1}, nr_{g-1}) \).
- If \( h = \delta_{c_k} \) for \( k \in \{1, \ldots, g - 1\} \), change \( (nl, c_{g+1}) \) by \( (p_{k+1}, nr_{k+1}) \).
- If \( h = \delta_{c_k} \), change \( (nl, c_{g+1}) \) by \( (nl_{g-1}, p_{g}) \).

To finish the proof, we need to consider the case \( h = \iota \). In this case, \( F_1 \cap h(F_1) \) contains the trivially pointwise stabilized set \( P \cup C \cup A_1 \), so it follows immediately that \( f = 1 \in \text{Mod}^+(S) \).

**Proof of Theorem 1.2 for closed surfaces.** Let \( S \) be a closed surface of genus \( g \geq 3 \). Consider \( F_1 := F_R \cup A \) and the set of generators \( H \cup \{i\} \) of the extended mapping class group \( \text{Mod}^+(S) \). By Lemma 5.1, we have the desired exhaustion, where the hypotheses have been checked in Lemmas 5.2 and 5.4.
Curves in \( C \) for \( S_{g,3} \).

Figure 8: Curves in \( F'_R \). Punctures are marked as small crosses on the top of the surface.

(see Figure 8 (a)). We will say that the puncture contained in the annulus bounded by \( p^+_{2,k-1} \) and \( p^+_{2,k} \) is the \( k \)-th puncture.

In this setting, we also define the back-front (orientation reversing) involution as the unique non-trivial element \( \iota \in \text{Mod}^d(S) \) that fixes every curve \( P \cup C \).

Let \( p_1 \) be the unique curve in the punctured pair of pants bounded by \( p_1, p_2, p^+_{2,1} \) that is disjoint from \( c_2 \) and is not a curve already in \( P \). Similarly, let \( p_{2,k} \) be the unique curve in the punctured pair of pants bounded by \( p_{k-1}, p_2, p^+_{2,k} \) that is disjoint from \( c_2 \) and is not already a curve in \( P \) (see Figure 8 (d)). We set

\[
P := \{ p_1, \ldots, p_{n_2} \}.
\]

Let \( pr_n \) be the unique curve in the punctured pair of pants bounded by \( p^+_{2,n-1}, p_3, p^+_{2,3} \) that is disjoint from \( c_2 \) and is not already a curve in \( P \). Inductively, define \( pr_k \) to be the unique curve in the punctured pair of pants bounded by \( p^+_{2,k-1}, p_3, p^+_{2,3} \) that is disjoint from \( c_2 \) and is not already a curve in \( P \) (see Figure 8 (e)). We set

\[
Pr := \{ pr_1, \ldots, pr_{n_2} \}.
\]

Lastly, we define the set

\[
C_* = \partial(c_1, p_2, c_2) \cup \partial(c_2, p_2, \ldots, p_g, c_g).
\]

Note that the set \( \partial(c_1, p_2, c_2) \) has two curves, one curve in the front component which we will denote by \( c_{1,g} \), and one curve in the back component which we will denote by \( \iota(c_{1,g}) \). Also, the set \( \partial(c_2, p_2, \ldots, p_g, c_g) \) has two curves; we denote by \( c_{2,g} \) the curve in the front component and by \( \iota(c_{2,g}) \) the curve in the back component (see Figure 8 (f)).

The finite rigid set \( F'_R \) is the maximal subcomplex of \( N(S) \) spanned by the vertices

\[
P \cup C \cup U \cup D \cup L \cup R \cup N \cup Pl \cup Pr \cup C_*.
\]

**Remark 6.1.** Note that the subcomplex \( F'_R \) has diameter two. Therefore, any locally injective simplicial map \( \phi : F'_R \to N(S) \) is injective.

The following lemmas prove the finite rigidity of \( F'_R \).

**Lemma 6.2.** Let \( S \) be a punctured surface of genus \( g \geq 3 \). For any locally injective simplicial map \( \phi : F'_R \to N(S) \), there exists \( h \in \text{Mod}^d(S) \) such that \( h|_P = \phi|_P \).
Proof. First, notice that we can extend Lemma 4.3 to the case of punctured surfaces, and the proof works with minor changes.

Lemma 4.4 can also be extended to punctured surfaces. The proof for once punctured surfaces is similar to the closed case. Here we prove it for surfaces with \( n \geq 1 \) punctures.

Using Remark 2.1, it is straightforward to deduce that the curve \( \phi(p_{2,1}^i) \) is adjacent rel to \( P \) to \( \phi(p_{2,i-1}^n) \) and \( \phi(p_{2,i-1}^n) \) for \( i \in \{1, \ldots, n-1\} \). As a consequence of these adjacencies, we deduce that \( \phi(p_{2,0}^i) \) and \( \phi(p_{2,n}^i) \) bound a subsurface with two boundary components and \( n \) punctures. It follows that \( \phi(p_{2,0}^i) \) is adjacent rel to \( P \) to at least three curves. On the other hand, the non-adjacency rel to \( P \) of \( \phi(p_{2,0}^i) \) implies that it is adjacent to at most three curves. Thus, \( \phi(p_{2,0}^i) \) is adjacent to exactly three curves, namely \( \phi(p_{2,1}^1), \phi(p_1) \) and \( \phi(p_2) \). The rest of the adjacencies follow just as in the proof of Lemma 4.4.

Once we know that \( \phi \) preserves adjacency rel to \( P \), we can use Corollary 4.5 to produce a mapping class \( h \in \text{Mod}^g(S) \) with \( h|_P = \phi|_P \).

The next lemma proves that we may choose \( h \in \text{Mod}^g(S) \) to coincide with \( \phi \) on all curves of \( \Gamma_R' \).

**Lemma 6.3.** Let \( S \) be a punctured surface of genus \( g \geq 3 \). For any locally injective simplicial map \( \phi : \Gamma_R' \to \mathcal{N}(S) \), there exists \( h \in \text{Mod}^g(S) \) such that \( \phi = h|_{\Gamma_R'} \).

**Proof.** The idea is to progressively detect intersections between curves and, by composing with Dehn twists along \( P \), construct a mapping class that coincides with \( \phi \) on \( \Gamma_R' \).

First, by Lemma 6.2, there exists \( h \in \text{Mod}^g(S) \) such that \( h \circ \phi \) fixes \( P \) pointwise; rename \( h \circ \phi \) to \( \phi \).

Next, using the same arguments as in Lemmas 4.6, 4.8 and 4.9, we detect the following intersections:

- Intersections of curves in \( U \cup D \) with curves in \( P \).
- Intersections of curves in \( P \cup \overline{L} \) with curves in \( P \).
- Intersections of curves in \( P \cup R \) with curves in \( P \).
- \( c_k \) with curves in \( P \) for \( k \in \{2, \ldots, g-1\} \).

Notice that we can now use the proof of Lemma 4.10 to produce a mapping class that coincides with \( \phi \) on the curves of \( U, D \) and \( \{c_2, \ldots, c_{g-1}\} \). Thus, we may assume that these curves are fixed by \( \phi \).

It follows that \( \phi \) fixes the curves in \( P \). To see this, note that \( \phi(pl_1) \) is a curve contained in the punctured sphere \( S' \) bounded by \( p_1, u_2, p_{2,1}^* \) and that it is disjoint from \( p_{2,1}^* \) for \( i > 0 \). Thus, \( \phi(pl_1) \) is contained in the once punctured annulus which is a component of \( S' \setminus p_{2,1}^* \). As there is only one curve in that annulus that is not already in \( P \), it follows that \( \phi(pl_1) = pl_1 \). In the same way, \( \phi(pl_k) \) is contained in the punctured sphere \( S_k' \) bounded by \( pl_{k-1}, u_2, p_{2,1}^* \) and it is disjoint from \( p_{2,1}^* \) for \( i > k - 1 \). Thus, \( \phi(pl_k) \) is contained in the once punctured annulus which is a component of \( S_k' \setminus p_{2,k}^* \) and since there is only one curve in that annulus which is not in \( P \), we conclude that \( \phi(pl_k) = pl_k \). Naturally, an analogous argument works to prove that \( \phi \) fixes every curve in \( Pr \).

Now, note that \( \phi \) detects the following intersections:

- Curves in \( c_1 \) with curves in \( P \).
- Curves in \( \{c_1, c_g\} \) with curves in \( P \).

For instance, consider \( c_{1,2} \in C_s \) and \( p \in P \) disjoint from \( c_{1,2} \). If \( \phi(c_{1,2}) \) was disjoint from \( p \), then \( \phi(c_{1,2}) \) would have to intersect either the curve \( \phi(c_1), \phi(c_2) \) or \( \phi(\iota(c_{1,2})) \), leading to a contradiction. Thus, \( \phi(c_{1,2}) \) and \( p \) intersect.

Using the above intersections and fixed curves, we now focus on finding a homeomorphism that agrees with \( \phi \) on the curves in \( C_s \).

Observe that \( \phi(c_1) \) (resp. \( \phi(c_g) \)) is contained in the subsurface \( S' = S_{1,2} \) bound by the curves \( p_{2,1}^*, p_{2,1}^* \) (resp. \( p_{g-1}^*, p_{g-1}^* \)). Additionally, since \( \phi(c_1) \) is disjoint from the arcs \( a_1 = S' \cap \phi(c_{1,2}) \) and \( a_2 = S' \cap \phi(c_2) \) (resp. \( S' \cap \phi(c_{1,2}) \) and \( S' \cap \phi(c_{g-1}) \)), we have that \( \phi(c_1) \) (resp. \( \phi(c_g) \)) is contained in the annulus \( S' \setminus (a_1 \cup a_2) \). It follows that \( \phi(c_1) \) (resp. \( \phi(c_g) \)) is the unique curve in the annulus. We may again consider a mapping class \( h \in \text{Mod}^g(S) \) that is a composition of twists along curves in \( P \) such that \( h \circ \phi(c) = c \) for \( c \in \{c_1, c_g\} \); we rename \( h \circ \phi \) to \( \phi \) (the same argument with more details is given in Lemma 4.10).

For \( c_{1,2} \in C_s \), note that \( \phi(c_{1,2}) \) is contained in a subsurface \( S' = S_{2,2} \) bound by \( pr_1 \) and \( p_3 \) (or \( p_4 \) if \( g = 3 \)). Even more, \( \phi(c_{1,2}) \) is contained in the pair of pants \( S' \setminus (c_1 \cup c_2 \cup c_3 \cup p_2) \), and therefore it is one of the boundary components. One of the boundaries is a separating curve, so either \( \phi(c_{1,2}) = c_{1,2} \) or \( \phi(c_{1,2}) = \iota(c_{1,2}) \). These two alternatives are related by the involution \( \iota \). Thus, by precomposing with \( \iota \), we can assume that \( \phi \) fixes \( c_{1,2} \).
Lemma 6.3 provides the mapping class inducing $\phi$. The uniqueness of the mapping class follows as in the closed case.

We have essentially completed the proof of Theorem 1.1.

Proof of Theorem 1.1 for surfaces with punctures. Let $\phi : F_R \to N(S)$ be a locally injective simplicial map. Then Lemma 6.3 provides the mapping class inducing $\phi$. The uniqueness of the mapping class follows as in the closed case.

7 Finite rigid exhaustion for punctured surface

Let $S$ be a punctured surface of genus $g \geq 3$. In this section, we construct a sequence $F'_1 \subset F'_2 \subset \cdots \subset N(S)$ such that each $F'_i$ is a finite rigid set with trivial pointwise stabilizer and

$$\bigcup_{i=1}^{\infty} F'_i = N(S).$$

The strategy to construct the exhaustion is the same as in the closed case (see Section 5). First, we are going to enlarge the finite rigid set $F'_R$ to $F'_1$, and then use Lemma 5.1 to construct the exhaustion.

7.1 Enlarging the finite rigid set

First, we enlarge $F'_R$.

Consider the set of curves $A_1$ and $A_2$ in the closed surface. Via the same procedure described in Section 6.1, we remove $n$ points from the interior of the closed surface $S_p$, and so we obtain the punctured surface $S$. This produces natural analogues of the set of curves $A_1$ and $A_2$ in the surface $S$.

We define the set of curves $A_3 := \{a_1, \ldots, a_n\}$, where $a_k$ is the unique curve in the torus bounded by $p_{l_k-1}$, $pr_{k+1}$ and $d_2$, which is disjoint from $c_1$, $c_2$, $c_3$ and $\iota(c_{1,2})$ (see Figure 9).

We set $A' := A_1 \cup A_2 \cup A_3$.

Lemma 7.1. The set $F'_R \cup A'$ is finite rigid with trivial pointwise stabilizer.

Proof. Let $\phi : F'_R \cup A' \to N(S)$ be a locally injective simplicial map. By precomposing with a mapping class, we may assume that $\phi$ fixes $F'_R$ pointwise.

First, we prove that $A_1$ is also fixed by $\phi$. The curves $c_{1,2}$, $\iota(c_{1,2}) \in A_1$ are already in $C_\ast \subset F'_R$, so they are fixed by $\phi$. For $c_{2,3} \in A_1$, notice that $\phi(c_{2,3})$ is contained in the sphere $S'$ bounded by $p_1$, $pl_n$, $c_2$, $c_3$, $p'_4$, and $p_4$. Moreover, $\phi(c_{2,3})$ is contained in a pair of pants $S' \setminus c_1 \cup p_3 \cup \iota(c_{2,3})$. But there is only one curve in that pair of pants that is non-separating, i.e., $\phi(c_{2,3}) = c_{2,3}$. Slight modifications yield that $\phi(c_{2,3}) = t(c_{2,3})$. For the rest of $A_1$, one can proceed as in the closed case (see the proof of Lemma 5.2).

To prove that $A_2$ is fixed, we can just repeat the argument as in the closed case (see Lemma 5.2).
To finish the proof, we must show that $a_k \in A_3$ is fixed. To check this, note that $\phi(a_k)$ is contained in the torus $T$ bounded by $pl_{k-1}$, $pr_{k+1}$ and $d_2$. Note that $\phi(a_k)$ is the unique curve in

$$T \setminus (c_1 \cup c_2 \cup c_3 \cup \{c_{1,2}\})$$

that is not $c_2$, that is, $\phi(a_k) = a_k$. Thus, $A'$ is fixed and $F'_R \cup A'$ is a finite rigid set with trivial pointwise stabilizer.

We define $F'_1 := F'_R \cup A'$.

### 7.2 Constructing the exhaustion for punctured surfaces

The goal of this section is to construct an exhaustion of $\mathcal{N}(S)$ by finite rigid sets with trivial pointwise stabilizers. In this direction, we will consider a set of generators of $\text{Mod}^3(S)$ such that the subcomplex $F'_1$ satisfies the hypotheses of Lemma 5.1. We are going to assume $S = S_{g,n}$ with $n > 0$ punctures.

Let $\iota$ be the back-front orientation reversing involution. Consider the usual Humphries generators

$$H' = \{\delta_{p_1}, \ldots, \delta_{p_{2g-1}}, \delta_{p_2}\} \cup \{\delta_{p_{2i}} | 0 \leq i \leq n\} \cup \{\delta_1, \ldots, \delta_c\}$$

and the half twists

$$\{h_{(k,k+1)} | 1 \leq k \leq n - 1\},$$

where $h_{(k,k+1)}$ is the half twist that permutes the puncture $k$ with the puncture $k + 1$. It is well known that

$$H' \cup \{\iota\} \cup \{h_{(k,k+1)} | 1 \leq k \leq n - 1\}$$

generates $\text{Mod}^3(S)$ (see [4, Chapter 4.4.4]).

**Lemma 7.2.** For every

$$h \in H' \cup \{\iota\} \cup \{h_{(k,k+1)} | 1 \leq k \leq n - 1\},$$

the set $F'_1 \cup h(F'_1)$ is finite rigid with trivial pointwise stabilizer.

**Proof.** The proof is analogous to the closed case (see Lemma 5.4) and works directly for $h = \iota$, $h = \delta_{p_{2i}}$, $h = \delta_{c_i}$ and $h = \delta_{p_2}$. We consider here the rest of the cases.

Recall that, given $h = \delta_x$ and $\phi : F'_1 \cup h(F'_1) \to \mathcal{N}(S)$, we can assume that $\phi$ fixes $F'_1$ and

$$\phi|_{h(F_1)} = \delta_x|_{h(F_1)}.$$

Thus, the proof is a matter of checking that $k = 0$.

For

$$h = \delta_{p_{2i}}$$

we can consider the curves $l_j$, $a_j$ and plug them into equation (5.1). It follows that this is satisfied if and only if $k = 0$. For $h = \delta_{p_{2i}}$, one uses the curves $r_n$, $a_n$ in the same equation.

The last case to prove is $h = h_{(k-1,k)}$, which can be proved using the curves $a_k, r_k$ in equation (5.1).

We now complete the proof of Theorem 1.2.

**Proof of Theorem 1.2 for punctured surfaces.** Lemmas 7.1 and 7.2 ensure that $F'_1$ satisfies the hypothesis of Lemma 5.1, which in turn produces the desired exhaustion.

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