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A formal Riemannian structure on conformal classes and the inverse Gauss curvature flow

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Abstract: We define a formal Riemannian metric on a given conformal class of metrics with signed curvature on a closed Riemann surface. As it turns out this metric is the well-known Mabuchi-Semmes-Donaldson metric of Kähler geometry in a different guise. The metric has many interesting properties, and in particular we show that the classical Liouville energy is geodesically convex. This suggests a different approach to the uniformization theorem by studying the negative gradient flow of the normalized Liouville energy with respect to this metric, a new geometric flow whose principal term is the inverse of the Gauss curvature. We prove long time existence of solutions with arbitrary initial data and weak convergence to constant scalar curvature metrics by exploiting the metric space structure.

Keywords: Inverse Gauss curvature flow; Uniformization Theorem

MSC: 53C44, 58E11

1 Introduction

In this paper we define a formal Riemannian metric on the space of metrics in a conformal class with positive (or negative) curvature. This metric is related to the Mabuchi-Semmes-Donaldson [16, 25, 33] metric of Kähler geometry, as well as the Weil-Peterson-type metric introduced by Donaldson for the space of volume forms [17]. Nevertheless, its interpretation as a metric defined on the space of conformal metrics leads to some interesting points of departure from the Kähler setting. In addition, our definition has a natural extension to higher dimensions where there is no connection at all to Kähler geometry; this is explored in [21], [22].

To begin, let (M, g_0) be a compact Riemannian surface with positive Gauss curvature $K_0 > 0$, and let $[g_0]$ denote the conformal class of g_0 . Given a conformal metric $g_u = e^{2u}g_0$, let K_u denote the Gauss curvature of g_u , and define

$$\Gamma_1^+ = \{g_u = e^{2u}g_0 \in [g_0] : K_u = K_{g_u} > 0\}, \quad (1.1)$$

the space conformal metrics with positive Gauss curvature. The tangent space to $[g_0]$ at any metric $g_u \in [g_0]$ is given by $C^\infty(M)$. For $\phi, \psi \in C^\infty(M)$ we define

$$\langle\langle \phi, \psi \rangle\rangle_u = \int_M \phi \psi K_u dA_u, \quad (1.2)$$

where dA_u is the area form of g_u . In other words, we weight the standard L^2 metric with the Gauss curvature of the given conformal metric. If the Gauss curvature of g_0 is negative, we define

$$\Gamma_1^- = \{g_u = e^{2u}g_0 \in [g_0] : K_u = K_{g_u} < 0\}, \quad (1.3)$$

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and the metric associated to this space is given by

$$\langle\langle \phi, \psi \rangle\rangle_u = \int_M \phi \psi (-K_u) dA_u. \quad (1.4)$$

We note here that these metrics have appeared implicitly in [2]. By standard formulas, the Gauss curvature and volume forms of g_u and g_0 are related by

$$K_u = e^{-2u}(K_0 - \Delta_0 u), \quad dA_u = e^{2u} dA_0.$$

Therefore, the inner product (1.2) can also be expressed as

$$\langle\langle \phi, \psi \rangle\rangle_u = \int_M \phi \psi (K_0 - \Delta_0 u) dA_0. \quad (1.5)$$

The Riemann surface (M, g_0) determines a unique integrable complex structure J_0 , and the structure (M, ω_0, J_0) is Kähler. Since the Gauss curvature $K_0 > 0$, it follows that the $(1, 1)$ -form $K_0 \omega_0$ is positive. It is also closed since we are in dimension two. Now consider the space of Kähler metrics cohomologous to $K_0 \omega_0$,

$$\mathcal{H}_{K_0 \omega_0} = \{\phi \in C^\infty(M) \mid K_0 \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi = (K_0 + \Delta_{g_0} \phi) g_0 > 0\}.$$

It follows that the map

$$\begin{aligned} \mathcal{B} : \Gamma_1^+ &\rightarrow \mathcal{H}_{K_0 \omega_0} \\ u &\rightarrow -u \end{aligned} \quad (1.6)$$

is a well-defined bijection¹. Note that $\mathcal{B}_* = -\text{Id}$. The Mabuchi metric on $\mathcal{H}_{K_0 \omega_0}$ takes the form

$$\langle \phi, \psi \rangle_u^{\text{Mab}} := \int_M \phi \psi (K_0 + \Delta_{g_0} u) \omega_{g_0}.$$

Using the above discussion we compute

$$\begin{aligned} \langle\langle \phi, \psi \rangle\rangle_u &= \int_M \phi \psi K_u dA_u \\ &= \int_M \phi \psi (K_0 - \Delta_{g_0} u) \omega_{g_0} \\ &= \langle \mathcal{B}_* \phi, \mathcal{B}_* \psi \rangle_{\mathcal{B}(u)}^{\text{Mab}}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle^{\text{Mab}}$ denotes the Mabuchi L^2 -metric. One can define an analogous bijection for Γ_1^- . Therefore, we have an isometry between $(\Gamma_1^+, \langle\langle \cdot, \cdot \rangle\rangle)$ and $(\mathcal{H}_{K_0 \omega_0}, \langle \cdot, \cdot \rangle^{\text{Mab}})$.

In [17], Donaldson defined a Weil-Peterson-type for the space of volume forms. In the case where the background metric g_0 has constant Gauss curvature, this metric corresponds with the definition in (1.2). Subsequently, Chen-He [8] proved the partial regularity of geodesics associated to this Riemannian structure, and showed that the space has non-positive curvature in the sense of Alexandrov. Of course, there is an obvious identification between the space of volume forms and the space of conformal metrics.

One of the main themes of this article is the global variational properties of the regularized determinant, or normalized Liouville energy, with respect to the ambient geometry induced by (1.2). Given $g_u = e^{2u} g_0 \in \Gamma_1^\pm$, define

$$F[u] = \int_M |\nabla_0 u|^2 dA_0 + 2 \int_M K_0 u dA_0 - \left(\int_M K_0 dA_0 \right) \log \left(\int_M e^{2u} dA_0 \right).$$

Polyakov [29] proved that if $g_u = e^{2u} g_0$ and g_0 have the same area, then

$$\log \frac{\det(-\Delta_{g_u})}{\det(-\Delta_0)} = -\frac{1}{12\pi} \int_M (|\nabla_0 u|^2 + 2K_0 u) dA_0, \quad (1.7)$$

¹ We would like to thank T. Darvas for pointing this out to us.

where the determinant is defined via Ray-Singer regularization [30]. It is well known that u is a critical point of F if and only if $g_u = e^{2u}g_0$ has constant Gauss curvature. Under the identification with Kähler metrics explained above, the functional F corresponds to Ding's functional (cf. Remark 4.1). It was observed by Berndtson [3] that the Ding functional is geodesically convex, thus since the map $\mathcal{B}$ in (1.6) is an isometry, it follows that $F : \Gamma_1^\pm \rightarrow \mathbb{R}$ is geodesically convex. In Section 4 we provide a direct alternate proof of this fact, which highlights the role of a sharp inequality of Ben Andrews (see Proposition 4.3). This inequality also has interesting consequences in the study of the gradient flow of F with respect to the metric (1.2), which is the primary focus of this article and seemingly has no counterpart in the literature in the Kähler setting.

It follows from a standard first variation calculation that

$$\begin{aligned}(F^{[\square]})_u(u^{[\square]}) &= \frac{d}{d\epsilon} F[u + \epsilon u^{[\square]}] \Big|_{\epsilon=0} \\ &= 2 \int_M u^{[\square]} (K_u - \bar{K}_u) dA_u,\end{aligned}$$

where $\bar{K}$ is the average Gauss curvature. Consequently,

$$\begin{aligned}(F^{[\square]})_u(u^{[\square]}) &= 2 \int_M u^{[\square]} \frac{(K_u - \bar{K}_u)}{K_u} K_u dA_u \\ &= 2 \left\langle u^{[\square]}, \frac{(K_u - \bar{K}_u)}{K_u} \right\rangle_{\Gamma_1^+}.\end{aligned}$$

It follows that the (negative) gradient flow for F (up to a factor of 2) is given by

$$\frac{\partial u}{\partial t} = \frac{(\bar{K}_u) - K_u}{K_u},$$

or written in metric terms,

$$\frac{\partial}{\partial t} g = 2 \left(\frac{\bar{K} - K}{K} \right) g.$$

We will refer this as the *inverse Gauss curvature flow*, or IGCF. For the cone of metrics with negative curvature, the flow is defined by

$$\frac{\partial u}{\partial t} = \frac{(K_u - \bar{K}_u)}{K_u},$$

or

$$\frac{\partial}{\partial t} g = 2 \left(\frac{K - \bar{K}}{K} \right) g.$$

Our main result gives a number of interesting formal properties of the IGCF, a complete picture of the long time existence, and a nearly complete picture of the convergence at infinity.

Theorem 1.1. Fix (M^2, g) a compact Riemann surface and $u \in \Gamma_1^\pm$.

1. The solution to IGCF with initial condition u exists on $[0, \infty)$.
2. The normalized Liouville energy is convex in time along the flow line, i.e.

$$\frac{d^2}{dt^2} F[u(t)] \geq 0.$$

3. Given $v(x, t)$ another solution to IGCF, the distance between flow lines is nonincreasing, i.e.

$$\frac{d}{dt} d(u(t), v(t)) \leq 0.$$

4. The entropy is non-increasing along the flow:

$$\frac{d}{dt} \int K_u \log K_u dA_u \leq 0.$$

5. If $u \in \Gamma_1^-$, then the solution converges as $t \rightarrow \infty$ in the C^∞ topology to the unique conformal metric of constant scalar curvature.
6. If $u \in \Gamma_1^+$ and $(M^2, g) \cong (S^2, g_{S^2})$, then the solution converges weakly in H_1^2 to a smooth metric of constant curvature.

Remark 1.2.

- In the setting of Kähler geometry in any dimension, properties (2) and (3) are in analogy with results relating the K -energy, Mabuchi metric, and Calabi flow (cf. [5]). Note however that in our setting, even with the equivalence to Kähler geometry discussed above, the claims are new because the flow is not the Calabi flow.
- We emphasize that the point of the hypothesis $(M^2, g) \cong (S^2, g_{S^2})$ is that we are not yet able to use the IGCF to provide an a priori proof of the Uniformization Theorem. We require the existence of a constant scalar curvature metric to ensure the weak convergence of the flow.
- The negative gradient flow for F with respect to the L^2 -metric is the Ricci flow, however, the functional F is not in general convex along the Ricci flow.
- In [27] Osgood-Phillips-Sarnak studied a gradient flow of F with respect to a different inner product (essentially including the conformal factor as a weight).

As we mentioned above, the definition (1.2) is actually a special case of a more general metric defined on higher (even-) dimensional manifolds. In dimensions $n \geq 4$, one can define a Riemannian structure on subsets of conformal classes satisfying an admissibility condition which naturally arises in the study of the $\sigma_{\frac{n}{2}}$ -Yamabe problem. As in the case of surfaces, the underlying metric is closely associated to a functional whose critical points ‘uniformize’ the conformal class. Moreover, in four dimensions the functional which generalizes F is also spectrally defined. These results are discussed in [21], [22].

2 Metric, connection, and curvature

In this section we summarize some of the basic geometric properties of the metric (1.2). As explained in the Introduction, most of these follow from known properties of the Mabuchi metric or the metric on the space of volume forms introduced by Donaldson. However, in some cases we will give proofs which are more natural in the conformal setting.

The metric (1.2) defines an induced connection on curves: Given a path of conformal factors² $u : [a, b] \rightarrow \Gamma_1^+$ and a vector field $\alpha = \alpha(\cdot, t)$ along u , we define

$$\frac{D}{dt}\alpha = \frac{\partial}{\partial t}\alpha + \frac{1}{K_u} \langle \nabla_u \alpha, \nabla_u \frac{\partial u}{\partial t} \rangle_u, \quad (2.1)$$

where $\langle \cdot, \cdot \rangle_u$ denotes the inner product with respect to g_u .

Lemma 2.1. *The connection defined by (2.1) is metric-compatible and torsion-free.*

Proof. Let α, β be vector fields along the path $u : [a, b] \rightarrow \Gamma_1^+$. To simplify notation we will drop the subscript u , and all metric-dependent quantities (curvature, area form, etc.) will be understood to be with respect to g_u .

² To simplify notation, we will often write $u \in \Gamma_1^+$ to mean $g_u = e^{2u}g_0 \in \Gamma_1^+$.

We first prove compatibility. This will require us to record the standard variational formulas for a path of conformal metrics $g = g(t) = e^{2u}g_0$:

$$\begin{aligned}\frac{\partial}{\partial t}K &= -\Delta\left(\frac{\partial u}{\partial t}\right) - 2K\frac{\partial u}{\partial t}, \\ \frac{\partial}{\partial t}dA &= 2\frac{\partial u}{\partial t}dA,\end{aligned}\tag{2.2}$$

where here and below Δ refers to the Laplacian with respect to the moving metric. We furthermore compute

$$\begin{aligned}\frac{d}{dt}\langle\langle\alpha, \beta\rangle\rangle &= \frac{d}{dt}\int_M \alpha\beta K dA \\ &= \left\langle\left\langle\frac{\partial}{\partial t}\alpha, \beta\right\rangle\right\rangle_u + \left\langle\left\langle\alpha, \frac{\partial}{\partial t}\beta\right\rangle\right\rangle_u + \int_M \alpha\beta \frac{\partial}{\partial t}(K dA) \\ &= \left\langle\left\langle\frac{\partial}{\partial t}\alpha, \beta\right\rangle\right\rangle_u + \left\langle\left\langle\alpha, \frac{\partial}{\partial t}\beta\right\rangle\right\rangle_u - \int_M \alpha\beta \Delta\left(\frac{\partial u}{\partial t}\right) \\ &= \left\langle\left\langle\frac{\partial}{\partial t}\alpha, \beta\right\rangle\right\rangle_u + \left\langle\left\langle\alpha, \frac{\partial}{\partial t}\beta\right\rangle\right\rangle_u + \int_M \langle\nabla\alpha\beta + \alpha\nabla\beta, \nabla\frac{\partial u}{\partial t}\rangle dA \\ &= \left\langle\left\langle\frac{D}{dt}\alpha, \beta\right\rangle\right\rangle_u + \left\langle\left\langle\alpha, \frac{D}{dt}\beta\right\rangle\right\rangle_u.\end{aligned}$$

To compute the torsion, let $u = u(\cdot, s, t)$ be a two-parameter family of conformal factors in Γ_1^+ . Then

$$\begin{aligned}\frac{D}{\partial s}\frac{\partial u}{\partial t} - \frac{D}{\partial t}\frac{\partial u}{\partial s} &= \frac{\partial^2 u}{\partial s\partial t} + \frac{1}{K_u}\langle\nabla\frac{\partial u}{\partial t}, \nabla\frac{\partial u}{\partial s}\rangle_u - \frac{\partial^2 u}{\partial s\partial t} - \frac{1}{K_u}\langle\nabla\frac{\partial u}{\partial s}, \nabla\frac{\partial u}{\partial t}\rangle_u \\ &= 0.\end{aligned}$$

□

Proposition 2.2. *Given $\phi, \psi \in T_u\Gamma_1^+$, the sectional curvature of the plane in $T_u\Gamma_1^+$ spanned by ϕ, ψ is given by*

$$\begin{aligned}K(\phi, \psi) &= \int \frac{1}{K_u}\left\{-|\nabla\phi|_u^2|\nabla\psi|_u^2 + g_u(\nabla\phi, \nabla\psi)^2\right\}dA_u \\ &= -\int \frac{1}{K_u}|d\phi \wedge d\psi|_u^2 dA_u \\ &\leq 0.\end{aligned}\tag{2.3}$$

Proof. The non-positivity of the sectional curvature was proved for the Mabuchi metric in [16, 25, 33]; see also Theorem 1 of [17]. While it possible to compute (2.3) directly, one also recovers the formula using the isomorphism $\mathcal{B}$ of (1.6) one recovers the claimed formula, and so we omit the proof. □

2.1 The negative cone

Now assume (M, g_0) is a closed surface with $K_0 < 0$, and let

$$\Gamma_1^- = \{g_u = e^{2u}g_0 : K_u = K_{g_u} < 0\}\tag{2.4}$$

denote the space conformal metrics with negative Gauss curvature. The formal tangent space at g_u is

$$T_u\Gamma_1^- \cong C^\infty(M).\tag{2.5}$$

Definition 2.3. For $\alpha, \beta \in T_u\Gamma_1^- \cong C^\infty(M)$, define

$$\langle\langle\alpha, \beta\rangle\rangle_u = \int_M \alpha\beta(-K_u)dA_u,\tag{2.6}$$

where dA_u is the area form of the metric $g_u = e^{2u}g_0$.

As before, we write $u \in \Gamma_1^-$ to mean $g_u = e^{2u}g_0 \in \Gamma_1^-$. Given a path of conformal factors $u : [a, b] \rightarrow \Gamma_1^+$ and a vector field $\alpha = \alpha(\cdot, t)$ along u , we now define

$$\frac{D}{dt}\alpha = \frac{\partial}{\partial t}\alpha + \frac{1}{K_u}\langle \nabla_u \alpha, \nabla_u \frac{\partial u}{\partial t} \rangle_u. \quad (2.7)$$

The proof of the next two results are essentially the same as in the case of the positive cone:

Lemma 2.4. *The connection defined by (2.7) is metric-compatible and torsion-free.*

Proposition 2.5. *Given $\phi, \psi \in T_u \Gamma_1^-$, we have*

$$K(\phi, \psi) = \int \frac{1}{(-K_u)} \left\{ -|\nabla \phi|_u^2 |\nabla \psi|_u^2 + g_u(\nabla \phi, \nabla \psi)^2 \right\} dA_u \leq 0. \quad (2.8)$$

2.2 Metric space structure

Finally, we summarize the metric space properties. These follow from the known properties of the Kähler cone, but more directly from results in Donaldson [17] and Chen-He [8]:

Theorem 2.6. *$(\Gamma_1^\pm, d)$ are metric spaces of nonpositive curvature in the sense of Alexandrov.*

3 Geodesics, length, and energy

Using the definition of the metric in the positive cone we can also define the associated notions of energy and length.

Definition 3.1. Given a path $u : [a, b] \rightarrow \Gamma_1^+$, the *energy density* of u is

$$E_u(t) = \left\| \frac{\partial u}{\partial t} \right\|_u^2 = \int_M \left(\frac{\partial u}{\partial t} \right)^2 K_u dA_u. \quad (3.1)$$

The *energy* of u is

$$E[u] = \frac{1}{2} \int_a^b E_u(t) dt = \frac{1}{2} \int_a^b \int_M \left(\frac{\partial u}{\partial t} \right)^2 K_u dA_u dt. \quad (3.2)$$

The *length* of u is

$$L[u] = \int_a^b \left\| \frac{\partial u}{\partial t} \right\|_u dt = \int_a^b \left[\int_M \left(\frac{\partial u}{\partial t} \right)^2 K_u dA_u \right]^{\frac{1}{2}} dt. \quad (3.3)$$

By taking the first variation of the energy we arrive at the geodesic equation:

Lemma 3.2. *$u : [a, b] \rightarrow \Gamma_1^+$ is a geodesic if and only if*

$$0 = \frac{D}{dt} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2} + \frac{1}{K_u} |\nabla_u \frac{\partial u}{\partial t}|^2. \quad (3.4)$$

Suppose $u : [a, b] \rightarrow \Gamma_1^+$ is a geodesic, and write $g_u = e^{2u}g_0$, where $g_0 \in \Gamma_1^+$. By the Gauss curvature equation,

$$K_u = e^{-2u}(K_0 - \Delta_0 u), \quad (3.5)$$

where K_0 is the Gauss curvature of g_0 . Therefore, we can rewrite (3.4) as

$$\frac{\partial^2 u}{\partial t^2} + \frac{|\nabla_0 \frac{\partial u}{\partial t}|^2}{K_0 - \Delta_0 u} = 0. \quad (3.6)$$

In Chen-He [8] the geodesic equation is given by

$$\frac{\partial^2 \Phi}{\partial t^2} + \frac{|\nabla_0 \frac{\partial \Phi}{\partial t}|^2}{1 + \Delta_0 \Phi} = 0, \quad (3.7)$$

hence by substituting $u \leftrightarrow \Phi$, the two equations are the same up to a zeroth order (background curvature) term. Chen-He proved ([8] Theorem 1.4) the existence of weak solutions to this equation. More precisely, they proved *a priori* estimates for solutions of a regularized version of (3.7). These estimates are independent of the regularization parameters up to the C^2 -level, hence by taking limits one obtains $C^{1,1}$ solutions of the original equation. Since equations (3.6) and (3.7) only differ by a zeroth order (background curvature) term³, the estimates of Chen-He ([8] Theorem 1.4) hold for solutions of the former equation.

Proposition 3.3. *Given $u_0, u_1 \in \Gamma_1^+$, there is a unique $C^{1,1}$ -geodesic with*

$$\begin{aligned} u(x, 0) &= u_0(x), \\ u(x, 1) &= u_1(x). \end{aligned} \quad (3.8)$$

An analogous statement holds for geodesics in the negative cone.

We will need some additional properties of geodesics that easily follow from the defining equation. As a preface, we remark that there is a canonical isometric splitting of $T\Gamma_1^+$ with respect to the metric. In particular, the real line $\mathbb{R} \subset T_u\Gamma_1^+$ given by constant functions is orthogonal (w.r.t. the metric in (1.2)) to

$$T_u^0\Gamma_1^+ := \left\{ \alpha \mid \int_M \alpha K_u dA_u = 0 \right\}.$$

As in the setting of the Mabuchi metric, we will see that geodesics preserve this isometric splitting, and are automatically parameterized with constant speed:

Lemma 3.4. *Let $\phi \in C^1(\mathbb{R})$, and $u : [a, b] \rightarrow \Gamma_1^+$ a geodesic. Then*

$$\frac{d}{dt} \int_M \phi\left(\frac{\partial u}{\partial t}\right) K_u dA_u = 0. \quad (3.9)$$

In particular,

$$\begin{aligned} \frac{d}{dt} \int_M \frac{\partial u}{\partial t} K_u dA_u &= 0, \\ \frac{d}{dt} \int_M \left(\frac{\partial u}{\partial t}\right)^2 K_u dA_u &= 0. \end{aligned} \quad (3.10)$$

Proof. Differentiating, integrating by parts, and using the geodesic equation gives

$$\begin{aligned} \frac{d}{dt} \int_M \phi\left(\frac{\partial u}{\partial t}\right) K_u dA_u &= \int_M \left\{ \frac{\partial}{\partial t} \left[\phi\left(\frac{\partial u}{\partial t}\right) \right] K_u dA_u + \phi\left(\frac{\partial u}{\partial t}\right) \frac{\partial}{\partial t} (K_u dA_u) \right\} \\ &= \int_M \left\{ \phi^{[\cdot]} \left(\frac{\partial u}{\partial t}\right) \frac{\partial^2 u}{\partial t^2} K_u - \phi\left(\frac{\partial u}{\partial t}\right) \Delta \left(\frac{\partial u}{\partial t}\right) \right\} dA_u \\ &= \int_M \left\{ \phi^{[\cdot]} \left(\frac{\partial u}{\partial t}\right) \frac{\partial^2 u}{\partial t^2} K_u + \phi^{[\cdot]} \left(\frac{\partial u}{\partial t}\right) |\nabla_u \frac{\partial u}{\partial t}|_u^2 \right\} dA_u \\ &= \int_M \phi^{[\cdot]} \left(\frac{\partial u}{\partial t}\right) \left\{ \frac{\partial^2 u}{\partial t^2} + \frac{1}{K_u} |\nabla_u \frac{\partial u}{\partial t}|_u^2 \right\} K_u dA_u \\ &= 0. \end{aligned}$$

□

³ In fact, assuming the curvature of g_0 is constant the equations coincide.

Choosing

$$\phi(t) = \begin{cases} t^p & t \geq 0 \\ 0 & t < 0, \end{cases}$$

with $p \gg 1$ large and apply (3.9), then in the limit as $p \rightarrow \infty$ we have the following corollary of Lemma 3.4:

Corollary 3.5. *If $u : [a, b] \rightarrow \Gamma_1^+$ is a geodesic, then $\sup_M \frac{\partial u}{\partial t}$ and $\inf_M \frac{\partial u}{\partial t}$ are constant in time.*

Example 3.6. Let (S^2, g_0) denote the round sphere. Using stereographic projection $\sigma : S^2 \setminus \{N\} \rightarrow \mathbb{R}^2$, where $N \in S^2$ denotes the north pole, one can define a one-parameter of conformal maps of S^2 by conjugating the dilation map $\delta_\alpha : x \mapsto \alpha^{-1}x$ on the plane with σ :

$$\varphi_\alpha = \sigma^{-1} \circ \delta_\alpha \circ \sigma : S^2 \rightarrow S^2.$$

Taking $\alpha(t) = e^{\lambda t}$, where λ is a fixed real number, we can define the path of conformal metrics

$$g(t) = e^{2u} g_0 = \phi_\alpha^* g_0 = \left[\frac{2\alpha(t)}{(1+\xi) + \alpha(t)^2(1-\xi)} \right]^2 g_0, \quad (3.11)$$

where $\xi = x_3$ is the coordinate function (see [24]).

Proposition 3.7. *The path $u : (-\infty, +\infty) \rightarrow \Gamma_1^+$ is a geodesic.*

Proof. By (3.11),

$$u = \log 2\alpha - \log [(1+\xi) + \alpha^2(1-\xi)].$$

Letting subscripts denote differentiation in t , we have

$$u_t = \frac{\alpha_t}{\alpha} - \frac{2\alpha\alpha_t(1-\xi)}{(1+\xi) + \alpha^2(1-\xi)},$$

and hence

$$u_{tt} = \frac{\alpha_{tt}}{\alpha} - \left(\frac{\alpha_t}{\alpha} \right)^2 - \frac{[(1+\xi) + \alpha^2(1-\xi)](2\alpha\alpha_{tt} + 2\alpha_t^2)(1-\xi) - 4\alpha^2\alpha_t^2(1-\xi)^2}{[(1+\xi) + \alpha^2(1-\xi)]^2}.$$

Since $\alpha = e^{\lambda t}$, this simplifies to

$$u_{tt} = \frac{-4\lambda^2\alpha^2(1-\xi)^2}{[(1+\xi) + \alpha^2(1-\xi)]^2}. \quad (3.12)$$

Also, if ∇ denotes the connection with respect to the round metric,

$$\begin{aligned} \nabla u_t &= \frac{2\alpha\alpha_t\nabla\xi}{(1+\xi) + \alpha^2(1-\xi)} + \frac{2\alpha\alpha_t(1-\xi)}{[(1+\xi) + \alpha^2(1-\xi)]^2} [(1-\alpha^2)\nabla\xi] \\ &= \frac{2\alpha\alpha_t\nabla\xi}{[(1+\xi) + \alpha^2(1-\xi)]^2} [(1+\xi) + \alpha^2(1-\xi) + (1-\xi)(1-\alpha^2)] \\ &= \frac{4\alpha\alpha_t\nabla\xi}{[(1+\xi) + \alpha^2(1-\xi)]^2}. \end{aligned} \quad (3.13)$$

Using the fact that ξ satisfies

$$|\nabla\xi|^2 = 1 - \xi^2, \quad (3.14)$$

it follows from (3.13) and (3.14) that

$$|\nabla u_t|^2 = \frac{4\lambda^2\alpha^2(1-\xi)^2}{[(1+\xi) + \alpha^2(1-\xi)]^2}. \quad (3.15)$$

Since $K_u = 1$ for all t , comparing (3.12) and (3.15) we see that u satisfies the geodesic equation (3.4). $\square$

4 Functional determinant and the inverse Gauss curvature flow

As pointed out in the introduction, in [29], Polyakov gave a remarkable formula for the ratio of regularized determinants of two conformal metrics (1.7). The integral in this formula is often referred to as the *Liouville energy*, and we will denote it by J :

$$J[u] = \int_M |\nabla_0 u|^2 dA_0 + 2 \int_M K_0 u dA_0. \quad (4.1)$$

Since J is not scale-invariant, it is convenient to also consider the normalized version of J , which we denote by F :

$$F[u] = \int_M |\nabla_0 u|^2 dA_0 + 2 \int_M K_0 u dA_0 - \left(\int_M K_0 dA_0 \right) \log \left(\int_M e^{2u} dA_0 \right), \quad (4.2)$$

where $\int$ denotes the normalized integral

$$\int_M f dA_0 = \frac{\int_M f dA_0}{\int_M dA_0}.$$

Note that $F[u + c] = F[u]$ for any real number c .

Remark 4.1. Under the map $\mathcal{B}$ of (1.6), the functional F corresponds to a functional defined by Ding ([15] pg. 469).

A first variation calculation of J at u is

$$\begin{aligned} (J^{[\square]})_u(u^{[\square]}) &= \frac{d}{ds} J[u + su^{[\square]}] \Big|_{s=0} \\ &= 2 \int_M u^{[\square]} (-\Delta_0 u + K_0) dA_0 \\ &= 2 \int_M u^{[\square]} K_u dA_u, \end{aligned} \quad (4.3)$$

and a first variation of F is

$$(F^{[\square]})_u(u^{[\square]}) = 2 \int_M u^{[\square]} (K_u - \bar{K}_u) dA_u, \quad (4.4)$$

where

$$\bar{K}_u = \frac{\int_M K_u dA_u}{\int_M dA_u}.$$

As described in the introduction, we want to consider the (negative) gradient flow for F with respect to the metric we have defined for Γ_1^+ . In view of (4.4) we have

$$\begin{aligned} (F^{[\square]})_u(u^{[\square]}) &= 2 \int_M u^{[\square]} (K_u - \bar{K}_u) dA_u \\ &= 2 \int_M u^{[\square]} \frac{(K_u - \bar{K}_u)}{K_u} K_u dA_u \\ &= 2 \left\langle u^{[\square]}, \frac{(K_u - \bar{K}_u)}{K_u} \right\rangle_{\Gamma_1^+}. \end{aligned} \quad (4.5)$$

It follows that the (negative) gradient flow for F (up to a factor of 2) is given by

$$\frac{\partial u}{\partial t} = \frac{(\bar{K}_u) - K_u}{K_u}, \quad (4.6)$$

or written in metric terms,

$$\frac{\partial}{\partial t} g = 2 \left(\frac{\bar{K} - K}{K} \right) g. \quad (4.7)$$

We will refer to (4.6) and (4.7) as the *inverse Gauss curvature flow*, or IGCf. For the cone of metrics with negative curvature, the flow is defined by

$$\frac{\partial u}{\partial t} = \frac{(K_u - \bar{K}_u)}{K_u}, \quad (4.8)$$

or

$$\frac{\partial}{\partial t} g = 2 \left(\frac{K - \bar{K}}{K} \right) g. \quad (4.9)$$

4.1 Formal Properties

In this section we establish the geodesic convexity of the Liouville energy, the convexity of the normalized Liouville energy along flow lines as well as the monotonicity of distances along flow lines. Remarkably, all three properties rely on a sharp application of a curvature-weighted Poincaré inequality [1]. We include the short proof as this result seems to not be well-known. We also record an entropy monotonicity formula for the inverse Gauss curvature flow on the sphere.

Proposition 4.2. (Andrews [1], cf. [11] pg. 517) *Let (M^n, g) be a closed Riemannian manifold with positive Ricci curvature. Given $\phi \in C^\infty(M)$ such that $\int_M \phi dV = 0$, then*

$$\frac{n}{n-1} \int_M \phi^2 dV \leq \int_M (\text{Rc}^{-1})^{ij} \nabla_i \phi \nabla_j \phi dV,$$

with equality if and only if $\phi \equiv 0$ or (M^n, g) is conformal to the round sphere.

Proof. Since $\int_M \phi dV = 0$ there exists ψ such that $\Delta \psi = \phi$. Observe that

$$\begin{aligned} \int_M \left| \nabla^2 \psi - \frac{\Delta \psi}{n} g \right|^2 &= \int_M \nabla_i \nabla_j \psi \nabla_i \nabla_j \psi - \frac{\phi^2}{n} \\ &= \frac{n-1}{n} \int_M \phi^2 dV - \int_M \text{Rc}_{ij} \nabla_i \psi \nabla_j \psi. \end{aligned}$$

Moreover

$$\begin{aligned} &\int_M (\text{Rc}^{-1})^{ij} [\nabla_i \phi + b \text{Rc}_{ik} \nabla_k \psi] [\nabla_j \phi + b \text{Rc}_{jl} \nabla_l \psi] \\ &= \int_M (\text{Rc}^{-1})^{ij} \nabla_i \phi \nabla_j \phi - 2b \phi^2 + b^2 \text{Rc}_{kl} \nabla_k \psi \nabla_l \psi. \end{aligned}$$

Combining these and choosing $a = b^{-1} = \frac{n-1}{n}$ yields

$$\begin{aligned} 0 &\leq \int_M \left| \nabla^2 \psi - \frac{\Delta \psi}{n} g \right|^2 + a^2 \int_M (\text{Rc}^{-1})^{ij} [\nabla_i \phi + b \text{Rc}_{ik} \nabla_k \psi] [\nabla_j \phi + b \text{Rc}_{jl} \nabla_l \psi] \\ &= \left[\frac{n-1}{n} - 2a^2 b \right] \int_M \phi^2 + [-1 + a^2 b^2] \int_M \text{Rc}_{ij} \nabla_i \psi \nabla_j \psi + a^2 \int_M (\text{Rc}^{-1})^{ij} \nabla_i \phi \nabla_j \phi \\ &= -\frac{n-1}{n} \int_M \phi^2 + \left(\frac{n-1}{n} \right)^2 \int_M (\text{Rc}^{-1})^{ij} \nabla_i \phi \nabla_j \phi. \end{aligned}$$

The inequality follows. In the case of equality, one observes that in fact ψ is a solution of

$$\nabla^2 \psi - \frac{\Delta \psi}{n} g \equiv 0.$$

Since M is compact, it follows from [37] that either $\psi \equiv 0$ or (M^n, g) is conformal to the round sphere. $\square$

Proposition 4.3. *The functional F is geodesically convex.*

Proof. In the positive cone, we use Lemma 3.4 and the geodesic equation to compute along a geodesic,

$$\begin{aligned}\frac{d^2}{dt^2}F[u] &= \frac{d}{dt}2 \int_M u_t [K_u - \bar{K}_u] dA_u \\ &= -4\pi \frac{d}{dt} \int_M u_t A_u^{-1} dA_u \\ &= -4\pi \int_M \left[u_{tt} A_u^{-1} - A_u^{-2} u_t \left(\int_M 2u_t dA_u \right) + 2A_u^{-1} u_t^2 \right] dA_u \\ &= 4\pi A_u^{-1} \left[\int_M \frac{1}{K_u} |\nabla u_t|^2 dA_u - 2 \left(\int_M u_t^2 dA_u - A_u^{-1} \left(\int_M u_t dA_u \right)^2 \right) \right] \\ &\geq 0,\end{aligned}$$

where the last line follows from Proposition 4.2. In the negative cone an analogous calculation yields

$$\begin{aligned}\frac{d^2}{dt^2}F[u] &= 2\pi(-\chi)A_u^{-1} \left[\int_M \frac{1}{-K_u} |\nabla u_t|^2 dA_u + 2 \left(\int_M u_t^2 dA_u - A_u^{-1} \left(\int_M u_t dA_u \right)^2 \right) \right] \\ &\geq 0.\end{aligned}$$

□

Proposition 4.4. *Given u a solution to IGCF, one has*

$$\frac{d^2}{dt^2}F[u] \geq 0.$$

Proof. If $u \in \Gamma_1^+$, using (4.4) we have for a solution to IGCF flow

$$\begin{aligned}\frac{d}{dt}F &= - \int_M \left(1 - \frac{\bar{K}_u}{K_u} \right)^2 K_u dA_u \\ &= - \int_M \left[1 - \frac{2\bar{K}_u}{K_u} + \frac{\bar{K}_u^2}{K_u^2} \right] K_u dA_u \\ &= 2\pi\chi(M) - \bar{K}_u^2 \int_M \frac{1}{K_u} dA_u.\end{aligned}$$

Hence we have

$$\begin{aligned}\frac{d^2}{dt^2}F &= \frac{d}{dt} \left[-\frac{(2\pi\chi)^2}{A_u^2} \int_M \frac{1}{K_u} dA_u \right] \\ &= \frac{(2\pi\chi)^2}{A_u^3} \left[2 \frac{d}{dt} A_u \int_M \frac{1}{K_u} dA_u - A_u \frac{d}{dt} \int_M \frac{1}{K_u} dA_u \right] \\ &= \frac{(2\pi\chi)^2}{A_u^3} \left[2 \left(-2A_u + 2\bar{K}_u \int_M \frac{1}{K_u} dA_u \right) \int_M \frac{1}{K_u} dA_u \right. \\ &\quad \left. - A_u \int_M 2 \left(-1 + \frac{\bar{K}_u}{K_u} \right) \frac{1}{K_u} dA_u \right. \\ &\quad \left. - A_u \int_M \left(-\frac{2}{K_u} + \frac{2\bar{K}_u}{K_u^2} + \frac{\bar{K}_u}{K_u^2} \Delta \frac{1}{K_u} \right) dA_u \right] \\ &= \frac{(2\pi\chi)^2}{A_u^3} \left[4\bar{K}_u \left(\int_M \frac{1}{K_u} dA_u \right)^2 - 4A_u \bar{K}_u \int_M \frac{1}{K_u^2} dA_u - A_u \bar{K}_u \int_M \frac{1}{K_u^2} \Delta \frac{1}{K_u} dA_u \right] \\ &= \frac{(2\pi\chi)^2}{A_u^3} \left[4\bar{K}_u \left(\int_M \frac{1}{K_u} dA_u \right)^2 - 4A_u \bar{K}_u \int_M \frac{1}{K_u^2} dA_u + 2A_u \bar{K}_u \int_M \frac{1}{K_u} \left| \nabla \frac{1}{K_u} \right|^2 dA_u \right] \\ &= \frac{16\pi^3\chi^3}{A_u^3} \left[-2 \int_M \left(\frac{1}{K_u} - A_u^{-1} \int_M \frac{1}{K_u} \right)^2 dA_u + \int_M \frac{1}{K_u} \left| \nabla \frac{1}{K_u} \right|^2 dA_u \right] \\ &\geq 0,\end{aligned}$$

where the last line follows from Proposition 4.2. A similar calculation when $u \in \Gamma_1^-$ yields

$$\begin{aligned} \frac{d^2}{dt^2} F &= \frac{16\pi^3 \chi^3}{A_u^3} \left[2 \int_M \left(\frac{1}{K_u} - A_u^{-1} \int_M \frac{1}{K_u} \right)^2 + \int_M \frac{1}{K_u} \left| \nabla \frac{1}{K_u} \right|^2 dA_u \right] \\ &\geq 0, \end{aligned}$$

since both terms on the right hand side are nonnegative. $\square$

Remark 4.5. The inverse Gauss curvature flow also has the remarkable quality of monotonically decreasing distances in Γ_1^+ . This is in direct analogy with the fact that the Calabi flow decreases distances in the Mabuchi metric [5].

Proposition 4.6. Let $u(s, t)$ $s \in [0, 1]$, $t \in [0, T]$ be a smooth two-parameter family of conformal factors such that for all $s \in [0, 1]$, the family $u(s, \cdot)$ is a solution to IGCF. Then

$$\frac{d}{dt} L(u(\cdot, t)) \leq 0.$$

Proof. For the positive cone, we directly compute

$$\begin{aligned} \frac{d}{dt} L(u(\cdot, t)) &= \frac{d}{dt} \int_0^1 \left[\int_M \left(\frac{\partial u}{\partial s} \right)^2 K_u dA_u \right]^{\frac{1}{2}} ds \\ &= \frac{1}{2} \int_0^1 |u_s|_u^{-1} \int_M [2u_{ts} u_s K_u - u_s^2 \Delta u_t] dA_u ds. \end{aligned}$$

Now we compute

$$\begin{aligned} u_{ts} &= \frac{\partial}{\partial s} \left[\frac{\bar{K}_u}{K_u} - 1 \right] \\ &= \bar{K}_u \left[-\frac{1}{A_u K_u} \frac{\partial}{\partial s} A_u + \frac{1}{K_u^2} (\Delta u_s + 2u_s K_u) \right] \\ &= \bar{K}_u \left[-\frac{1}{A_u K_u} \int_M 2u_s dA_u + \frac{1}{K_u^2} \Delta u_s + \frac{2u_s}{K_u} \right]. \end{aligned}$$

Hence

$$\begin{aligned} &\int_M 2u_{ts} u_s K_u dA_u \\ &= 2\bar{K}_u \int_M \left[-2A_u^{-1} K_u^{-1} \int_M u_s dA_u + K_u^{-2} \Delta u_s + 2K_u^{-1} u_s \right] u_s K_u dA_u \\ &= 4\bar{K}_u A_u^{-1} \left\{ A_u \int_M u_s^2 dA_u - \left[\int_M u_s dA_u \right]^2 \right\} + 2\bar{K}_u \int_M K_u^{-1} u_s \Delta u_s dA_u \\ &= 4\bar{K}_u A_u^{-1} \left\{ A_u \int_M u_s^2 dA_u - \left[\int_M u_s dA_u \right]^2 \right\} \\ &\quad - 2\bar{K}_u \int_M \left[K_u^{-1} |\nabla u_s|^2 + u_s \langle \nabla u_s, \nabla K_u^{-1} \rangle \right] dA_u. \end{aligned}$$

Also

$$\begin{aligned} - \int_M u_s^2 \Delta u_t dA_u &= 2 \int_M u_s \langle \nabla u_t, \nabla u_s \rangle dA_u \\ &= 2\bar{K}_u \int_M u_s \langle \nabla u_s, \nabla K_u^{-1} \rangle dA_u. \end{aligned}$$

Collecting these calculations yields

$$\begin{aligned} \frac{d}{dt} L(u(\cdot, t)) &= - \int_0^1 \bar{K}_u |u_s|_u^{-1} \left[\int_M K_u^{-1} |\nabla u_s|^2 dA_u - 2 \int_M u_s^2 + 2A_u^{-1} \left[\int_M u_s dA_u \right]^2 \right] ds \\ &\leq 0, \end{aligned}$$

using Proposition 4.2. In the negative cone a closely related calculation yields

$$\begin{aligned} \frac{d}{dt} L(u(\cdot, t)) &= - \int_0^1 (-\bar{K}) |u_s|_u^{-1} \left[\int_M (-K_u)^{-1} |\nabla u_s|^2 dA_u + 2 \int_M u_s^2 - 2A_u^{-1} \left[\int_M u_s dA_u \right]^2 \right] ds \\ &\leq 0, \end{aligned}$$

using the Cauchy-Schwarz inequality. $\square$

In [10] a natural scalar curvature entropy functional was shown to be monotone under Ricci flow. This entropy is also monotone under IGCF.

Proposition 4.7. *Given u a solution to IGCF with $K_u > 0$, one has*

$$\frac{\partial}{\partial t} \int_M K_u \log K_u dA_u = -\bar{K}_u \int_M K_u |\nabla K_u^{-1}|^2 dA_u \leq 0.$$

Proof. We compute

$$\begin{aligned} &\frac{\partial}{\partial t} \int_M K_u \log K_u dA_u \\ &= \int_M [1 + \log K_u] \dot{K}_u dA_u - \int_M K_u \log K_u (2\dot{u}) dA_u \\ &= \int_M [1 + \log K_u] \left[-\Delta \frac{\bar{K}_u}{K_u} - 2[K_u - \bar{K}_u] \right] dA_u + 2 \int_M K_u \log K_u \left[1 - \frac{\bar{K}_u}{K_u} \right] dA_u \\ &= \bar{K}_u \int_M K_u^{-1} \langle \nabla K_u, \nabla K^{-1} \rangle dA_u \\ &= -\bar{K}_u \int_M K_u |\nabla K_u^{-1}|^2 dA_u, \end{aligned}$$

as required. $\square$

4.2 Higher genus surfaces

In this section we analyze the IGCF in the space of conformal metrics of negative curvature. To simplify notation we drop the subscript u and write (4.8) as

$$\frac{\partial}{\partial t} u = \frac{K - \bar{K}}{K}. \quad (4.10)$$

In the following, the curvature and other metric-dependent quantities will be understood to be with respect to g ; all quantities with a subscript 0 are with respect to a fixed background metric g_0 which we may take to be the initial metric.

Lemma 4.8. *Let u be a solution to (4.10). Then*

$$\frac{\partial}{\partial t} K = -\frac{\bar{K}}{K^2} \Delta K + 2\frac{\bar{K}}{K^3} |\nabla K|^2 + 2(\bar{K} - K).$$

Proof. We directly compute

$$\begin{aligned} \frac{\partial}{\partial t} K &= \frac{\partial}{\partial t} e^{-2u} [-\Delta_{g_0} u + K_0] \\ &= -2\dot{u}K - e^{-2u} \Delta_{g_0} \dot{u} \\ &= -2K \left[\frac{K - \bar{K}}{K} \right] - \Delta_{g_u} \left[\frac{K - \bar{K}}{K} \right] \\ &= \bar{K} \Delta \frac{1}{K} + 2(\bar{K} - K) \\ &= -\frac{\bar{K}}{K^2} \Delta K + 2\frac{\bar{K}}{K^3} |\nabla K|^2 + 2(\bar{K} - K). \end{aligned}$$

□

Proposition 4.9. *Let u be a solution to (4.10). Then for $0 \leq t < T$,*

$$\inf_M K_0 \leq K \leq \sup_M K_0.$$

Proof. We apply the maximum principle to the result of Lemma 4.8. At a minimum point, certainly $-\frac{\bar{K}}{K^2}\Delta K \geq 0$ (since $\bar{K} < 0$), $\nabla K = 0$, and $K < \bar{K}$. It follows that the minimum of K is nondecreasing along the flow. A similar argument shows that the maximum is nonincreasing. □

Proposition 4.10. *Let u be a solution to (4.10). There exists a constant $C = C(M, u_0)$ such that for all $0 \leq t < T$,*

$$|u| \leq C.$$

Proof. We apply the maximum principle directly to (4.10). At a spacetime minimum point for u , we have $\Delta_{g_0} u \geq 0$, and hence

$$\begin{aligned} \frac{\partial}{\partial t} u &= 1 - \frac{\bar{K}}{K} \\ &= 1 - \frac{\bar{K}}{e^{-2u}(-\Delta_0 u + K_0)} \\ &\geq 1 - \frac{\bar{K}}{e^{-2u}K_0} \\ &= 1 - \frac{\bar{K}}{K_0} e^{2u} \\ &\geq 0, \end{aligned}$$

if $u(x, t) \leq \frac{1}{2} \log \frac{\sup K_0}{\bar{K}}$. Similarly, at a spacetime maximum one has $\Delta_{g_0} u \leq 0$ and hence

$$\begin{aligned} \frac{\partial}{\partial t} u &= 1 - \frac{\bar{K}}{K} \\ &\leq 1 - \frac{\bar{K}}{e^{-2u}K_0} \\ &= 1 - \frac{\bar{K}}{K_0} e^{2u} \\ &\leq 0, \end{aligned}$$

if $u(x, t) \geq \frac{1}{2} \log \frac{\inf K_0}{\bar{K}}$. The result follows. □

Next, we use the *a priori* estimates of Propositions 4.9 and 4.10 to prove higher order estimates. In what follows we will assume there is a constant Λ such that

$$-\Lambda \leq K \leq -\Lambda^{-1}, \quad |u| \leq \Lambda. \quad (4.11)$$

Furthermore, given a smooth solution to (4.10) satisfying these bounds, we adopt the notation that X_i refers to any quantity which is uniformly controlled along the flow in terms of Λ .

We begin with a gradient estimate for solutions:

Lemma 4.11. *Let u be a solution to (4.10) satisfying (4.11). Then*

$$\frac{\partial}{\partial t} |\nabla_0 u|^2 \leq -\frac{\bar{K}}{K^2} \Delta_0 |\nabla u|_0^2 + \frac{2\bar{K}}{K^2} e^{-2u} |\nabla_0^2 u|_0^2 + C_1 (|\nabla u|_0^2 + 1),$$

where C_1 depends on the initial data and Λ .

Proof. We first compute that

$$\begin{aligned}
 \frac{\partial}{\partial t} \nabla_0 u &= \nabla_0 \frac{\partial}{\partial t} u \\
 &= -\bar{K} \nabla_0 K^{-1} \\
 &= \frac{\bar{K}}{K^2} \nabla_0 \left[e^{-2u} (-\Delta_0 u + K_0) \right] \\
 &= \frac{\bar{K}}{K^2} e^{-2u} [-\nabla_0 \Delta_0 u] - 2 \frac{\bar{K}}{K} \nabla_0 u + \frac{\bar{K}}{K^2} e^{-2u} \nabla_0 K_0 \\
 &= -\frac{\bar{K}}{K^2} \Delta \nabla_0 u + X_1 * \nabla_0 u + X_2.
 \end{aligned}$$

□

Lemma 4.12. *Let u be a solution to (4.10) satisfying (4.11). Given Z a smooth vector field on M , one has*

$$\frac{\partial}{\partial t} \nabla_Z^0 \nabla_Z^0 u \leq -\frac{\bar{K}}{K^2} \Delta_0 \nabla_Z^0 \nabla_Z^0 u + \nabla_0 Z * \nabla_0^3 u + X_1 * \nabla^0 \nabla^0 u + X_2 * (\nabla u)^{*2} + X_3.$$

Proof. First we compute

$$\begin{aligned}
 \frac{\partial}{\partial t} \nabla_Z^0 \nabla_Z^0 u &= \nabla_Z^0 \nabla_Z^0 \left[\frac{K - \bar{K}}{K} \right] \\
 &= -\bar{K} \nabla_Z^0 \nabla_Z^0 K^{-1} \\
 &= \frac{\bar{K}}{K^2} \nabla_Z^0 \nabla_Z^0 K - \frac{2\bar{K}}{K^3} \nabla_Z K \otimes \nabla_Z K \\
 &= \frac{\bar{K}}{K^2} \nabla_Z^0 \nabla_Z^0 \left[e^{-2u} (-\Delta_0 u + K_0) \right] - \frac{2\bar{K}}{K^3} \nabla_Z K \otimes \nabla_Z K \\
 &= \frac{\bar{K}}{K^2} \left[\left(\nabla_Z^0 \nabla_Z^0 e^{-2u} \right) e^{2u} K - 2 \nabla_Z^0 e^{-2u} \nabla_Z (e^{2u} K) + e^{-2u} \nabla_Z^0 \nabla_Z^0 (-\Delta_0 u + K_0) \right] \\
 &\quad - \frac{2\bar{K}}{K^3} \nabla_Z K \otimes \nabla_Z K. \\
 &\leq -\frac{\bar{K}}{K^2} e^{-2u} \nabla_Z^0 \nabla_Z^0 \Delta_0 u + X_1 * \nabla_Z^0 \nabla_Z^0 u + X_2 * (\nabla u)^{*2} + X_3,
 \end{aligned}$$

where we applied the Cauchy-Schwarz inequality in the final line. Next, we commute derivatives to yield

$$\begin{aligned}
 \nabla_Z^0 \nabla_Z^0 \Delta_0 u &= \nabla_Z^0 \nabla_Z^0 \nabla_{e_i}^0 \nabla_{e_i}^0 u \\
 &= \nabla^4 u(Z, Z, e_i, e_i) \\
 &= \nabla^4 u(e_i, e_i, Z, Z) + K_0 * \nabla_0^2 u + \nabla K_0 * \nabla u \\
 &= \Delta_0 \left(\nabla_0^2 u(Z, Z) \right) + \nabla_0 Z * \nabla_0^3 u + \nabla_0^2 u * \left(\nabla_0^2 Z + (\nabla_0 Z)^{*2} \right) \\
 &\quad + K_0 * \nabla_0^2 u + \nabla K_0 * \nabla u.
 \end{aligned}$$

□

Proposition 4.13. *Let u be a solution to (4.10) satisfying (4.11) on $[0, T)$. There exists a constant $C = C(\Lambda, u_0)$ such that*

$$\sup_{M \times [0, T)} |\nabla_0^2 u| \leq C.$$

Proof. Fix some constant A and consider the function

$$\beta(p) = \max_{X \in T_p M \setminus \{0\}} \frac{\nabla_0^2 u(X, X)}{|X|^2}$$

This is continuous, and an upper bound for β implies an upper bound for the Hessian of u . Fix a constant A and consider the function

$$\Phi(x, t) = t\beta + A |\nabla u|_0^2.$$

We claim that for A chosen sufficiently large that there is an a priori bound for an interior maximum on $[0, 1]$. Suppose some point (x, t) is such an interior spacetime maximum for Φ . Fix a unit vector $Z \in T_x M$ realizing the supremum in the definition of $\beta(x)$. We may extend Z in a neighborhood of x by parallel transport along radial geodesics. This yields

$$\begin{aligned} |Z| &= 1 \text{ on } B_\epsilon(x) \\ \nabla_X^0 Z(x) &= 0 \text{ for all } X \\ \left| \nabla_0^2 Z \right|(x) &\leq C(g_0). \end{aligned}$$

We may extend Z to all of M by multiplying by a cutoff function. We observe that the function

$$\Psi(x, t) = t \nabla_0^2(Z, Z) + A |\nabla u|_0^2$$

also has a spacetime maximum at (x, t) . Combining Lemmas 4.11 and 4.12 yields that, at x ,

$$\begin{aligned} 0 &\leq \frac{\partial}{\partial t} \Psi + \frac{\bar{K}}{K^2} \Delta \Psi \\ &\leq \nabla_0 Z * \nabla_0^3 u + X_1 * \nabla_0 \nabla_0 u + X_2 * (\nabla_0 u)^{*2} + X_3 \\ &\quad + A \left[\frac{2\bar{K}}{K^2} e^{-2u} \left| \nabla_0^2 u \right|_0^2 + C |\nabla_0 u|_0^2 + C \right] \\ &\leq -\delta \left| \nabla_0^2 u \right|_0^2 + C, \end{aligned}$$

where the last line follows by choosing A sufficiently large with respect to Λ and using the a priori gradient bound. This implies an a priori upper bound for the Hessian of u at (x, t) , and hence since $t \leq 1$ an a priori bound for Ψ at (x, t) . This yields an a priori upper bound for $\nabla_Z^0 \nabla_Z^0 u$ for any interior time $t > \epsilon$. Combined with some ineffective estimate depending on the given solution for $[0, \epsilon)$ yields an a priori upper bound on $[0, T)$. Since the Laplacian of u is uniformly bounded below, this then yields the full a priori Hessian bound. $\square$

Proposition 4.14. *Let (M^2, g_0) be a compact Riemann surface such that $K_0 < 0$. Given $u \in \Gamma_1^-$, the solution to (4.10) exists for all time and converges exponentially in C^∞ to a metric of constant negative scalar curvature.*

Proof. Propositions 4.9 and 4.10 guarantee uniform estimates on u , K and $\frac{1}{K}$. Proposition 4.13 then implies a uniform estimate for the Hessian of u . We now observe that the operator $\Phi(u) = \frac{\bar{K}-K}{K}$ is convex, and with the uniform upper and lower bounds on curvature, is uniformly elliptic. Thus the Evans-Krylov theorem [18], [23] yields an a priori $C^{2,\alpha}$ estimate for u . Schauder estimates can then be applied to obtain estimates of every $C^{k,\alpha}$ norm of u . It follows that the solution exists on $[0, \infty)$ and every sequence of times approaching infinity admits a subsequence such that $\{u_{t_i}\}$ converges to a limiting function u_∞ . Using that the flow is the gradient flow for F , which is bounded below, it follows that the limiting metric u_∞ is a critical point for F , and hence has constant curvature, and since this metric is unique by the maximum principle, the whole flow converges to u_∞ . $\square$

4.3 The sphere

We now consider the case of $K_0 > 0$, and so $M \cong S^2$. In this case we are studying the flow

$$\frac{\partial}{\partial t} u = \frac{\bar{K} - K}{K}. \quad (4.12)$$

4.3.1 Evolution Equations

To begin we build up some evolution equations. First we rewrite the evolution of u in terms of the linearized operator.

Lemma 4.15. *Given u a solution to (4.12) we have*

$$\frac{\partial}{\partial t} u = \frac{\bar{K}}{K^2} \Delta u - 1 + \frac{2\bar{K}}{K} - \frac{e^{-2u} K_0 \bar{K}}{K^2}.$$

Proof. To begin we compute

$$-\frac{\bar{K}}{K^2} \Delta u = \frac{\bar{K}}{K^2} \left[K - e^{-2u} K_0 \right] = \frac{\bar{K}}{K} - \frac{e^{-2u} K_0 \bar{K}}{K^2}.$$

Hence using (4.12) we have

$$\begin{aligned} \frac{\partial}{\partial t} u - \frac{\bar{K}}{K^2} \Delta u &= \frac{\bar{K} - K}{K} + \frac{\bar{K}}{K} - \frac{e^{-2u} K_0 \bar{K}}{K^2} \\ &= -1 + \frac{2\bar{K}}{K} - \frac{e^{-2u} K_0 \bar{K}}{K^2}, \end{aligned}$$

as required. $\square$

Lemma 4.16. *Let u be a solution to (4.12). Then*

$$\frac{\partial}{\partial t} K = \frac{\bar{K}}{K^2} \Delta K - 2 \frac{\bar{K} |\nabla K|^2}{K^3} + 2(K - \bar{K}). \quad (4.13)$$

Proof. We directly compute

$$\begin{aligned} \frac{\partial}{\partial t} K &= \frac{\partial}{\partial t} e^{-2u} [-\Delta_{g_0} u + K_0] \\ &= -2\dot{u}K - e^{-2u} \Delta_{g_0} \dot{u} \\ &= -2K \left[\frac{\bar{K} - K}{K} \right] - \bar{K} \Delta \frac{1}{K} \\ &= \frac{\bar{K}}{K^2} \Delta K - 2 \frac{\bar{K} |\nabla K|^2}{K^3} + 2(K - \bar{K}), \end{aligned}$$

as required. $\square$

Lemma 4.17. *Let u be a solution to (4.12). Then*

$$\frac{\partial}{\partial t} K^{-1} = \frac{\bar{K}}{K^2} \Delta K^{-1} - 2K^{-1} + 2 \frac{\bar{K}}{K^2}.$$

Proof. We compute

$$\begin{aligned} \frac{\partial}{\partial t} K^{-1} &= -\frac{\frac{\partial}{\partial t} K}{K^2} \\ &= \frac{1}{K^2} \left[2u_t K + e^{-2u} \Delta_{g_0} u_t \right] \\ &= \frac{\bar{K}}{K^2} \Delta K^{-1} - 2K^{-1} + 2 \frac{\bar{K}}{K^2}, \end{aligned}$$

as required. $\square$

Lemma 4.18. *Let u be a solution to (4.12). Then*

$$A(t) = A(0) e^{2(F[u_0] - F[u_t])}.$$

Proof. We observe

$$\begin{aligned}\frac{d}{dt}A &= \frac{\partial}{\partial t} \int_M e^{2u} dA_0 \\ &= 2 \int_M \frac{\bar{K} - K}{K} dA_u \\ &= A \left[-2 + \frac{\bar{K}}{A^2} \int_M \frac{1}{K} dA_u \right] \\ &= -2A \frac{dF}{dt}.\end{aligned}$$

Integrating this ODE we conclude the result. $\square$

Lemma 4.19. *Let u be a solution to (4.12). Then*

$$\int_M (|\nabla u|^2 + 2K_0 u) dA_0 = \int_M (|\nabla u_0|^2 + 2K_0 u_0) dA_0.$$

Proof. One observes that

$$\int_M (|\nabla u|^2 + 2K_0 u) dA_0 = F[u] + \log A,$$

and the result follows directly from the calculation of Lemma 4.18. $\square$

4.3.2 A priori estimates

Proposition 4.20. *Let u be a solution to (4.12). For all smooth existence times t of the flow one has*

$$\sup_{M \times \{t\}} K \leq \sup_M K_0 e^{2t}.$$

Proof. This follows directly from the maximum principle applied to (4.13). $\square$

Proposition 4.21. *Let u be a solution to (4.12). For all smooth existence times t of the flow one has*

$$\inf_{M \times \{t\}} u \geq \inf_M u_0 - t.$$

Proof. We observe that at a spacetime minimum for u , one has $\Delta_{g_0} u \geq 0$, and hence

$$\begin{aligned}\frac{\partial}{\partial t} u &= -1 + \frac{\bar{K}}{K} \\ &\geq -1 + \frac{\bar{K} e^{2u}}{K_0} \\ &\geq -1.\end{aligned}$$

The result follows from the maximum principle. $\square$

Proposition 4.22. *Let u be a solution to (4.12). There exists a constant $C = C(u_0)$ such that for all smooth existence times t of the flow one has*

$$\|\nabla u(t)\|_{L^2}^2 \leq C[1+t]$$

Proof. We use Lemma 4.19 with the estimate of Proposition 4.21 to yield

$$\begin{aligned}\|\nabla u\|_{L^2}^2 &= \int_M (|\nabla u|^2 + 2K_0 u - 2K_0 u) dA_0 \\ &= \int_M (|\nabla u_0|^2 + 2K_0 u_0) dA_0 - 2K_0 \int_M u dA_0 \\ &\leq C - C \inf u \\ &\leq C(1+t).\end{aligned}$$

$\square$

Proposition 4.23. *Let u be a solution to (4.12) on $[0, T)$. There exists a constant $C = C(u_0, T)$ such one has*

$$\sup_{M \times [0, T)} |u| \leq C.$$

Proof. First from Proposition 4.22 there is a time-dependent bound on $\|\nabla u\|_{L^2}$. By the Moser-Trudinger inequality we obtain a uniform estimate of $\int_M e^{4|u|} dA_0$. We now claim that there is a uniform constant $R > 0$ so that

$$\sup_{x \in S^2, t \in [0, T)} \int_{B_R(x_0)} |K_t| dA_t < 2\pi.$$

Using Proposition 4.20 we estimate

$$\begin{aligned} \int_{B_R(x_0)} |K_t| dA_t &= \int_{B_R(x_0)} K_t e^{2u_t} dA_0 \\ &\leq C \int_{B_R(x_0)} e^{2u_t} dA_0 \\ &\leq C \left[\int_{B_R(x_0)} e^{4|u_t|} dA_0 \right]^{\frac{1}{2}} \left[\int_{B_R(x_0)} dA_0 \right]^{\frac{1}{2}} \\ &\leq CR \\ &< 2\pi, \end{aligned}$$

where the last inequality follows by choosing R small with respect to C . Invoking [36] Theorem 3.2 we conclude a uniform H^2_2 bound for u , which by the Sobolev inequality implies the uniform bound for u . $\square$

Proposition 4.24. *Let u be a solution to (4.12) on $[0, T)$ such that $|u| \leq \Lambda$. There exists a constant $C = C(T, u_0)$ such that*

$$K^{-1} \leq C.$$

Proof. Let $\Phi = tK^{-1} + Au$, where $A > 0$ is a constant yet to be determined. Using Lemmas 4.15, 4.17 and Proposition 4.23 we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \frac{\bar{K}}{K^2} \Delta \right) \Phi &= -2K^{-1} + 2\bar{K}K^{-2} + A \left[-1 + 2\bar{K}K^{-1} - e^{-2u} K_0 \bar{K}K^{-2} \right] \\ &\leq K^{-2} [C + CAK - \delta A]. \end{aligned}$$

where the constant $\delta > 0$ is determined by the upper bound for u . If we choose A sufficiently large with respect to δ , then at a sufficiently large maximum for K^{-1} we obtain

$$C + CAK - \delta A \leq C - \frac{\delta}{2} A \leq 0.$$

The result follows from the maximum principle. $\square$

4.3.3 Main proofs

Proposition 4.25. *Let (M^2, g_0) be a compact Riemann surface such that $K_0 > 0$. Given $u \in C^\infty(M)$, the solution to (4.12) exists for all time.*

Proof. Combining Propositions 4.23, 4.20, and 4.24, we obtain uniform estimates on u , K , and K^{-1} for any finite existence time T . Given these the higher order estimates follow as in the proof of Proposition 4.13 and Theorem 4.14, and so the long time existence follows. $\square$

Proposition 4.26. *Given (S^2, g_{S^2}) , and $u \in \Gamma_1^+$, the solution to (4.12) exists for all time and converges weakly in H^2_1 to a smooth metric of constant curvature.*

Proof. The global existence of the solution follows from Proposition 4.25. As the metric g_{S^2} , corresponding to $u \equiv 0$, is a fixed point of (4.12), it follows from Proposition 4.6 that the distance between $u(\cdot, t)$ and 0 is nonincreasing, and so uniformly bounded for all t . It follows from ([13] Theorem 3) that there is a uniform H_1^2 estimate for $u(\cdot, t)$.

Since the flow is the gradient flow for the energy functional F , which is bounded below, we have the estimate

$$\int_0^\infty \int_M \frac{(K - \bar{K})^2}{K} dA dt \leq C.$$

Therefore, for a sequence of times $t_i \rightarrow \infty$, the metrics $g_i = e^{2u(\cdot, t_i)} g_0 = e^{2u_i} g_0$ satisfy

$$\int_M \frac{(K_i - \bar{K}_i)^2}{K_i} dA_i \rightarrow 0. \quad (4.14)$$

We may also assume using standard compactness results and Trudinger's inequality that

$$\begin{aligned} u_i &\rightarrow u_\infty \text{ in } H^{1,2}, \\ u_i &\rightarrow u_\infty \text{ in } L^2, \\ e^{u_i} &\rightarrow e^{u_\infty} \text{ in } L^p, \end{aligned} \quad (4.15)$$

for any $p \geq 1$. By extracting a subsequence if necessary, we may also assume

$$\bar{K}_i \rightarrow \lambda > 0.$$

We claim that u_∞ is a weak solution of

$$\Delta_0 u_\infty + \lambda e^{2u_\infty} = K_0,$$

which is smooth by elliptic regularity using that e^{u_∞} is in L^p , and hence defines a constant curvature metric. To establish this fix a test function ϕ and estimate

$$\begin{aligned} \int \phi(\Delta_0 u_i + \bar{K}_i e^{2u_i} - K_0) dA_0 &= \int \phi(-K_i + \bar{K}_i) dA \\ &\leq \left(\int_M \frac{(K_i - \bar{K}_i)^2}{K_i} dA_i \right)^{1/2} \left(\int_M \phi K_i dA_i \right)^{1/2} \\ &= \left(\int_M \frac{(K_i - \bar{K}_i)^2}{K_i} dA_i \right)^{1/2} \left(\int_M \phi(-\Delta_0 u_i + K_0) dA_0 \right)^{1/2} \\ &\leq \left(\int_M \frac{(K_i - \bar{K}_i)^2}{K_i} dA_i \right)^{1/2} \left(\int_M (-\langle \nabla_0 \phi, \nabla_0 u_i \rangle + K_0 \phi) dA_0 \right)^{1/2} \\ &\leq C \left(\int_M \frac{(K_i - \bar{K}_i)^2}{K_i} dA_i \right)^{1/2}, \end{aligned}$$

where $C = C(\|\phi\|_{H^{1,2}}, \|\nabla_0 u\|_{L^2})$. It follows from (4.14) that

$$\int (-\langle \nabla_0 \phi, \nabla_0 u_i \rangle + \bar{K}_i \phi e^{2u_i} - K_0 \phi) dA_0 = \int \phi(\Delta_0 u_i + \bar{K}_i e^{2u_i} - K_0) dA_0 \rightarrow 0 \quad (4.16)$$

as $i \rightarrow \infty$. By (4.15), this implies

$$\int (-\langle \nabla_0 \phi, \nabla_0 u_\infty \rangle + \lambda \phi e^{2u_\infty} - K_0 \phi) dA_0 = 0, \quad (4.17)$$

as required. $\square$

Proof of Theorem 1.1. The required results follow from Propositions 4.4, 4.6, 4.7, 4.14, 4.25 and 4.26. $\square$

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References

- [1] B. Andrews, unpublished.
- [2] M. Bauer, P. Harms, P. Michor, *Sobolev metrics on the manifold of all Riemannian metrics*, J. Diff. Geom. 94, 2 (2013), 187-208.
- [3] B. Berndtsson, *A Brunn-Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry*, Invent. Math. 200 (2015), 149-200.
- [4] Z. Blocki, *On geodesics in the space of Kähler metrics*, Advanced Lectures in Mathematics 21, p. 3-20, International Press, 2012.
- [5] E. Calabi, X.X. Chen, *The space of Kähler metrics II*, J. Diff. Geom. 61 (2002), 173-193.
- [6] X.X. Chen, *The space of Kähler metrics*, J. Diff. Geom. 56 (2000), 189-234.
- [7] X.X. Chen, *The space of Kähler metrics III - On the lower bound of the Calabi energy and geodesic distance*, Invent. Math. 175 (2009) 453-503.
- [8] X.X. Chen, Weiyong He, *The space of volume forms*, Int. Math. Res. Not. IMRN (2011), no. 5, 967-1009.
- [9] X.X. Chen, G. Tian, *Geometry of Kähler metrics and foliations by holomorphic discs* Publ. Math. de L'IHES, Vol. 107, No. 1, 1-107.
- [10] B. Chow, *On the entropy estimate for the Ricci flow on compact 2-orbifolds*, J. Differential Geom., 33 (1991), no. 2, 597-600.
- [11] B. Chow, P. Lu, L. Ni, *Hamilton's Ricci flow*, Lectures in Contemporary Mathematics, Science Press, Beijing.
- [12] T. Darvas, private communication
- [13] T. Darvas, *The Mabuchi geometry of finite energy classes*, Adv. Math. 285 (2015), 182-219.
- [14] T. Darvas, L. Lempert, *Weak geodesics in the space of Kähler metrics*, Math. Res. Lett. 19 (2012), no. 5.
- [15] W. Ding, *Remarks on the existence problem of positive Kähler-Einstein metrics*, Math. Ann. 282, 463-471 (1988).
- [16] S. K. Donaldson, *Symmetric spaces, Kähler geometry and Hamiltonian dynamics*, in Northern California Symplectic Geometry Seminar, Amer. Math. Soc. Transl. Ser. 2, 196, Amer. Math. Soc., Providence, 1999, 13-33.
- [17] S. K. Donaldson, *Nahm's equations and free-boundary problems. The many facets of geometry*, 71-91, Oxford Univ. Press, Oxford, 2010.
- [18] L. C. Evans, *Classical solutions of fully nonlinear, convex, second-order elliptic equations*, Comm. Pure Appl. Math., 35 (1982), 333-363.
- [19] D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order*, Springer, 1998.
- [20] V. Guedj, *The metric completion of the Riemannian space of Kähler metrics*, arXiv:1401.7857.
- [21] M. J. Gursky, J. Streets, *A formal Riemannian structure on conformal classes and the σ_2 -Yamabe problem*, Geom. Topol. 22 (2018), no. 6, 3501-3573.
- [22] M. J. Gursky, J. Streets, *Variational structure of the $v_{n/2}$ -Yamabe problem*, Differential Geom. Appl. 56 (2018), 187-201.
- [23] N. V. Krylov, *Boundedly inhomogeneous elliptic and parabolic equations in a domain*, Izv. Akad. Nauk SSSR Ser. Mat. 47 (1983), 75-108.
- [24] J. Lee, T. Parker, *The Yamabe problem*, Bull. AMS. Vol 17, No. 1, (1987).
- [25] T. Mabuchi, *Some symplectic geometry on compact Kähler manifolds (I)*, Osaka J. Math 24 (1987), 227-252.
- [26] U. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic maps*, Comm. Anal. Geom., Vol. 6, No. 2, 199-253, 1998.
- [27] B. Osgood, R. Phillips, P. Sarnak, *Extremals of determinants of Laplacians*, J. Func. Anal. 80, (1988), 148-211.
- [28] D. Phong, J. Sturm, *Test configurations for K-stability and geodesic rays*, J. Symp. Geom. Vol. 5, No. 2 (2007), 221-247.
- [29] A. M. Polyakov *Quantum geometry of bosonic strings*, Phys. Lett. B 103 (1981), 207-210.
- [30] D. B. Ray, I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Math. 7 (1971), 145-210.
- [31] J. Ross, D. Witt Nyström, *Analytic test configurations and geodesic rays*, J. Symp. Geom. Vol. 12, No. 1 (2014), 125-169.
- [32] Y. A. Rubinstein, *On energy functionals, Kähler-Einstein metrics, and the Moser-Trudinger-Onofri neighborhood*, J. Funct. Anal. 255 (2008), no. 9, 2641-2660.
- [33] S. Semmes, *Complex Monge-Ampère equations and symplectic manifolds*, Amer. J. Math. 114 (1992), 495-550.
- [34] J. Streets, *Long time existence of minimizing movement solutions of Calabi flow*, Adv. Math., Vol. 259, 2014, 688-729.
- [35] J. Streets, *Consistency and convergence of K-energy minimizing movements*, Trans. Amer. Math. Soc. 368 (2016), no. 7, 5075-5091.
- [36] M. Struwe, *Curvature flows on surfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 1 (2002), no. 2, 247-274.
- [37] Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc 117 (1965), 251-275.